ACTIVITY FROM MATROIDS TO ROOTED TREES AND BEYOND

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ABSTRACT. The interior and exterior activities of bases of a matroid are well-known notions that for instance permit one to define the Tutte polynomial. Recently, we have discovered correspondences between the regions of gainic hyperplane arrangements and colored labeled rooted trees. Here we define a general activity theory that applies in particular to no-broken circuit (NBC) sets and labeled colored trees. The special case of activity 0 was our motivating case. As a consequence, in a gainic hyperplane arrangement the number of bounded regions is equal to the number of the corresponding colored labeled rooted trees of activity 0.

1. INTRODUCTION

An *integral affinographic hyperplane* is a hyperplane of the form $x_j - x_i = g$ in real affine space, where g is an integer. Many authors have been interested in this type of hyperplane arrangement (see, for example, [2, 9, 12]). Some familiar examples are (taking complete graphs) the *braid* arrangement, where $g = 0$; the *graphic arrangements*, subarrangements of the braid arrangement; the *Shi* arrangement, where $g = 0$ or $g = 1$ with $1 \le i < j \le n$; the Catalan arrangement, where $g = -1$, 0, or 1; the *Linial arrangement*, where $g = 1$ with $1 \le i < j \le n$ (denoted by $\mathscr{L}_n$). The general type is called *gainic arrangements*; they correspond with gain graphs.

The basic definitions and the background given here are based on what is given more generally in [9,16]. Let Γ be a graph and take the group $\mathbb{Z} := (\mathbb{Z}, +)$, where $-a$ is the inverse of a. Orient the edges of Γ, and let $\vec{E}$ be the set of oriented edges. Denote by $-e$ the edge e with its opposite orientation. An *integral gain graph* Φ is a pair (Γ, φ), where $\varphi : \vec{E} \to \mathbb{Z}$ satisfies $\varphi(-e) = -\varphi(e)$ for all oriented edges e. The function φ is called the *gain mapping* and $\varphi(e)$ is the *gain* of e. The gain of a walk $e_1 e_2 \cdots e_l$ is

$$\varphi(e_1 e_2 \cdots e_l) = \varphi(e_1) + \varphi(e_2) + \cdots + \varphi(e_l).$$

A circle C is a 2-regular connected graph, and it is *balanced* if $\varphi(C) = 0$. For the sake of simplicity in this paper we use $V := [n] = \{1, 2, \ldots, n\}$ as the set of vertices of Φ and $g(i, j)$ to represent an edge of Φ with orientation from i to j. We associate $g(i, j)$ to the hyperplane $x_j - x_i = g$.

Let K_n be the complete graph and let $a \le b$ be integers. We denote by $K_n^{[a,b]}$ the gain graph with vertex set $V = [n]$ and edges $k(i, j)$ with $a \le k \le b$ and $1 \le i < j \le n$. Some familiar examples are the *braid gain graph* $B_n := K_n^{[0,0]}$, the *Linial gain graph* $L_n := K_n^{[1,1]}$, the *Shi gain graph* $S_n := K_n^{[0,1]}$, and the *Catalan gain graph* $C_n := K_n^{[-1,1]}$.

The gain graphs and affinographic hyperplane arrangements can be easily associated by matching the hyperplane with equation $x_j - x_i = g$ and the edge (i, j) with gain g. Therefore, we will accept to say the hyperplane arrangement $K_n^{[a,b]}$ when it should be the hyperplane arrangement corresponding to the gain graph $K_n^{[a,b]}$.

Given a linear order $<_O$ on the set of edges E, a *broken circuit* is the set of edges obtained by deleting the smallest element in a balanced circle. A set of edges, $N \subseteq E$, is a *no-broken-circuit set*, denoted NBC, if it does not contain a broken circuit. This concept is from matroid theory (see for example [4]). It is well-known that this set depends on the choice of the order. However, the cardinality of the set of NBC sets of the gain graph does not depend on choice of an order.

There are still many questions related to these families of hyperplane arrangements. For instance, is it possible to find their characteristic polynomials and their number of regions? This may help us to answer other questions. In fact, when the characteristic polynomial is evaluated at $x = -1$ and $x = 1$, it gives the number of regions and bounded regions (see, for example, Zaslavsky [17]). Once the number of regions is found, another good question could be, is it possible to find a bijective proof, and what is the target set of a bijection? Of course these are not easy questions. For example, it is known that the regions of the Shi arrangement are in correspondence with parking functions (a target set) and the regions of the Linial arrangement are in correspondence with local binary search trees (a target set).

Let $b(\mathscr{L}_n)$ be the number of bounded regions of the Linial arrangement $\mathscr{L}_n$: $x_i - x_j = 1$ for $1 \le i < j \le n$. Athanasiadis [2] found a closed formula for $b(\mathscr{L}_n)$. He proved (based on generating functions) that

$$b(\mathscr{L}_n) = \frac{1}{2^n} \sum_{j=0}^{n} \binom{n}{j} (j-1)^{n-1} \tag{1}$$

and stated that "it would be interesting to find a combinatorial interpretation for the numbers $b(\mathscr{L}_n)$". Here in Theorem 4.3 we give a combinatorial interpretation for this formula. (Independently, Tewari [15] gave a different interpretation.)

The Tutte polynomial is an important combinatorial invariant of a matroid that is more general than the characteristic polynomial of a hyperplane arrangement. There is an expansion of the Tutte polynomial using the bases and their interior and exterior activities. The no-broken-circuit (NBC) sets of the arrangement have exterior activity 0. Thus, only interior activity applies to the NBC sets. The restriction of the Tutte polynomial to the NBC bases gives an expansion of the activity polynomial (to be defined shortly) very close to that of the characteristic polynomial.

Some bijections from the NBC sets of gainic arrangements to different families of colored labeled rooted trees have been given by Forge et al. in [5,6,8]. Bernardi [3] found a bijection between regions of the same hyperplane arrangements using the trees as defined in [5]. Levear [11] uses the same trees to find bijections for the k-dimensional faces of the same arrangements. In this paper, we define an activity on those families of target trees that corresponds to the activity of the NBC bases. This will give a relation between the NBC bases of activity 0 and some special subsets of our target set. Since the NBC bases with activity 0 correspond to the bounded regions of the arrangement, this gives a proof that the number of bounded regions is equal to the number of trees of activity 0.

In section 2 we give some background results which brought us to what will follow in the next sections. Specially Theorem 2.1 and Theorem 2.2 are the most important ones.

In section 3, we define covering systems a generalization of matroids where both Theorem 2.1 and Theorem 2.2 are preserved. A covering system S is a generalization of the concept of a matroid. We generalize activity of NBC bases to an activity on a covering system S. This activity gives rise to a polynomial, the activity polynomial. We show both some applications of the activity polynomial and its relations with the Tutte polynomial.

In Section 4, we apply the results from Section 3 to the set of rooted colored forests that was our motivation. Theorem 4.3 gives that the number of bounded regions of an affinographic arrangement is equal to the number of the corresponding colored forest with activity 0.

Finally we give some examples and constructive proofs for the number of NBC sets of a given activity for the braid arrangement, Shi arrangement, and Lineal arrangement. The results in this paper give rise to a conjecture that we state at the end of the paper.

2. Some background and motivating results

As a motivation for the study of NBC bases of gain graphs, in this section we present two theorems that show relationships between the NBC bases of gain graphs and other areas of combinatorics.

2.1. **Rooted colored trees.** We color the edges of trees with the numbers $[1, k]$. Let T be a rooted k-colored tree. The edges of T are represented by (u, w), or by uw (for simplicity) if there is no ambiguity, where w is a child of u. For a fixed internal vertex v of T, we set c_v to mean the minimum color used by the edges going out of v. We say that T is *decreasing* (*increasing*) if for any internal vertex $v \in T$, the label of v is larger (smaller) than the labels of each of its children w_i such that vw_i is colored with c_v. Note that the edges vw_i colored with a bigger that c_v are not considered. We say that T is *non-increasing* (*non-decreasing*) if for any internal vertex $v \in T$, the label of v is larger (smaller) than the label of at least of one of its children w_i such that vw_i is colored with c_v. Similarly, for $k_1 + k_2 = k$, we say that T is (k_1, k_2)-colored decreasing if for any internal vertex v such that $c_v \leq k_1$, the label of v is larger than the label of each of its child w_i and such that the edge vw_i is colored with c_v. Note that in this definition the edges colored with the k_2 colors in $[k_1 + 1, k]$ never are checked. Therefore, these k_2 colors are called free colors and the k_1 first colors are the non-free colors. The definitions of (k_1, k_2)-colored increasing, (k_1, k_2)-colored non-increasing, and (k_1, k_2)-colored non-decreasing are similar and we omit them.

Example 2.1 (decreasing tree using 4 colors). Figure 1 Part (a) depicts a decreasing tree colored with 4 colors. In this figure we use v_i to mean the vertex with label i. For example, the vertex v_{10}, with label 10, is the parent of v_3, v_5, and v_7. The minimum color used by $v_{10}v_3$, $v_{10}v_5$ and $v_{10}v_7$ is $c_{v_{10}} = 1$. We need to check decreasing condition only for the edge $v_{10}v_3$. From the figure can see that $v_3 < v_{10}$, so we check it as OK. Similarly, from the figure we can see that the edges v_3v_1 and v_3v_2 are colored with 1 and the edges v_3v_4 and v_3v_{11} are colored with 2 and 3, respectively. Therefore, $c_{v_3} = 1$. Thus, we only need to check that in fact both satisfy that $v_1 < v_3$ and $v_2 < v_3$. Finally, we observe that $c_{v_7} = 2$. Since $v_6 < v_7$, we check it as OK.

Example 2.2 ($(2, 2)$-colored decreasing tree). Figure 1 Part (b) depicts a $(2, 2)$ colored decreasing tree. In this example we have two non-free colors 1 and 2, and two free colors 3 and 4. The vertex v_6 is the parent of v_3, v_5, and v_{11}. The colors used by the edges v_6v_3, v_6v_5, and v_6v_{11} give that the minimum color is $c_{v_6} = \min\{1, 2, 3\} = 1$. To check that the

tree is $(2,2)$ colored decreasing at v_6, we need to check only the vertices of the edges using the color 1. So, in this case $v_3 < v_6$. Therefore we check it as OK. Similarly, since $c_{v_3} = 1$, we need to check that in fact these hold $v_1 < v_3$ and $v_2 < v_3$. Note that $c_{v_5} = 3$, so there is nothing to check since 3 is a free color. Finally, we observe that $c_{v_{11}} = 2$, so we check that in fact the condition $v_7 < v_{11}$ holds.
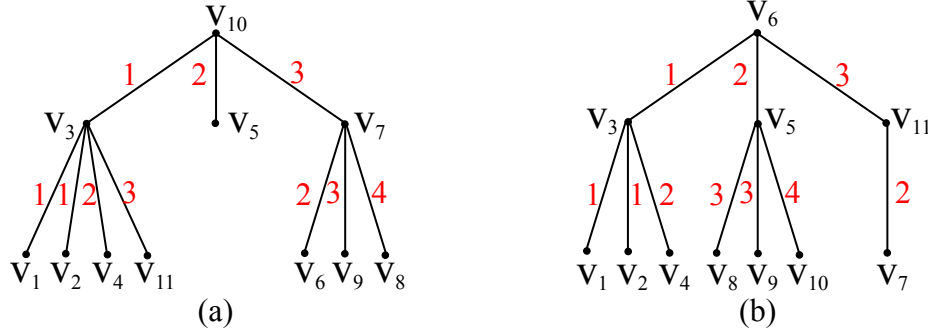

FIGURE 1.   (a) Decreasing tree using 4 colors   (b) $(2,2)$-colored decreasing tree.

2.2. **Activity of NBC sets.** Given a matroid M on an ordered set E, an element e is exteriorly (respectively, interiorly) active relative to a base B if e is the smallest element of the fundamental circuit $C(B,e)$ (respectively, cocircuit $C^*(B,e)$) [10]. We use $\iota(B)$ and $\varepsilon(B)$ to denote the sets of interiorly and exteriorly active elements of a base B, and use $|\iota(B)|$ and $|\varepsilon(B)|$ to denote the cardinality of those sets (these two cardinalities are called the *interior* and *exterior activity numbers*). These definitions were introduced by Tutte for graphs and extended by Crapo [7] to matroids. The activity numbers give a formula for the Tutte polynomial:

$$\text{Tutte}_A(x,y) = \sum_{B \text{ a basis}} x^{|\iota(B)|} y^{|\varepsilon(B)|}. \tag{2}$$

The *NBC bases* are exactly those of exterior activity 0. The activity of an NBC basis means only its interior activity. Any subset of an NBC basis is called *NBC set*. The following classic partition theorem is at the origin of our definition of a generalized activity. For simplicity, in this paper, when we refer to an activity number we mean the interior activity number.

Theorem 2.1 ([4]). *Let M be a matroid on a linearly ordered set E and let $\mathfrak{I}$ be the set of all independent sets. Then:*

(1) *Every independent set I of M can be uniquely written in the form $I = B \backslash Y$ for some basis B and some subset $Y \subset \iota(B)$. Equivalently, the intervals $[B \backslash \iota(B), B]$, form a partition of the set $\mathfrak{I}$.*

(2) *Every NBC set A of M can be uniquely written in the form $A = B \backslash Y$ for some NBC basis B and some subset $Y \subset \iota(B)$. Equivalently, the intervals $[B \backslash \iota(B), B]$ form a partition of the set $\mathfrak{N}$ of NBC sets.*

All these definitions are also valid for a semimatroid. An (affine) hyperplane arrangement defines the semilattice of flats of a semimatroid by considering all the non-empty intersection sets of hyperplanes. So, for simplicity, instead to say basis of the semimatroid defined by an arrangement we say basis of the arrangement. Similarly, whenever we need to say the

basis of the semimatroid defined by NBC sets we say NBC basis and so on. For details on semimatroids see Ardila [1]. The following theorem is a classic result, see for example Crapo [7], Tutte, and Zaslavsky's Theorem in the vocabulary of regions.

Theorem 2.2. *The Tutte polynomial of an arrangement A restricted to $y = 0$ gives the activity polynomial, i.e., $\alpha(x) = \text{Tutte}_A(x, 0) = \sum_{i=0}^{n} a_i x^i$. Furthermore, these hold:*

(1) *the activity polynomial is related to the characteristic polynomial χ by*
$$\alpha(x) = (-1)^n \chi(1 - x).$$

(2) *The coefficient a_i is equal to the number of NBC bases of activity i of the arrangement.*

(3) *The value $\alpha(0)$ gives the number of bounded regions of the arrangements and the number of NBC bases of activity 0 of the arrangement.*

(4) *The value $\alpha(1)$ gives the number of NBC bases of the arrangement.*

(5) *The value $\alpha(2)$ gives the number of regions and the number of NBC sets of the arrangement.*

3. Activity of a covering system

We define an activity which is a generalization of the NBC activity using the partition property of Theorem 2.1. The power set of a finite set E is denoted by 2^E. An r-set is a set with r elements, where the empty set is the 0-set. Let $X \subseteq Y$ be finite sets; we use $[X, Y]$ to denote the set $\{Z \mid X \subseteq Z \subseteq Y\}$, called an *interval*. An *r-covering system* $\mathcal{S}$ is an ordered pair $(E, \mathcal{I})$ consisting of a finite set E and a collection $\mathcal{I}$ of subsets of E of cardinality less than or equal to r, with the condition that for every $I \in \mathcal{I}$ there is an r-set $B \in \mathcal{I}$ such that $[I, B] \subseteq \mathcal{I}$. An r-set in $\mathcal{I}$ is called a *basis* and the set of all bases is denoted by $\mathcal{B}$. The elements of $\mathcal{I}$ are called *independent sets*. Note that we borrowed this terminology from matroids.

For an r-covering system $\mathcal{S}$ the function $\boldsymbol{a} : \mathcal{B} \to 2^E$ is an *activity*, if for every $B \in \mathcal{B}$ it holds that $\boldsymbol{a}(B) \subseteq B$ with $[B \setminus \boldsymbol{a}(B), B] \subseteq \mathcal{I}$ and for every $I \in \mathcal{I}$ there is a unique $B \in \mathcal{B}$ such that $B \setminus \boldsymbol{a}(B) \subseteq I \subseteq B$. When these conditions are expressed by the intervals of the form $[B \setminus \boldsymbol{a}(B), B]$, they give rise to a partition of $\mathcal{I}$. The set $\boldsymbol{a}(B)$ and its cardinality $|\boldsymbol{a}(B)|$ are called the set of *active elements* of B and the *activity number* of B, respectively.

The vector $A = (a_0, \ldots, a_r)$ is called the *activity vector* of $\boldsymbol{a}$, where a_i represents the number of sets of $\mathcal{B}$ of activity number i (clearly, $|\mathcal{B}| = \sum_{k=0}^{r} a_k$). The *activity polynomial* is $\alpha(x) = \sum_{i=0}^{r} a_i x^i$, where a_i is an element of the activity vector.

We say that $C = (c_0, \ldots, c_r)$ is the *cardinality vector* of $\mathcal{S}$ if c_i is the number of sets in $\mathcal{I}$ of cardinality i. In particular, $c_r = |\mathcal{B}|$ and c_0 is equal to 1 if $\emptyset \in \mathcal{I}$ and 0 otherwise.

For example, the ordered pair $\mathcal{S} := (E, \mathcal{I})$ is a covering system with three bases, where $E = \{1, 2, 3, 4, 5\}$ and

$$\mathcal{I} = \{\{1, 2, 3, 4\}, \{1, 2, 4\}, \{2, 3, 4\}, \{2, 4\}, \{1, 2, 3, 5\}, \{1, 2, 3\}, \{2, 3, 5\}, \{2, 3\}, \{1, 3, 4, 5\},$$
$$\{1, 3, 4\}, \{1, 3, 5\}, \{1, 4, 5\}, \{1, 3\}, \{1, 4\}, \{1, 5\}, \{1\}, \{2, 3, 4, 5\}\}.$$

An activity is given by

$$\boldsymbol{a}(\{1, 2, 3, 4\}) = \{1, 3\}, \qquad\qquad \boldsymbol{a}(\{1, 2, 3, 5\}) = \{1, 5\},$$
$$\boldsymbol{a}(\{1, 3, 4, 5\}) = \{3, 4, 5\}, \qquad \text{and} \qquad \boldsymbol{a}(\{2, 3, 4, 5\}) = \emptyset.$$

In this case we have $A = (1, 0, 2, 1, 0)$, $\alpha(x) = 1 + 2x^2 + x^3$ and $C = (0, 1, 5, 7, 4)$. We show the intervals in the poset determined by the antichains of NBC bases in Figure 2.
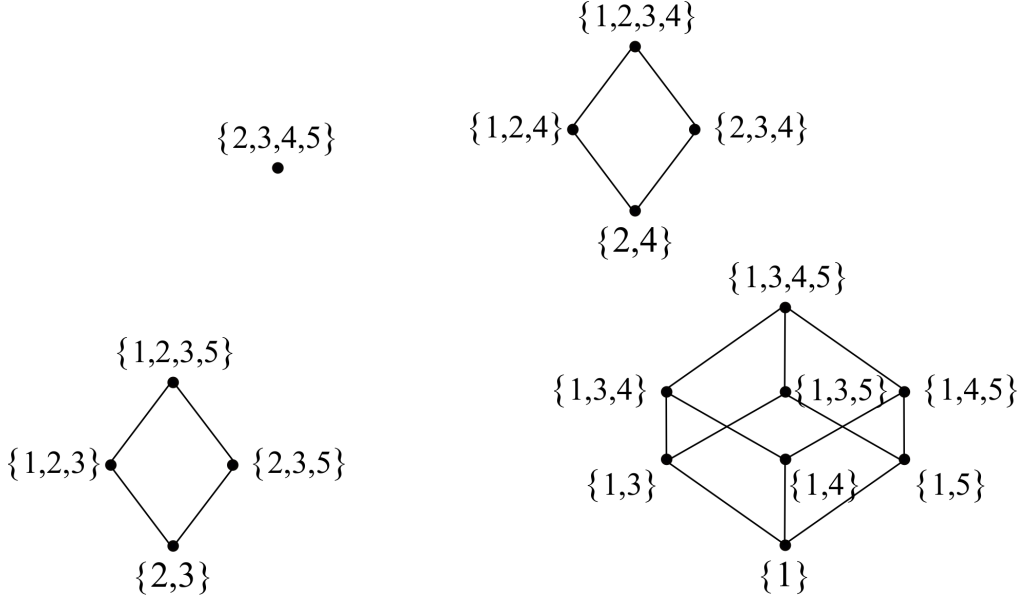
FIGURE 2. Antichains of NBC bases.

The following result is a natural generalization from matroid theory. So, its proof is straightforward.

Proposition 3.1. *If $(E, \mathfrak{I})$ is a covering system with activity polynomial α, then*

(1) *the number of bases of activity 0 is given by $\alpha(0)$,*
(2) *the number of bases is given by $\alpha(1)$, and*
(3) *the number of sets in $\mathfrak{I}$ is given by $\alpha(2)$.*

Proof. The proofs of Part (1) and Part (2) are straightforward from the definition of a_i (the number of bases of activity i).

Proof of Part (3). Since a base of activity i covers exactly 2^i sets of $\mathfrak{I}$, the conclusion follows from the partition property of the activity.   □

Theorem 3.2. *Let $\mathcal{S} := (E, \mathfrak{I})$ be an r-covering system with $C = (c_0, \ldots, c_r)$ its cardinality vector. Let $\boldsymbol{a}$ be an activity for $\mathcal{S}$ and let $A = (a_0, \ldots, a_r)$ be the corresponding activity vector. Then A can be obtained from C by setting $a_r = c_0$ and $a_{r-i} = c_i - \sum_{j=r-i+1}^{r} a_j \binom{j}{r-i}$. In particular, in every case there is at most one basis of activity r.*

Proof. From the definition of activity we know that a base B of activity $\boldsymbol{a}(B)$ covers the interval $[B \setminus \boldsymbol{a}(B), B]$. Therefore, a base of activity $\boldsymbol{a}(B)$ covers $2^{|\boldsymbol{a}(B)|}$ independents, more precisely there are $\binom{|\boldsymbol{a}(B)|}{r-i}$ independents of cardinality i. Summing over all bases and using the fact that the intervals $[B \setminus \boldsymbol{a}(B), B]$ form a partition of all independent sets, we get that the number of independent sets of cardinality i is $\sum_{j=r-i+1}^{r} a_j \binom{j}{r-i}$. This completes the proof.   □

From Theorem 3.2 we obtain a relation between the activity polynomial $\alpha(x)$ and the cardinality polynomial $\mathrm{card}(x) = \sum c_i x^i$. We state it formally in this corollary.

Corollary 3.3. *If $A = (a_0, \ldots, a_n)$ is an activity vector of $\boldsymbol{a}$, then $\mathrm{card}(x) = x^n \alpha(1/x + 1)$.*

The following result is a corollary of Corollary 3.3[1].

Corollary 3.4. *All activities of a given r-covering system $\mathcal{S}$ have the same activity vector.*

Two r-covering systems $\mathcal{S}_1 := (E_1, \mathcal{I}_1)$ and $\mathcal{S}_2 := (E_2, \mathcal{I}_2)$, with $|E_1| = |E_2|$, are *similar* if there is a bijection ψ from $\mathcal{I}_1$ to $\mathcal{I}_2$ such that for every $X \in \mathcal{I}_1$, $|\psi(X)| = |X|$. Two similar covering systems have necessarily the same rank.

The following theorem makes the definition of activities works. Thus, the sizes of the various intervals are invariant. The proof of this theorem justifies defining polynomials that do not depend on the specific interval partition. Thus, it justifies the partition definition of activities.

Theorem 3.5. *If $\mathcal{S}$ and $\mathcal{S}'$ are similar r-covering systems with activities $\boldsymbol{a}$ and $\boldsymbol{a}'$, then their activity vectors A and A' are equal.*

Proof. From the definition of similarity, the number of sets of a given cardinality k is the same in $\mathcal{S}$ and in $\mathcal{S}'$, i.e., $c_k = c'_k$ for every $1 \leq k \leq n$. The conclusion follows from Theorem 3.2. $\square$

Proposition 3.6. *Let $\mathcal{S} := (E, \mathcal{I})$ be an r-covering system with bases $\mathcal{B}$ and with activity function $\boldsymbol{a}$. For nonempty $\mathcal{B}' \subseteq \mathcal{B}$ let $\mathcal{I}' = \{I \in \mathcal{I} \mid \exists B \in \mathcal{B}' \text{ such that } B \setminus \boldsymbol{a}(B) \subseteq I \subseteq B\}$. Then $\mathcal{S}' := (E, \mathcal{I}')$ is an r-covering system and $\boldsymbol{a}' = \boldsymbol{a}|_{\mathcal{B}'}$ is an activity.*

Proof. The intervals $[B \setminus \boldsymbol{a}'(B), B] = [B \setminus \boldsymbol{a}(B), B]$ for $B \in \mathcal{B}'$ form a partition of $\mathcal{I}'$. $\square$

Proposition 3.7. *Let $\mathcal{S}_1 := (E, \mathcal{I}_1)$ and $\mathcal{S}_2 := (E, \mathcal{I}_2)$ be r-covering systems with activities $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$. If $\mathcal{I}_1 \cap \mathcal{I}_2 = \emptyset$, then $\mathcal{S} := (E, \mathcal{I}_1 \cup \mathcal{I}_2)$ is a covering system with activity defined by*

$$
\boldsymbol{a}(B) = \begin{cases} \boldsymbol{a}_1(B), & \text{if } B \in \mathcal{I}_1, \\ \boldsymbol{a}_2(B), & \text{if } B \in \mathcal{I}_2. \end{cases}
$$

Proof. The intervals $[B \setminus \boldsymbol{a}_1(B), B]$ for $B \in \mathcal{B}_1$ together with the intervals $[B \setminus \boldsymbol{a}_2(B), B]$ for $B \in \mathcal{B}_2$ form a partition of $\mathcal{I}_1 \cup \mathcal{I}_2$. $\square$

Let $\mathcal{S} := (E, \mathcal{I})$ be an r-covering system with an activity $\boldsymbol{a}$ and let X be an element of $\mathcal{I}$. We define $\mathcal{I}_X := \{I \in \mathcal{I} : X \subseteq I\}$, $\mathcal{B}_X := \{X \in \mathcal{B} : X \subseteq B\}$, and $\boldsymbol{a}_X(B) := \boldsymbol{a}(B) \setminus X$, for every $B \in \mathcal{B}_X$.

Proposition 3.8. *Let $\mathcal{S} := (E, \mathcal{I})$ be an r-covering system with an activity $\boldsymbol{a}$ and let $X \in \mathcal{I}$. Then $\mathcal{S}_X := (E, \mathcal{I}_X)$ is an r-covering system and $\boldsymbol{a}_X$ is an activity of $\mathcal{S}_X$.*

Proof. The set $\mathcal{I}_X$ is the intersection of $\mathcal{I}$ and the interval $[X, E]$. Therefore, every interval $[B \setminus \boldsymbol{a}(B), B]$ in the partition of $\mathcal{I}$ that intersects $[X, E]$ gives the interval $[(B \setminus \boldsymbol{a}(B)) \cup X, B]$, if $X \subseteq B$. (Note that $(B \setminus \boldsymbol{a}(B)) \cup X = B \setminus (\boldsymbol{a}(B) \setminus X)$ if $X \subseteq B$.) They form a partition of $\mathcal{I}_X$. $\square$

3.1. Examples of covering systems with activities.

Example 3.1 (*r-pure independence system*)**.** Let E be a finite set. We say that a collection $\mathcal{I}$ of subsets of E is an *r-pure independence system* if every maximal element in $\mathcal{I}$ has cardinality r and if for each $I \in \mathcal{I}$, every subset of I is also in $\mathcal{I}$. Clearly, every r-pure independence system is an r-covering system. But an r-pure independence system does not

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[1]T. Zaslavsky suggested to call it "corollorollary"

necessarily have an activity. For example, let $(E, \mathfrak{I})$ be a 2-pure independence system with $E = \{1, 2, 3, 4\}$, where $\mathfrak{I}$ has exactly two maximal elements, $B_1 = \{1, 2\}$ and $B_2 = \{3, 4\}$. We claim that $(E, \mathfrak{I})$ does not have an activity. Indeed, suppose that $\boldsymbol{a}$ is such an activity; then to "cover" the empty set we need $\boldsymbol{a}(B_i) = B_i$ for either $i = 1$ or $i = 2$. Let us suppose that $\boldsymbol{a}(\{1, 2\}) = \{1, 2\}$ (the other case is similar); to cover $\{3\}$ we need $\boldsymbol{a}(\{3, 4\}) = \{4\}$, but then we do not have any possibility to cover $\{4\}$, which is a contradiction.

Another way to prove that there is not an activity is by Theorem 3.2. Here the cardinality vector of $(E, \mathfrak{I})$ is given by $C = (1, 4, 2)$. From Theorem 3.2 we know that if there is an activity vector it is $A = (-1, 2, 1)$. Since the first entry is negative, there is no such activity.

Example 3.2 (matroids). The set of independent sets of a matroid and the set of NBC sets of a matroid are examples of r-pure independence systems. The activity defined by the order of the elements as explained in Section 2.2 is an activity for both of them by Theorem 2.1.

For the set of NBC sets the activity polynomial α_0 is related to the Tutte polynomial by $\alpha_0(x) = \text{Tutte}(x, 0)$. So, $\alpha_0(1) = \text{Tutte}(1, 0)$ is the number of NBC bases and $\alpha_0(2) = \text{Tutte}(2, 0)$ is the number of NBC sets. The number $\alpha_0(2)$ is, at the same time, the number of regions of the arrangement when the matroid is defined by an arrangement of hyperplanes.

For the set of independent sets the activity polynomial α_1 is related to the Tutte polynomial by $\alpha_1(x) = \text{Tutte}(x, 1)$. So, $\alpha_1(1) = \text{Tutte}(1, 1)$ is the number of bases and $\alpha_1(2) = \text{Tutte}(2, 1)$ is the number of independents sets of the matroid.

Example 3.3 (colored rooted forests). Let E be the set of all directed edges with vertices in the set $V = \{1, 2, \ldots, n\}$. Let $\mathfrak{I}$ be the set of rooted forests on the set V. The rooted forests are such that every vertex has at most one incoming edge; the roots are the vertices with indegree zero. In a rooted forest, each component is a directed tree and has exactly one root. The set B of edges of a connected acyclic graph with n vertices is defined as a base in $\mathcal{B}$. The couple $(E, \mathfrak{I})$ is an $(n-1)$-covering system on the set of ordered edges $\{(i, j) : 1 \le i \ne j \le n\}$. It is a pure independence system. The subset of decreasing rooted forests is still a pure independence system but the set of non-increasing rooted forests is not. Indeed, there are some edges that cannot be deleted while keeping the non-increasing property. For the example, the tree defined with a root $r = 2$ and edges $(2, 1)$ and $(2, 3)$ is non-increasing tree, and if we remove the edge $(2, 1)$ (but not its vertices) we obtain the forest formed by the vertex 1 and the edge $(2, 3)$. So, it does not give a non-increasing forest. Therefore, the set of non-increasing forest is not a pure independent system. Nevertheless, it is still a covering system, since it is possible to add some edges to any non-increasing trees. A potential activity is the set of edges (n, v) in a tree. This will be discussed later, in the next section.

Example 3.4 (paths connecting two vertices). We now give an application of Propositions 3.7 and 3.8. Let $G = (V, E)$ be a connected graph, where $|V| = n$, and let a and b be distinct vertices. Let $\mathcal{F}_{ab}$ be the set all forests containing a path connecting a and b. This set forms an $(n-1)$-covering system. Let $\mathcal{P} = \{P : P$ is a path connecting a and $b\}$ and for a fixed path P connecting a and b, let $\mathcal{F}_P$ the set of all forests containing P. We have $\mathcal{F}_{ab} = \biguplus_{P \in \mathcal{P}} \mathcal{F}_P$ ($\uplus$ means disjoint union). From Propositions 3.7 and 3.8 we see that $\mathcal{F}_{ab}$ is an $(n-1)$-covering system and it also has an activity. Note that in this example the empty set is not an independent set.

4. ACTIVITY OF COLORED ROOTED TREES

Corteel et al. [5] define some classes of colored rooted trees and show that they correspond to the NBC sets of the deformations of the braid arrangement with an interval $[a, b]$ containing 0 or 1. The original motivating examples were the decreasing labeled trees and the local binary search trees. Following Stanley [14], a *local binary search tree* —LBS tree, for short— is a labeled rooted binary tree, with vertex set $[n]$, such that every left child of a vertex is less than its parent, and every right child is greater than its parent. Note that a parent may have only one child. The number of LBS trees on the set $[n]$ is known to be equal to the number of regions of the Linial arrangement in dimension n; see [12].

In general, if T is (k_1, k_2)-decreasing (increasing), then T is $(k_1 - \alpha, k_2 + \alpha)$-decreasing (increasing) if $k_1 - \alpha \geq 0$.

The following theorem shows a relation between the number of NBC bases and the total number of decreasing (non-increasing) colored trees. Part (1) corresponds to the Shi arrangement with interval $[0, 1]$ and to the generalizations $[-k_2, 1 + k_2]$, Part (2) corresponds to the so called Catalan arrangements $[-k_2, k_2]$; and to some generalization $[-k_2, k_1 + k_2]$; Part (3) corresponds to the Linial arrangement $[1, 1]$ and to some generalizations $[1, k_1]$, and $[1 - k_1, k_1 + k_2]$.

The proofs of Theorem (4.1) Parts (1) and (2) are in [6] and the proof of Part (3) is in [8].

Theorem 4.1. *Let k_1 and $k_2 \geq 0$. If $K_n^{[a,b]}$ is the gain graph with vertices $V = [n]$, then:*

(1) *there is a bijection between the set of NBC sets of $K_n^{[-k_2, 1+k_2]}$ and the set of forests with k_2 free colors.*

(2) *There is a bijection between the set of NBC sets of $K_n^{[-k_2, k_1+k_2]}$ and the set of $(k_1 + 1, k_2)$-decreasing colored forests.*

(3) *There is a bijection between the set of NBC sets of $K_n^{[1-k_2, k_1+1+k_2]}$ and the sets of $(k_1 + 1, k_2)$-non-increasing colored forests.*

This theorem is the main research object by Corteel et al. in [5]. From Theorem 4.1 we can deduce many unexpected correspondences. For example, there is a bijection between the NBC sets of $K_n^{[0,2]}$ and both the $(3, 0)$ decreasing trees (setting $k_1 = 2$ and $k_2 = 0$ in Part 2) and the $(1, 1)$ non-increasing trees (setting $k_1 = 1$ and $k_2 = 1$ in Part 3). In [5] the authors gave a bijective proof in addition to general proof.

We recall that given an edge e in a tree, we distinguish one vertex of e as the parent of the second vertex (called child) if it is closer to the root. Let k be a positive integer with $k = k_1 + k_2$, where k_1 and $k_2 \geq 0$ and let T be a tree with vertex set equal to $[n]$. If T is either a (k_1, k_2)-decreasing k-colored tree or is a (k_1, k_2)-non-increasing k-colored tree, then we define $\boldsymbol{a}(T)$ as the set of edges having n as their parent and colored with 1.

Proposition 4.2. *Let k be a positive integer with $k = k_1 + k_2$, where k_1 and $k_2 \geq 0$. If T is either a (k_1, k_2)-decreasing k-colored tree or is a (k_1, k_2)-non-increasing k-colored tree, then $\boldsymbol{a}(T)$ is an activity.*

Proof. We prove the case in which a forest F is k-colored and (k_1, k_2)-decreasing with vertex set $[n]$. The other case is similar and we omit it.

We want to show that the sets $[T \setminus \boldsymbol{a}(T), T]$ form a partition of the set of forests. For any forest F with d components $T_1, T_2, \ldots, T_d$ not containing the vertex n. (There is a $(d+1)$th component T_0 containing the vertex n.) The unique tree T such that $F \in [T \setminus \boldsymbol{a}(T), T]$ is

the tree obtained by adding the d edges (n, r_i) to F, for $1 \leq i \leq d$ where r_i is the root of the tree T_i. □

Our motivation for the rest of this paper is to provide a bijective proof of the formula of Athanasiadis (see equation (1)). He wanted an interpretation similar to the one given by [13, Theorem 4.1] for the number of regions $b(\mathscr{L}_n)$. The following theorem answers that question.

For the following theorem we use $K_n^{[a,b]}$ for the general case. But for each part we give the details how a and b will be.

Theorem 4.3. *Let k_1 and $k_2 \geq 0$ and $0 \leq \alpha \leq n - 1$. Then*
  (1) *The number of NBC bases of $K_n^{[-k_2, 1+k_2]}$ with activity α is equal to the number of trees with k_2 colors having α edges of the form (n, i) colored with 1. In Particular, the number of bounded regions of the arrangement $K_n^{[-k_2, 1+k_2]}$ is equal to the number of trees with k_2 colors having no-edge of the form (n, i) colored with 1.*
  (2) *The number of NBC bases of $K_n^{[-k_2, k_1+k_2]}$ with activity α is equal to the number of $(k_1 + 1, k_2)$ colored decreasing trees with α edges of the form (n, i) colored with 1. In particular, the number of bounded regions of the arrangement $K_n^{[-k_2, k_1+k_2]}$ is equal the number of $(k_1 + 1, k_2)$ decreasing trees with no-edge on the form $(n, 1)$ colored with 1.*
  (3) *The number of NBC bases of $K_n^{[1-k_2, k_1+k_2]}$ of activity α is equal to the number of (k_1, k_2) colored non-increasing tress with α edges of the form (n, i) colored with 1. In particular the number of bounded regions of $K_n^{[1-k_2, k_1+k_2]}$ is equal to the number of (k_1, k_2) colored non-increasing trees with no-edge of the form (n, i) colored with 1.*

Proof. We prove Part (1). From Theorem 4.1 Part (1) we have a bijection between the set of NBC sets of $K_n^{[-k_2, 1+k_2]}$ and the set of forests with k_2 colors. Since these two sets are covering systems, they are similar covering systems. From Proposition 4.2 we know that for the sets of trees with k_2 colors, the number of edges of the form (n, i) colored with 1, is an activity. Therefore, Theorem 3.5 imply that their activity vectors are equal.

We now prove the bounded regions part. It follows, just recalling that the number of bounded NBC bases of activity 0 of a gain graph is equal to the number of bounded regions of the corresponding hyperplane arrangements (see for example, Theorem 2.2 Part (3)).

Note that the proofs of Parts (2) and (3) are identical —replace in above argument Theorem 4.1 Part (1) by Theorem 4.1 Part (2) and (3), respectively. □

Example 4.1 (activity numbers of $\mathscr{L}_3$). In Figure 3 we represent all non-increasing trees with three vertices. From Theorem 4.1 we know that they correspond to the NBC bases of the gain graph $K_n^{[1,1]}$ corresponding to the Linial arrangement.

The activity number of a base is simply the number of children of vertex number 3. We do not color the edges of the trees in Figures 3 and 4 because there is only one color (the color 1). The numbers are, from left to right, 2, 1 and 0. It is easy to see that $\alpha(x) = x^2 + x + 1$. Note that

$$b(\mathscr{L}_3) = (1/8) \left(\binom{3}{0} \cdot (-1)^2 + \binom{3}{2} \cdot 1^2 + \binom{3}{3} \cdot 2^2 \right) = 1.$$

Example 4.2 (activity numbers of $\mathscr{L}_4$). In Figure 4, we represent all non-increasing trees with four vertices. By Theorem 4.1 we know that they correspond to the NBC bases of the Linial arrangement. The activity number of a base is simply the number of children of vertex

number 4. There are one tree of activity number 3, three trees of activity number 2, six trees of activity number 1 and four tress of activity number 0. Therefore, $\alpha(x) = x^3 + 3x^2 + 6x + 4$. Finally, we note that there are four trees of activity 0. That is, there are four bounded regions. We also note, from the equation (1), that we have

$$b(\mathscr{L}_4) = (1/16) \left(\binom{4}{0} \cdot (-1)^3 + \binom{4}{2} \cdot 1^3 + \binom{4}{3} \cdot 2^3 + \binom{4}{4} \cdot 3^3 \right) = 4.$$
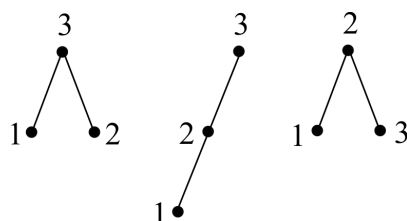
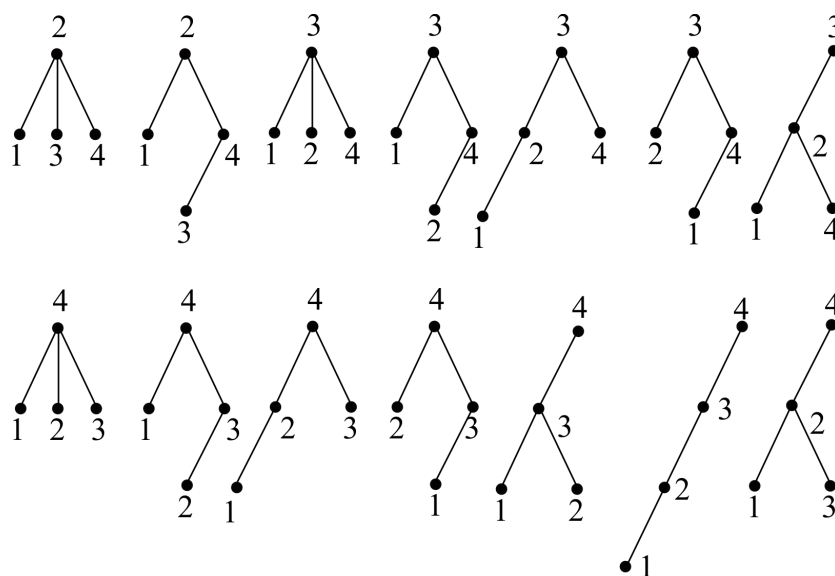FIGURE 3. The activities are, in order from left to right, 2, 1 and 0.

FIGURE 4. The activities are, from left to right, 0, 1, 0, 1, 0, 1, 0, 3, 2, 2, 2, 1, 1, 1.

5. THE LOOSING ACTIVITY

We started this research with the aim of finding something in the binary search trees that would correspond to activity in the NBC trees. We wanted to use NBC trees of the gain graphs as done by Forge in [8]. After some experimentation, we figure out that the number of edges of the form $(i + 1, i)$ were this something that we where looking for. We thought to call it "activity". But now it is only the loosing activity, since it does not work as nicely as the edges of the form (n, i). Nevertheless, we manage to prove some interesting results about the number of trees with k edges of the form $(i + 1, i)$. We conjecture a similar result in the case of LBS trees.

In the previous section, we gave a general definition for an activity. We showed that the set of edges of the form (n, i) is an activity for the general rooted trees (Shi), the decreasing trees (braid), and the non-decreasing trees. We were searching to find out if the number of edges of the form $(i + 1, i)$ was the correct activity. However, we realized that the correct one was the number of edges of the form (n, i). Here we give some results obtained from the original question. So, the aim of this section is to give a count (equivalent to an activity) for rooted labeled trees that fits the activity of the corresponding NBC bases. With this objective in mind, we start with some classical arrangements —the braid arrangement, the Shi arrangement, and the Linial arrangement. We analyze these arrangements with rooted trees —decreasing trees, general trees, and local binary search trees (LBS trees).

We knew the activity numbers for every n, so we could try different possible definitions. By looking at the first cases, we found two different candidates for activity for trees on n vertices:

(1) the edges (n, i) in the tree,
(2) the edges $(i + 1, i)$ in the tree.

In this section we prove that these two activities give the same numbers for these two cases of increasing trees and of general trees.

5.1. **The braid case.** The NBC set in B_n, the braid arrangement in n dimensions, corresponds to the set of increasing labeled trees. Internally active edges in such a tree are just the edges (n, i). We recall that the rising factorial is defined as $x^{(n)} = (x+1)(x+2)\cdots(x+n-1)$. The coefficient of x^k in $x^{(n)}$ is an unsigned Stirling number of the first kind denoted by $(-1)^{n-k} s(n, k)$.

Theorem 5.1. *The number of NBC sets of activity k in B_n is equal to these:*

(1) *the number of increasing trees on n vertices where the vertex 1 has degree k.*
(2) *The number of decreasing trees on n vertices where the vertex n has degree k.*
(3) *The number of decreasing trees on n vertices with k edges of the form $(i + 1, i)$.*
(4) $(-1)^{n-k} s(n, k)$.

Proof. First of all, we observe that the number of NBC sets of activity k in B_n is equal to the Tutte polynomial given in Equation (1) on Page 12. This follows by Theorem 4.1 Part (1) with $k_2 = 0$ and Theorem 4.3.

It is straightforward to see that Parts (1) and (2) are equivalent. Note that Part (2) is equivalent to Part (4) (it follows from Theorems 4.1 Part (1) and from Theorem 4.3).

We now prove that Part (2) and Part (3) are equivalent. We just define an involution on the set of decreasing trees on n vertices that send a tree with with k edges of the form (n, i) to a tree with k edges of the form $(i + 1, i)$. For a tree T, we just need to replace every edge in T of the form (n, i) by the edge $(i + 1, i)$ and every edge in T of the form $(i + 1, i)$ by the edge (n, i). This completes the proof. □

Note that n is the root of any decreasing tree and that therefore there are no decreasing tree of activity 0. This shows that the braid arrangement has no bounded regions. We also note that both the sets of edges (n, i) and of edges $(i + 1, i)$ define an activity in the system of increasing trees. This gives a second proof that Part (2) and Part (3) are equivalent.

5.2. **The Shi case.** The NBC bases of the Shi arrangement correspond to the rooted trees. The aim of Theorem 5.2 is to show that (n, i) or the edges $(i + 1, i)$ give rise to the same

activities as the NBC bases of the Shi arrangement. Part (1) is a special case of Theorems 4.1 and 4.3. However, Part (2) is not a special case, because the edges $(i+1, i)$ do not define an activity anymore. For the proof of the following theorem see the Appendix.

Theorem 5.2. *The number of NBC sets of activity k in the Shi arrangement S_n is equal to these:*

(1) *the number of rooted trees on n vertices where the vertex n has degree k.*
(2) *The number of rooted trees on n vertices with k edges of the form $(i+1, i)$.*
(3) $\binom{n-1}{k}(n-1)^{n-1-k}$.

5.3. **Examples.** The examples in this section are based on the algorithms from the Appendix. Consider the tree given in Figure 5.

Example 5.1 (Prüffer decodings). We decode $w_p = 265256$ with $n = 7$.
Step i=1. Find the first letter in w_p after the last 1, and then replace it in w_p by $(2, 1)$. Set $w_p := (2, 1)65256$.
Step i=2. Find the first letter in w_p after the last 2, and then replace it in w_p by $(5, 2)$. Set $w_p := (2, 1)652(5, 2)6$.
Step i=3. Find the first letter in w_p after the last 3, and then replace it in w_p by $(6, 3)$. Set $w_p := (2, 1)(6, 3)52(5, 2)6$.
Step i=4. Find the first letter in w_p after the last 4, and then replace it in w_p by $(5, 4)$. Set $w_p := (2, 1)(6, 3)(5, 4)2(5, 2)6$.
Step i=5. Find the first letter in w_p after the last 5, and then replace it in w_p by $(6, 5)$. Set $w_p := (2, 1)(6, 3)(5, 4)2(5, 2)(6, 5)$.
Step i=6. The root is equal to 6.
Step i=7. Find the first letter in w_p after the last 6, and then replace it in w_p by $(2, 7)$. Set $w_p := (2, 1)(6, 3)(5, 4)(2, 2)(5, 2)(6, 5)$.
Output. The tree has edges $(2, 1)$, $(6, 3)$, $(5, 4)$, $(2, 7)$, $(5, 2)$, $(6, 5)$. See Figure 5.

Example 5.2 (Blue coding and decoding). This coding is given by $w = b5b24b$ with $n = 7$ with output depicted in Figure 5.
The decoding is:
Step i=1. The parent of 1 is given by the first letter (left-to-right), in this case it is b. Replace b by $(2, 1)$, as given in the tree in Figure 5, and set $w = (2, 1)5b24b$.
Step i=2. The parent of 2 is given by the first letter after the last 2, in this case it is 4. Since $4 > 2$, from the tree in Figure 5 we can see that 4 must be replaced by $(5, 2)$. Now set $w = (2, 1)5b2(5, 2)b$.
Step i=3. The parent of 3 is given by the first regular letter, in this case it is 5. Since $5 > 3$, from the tree in Figure 5 we can see that 5 must be replaced by $(6, 3)$. Now set $w = (2, 1)(6, 3)b2(5, 2)b$.
Step i=4. The parent of 4 is given by the first regular letter, in this case it is b. Replace b by $(5, 4)$ and set $w = (2, 1)(6, 3)(5, 4)2(5, 2)b$.
Step i=5. The parent of 5 is given by the first letter after the last 5, in this case it is b. Replace b by $(6, 5)$ and set $w = (2, 1)(6, 3)(5, 4)2(5, 2)(6, 5)$.
Note that 6 is the last letter in the last outcome of w, so we conclude that 6 is the root.
Step i=6. The parent of the vertex 7 is the vertex 2. Then replace 2 by $(2, 7)$ and set $w = (2, 1)(6, 3)(5, 4)(2, 7)(5, 2)(6, 5)$.
Output. The tree has edges $(2, 1)$, $(6, 3)$, $(5, 4)$, $(2, 7)$, $(5, 2)$, $(6, 5)$. See, Figure 5.
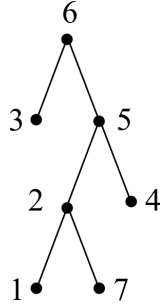We can check that the coding of this tree is indeed the starting word.

FIGURE 5. Tree with root 6, and edges $(2,1)$, $(6,3)$, $(5,4)$, $(2,7)$, $(5,2)$, $(6,5)$.

6. Some conclusions and remarks

A *local binary search tree* (LBS tree, for short) is a planar rooted tree for which every vertex v has two possible children, the left one denoted by l and the right one denoted by r, ordered so that $l < v$ (in case that it exists) and $r > v$ (in case it exists). A left LBS tree is a LBS tree whose root has no right child. We believe that the number of left LBS trees with k edges of the form $(i+1, i)$ (recall that the vertices are $[n] = \{1, 2, \ldots, n\}$) is equal to the number of left LBS trees with k edges of the form (n, i). This is stated formally in Conjecture 6.1.

Motivated by the Athanasiadis' question [2], our main interest from the beginning of this project was the Linial case. The number of regions of the Linial arrangement is known to be equal to the number of LBS trees. So, we were expecting to give some kind of a activity on the edges of the LBS trees, which match with the activity of NBC trees in $K_n^{[1,1]}$. We were looking only at the left LBS trees which correspond to NBC trees as all LBS trees correspond to all NBC sets. At some point we thought that in left LBS trees the number of edges of the form $(i+1, i)$ should be the desired "activity". A computer exploration give some confirmation on this belief. So, we left it as a conjecture (see Conjecture 6.1).

Let us first see that the left LBS trees are actually equivalent to non-increasing rooted forests. That correspondence is directly described by a *rotation* of the tree. A *local binary search tree* (LBS tree, for short) is a planar rooted tree for which every vertex v has two possible children, the left one denoted by l and right one denoted by r, such that $l < v$ (in case it exists) and $r > v$ (in case it exists). A left LBS tree is an LBS tree whose root has no right child. The correspondence holds by replacing all right edges of an LBS tree as follows; take a right edge (v, r) and find the first left ancestor x of v, and then replace (v, r) by (x, r).

In Figure 6 we can see that

$$(2, 5) \to (7, 5);\ (5, 9) \to (7, 9);\ (1, 8) \to (9, 8);\ (4, 6) \to (5, 6);\ \text{ and }\ (6, 10) \to (5, 10).$$

Conjecture 6.1. *For $0 \leq k \leq n-1$, the number of left LBS trees with k edges of the form $(i+1, i)$ is equal to the number of non-increasing trees with k edges of the form (n, i).*

In Figure 7 we list all 14 left LBS with four vertices. We also give, for each one, the number of edges of the from $(i+1, i)$ and we find these activities: one of activity 3, three of activity 2, six of activity 1, and four of activity 0.

The conjecture could be stated using non-increasing trees instead of LBS trees. The edges $(i+1, i)$ would be only those in which i is the smallest child of $i+1$. So, the activity would be the number of vertices v such that their smallest child is $v-1$.
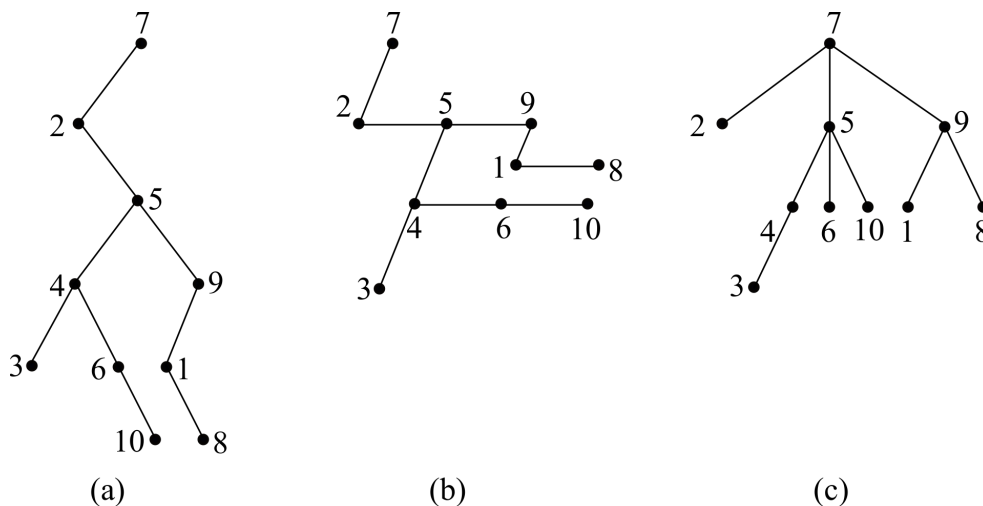
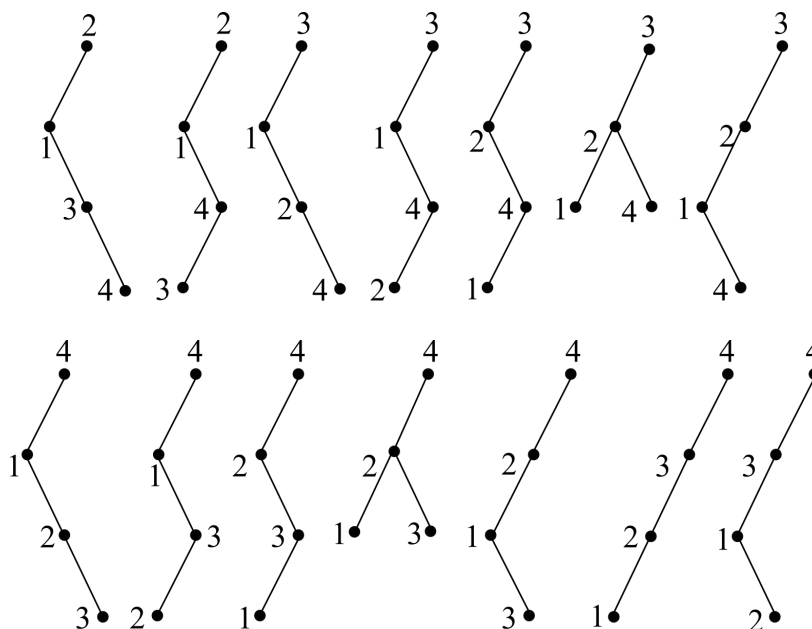FIGURE 6.   (a) Left LBS.   (b) Rotation   (c) Non-increasing tree

FIGURE 7.   'Loosing' activities. From left to right: 1, 2, 0, 0, 1, 2, 2, 0, 1, 0, 1, 1, 3, 1.

7. APPENDIX. ALGORITHMS.

Proof of Theorem 5.2. We prove that Part (1) is equivalent to Part (3). From Theorems 4.1 and 4.3 we obtain a straightforward proof (because the edges (n, i) define an activity) that the number of rooted trees on n vertices where the vertex 1 has degree k is given by $\binom{n-1}{k}(n-1)^{n-1-k}$.

We give here a second constructive proof of this fact that may lead to a better understanding of this property.

The Prüffer code. The coding algorithm (see Algorithm 1) takes as an input a rooted labeled tree T on n vertices and gives as an output a word w of length $n-1$ on the alphabet $\{1, 2, \ldots, n\}$. In this algorithm λ represents the empty word and l_i represents the smallest leaf of T.

Algorithm 1 The Prüffer code

---

procedure PRÜFFER CODE(T)
    $w \leftarrow \lambda$ (the empty word)
    for $i = 1 \rightarrow n-1$ **do**
        $w \leftarrow wl_i$ $\triangleright$ l_i= smallest leaf of T
        $T \leftarrow T \setminus l_i$ $\triangleright$ delete l_i from T
    Return w

The decoding algorithm (see Algorithm 2) takes as an input a length $n-1$ word on the alphabet $\{1, 2, \ldots, n\}$, and gives as an output a tree.

Running Algorithm 2 we replace a letter x by a pair (x, i). So, the word that contains only letter at the beginning, during the process will contain both regular letters and pairs of letters and at the end only pairs. The Algorithm 4, that is very similar to Algorithm 2, does the same replacement of letters by pairs.

Algorithm 2 The Prüffer decoding

---

procedure PRÜFFER DECODING(w)
    for $i = 1 \rightarrow n$ **do**
        if i is the last letter of w **then** root $\leftarrow i$
        else
            Search for the last occurrence of i in w
            $x \leftarrow$ the first regular letter after the last i. $\triangleright$ (Suppose that
the last occurrence of i is in a position j. That is, $w_j = i$ and the first regular letter x is at position j', i.e. $w'_j = x$ with $j' > j$.)
            $w_j \leftarrow (x, i)$
    $T \leftarrow$ root $\triangleright$ T is the tree
    while $w \neq \emptyset$ **do**
        Let (x, i) be the first letter of ω
        $T \leftarrow T \cup (x, i)$
        $w \leftarrow w \setminus (x, i)$
    Return T

Proof that Part (3) **is equivalent to Part** (2).

The Blue code. The coding algorithm (see Algorithm 3) takes as an input a rooted labeled tree T on n vertices and gives as an output a word w of length $n-1$ on the alphabet $\{b, 1, 2, \ldots, n-1\}$. In this algorithm l_i represents the smallest leaf of T and p_i its parent.

The decoding of the word w needs to correct the two transformations of the coding. The procedure is like in our Prüffer decoding to find the parents of the vertices from 1 to n. $\quad\square$

Algorithm 3 The Blue code for a rooted tree

---

procedure BLUE CODING(T)
    $w \leftarrow \lambda$ (the empty word)
    while T has a leaf **do**   ▷ stops when T has no leaf
        for $i = 1 \to n-1$ **do**
            if $p_i = l_i + 1$ **then** $w \leftarrow wb$ ▷ l_i is the smallest leaf with parent p_i
            else
                if $p_i > l_i$ **then** $w \leftarrow w(p_i - 1)$
                else $w \leftarrow wp_i$
                    $T \leftarrow T \setminus l_i$
    Return w

Algorithm 4 The Blue decoding for a word

---

procedure BLUE DECODING(w)
    $\omega \leftarrow w$
    for $i = 1 \to n$ **do**
        if i is the last letter of w **then** root $\leftarrow i$
            if $i \notin w$ **then** $0 \leftarrow j$
        else
            Search the last occurrence of i in ω
            $x \leftarrow$ the first regular letter after i ▷ (Suppose that last occurrence of i is
at position j (so, $\omega_j = i$) and the first regular letter x is at position $j\prime$ (so, $\omega_{j\prime} = x$) with
$j\prime > j$.)
                if $x = b$ **then** $\omega_{j\prime} \leftarrow ((1+i), i)$
                else
                    if $x > i$ **then** $w_j \leftarrow ((x-1), i)$
                    else $\omega_{j\prime} \leftarrow (x, i)$
    $T \leftarrow$ root
    while $\omega \neq \emptyset$ **do**
        Let (x, i) be the first letter of ω
        Edges $\leftarrow$ edges $\cup (x, i)$
        $\omega \leftarrow \omega \setminus (x, i)$
    Return T ▷ defined by roots and edges

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