

Independent domination of graphs with bounded maximum degree

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Abstract

An independent dominating set of a graph, also known as a maximal independent set, is a set S of pairwise non-adjacent vertices such that every vertex not in S is adjacent to some vertex in S . We prove that for $\Delta = 4$ or $\Delta \geq 6$, every connected n -vertex graph of maximum degree at most Δ has an independent dominating set of size at most $(1 - \frac{\Delta}{\lfloor \Delta^2/4 \rfloor + \Delta})(n - 1) + 1$. In addition, we characterize all connected graphs having the equality and we show that other connected graphs have an independent dominating set of size at most $(1 - \frac{\Delta}{\lfloor \Delta^2/4 \rfloor + \Delta})n$.

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1 Introduction

All graphs in this paper are simple, meaning that they have no loops and no parallel edges. For a graph G , we write $V(G)$ for its vertex set and $E(G)$ for its edge set. For $v \in V(G)$, we use $d_G(v)$ to denote the degree of v in G . For $v \in V(G)$, we use $N_G(v)$ to denote the set of neighbors of v in G , and $N_G[v] = N_G(v) \cup \{v\}$. For $X \in V(G)$, we use $G - X$ to denote the subgraph of G induced by $V(G) \setminus X$. We write $\Delta(G)$ to denote the maximum degree of vertices in a graph G . A vertex of degree 0 is called *isolated*.

An *independent dominating set* of a graph is a set S of vertices that are pairwise non-adjacent, which is called *independent*, and every vertex not in S has a neighbor in S , which is called *dominating*. Independent dominating sets appear as early as 1962 in Berge [2, Chapter 5] and Ore [7, Chapter 13] under the names *kernels* and maximal independent sets, respectively. It is easy to see that independent dominating sets are precisely maximal independent sets, which is stated in both Berge [2, Chapter 5] and Ore [7, Chapter 13]. We write $i(G)$ to denote the size of the minimum independent dominating set in a graph G . Haynes, Hedetniemi, and Slater discussed the relation between the domination parameters and maximum degree in their book [6, Subsection 9.3.2].

We are interested in bounding $i(G)$ in terms of the number of vertices in a graph G of maximum degree at most Δ . It is trivial to see the inequality

$$i(G) \leq |V(G)| - \Delta$$

because a maximal independent set containing a vertex of maximum degree has at most $|V(G)| - \Delta$ vertices. Domke, Dunbar, and Markus [5] showed the above inequality and determined connected bipartite graphs and trees having the equality. As a refinement of the trivial inequality, Blidia, Chelali, and Maffray [3] showed that $i(G) \leq |V(G)| - \Delta'(G)$, where $\Delta'(G) = \max_{v \in V(G)} (d_G(v) + \nu(G - N_G[v]))$ and $\nu(G - N_G[v])$ is the size of a largest matching in $G - N_G[v]$.

Akbari et al. [1] showed that $i(G) \leq \frac{1}{2}|V(G)|$ for graphs G of maximum degree at most 3 with no isolated vertex and characterized all graphs achieving the equality. Cho, Choi, and Park [4, Theorem 1.5] proved that $i(G) \leq \frac{5}{9}|V(G)|$ for graphs G of maximum degree at most 4 with no isolated vertices.

Our main theorem provides a tight upper bound of $i(G)$ in terms of $|V(G)|$ for graphs G of maximum degree at most Δ .

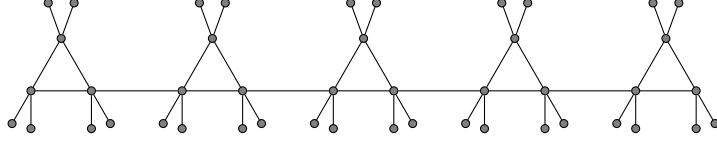


Figure 1: A connected graph G satisfying $i(G) = \frac{5}{9}|V(G)|$ for $\Delta = 5$.

Theorem 1.1. *Let $\Delta \geq 4$ be an integer and let G be a connected graph of maximum degree at most Δ .*

- *If $\Delta \neq 5$, then G has an independent dominating set of size at most $(1 - \frac{\Delta}{\lfloor \Delta^2/4 \rfloor + \Delta})(|V(G)| - 1) + 1$.*
- *If $\Delta = 5$, then G has an independent dominating set of size at most $\max\left(\frac{5}{9}|V(G)|, (1 - \frac{\Delta}{\lfloor \Delta^2/4 \rfloor + \Delta})(|V(G)| - 1) + 1\right)$.*

In addition, we characterize all connected graphs achieving the equality of Theorem 1.1 when $\Delta \neq 5$. They will be called Δ -special graphs. For an integer $\Delta \geq 4$, a graph is Δ -special if it is isomorphic to one of the following.

- (a) K_1 .
- (b) A graph obtained from $K_{\lfloor \Delta/2 \rfloor + 1}$ or $K_{\lfloor \Delta/2 \rfloor + 1}$ by attaching degree-1 vertices so that each vertex in the initial clique has degree Δ .
- (c) A graph obtained from an odd cycle by attaching two vertices of degree 1 at each vertex when $\Delta = 4$.
- (d) A graph obtained from the union of two Δ -special graphs G_1 and G_2 with $|V(G_1) \cap V(G_2)| = 1$ where the vertex in the intersection does not belong to any cycles in G_1 or G_2 .

A precise and equivalent definition will be discussed in Section 3.

Now we present a stronger theorem, implying Theorem 1.1.

Theorem 1.2. *Let $\Delta \geq 4$ be an integer and let G be a connected graph of maximum degree at most Δ .*

- (i) *If G is Δ -special, then $i(G) = \left(1 - \frac{\Delta}{\lfloor \Delta^2/4 \rfloor + \Delta}\right) |V(G)| + \frac{\Delta}{\lfloor \Delta^2/4 \rfloor + \Delta}$.*

(ii) If G is not Δ -special and $\Delta \neq 5$, then $i(G) \leq \left(1 - \frac{\Delta}{\lfloor \Delta^2/4 \rfloor + \Delta}\right) |V(G)|$.

(iii) If G is not Δ -special and $\Delta = 5$, then $i(G) \leq \frac{5}{9} |V(G)|$.

There is an infinite family of non-special connected graphs achieving the bound of Theorem 1.2. When $\Delta \neq 5$, let G be a graph obtained from any non-trivial Δ -special graph by removing one vertex of degree 1. It is straightforward to verify that $i(G) = \left(1 - \frac{\Delta}{\lfloor \Delta^2/4 \rfloor + \Delta}\right) |V(G)|$.

For $\Delta = 5$, let G be a graph obtained from a path $v_1 v_2 \dots v_{3k}$ on $3k$ vertices by adding k edges $v_{3i-2} v_{3i}$ for all $i = 1, 2, \dots, k$ and attaching 2 vertices of degree 1 to each vertex on the path, see Figure 1. Let S be a set consisting of v_3, v_6, \dots, v_{3k} and all vertices of degree 1 adjacent to v_i for some $i \not\equiv 0 \pmod{3}$. Then it is easy to verify that S is a minimum independent dominating set of G and $i(G) = |S| = 5k$ and therefore $i(G) = \frac{5}{9} |V(G)|$.

Theorem 1.2 proves the following corollary, which was conjectured by Cho, Choi, and Park [4]. They claimed to have verified the conjecture for $\Delta \leq 8$ and presented its proof for $\Delta = 4$. Let $H(p, q)$ be the graph obtained by attaching q vertices of degree 1 to every vertex of K_p .

Corollary 1.3. *Let Δ be a positive integer. Every graph G with maximum degree at most Δ and no isolated vertices has an independent dominating set having at most $(1 - \frac{\Delta}{\lfloor (\Delta+2)^2/4 \rfloor}) |V(G)|$ vertices. Furthermore, when $\Delta \geq 4$, the equality holds if and only if every component is isomorphic to $H(\lceil \Delta/2 \rceil + 1, \lfloor \Delta/2 \rfloor)$ or $H(\lfloor \Delta/2 \rfloor + 1, \lceil \Delta/2 \rceil)$.*

Proof of Corollary 1.3 assuming Theorem 1.2. Let $n = |V(G)|$. It is trivial if $\Delta \leq 2$. We may assume that G is connected. Akbari et al. [1, Theorem 3] showed that $i(G) \leq n/2$ if $\Delta = 3$. So we may assume that $\Delta \geq 4$. We may assume that G is Δ -special by (ii) and (iii) of Theorem 1.2, because $\frac{5}{9} < \frac{7}{12}$ and $\frac{\Delta}{\lfloor \Delta^2/4 \rfloor + \Delta} > \frac{\Delta}{\lfloor (\Delta+2)^2/4 \rfloor}$. Since G has no isolated vertex, $n \geq (\lfloor \Delta/2 \rfloor + 1)(\Delta - \lfloor \Delta/2 \rfloor + 1)$ or $n \geq (\lceil \Delta/2 \rceil + 1)(\Delta - \lceil \Delta/2 \rceil + 1)$. In both inequalities, the right-hand side is equal and we have

$$n \geq \lfloor (\Delta + 2)^2/4 \rfloor.$$

This is equivalent to $\left(1 - \frac{\Delta}{\lfloor \Delta^2/4 \rfloor + \Delta}\right) n + \frac{\Delta}{\lfloor \Delta^2/4 \rfloor + \Delta} \leq \left(1 - \frac{\Delta}{\lfloor (\Delta+2)^2/4 \rfloor}\right) n$, proving the inequality. Furthermore the equality holds if and only if G is Δ -special and $n = \lfloor (\Delta+2)^2/4 \rfloor$. Indeed, $H(\lceil \Delta/2 \rceil + 1, \lfloor \Delta/2 \rfloor)$ and $H(\lfloor \Delta/2 \rfloor + 1, \lceil \Delta/2 \rceil)$ are the only such graphs. This completes the proof. \square

Our proof is by induction on $|V(G)| + |E(G)|$. Here are two key inequalities on $i(G)$ for our proof.

- (1) $i(G) \leq i(G - N_G[v]) + 1$ for a vertex v of a graph G .
- (2) $i(G) \leq i(H)$ if H is obtained from a graph G by deleting all but one of the edges incident with a fixed vertex of degree at least 2.

The inequality (1) is commonly used when dealing with the independent domination, but the authors could not find papers using or mentioning (2). Perhaps it is because it is difficult to control the change of $i(G)$ after deleting edges.

The paper is organized as follows. Section 2 presents several useful lemmas, including the proof of (2). Section 3 discusses Δ -special graphs. Section 4 presents the proof of the main theorem.

2 Several lemmas

We write $n_0(G)$ to denote the number of isolated vertices of a graph G . We start with a few useful lemmas.

Lemma 2.1. *Let H be a graph and w be a vertex with degree at least 2. Let u_1, u_2, \dots, u_k be the list of all neighbors of w . Then $H' = H - \{wu_i : i \in \{1, 2, \dots, k-1\}\}$ satisfies $i(H) \leq i(H')$.*

Proof. Let S be a minimum independent dominating set of H' . Then S is a dominating set of H . If S is an independent set of H , then $i(H) \leq |S| = i(H')$ and therefore we may assume that S contains both w and u_j for some $j \in \{1, 2, \dots, k-1\}$. If $S \setminus \{w\}$ has a neighbor of u_k , then $S \setminus \{w\}$ is an independent dominating set of H so that $i(H) \leq i(H') - 1$. Therefore, we may assume that $S \setminus \{w\}$ has no neighbor of u_k . It follows that $(S \setminus \{w\}) \cup \{u_k\}$ is an independent dominating set of H , implying that $i(H) \leq |(S \setminus \{w\}) \cup \{u_k\}| = i(H')$. \square

Lemma 2.2. *Let G_1, G_2 be graphs with the unique common vertex v . If G_i has a minimum independent dominating set S_i containing v such that $S_i \setminus \{v\}$ is a dominating set of $G_i - v$ for some $i = 1, 2$, then*

$$i(G_1 \cup G_2) = i(G_1) + i(G_2) - 1.$$

Proof. Let $G = G_1 \cup G_2$. First let us prove $i(G) \geq i(G_1) + i(G_2) - 1$. Let S be an independent dominating set of G . If $S \cap V(G_1)$ and $S \cap V(G_2)$ are independent dominating sets of G_1 and G_2 , respectively, then

$$|S| \geq |S \cap V(G_1)| + |S \cap V(G_2)| - 1 \geq i(G_1) + i(G_2) - 1.$$

Note that we subtracted 1 because v may belong to S . Thus we may assume that $S \cap V(G_1)$ is not an independent dominating set of G_1 . Then $v \notin S$ and v is not dominated by $S \cap V(G_1)$. Then $(S \cap V(G_1)) \cup \{v\}$ is an independent dominating set of G_1 and therefore $|S \cap V(G_1)| \geq i(G_1) - 1$. In this case, $S \cap V(G_2)$ dominates v and therefore $|S \cap V(G_2)| \geq i(G_2)$. This again implies that

$$|S| = |S \cap V(G_1)| + |S \cap V(G_2)| \geq i(G_1) + i(G_2) - 1.$$

Now let us prove that $i(G) \leq i(G_1) + i(G_2) - 1$. Assume that G_2 has a minimum independent dominating set S_2 containing v such that $S_2 \setminus \{v\}$ is a dominating set of $G_2 - v$. Let S_1 be a minimum independent dominating set of G_1 . Then $S = S_1 \cup (S_2 \setminus \{v\})$ is an independent dominating set of G and therefore $i(G) \leq |S_1| + |S_2| - 1 = i(G_1) + i(G_2) - 1$. \square

Lemma 2.3. *Let $\Delta \geq 4$, $d_1 < \Delta$ be integers. Let G be a connected graph of maximum degree at most Δ . Let X be the set of all vertices of degree 1. If every vertex of degree at least 2 is adjacent to exactly d_1 neighbors of degree 1, then*

$$i(G) \leq \left(1 - \frac{\Delta - 1}{(d_1 + 1)(\Delta - d_1)}\right) |V(G)|,$$

unless $G - X$ is a complete graph on $\Delta - d_1 + 1$ vertices or is an odd cycle when $\Delta - d_1 = 2$.

Proof. Let $n = |V(G)|$. Let $m = n/(d_1 + 1)$ be the number of vertices of $G - X$. Observe that $G - X$ is connected. Let S be a maximum independent set of $G - X$ and let $\alpha = |S|$. Then $\alpha \geq \frac{m}{\chi(G-X)} = \frac{1}{(d_1+1)\chi(G-X)}n$. Observe that $S \cup \{v \in X : v \notin N_G(S)\}$ is an independent dominating set of G and therefore

$$\begin{aligned} i(G) &\leq \alpha + (m - \alpha)d_1 = md_1 - \alpha(d_1 - 1) \\ &\leq n - \frac{d_1 - 1 + \chi(G - X)}{(d_1 + 1)\chi(G - X)}n. \end{aligned}$$

By Brooks' theorem, we deduce that $\chi(G - X) \leq \Delta - d_1$. Therefore

$$i(G) \leq m(d_1 + 1) - m \frac{\Delta - 1}{\Delta - d_1} = \left(1 - \frac{\Delta - 1}{(d_1 + 1)(\Delta - d_1)}\right) n. \quad \square$$

Lemma 2.4. *Let Δ be an integer. Let G be a connected graph with at least one vertex of degree at least 2 such that the maximum degree is at most Δ and every vertex of degree at least 2 is adjacent to at least one vertex of degree 1. Let d_1 be the maximum number of degree-1 neighbors of a vertex. If some vertex has less than d_1 neighbors of degree 1, then there is a vertex v such that*

$$\frac{n_0(G - N_G[v]) + 1}{d_G(v)} \leq \frac{\lfloor \Delta^2/4 \rfloor}{\Delta}.$$

Proof. Let X be the set of all vertices of degree 1 in G . Since $G - X$ is connected, there is an edge uv such that u has less than d_1 neighbors of degree 1 and v has exactly d_1 neighbors of degree 1. By the assumption, every isolated vertex of $G - N_G[v]$ is adjacent to exactly one neighbor of v and therefore each isolated vertex of $G - N_G[v]$ is a degree-1 neighbor of a vertex in $N_G(v)$. Since u has less than d_1 neighbors of degree 1, we have

$$n_0(G - N_G[v]) + 1 \leq d_1(d_G(v) - d_1).$$

Let $f(x, y) = \frac{x(y-x)}{y}$ for integers $1 \leq x < y \leq \Delta$. Observe that $f(x, y) \leq f(x, y+1)$ for $x < y < \Delta$ and therefore the maximum of f is achieved when $y = \Delta$. Since $\frac{n_0(G - N_G[v]) + 1}{d_G(v)} \leq f(d_1, d_G(v))$ and $x(\Delta - x) \leq \lfloor \Delta^2/4 \rfloor$ for any integer x , we deduce that $\frac{n_0(G - N_G[v]) + 1}{d_G(v)} \leq \frac{\lfloor \Delta^2/4 \rfloor}{\Delta}$. \square

Lemma 2.5. *Let $x < y$ be positive integers and $y \geq 5$. Then*

$$\frac{x(y-x)+1}{y} < \frac{x(y+1-x)}{y+1} \text{ or } \frac{x(y-x)+1}{y} < \frac{(x+1)(y-x)}{y+1}.$$

Proof. It is enough to show that

$$2 \frac{x(y-x)+1}{y} < \frac{x(y+1-x)}{y+1} + \frac{(x+1)(y-x)}{y+1},$$

which is equivalent to

$$\frac{2xy - 2x^2 + 2}{y} < \frac{2xy - 2x^2 + y}{y+1}.$$

As $y \geq 5$, we have

$$\begin{aligned}
& (y+1)(2xy - 2x^2 + 2) - y(2xy - 2x^2 + y) \\
&= 2y + 2xy - 2x^2 + 2 - y^2 \\
&= 2y + 2 - x^2 - (y-x)^2 \\
&\leq 2y + 2 - \frac{1}{2}y^2 \\
&< 0
\end{aligned}$$

by Cauchy-Schwarz inequality. \square

Lemma 2.6. *Let $\Delta \geq 4$, $d_1 > 0$ be integers. Let G be a connected graph with at least one vertex of degree at least 2 such that the maximum degree is at most Δ and every vertex of degree at least 2 has exactly d_1 neighbors of degree 1. Let v be a vertex of G of degree at least 2. If $d_1 \notin \{\lceil \Delta/2 \rceil, \lfloor \Delta/2 \rfloor\}$ or $d_G(v) < \Delta$, then*

$$\frac{n_0(G - N_G[v]) + 1}{d_G(v)} \leq \frac{\lfloor \Delta^2/4 \rfloor}{\Delta},$$

unless $\Delta = 5$, $d_G(v) = 4$, and $d_1 = 2$.

Proof. Let v be a vertex of degree at least 2. By the assumption, every isolated vertex of $G - N_G[v]$ is adjacent to exactly one neighbor of v and therefore each isolated vertex of $G - N_G[v]$ is a degree-1 neighbor of a vertex in $N_G(v)$. Therefore,

$$n_0(G - N_G[v]) = d_1(d_G(v) - d_1).$$

If $d_G(v) = \Delta$, then $d_1(\Delta - d_1) \leq \lfloor \Delta^2/4 \rfloor - 1$ because $d_1 \notin \{\lceil \Delta/2 \rceil, \lfloor \Delta/2 \rfloor\}$. Therefore $(n_0(G - N_G[v]) + 1)/d_G(v) \leq \lfloor \Delta^2/4 \rfloor/\Delta$. So we may assume that $d_G(v) < \Delta$.

If $d_1 = d_G(v) - 1$ or $d_1 = 1$, then $\frac{d_1(d_G(v)-d_1)+1}{d_G(v)} = 1 \leq \frac{1}{\Delta} \lfloor \Delta^2/4 \rfloor$. Thus we may assume that $1 < d_1 < d_G(v) - 1$. So $d_G(v) \geq 4$. If $d_G(v) = 4$, then $d_1 = 2$ and $\frac{d_1(d_G(v)-d_1)+1}{d_G(v)} = \frac{5}{4}$. If $\Delta \geq 6$, then $\frac{1}{\Delta} \lfloor \Delta^2/4 \rfloor \geq \frac{5}{4} = \frac{d_1(d_G(v)-d_1)+1}{d_G(v)}$. (If $\Delta = 5$, then $\frac{\lfloor \Delta^2/4 \rfloor}{\Delta} = \frac{6}{5}$, $\frac{d_1(d_G(v)-d_1)+1}{d_G(v)} = \frac{5}{4}$.)

Thus we may assume that $5 \leq d_G(v) < \Delta$. Then by Lemma 2.5, $\frac{n_0(G - N_G[v]) + 1}{d_G(v)} \leq \frac{(d_1+1)(d_G(v)-d_1)}{d_G(v)+1}$ or $\frac{n_0(G - N_G[v]) + 1}{d_G(v)} \leq \frac{d_1(d_G(v)+1-d_1)}{d_G(v)+1}$. By applying Lemma 2.5 inductively, we deduce that there is a positive integer $x < \Delta$ such that $\frac{n_0(G - N_G[v]) + 1}{d_G(v)} \leq \frac{x(\Delta-x)}{\Delta}$. We deduce the conclusion because $x(\Delta-x) \leq \lfloor \Delta^2/4 \rfloor$. \square

3 Special graphs

For an integer $\Delta \geq 4$, we say that a connected graph G is called Δ -*special* if it satisfies the following.

- (i) Every vertex that belongs to some cycle has degree Δ .
- (ii) Every edge has at least one end that belongs to some cycle.
- (iii) Each component of the subgraph induced by the set of all vertices that belong to cycles is isomorphic to one of the following.
 - (a) $K_{\lceil \Delta/2 \rceil + 1}$.
 - (b) $K_{\lfloor \Delta/2 \rfloor + 1}$.
 - (c) An odd cycle when $\Delta = 4$.

When Δ is clear from the context, we will say *special* instead of Δ -special for brevity. A special graph is called *non-trivial* if it has at least two vertices and *trivial* otherwise.

Proposition 3.1. *Let $\Delta \geq 4$ be an integer. If G is a Δ -special graph on n vertices, then*

$$i(G) = \left(1 - \frac{\Delta}{\lfloor \Delta^2/4 \rfloor + \Delta}\right) (n - 1) + 1.$$

Proof. Let $t = \frac{\Delta}{\lfloor \Delta^2/4 \rfloor + \Delta}$. Let X be the set of all vertices v of G such that there is a cycle containing v . If $X = \emptyset$, then $n = i(G) = 1$ and therefore we may assume that X is nonempty.

First, we will treat the case that $G[X]$ has only one component. Let S be an independent dominating set of G . If $G[X]$ has exactly one component, then $G[X]$ is isomorphic to $K_{\lceil \Delta/2 \rceil + 1}$ or $K_{\lfloor \Delta/2 \rfloor + 1}$, unless $\Delta = 4$ and it is isomorphic to an odd cycle. Let r be an integer such that $G[X]$ is r -regular. By the definition, $r = \lceil \Delta/2 \rceil$ or $r = \lfloor \Delta/2 \rfloor$. Furthermore, by (i) and (ii), G is isomorphic to a graph obtained from $G[X]$ by attaching $\Delta - r$ vertices of degree 1 to every vertex in X . Then $n = |X|(\Delta - r + 1)$. Suppose that $G[X]$ is isomorphic to $K_{\lceil \Delta/2 \rceil + 1}$ or $K_{\lfloor \Delta/2 \rfloor + 1}$. Then $|S \cap X| \leq 1$. Note that $n = \lfloor \Delta/2 + 1 \rfloor \lceil \Delta/2 + 1 \rceil = \lfloor (\Delta + 2)^2/4 \rfloor = \lfloor \Delta^2/4 \rfloor + \Delta + 1$. Since S is dominating,

$$\begin{aligned} |S| &= |S \cap X| + |X \setminus S|(\Delta - r) \geq 1 + (|X| - 1) \cdot (\Delta - r) = \lfloor \Delta^2/4 \rfloor + 1 \\ &= (1 - t)(n - 1) + 1. \end{aligned}$$

In this case, the equality holds if and only if $|S \cap X| = 1$ and such a set S exists. If $\Delta = 4$ and $G[X]$ is an odd cycle, then $|S \cap X| \leq \frac{|X|-1}{2}$ and since S is dominating,

$$|S| = |S \cap X| + |X \setminus S|(\Delta - r) \geq \frac{|X| - 1}{2} + \frac{|X| + 1}{2} \cdot 2 = \frac{3|X| + 1}{2} = \frac{n - 1}{2} + 1.$$

In this case, the equality holds if and only if $|S \cap X| = (|X| - 1)/2$ and such a set S exists. Thus $i(G) = (1 - t)(n - 1) + 1$ if $G[X]$ has one component.

So we may assume that $G[X]$ has at least two components C_1 and C_2 . Since G is connected, G has a shortest path from C_1 to C_2 . By (ii) and (iii), this path has length 2 whose middle vertex v is not in X . By the definition of X , $G - v$ is disconnected. Among all choices of v and a component F of $G - v$, we choose the one such that F is minimal. We claim that $G[V(F) \cap X]$ is connected. Suppose that $G[V(F) \cap X]$ contains two components D_1 and D_2 . Then F has a shortest path from D_1 to D_2 . By (ii) and (iii), this path has length 2 whose middle vertex w is not in X . By the definition of X , $F - w$ is disconnected. Since v is not in a cycle, v cannot have neighbors in distinct components of $F - w$. Let F' be a component of $F - w$ not having neighbors of v . Then F' is a component of $G - w$, contradicting the choice of v and F . This proves that $G[V(F) \cap X]$ is connected.

Let $G_1 = G - V(F)$ and $G_2 = G[V(F) \cup \{v\}]$. Observe that both G_1 and G_2 are special.

Note that $G_2[V(G_2) \cap X]$ is isomorphic to one of (a), (b), (c) and v has degree 1 in G_2 . Because of our analysis in the first part of the proof, G_2 has a minimum independent dominating set S_2 containing v and another vertex w such that $N_{G_2}(v) \subseteq N_{G_2}(w)$. Then $S_2 \setminus \{v\}$ is a dominating set of $G_2 - v$. By the induction hypothesis, $i(G_1) = (1 - t)(|V(G_1)| - 1) + 1$ and $i(G_2) = (1 - t)(|V(G_2)| - 1) + 1$. By Lemma 2.2, $i(G) = i(G_1) + i(G_2) - 1 = (1 - t)(|V(G_1)| - 1) + (1 - t)(|V(G_2)| - 1) + 1 = (1 - t)(n - 1) + 1$. \square

4 Main theorem

Let $n_\Delta(G)$ be the number of Δ -special components. We now prove our main theorem, which proves Theorem 1.2.

Theorem 4.1. *Let $\Delta \geq 4$ be an integer and let G be a graph with maximum*

degree at most Δ . Let

$$t = \begin{cases} \frac{4}{9} & \text{if } \Delta = 5, \\ \frac{\Delta}{\lfloor \Delta^2/4 \rfloor + \Delta} & \text{otherwise.} \end{cases}$$

Then $i(G) \leq (1 - t)|V(G)| + tn_\Delta(G)$.

Proof. We proceed by induction on $|V(G)| + |E(G)|$. By the induction hypothesis, we may assume that G is connected. It is true if G is special by Proposition 3.1 and therefore we may assume that G is not special. Thus $n_\Delta(G) = 0$. Note that if $\Delta = 5$, then $\frac{\Delta}{\lfloor \Delta^2/4 \rfloor + \Delta} > \frac{4}{9}$.

Claim 4.2. *If G has a vertex of degree at least 2 with no neighbors of degree 1, then $i(G) \leq (1 - t)|V(G)|$.*

Proof. Suppose that G has a vertex v of degree at least 2 with no neighbors of degree 1. Let u_1, u_2, \dots, u_k be the neighbors of v . For $j \in [k]$, let $H_j := G - \{vu_i : i \in [k], i \neq j\}$. By Lemma 2.1, $i(G) \leq i(H_j)$. Since $d_G(u_i) > 1$ for all $i \in [k]$, $n_0(H_j) = 0$ for all $j \in [k]$. Thus if H_j has no non-trivial special component for some $j \in [k]$, then $i(G) \leq i(H_j) \leq (1 - t)|V(G)|$ by the induction hypothesis. Thus we may assume that H_j has a non-trivial special component C_j for all $j \in [k]$. Since C_j is not a component of G , C_j contains a vertex x_j incident with a deleted edge in G . Since x_j has degree less than Δ in C_j , C_j has no cycle containing x_j by the definition of a special graph. By the definition of a special graph, every neighbor of x_j in C_j belongs to some cycle of C_j .

Suppose $x_j = u_i$ for some $j \in [k]$ and $i \in [k] \setminus \{j\}$. Let F_j be the set of edges of $G - u_i v$ incident with u_i . As u_i has degree at least 2 in G , F_j is nonempty. By Lemma 2.1, $i(G) \leq i(G - F_j)$. By the induction hypothesis, we may assume that $G - F_j$ contains a non-trivial special component C'_j . Since C'_j is not a component of G , there is a vertex y_j of C'_j that is incident with some edge in F_j . Since the degree of y_j in $G - F_j$ is less than Δ , y_j does not belong to any cycle of C'_j and therefore $y_j = x_j$. However in this case, $G = C_j \cup C'_j$, u_i is the unique vertex in $C_j \cap C'_j$, and u_i does not belong to any cycle in C_j or C'_j . This implies that G is special, contradicting the assumption.

Thus $x_j = v$ and $V(C_j) \cap \{u_1, \dots, u_k\} = \{u_j\}$ for all $j \in [k]$. Then G is obtained from the disjoint union of C_1, \dots, C_k by identifying one vertex from each component that does not belong to a cycle to become v . This implies that G is special, contradicting the assumption. \blacksquare

Suppose that $G - N_G[v]$ has a non-trivial special component C for some vertex v . Since G is connected, C has a vertex x adjacent to some neighbors of v and therefore the degree of x in C is less than Δ . By the definition of a special graph, all neighbors of x in C have degree Δ and therefore all neighbors of x in G have degree at least 2 in G , implying that $i(G) \leq (1-t)|V(G)|$ by Claim 4.2. Therefore, we may assume that $G - N_G[v]$ has no non-trivial special component for every vertex v , that is, $n_\Delta(G - N_G[v]) = n_0(G - N_G[v])$.

Let d_1 be the maximum number of degree-1 neighbors of a vertex of G . By Claim 4.2, we may assume that G has a vertex of degree 1 and therefore $d_1 > 0$. Suppose that there is a vertex of degree at least 2 having less than d_1 neighbors of degree 1. Let v be the vertex chosen by Lemma 2.4. By the induction hypothesis, we have

$$\begin{aligned}
i(G) &\leq i(G - N_G[v]) + 1 \\
&\leq (1-t)(|V(G)| - d_G(v) - 1) + tn_0(G - N_G[v]) + 1 \\
&= (1-t)|V(G)| - (1-t)d_G(v) + tn_0(G - N_G[v]) + t \\
&\leq (1-t)|V(G)| - d_G(v) + t \frac{\lfloor \Delta^2/4 \rfloor + \Delta}{\Delta} d_G(v) && \text{by Lemma 2.4} \\
&\leq (1-t)|V(G)|.
\end{aligned}$$

Thus we may assume that every vertex of degree at least 2 has exactly d_1 neighbors of degree 1. Let X be the set of all vertices of degree 1. Since G is connected, $G - X$ is connected. Note that $|V(G - X)| = \frac{1}{d_1+1}|V(G)|$.

If $\Delta = 5$, $d_1 = 2$, and G has a vertex of degree 2, then by Lemma 2.3, $i(G) \leq \frac{5}{9}|V(G)| = (1-t)|V(G)|$. Hence, we may assume that $\Delta \neq 5$, or $d_1 \neq 2$, or G has no vertex of degree 2.

Let v be a vertex of degree at least 2. If $d_1 \notin \{\lceil \Delta/2 \rceil, \lfloor \Delta/2 \rfloor\}$ or $d_G(v) < \Delta$, then by Lemma 2.6, $\frac{n_0(G - N_G[v]) + 1}{d_G(v)} \leq \frac{\lfloor \Delta^2/4 \rfloor}{\Delta}$, which implies that $i(G) \leq (1-t)|V(G)|$ as in the previous computation. Therefore, we may assume that $d_1 \in \{\lceil \Delta/2 \rceil, \lfloor \Delta/2 \rfloor\}$ and every vertex of degree at least 2 has degree exactly Δ .

Then $G - X$ is $(\Delta - d_1)$ -regular. Since G is not special, $G - X$ is not a complete graph.

If $G - X$ is not an odd cycle, then by Lemma 2.3,

$$i(G) \leq \left(1 - \frac{\Delta - 1}{(d_1 + 1)(\Delta - d_1)}\right) |V(G)|.$$

If $\Delta \geq 6$, then

$$\begin{aligned} \frac{1}{t} - \frac{(d_1 + 1)(\Delta - d_1)}{\Delta - 1} &\geq \frac{(\Delta^2 - 1)/4 + \Delta}{\Delta} - \frac{(\Delta + 1)^2/4}{\Delta - 1} \\ &= \frac{((\Delta^3 + 3\Delta^2 - 5\Delta + 1) - (\Delta^3 + 2\Delta^2 + \Delta))}{4\Delta(\Delta - 1)} \\ &= \frac{(\Delta^2 - 6\Delta + 1)}{4\Delta(\Delta - 1)} > 0 \end{aligned}$$

and therefore

$$i(G) \leq \left(1 - \frac{\Delta - 1}{(d_1 + 1)(\Delta - d_1)}\right) |V(G)| \leq (1 - t)|V(G)|.$$

If $\Delta = 5$, then $d_1 \in \{2, 3\}$ and so $\frac{\Delta - 1}{(d_1 + 1)(\Delta - d_1)} \geq \frac{4}{9} = t$. If $\Delta = 4$, then $d_1 = 2$ and $\frac{\Delta - 1}{(d_1 + 1)(\Delta - d_1)} = \frac{1}{2} = t$.

Now we may assume that $G - X$ is an odd cycle. Since G is not special, $\Delta = 5$. Since $G - X$ is $(\Delta - d_1)$ -regular, we have $d_1 = 3$. As $G - X$ is an odd cycle, $G - X$ has an independent set S of size $(|V(G - X)| - 1)/2 = \frac{|V(G)|}{2(d_1 + 1)} - \frac{1}{2}$. As in the proof of Lemma 2.3, $S \cup N_G(V(G) - X - S)$ is an independent dominating set and so $i(G) \leq |S| + d_1|V(G) - X - S| = \frac{|V(G)|}{2(d_1 + 1)} - \frac{1}{2} + d_1\left(\frac{|V(G)|}{2(d_1 + 1)} + \frac{1}{2}\right) = |V(G)|/2 + \frac{d_1 - 1}{2} = |V(G)|/2 + 1$. Since G is not special, the length of $G - X$ is at least 5 and $|V(G)| \geq 5(d_1 + 1) = 20$, and therefore $|V(G)|/2 + 1 \leq \frac{5}{9}|V(G)|$. \square

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