Lower Bounds for the Error of Quadrature Formulas for Hilbert Spaces

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Abstract

We prove lower bounds for the worst case error of quadrature formulas that use given sample points $\mathcal{X}_n = \{x_1, \ldots, x_n\}$. We are mainly interested in optimal point sets $\mathcal{X}_n$, but also prove lower bounds that hold with high probability for sets of independently and uniformly distributed points. As a tool, we use a recent result (and extensions thereof) of Vybíral on the positive semi-definiteness of certain matrices related to the product theorem of Schur. The new technique also works for spaces of analytic functions where known methods based on decomposable kernels cannot be applied.

Keywords: numerical integration in high dimensions, curse of dimensionality, positive definite matrices, Schur's product theorem.

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1 Introduction

We study error bounds for quadrature formulas and assume that the integrand is from a Hilbert space F of real valued functions defined on a set D. We assume that function evaluation is continuous and hence are dealing with a reproducing kernel Hilbert space (RKHS) F with a kernel K. We want to compute $S(f)$ for $f \in F$, where S is a continuous linear functional, hence $S(f) = \langle f, h \rangle$ for some $h \in F$. We consider, for $c \in \mathbb{R}^n$ and $\mathcal{X}_n = \{x_1, \ldots, x_n\} \subset D$, quadrature formulas $Q_{c,\mathcal{X}_n} : F \to \mathbb{R}$ defined by

$$Q_{c,\mathcal{X}_n}(f) = \sum_{j=1}^{n} c_j f(x_j).$$

Then the worst case error (on the unit ball of F) of $Q_{c,\mathcal{X}_n}$ is defined by

$$e(Q_{c,\mathcal{X}_n}, S) := \sup_{||f|| \leq 1} |S(f) - Q_{c,\mathcal{X}_n}(f)|.$$

If we fix a set $\mathcal{X}_n \subset D$ of sample points we may define the radius of information $e(\mathcal{X}_n, S)$ by

$$e(\mathcal{X}_n, S) = \inf_{c \in \mathbb{R}^n} e(Q_{c,\mathcal{X}_n}, S).$$

Our main interest is in the optimization of $\mathcal{X}_n$ as well as of the weights c. Then we obtain the nth minimal worst case error

$$e(n, S) = \inf_{\mathcal{X}_n \subset D} e(\mathcal{X}_n, S) = \inf_{c \in \mathbb{R}^n} \inf_{\mathcal{X}_n \subset D} e(Q_{c,\mathcal{X}_n}, S).$$

The minimal number of sample points that are needed to achieve a fixed error demand $\varepsilon \geq 0$ is described by the number

$$n(\varepsilon, S) = \min \{ n \in \mathbb{N} \mid e(n, S) \leq \varepsilon \}.$$

We are mainly interested in *tensor product problems*. We will therefore assume that F_i is a RKHS on a domain D_i with kernel K_i for all $i \leq d$ and that $\mathbf{F}_d$ is the tensor product of these spaces. That is, $\mathbf{F}_d$ is a RKHS on $\mathbf{D}_d = D_1 \times \cdots \times D_d$ with reproducing kernel

$$\mathbf{K}_d : \mathbf{D}_d \times \mathbf{D}_d \to \mathbb{R}, \quad \mathbf{K}_d(x, y) = \prod_{i=1}^{d} K_i(x^i, y^i).$$

If $h_i \in F_i$ and $S_i(f) = \langle f, h_i \rangle$ for $f \in F_i$, we will denote by $\mathbf{h}_d$ the tensor product of the functions h_i, i.e.,

$$\mathbf{h}_d(t) = (h_1 \otimes \cdots \otimes h_d)(t) = h_1(t^1) \cdot \ldots \cdot h_d(t^d), \quad t = (t^1, \ldots, t^d) \in \mathbf{D}_d.$$

We study the tensor product functional $\mathbf{S}_d = \langle \cdot, \mathbf{h}_d \rangle$ on $\mathbf{F}_d$. Note that in this paper we assume that $\mathbf{S}_d$ is a tensor product *functional*, but the results can also be applied to *operators*, see [17].

The complexity of the tensor product problem is given by the numbers $e(n, \mathbf{S}_d)$ and $n(\varepsilon, \mathbf{S}_d)$ and has been studied in many papers for a long time. Traditionally, the functional $\mathbf{S}_d$ and the dimension d was fixed and the interest was on large n. Here we are mainly interested in the *curse of dimensionality*: Do we need exponentially many (in d) function values to obtain an error ε when we fix the error demand and vary the dimension?

To answer this question one has to prove the corresponding upper or lower bounds. *Upper bounds* for specific problems can often be proved by quasi Monte Carlo methods, see [2]. In addition there exists a general method, the analysis of the Smolyak algorithm, see [16, 23] and the recent supplement [18].

In this paper we concentrate on *lower bounds*, again for a fixed error demand ε and (possibly) large dimension. Such bounds were first studied in [12] for certain special problems and later in [14] with the technique of *decomposable kernels*. This technique is rather general as long as we consider finite smoothness. The technique does not work, however, for analytic functions.

In contrast, the approach of [22] also works for polynomials and other analytic functions. We continue this approach since it opens the door for more lower bounds under general assumptions. One result of this paper (Theorem 10) reads as follows:

Theorem 10. *For all $i \leq d$, let F_i be a RKHS and let S_i be a bounded linear functional on F_i with unit norm and nonnegative representer h_i. Assume that there are functions f_i and g_i in F_i and a number $\alpha_i \in (0,1]$ such that (h_i, f_i, g_i) is orthonormal in F_i and $\alpha_i h_i = \sqrt{f_i^2 + g_i^2}$. Then the tensor product problem $\mathbf{S}_d = S_1 \otimes \ldots \otimes S_d$ satisfies for all $n \in \mathbb{N}$ that*

$$e(n, \mathbf{S}_d)^2 \geq 1 - n \prod_{i=1}^{d} (1 + \alpha_i^2)^{-1}.$$

In particular, if all the α_i's are equal to some $\alpha > 0$ and we want to achieve $e(n, \mathbf{S}_d) \leq \varepsilon$ for some $0 < \varepsilon < 1$, we obtain

$$n(\varepsilon, \mathbf{S}_d) \geq (1 - \varepsilon^2)(1 + \alpha^2)^d.$$

This implies the curse of dimensionality. As an application, we use this result to obtain lower bounds for the complexity of the integration problem on Korobov spaces with increasing smoothness, see Section 4.1. These lower bounds complement existing upper bounds from [16, Section 10.7.4].

We add in passing that lower bounds of this form are known and much easier to prove for quadrature formulas that only have positive weights, see Theorem 10.2 of [16].

The paper is organized as follows. We first provide a general connection between the worst case error of quadrature formulas and the positive semi-definiteness of certain matrices in Section 2. We then turn to tensor product problems. We start with homogeneous tensor products (i.e., all factors F_i and h_i are equal), see Section 3, where we also consider several examples. The non-homogeneous case is then discussed in Section 4. This section also contains the results for Korobov spaces with increasing smoothness. Section 3 and Section 4 are based on a recent generalization of Schur's product theorem from [22]. In Section 5, we discuss further generalizations of Schur's theorem and possible applications to numerical integration. Finally, in Section 6, we consider lower bounds for the error of quadrature formulas that use random point sets (as opposed to optimal point sets). This allows us to approach situations where we conjecture but cannot prove the curse of dimensionality for optimal point sets.

2 Lower bounds and positive definiteness

We begin with a somewhat surprising result: Lower bounds for the worst case error of quadrature formulas are equivalent to the statement that certain matrices are positive semi-definite.

Proposition 1. *Let F be a RKHS on D with kernel K and let $S = \langle \cdot, h \rangle$ for some $h \in F$ and $\alpha > 0$.*

(i) The following statements are equivalent for all $\mathcal{X}_n = \{x_1, \dots, x_n\} \subset D$.

- *The matrix $\big(K(x_j, x_k) - \alpha h(x_j) h(x_k)\big)_{j,k \le n}$ is positive semi-definite,*
- *$e(\mathcal{X}_n, S)^2 \ge \|h\|^2 - \alpha^{-1}$.*

(ii) The following statements are equivalent for all $n \in \mathbb{N}$.

- *The matrix* $\big(K(x_j,x_k)-\alpha h(x_j)h(x_k)\big)_{j,k\le n}$ *is positive semi-definite for all* $x_1,\dots,x_n\in D$,
- $e(n,S)^2 \ge \|h\|^2-\alpha^{-1}$.

Proof. To prove the first part, we fix $\mathcal{X}_n=\{x_1,\dots,x_n\}\subset D$. For $c\in\mathbb{R}^n$ consider the quadrature rule $Q_{c,\mathcal{X}_n}:F\to\mathbb{R}$ with

$$Q_{c,\mathcal{X}_n}(f)=\sum_{j=1}^n c_j f(x_j).$$

Clearly, we have

$$\begin{aligned} e(Q_{c,\mathcal{X}_n},S)^2 &= \sup_{\|f\|\le 1}\big|S(f)-Q_{c,\mathcal{X}_n}(f)\big|^2=\Big\|h-\sum_{j=1}^n c_jK(x_j,\cdot)\Big\|^2\\ &=\|h\|^2-2\sum_{j=1}^n c_jh(x_j)+\sum_{j,k=1}^n c_jc_kK(x_j,x_k). \end{aligned}$$

The function $g:\mathbb{R}\to\mathbb{R}$ with $g(a)=e(Q_{ac,\mathcal{X}_n},S)^2$ attains its minimum for

$$a=\frac{\sum_{j=1}^n c_jh(x_j)}{\sum_{j,k=1}^n c_jc_kK(x_j,x_k)},$$

where $0/0$ is interpreted as 0. This yields

$$e(\mathcal{X}_n,S)^2=\inf_{c\in\mathbb{R}^n}\inf_{a\in\mathbb{R}} e(Q_{ac,\mathcal{X}_n},S)^2=\|h\|^2-\sup_{c\in\mathbb{R}^n}\frac{\Big(\sum_{j=1}^n c_jh(x_j)\Big)^2}{\sum_{j,k=1}^n c_jc_kK(x_j,x_k)}.$$

The last expression is larger or equal to $\|h\|^2-\alpha^{-1}$ if, and only if,

$$\sum_{j,k=1}^n c_jc_kK(x_j,x_k)\ge\alpha\left(\sum_{j=1}^n c_jh(x_j)\right)^2$$

holds for all $c\in\mathbb{R}^n$, i.e. when the matrix $\big(K(x_j,x_k)-\alpha h(x_j)h(x_k)\big)_{j,k\le n}$ is positive semi-definite. This yields the statement.

The proof of the second part follows from the first part by taking the infimum over all $\mathcal{X}_n=\{x_1,\dots,x_n\}\subset D$. $\square$

The idea now is to use some properties of the Schur product (often also called Hadamard or entrywise product) of matrices. We denote by diag $M = (M_{1,1}, \dots, M_{n,n})^T$ the diagonal entries of M whenever $M \in \mathbb{R}^{n\times n}$. Moreover, if $A, B \in \mathbb{R}^{n\times n}$ are two symmetric matrices, we write $A \succeq B$ if $A - B$ is positive semi-definite. The Schur product of A and B is the matrix $A \circ B$ with

$$(A \circ B)_{i,j} = A_{i,j} B_{i,j} \quad \text{for} \quad i, j \le n.$$

The classical Schur product theorem states that the Schur product of two positive semi-definite matrices is again positive semi-definite. However, this statement can be improved [22] when $A = B$.

Proposition 2. *Let $M \in \mathbb{R}^{n\times n}$ be a positive semi-definite matrix. Then*

$$M \circ M \succeq \frac{1}{n} (\text{diag } M)(\text{diag } M)^T.$$

A direct proof of Proposition 2 may be found in [22]. As pointed out to the authors by Dmitriy Bilyk, the result follows also from the theory of positive definite functions on the spheres as developed in the classical work of Schoenberg [20]. To sketch this approach, let $(\mathbf{C}_k^{\lambda}(t))_{k=0}^{\infty}$ denote the sequence of Gegenbauer (or ultraspherical) polynomials. These are polynomials of order k on $[-1, 1]$, which are orthonormal with respect to the weight $(1 - t^2)^{\lambda - 1/2}$. Here, $\lambda > -1/2$ is a real parameter. By the Addition Theorem [1, Theorem 9.6.3], there is a positive constant $C_{k,n}$ and a natural number $c_{k,n}$, both only depending on k and n, such that

$$\mathbf{C}_k^{(n-2)/2}(\langle x, y\rangle) = C_{k,n} \sum_{l=1}^{c_{k,n}} S_{k,l}(x) S_{k,l}(y), \quad x, y \in \mathbb{S}^{n-1}. \tag{1}$$

Here, $\mathbb{S}^{n-1}$ stands for the unit sphere in $\mathbb{R}^n$ and $S_{k,1}, \dots, S_{k,c_{k,n}}$ form an orthonormal basis of the space of harmonic polynomials of degree k in $\mathbb{R}^n$.

If now $X = (x_{i,j})_{i,j=1}^n \in \mathbb{R}^{n\times n}$ is a positive semi-definite matrix with ones on the diagonal and $f(t) = \sum_{k=0}^{\infty} a_k \mathbf{C}_k^{(n-2)/2}(t)$ with $a_k \ge 0$, then $(f(x_{i,j}))_{i,j=1}^n$ is also positive semi-definite. Indeed, we can write $x_{i,j} = \langle x_i, x_j\rangle$ for some vectors $x_1, \dots, x_n \in \mathbb{S}^{n-1}$ and use (1) to compute for every $c \in \mathbb{R}^n$

$$\begin{aligned}\sum_{i,j=1}^{n} c_i c_j f(x_{i,j}) &= \sum_{i,j=1}^{n} c_i c_j \sum_{k=0}^{\infty} a_k \mathbf{C}_k^{(n-2)/2}(\langle x_i, x_j \rangle) \\ &= C_{k,n} \sum_{i,j=1}^{n} c_i c_j \sum_{k=0}^{\infty} a_k \sum_{l=1}^{c_{k,n}} S_{k,l}(x_i) S_{k,l}(x_j) \\ &= C_{k,n} \sum_{k=0}^{\infty} a_k \sum_{l=1}^{c_{k,n}} \Big(\sum_{i=1}^{n} c_i S_{k,l}(x_i) \Big)^2 \geq 0.\end{aligned}$$

For positive semi-definite matrices $M \in \mathbb{R}^{n \times n}$ with ones on the diagonal, Proposition 2 then follows by observing that $f(t) = t^2 - \frac{1}{n}$ is (up to a positive multiplicative constant) exactly the polynomial $\mathbf{C}_2^{(n-2)/2}(t)$. Finally, the general form of Proposition 2 is given by a simple scaling argument. $\square$

3 Homogeneous tensor products

We now use Propositions 1 and 2 in order to obtain the curse of dimensionality for certain tensor product (integration) problems. In this section, we consider homogeneous tensor products, i.e., $\mathbf{F}_d = F_1 \otimes \cdots \otimes F_1$ and $\mathbf{h}_d = h_1 \otimes \cdots \otimes h_1$.

Theorem 3. *Let F_1 be a RKHS on D_1. Assume that there are functions e_1 and e_2 on D_1 such that e_1^2, e_2^2 and $\sqrt{2} e_1 e_2$ are orthonormal in F_1 and let S_1 be a linear functional with $S_1(e_i^2) = \sqrt{2}/2$ and $S_1(e_1 e_2) = 0$. Then the tensor product problem $\mathbf{S}_d$ satisfies*

$$e(n, \mathbf{S}_d)^2 \geq 1 - n\, 2^{-d}.$$

In particular, it suffers from the curse of dimensionality since we need at least $2^d\,(1 - \varepsilon^2)$ function values to achieve the error ε.

Remark 1. We usually work with normalized problems, i.e., we assume that $e(0, S_1) = \|h_1\| = 1$ and thus $e(0, \mathbf{S}_d) = \|\mathbf{h}_d\| = 1$. Note that the only normalized functional S_1 that satisfies the above conditions is given by $S_1 = \langle \cdot, h_1 \rangle$ with the representer $h_1 = \frac{1}{2}\sqrt{2}(e_1^2 + e_2^2)$.

Proof. Without loss of generality, we may assume that F_1 is 3-dimensional, i.e., that e_1^2, e_2^2 and $\sqrt{2}e_1e_2$ form an orthonormal basis. Then also $b_1 = \frac{1}{2}\sqrt{2}(e_1^2 + e_2^2)$, $b_2 = \frac{1}{2}\sqrt{2}(e_1^2 - e_2^2)$, and $b_3 = \sqrt{2}e_1e_2$ is an orthonormal basis since b_1 and b_2 are just a rotation of e_1^2 and e_2^2. On the 3-dimensional space F_1, the functional S_1 is represented by $h_1 = b_1$. The function

$$M_1 \colon D_1 \times D_1 \to \mathbb{R}, \quad M_1(x,y) = \sum_{i=1}^{2} e_i(x)e_i(y),$$

is a reproducing kernel on D_1. The reproducing kernel K_1 of F_1 satisfies

$$K_1(x,y) = \sum_{i=1}^{3} b_i(x)b_i(y) = \left(\sum_{i=1}^{2} e_i(x)e_i(y)\right)^2 = M_1(x,y)^2$$

for all $x, y \in D_1$. Moreover, we have $h_1(x) = \frac{1}{2}\sqrt{2}M_1(x,x)$ for all $x \in D_1$. Therefore, also $\mathbf{K}_d(x,y) = M_d(x,y)^2$ and $\mathbf{h}_d(x) = 2^{-d/2}\,\mathbf{M}_d(x,x)$ for $x, y \in \mathbf{D}_d$, where $\mathbf{M}_d$ is the d-times tensor product of M_1 and $\mathbf{h}_d$ is the d-times tensor product of h_1. By Proposition 2 the matrix

$$\left(\mathbf{K}_d(x_j,x_k) - n^{-1}\,2^d\,\mathbf{h}_d(x_j)\mathbf{h}_d(x_k)\right)_{j,k\le n}$$

is positive semi-definite for all $x_1, \ldots, x_n \in \mathbf{D}_d$. Proposition 1 yields that

$$e(n, \mathbf{S}_d)^2 \ge 1 - n\,2^{-d}.$$

□

We now consider several applications of this result and start with a general remark. As in the proof of Theorem 3 these examples are finite-dimensional, F_1 has dimension three. Of course the lower bounds are valid for all larger Hilbert spaces with the same norm on the subspace from F_1. A RKHS is equivalently given by the scalar product or the kernel or a complete orthonormal system. Unfortunately, there is no simple way to compute the scalar product if the kernel is given, or vice versa. Because of the form of our result, it is convenient to start with three vectors b_i from F_1 and to claim that they are orthonormal. This *defines* the scalar product (and the kernel) for this three-dimensional space, though it is possible to extend the scalar product to larger spaces in many different ways. Because of the very specific form of the orthonormal system that is required in our result, we are *not* free to choose the scalar product and hence, later, will have to work with Sobolev spaces with a non-standard norm or scalar product.

This means that all examples of this section will be specified only by defining two functions e_1 and e_2. These functions immediately define a 3-dimensional space F_1 (with orthonormal basis e_1^2, e_2^2 and $\sqrt{2}e_1e_2$ or equivalently b_1, b_2 and b_3 as above) and a linear functional S_1 with representer $h_1 = b_1$ on F_1. There are always many ways of writing down the norm of F_1 (and thus many ways to interpret F_1 as a subspace of some larger space) and we will provide some of them. But since this is just for the purpose of interpretation, we will not always provide the (tedious) computations.

3.1 Trigonometric polynomials of degree 1

This example is already contained in Vybíral [22]; now we can see it as an application of the general Theorem 3. It is defined by the choice $e_1(x) = 2^{1/4}\cos(\pi x)$ and $e_2(x) = 2^{1/4}\sin(\pi x)$ on $[0, 1]$. Then one obtains $b_1 = h_1 = 1$ and $b_2(x) = \cos(2\pi x)$ and $b_3(x) = \sin(2\pi x)$. The functions b_i are orthonormal by definition and we have many ways to define matching norms on larger spaces. One way to write down the norm on the space F_1 of trigonometric polynomials of degree 1 on the interval $[0, 1]$ is

$$\|f\|^2 = \|f\|_2^2 + \frac{1}{4\pi^2}\|f'\|_2^2.$$

One only has to check that the b_i's indeed are orthonormal with respect to this Sobolev Hilbert space. For $d \in \mathbb{N}$ we take the tensor product space of the three-dimensional F_1 with the kernel

$$\mathbf{K}_d(x, y) = \prod_{i=1}^{d}(1 + \cos(2\pi(x_i - y_i))).$$

We obtain $\Gamma_d = \sup_x \mathbf{K}_d(x, x)^{1/2} = 2^{d/2}$ and Γ_d is the norm of the embedding of $\mathbf{F}_d$ into the space of continuous functions with the sup norm, see Lemma 5 of [13]. Hence functions in the unit ball of $\mathbf{F}_d$ may take large values if d is large, but the integral is bounded by one. By applying Theorem 3 we obtain the following result of [22] that solved an open problem of [11], see also [6].

Corollary 4. *Let F_1 be the RKHS on $[0, 1]$ with the orthonormal system 1, $\cos(2\pi x)$ and $\sin(2\pi x)$. Then the integration problem $\mathbf{S}_d = \langle \cdot, 1\rangle$ on the* *tensor product space* $\mathbf{F}_d$ *satisfies*

$$e(n, \mathbf{S}_d)^2 \geq 1 - n\, 2^{-d}.$$

In particular, it suffers from the curse of dimensionality.

Remark 2. The same vector space with dimension 3^d was studied by Sloan and Woźniakowski [21] who were mainly interested in the Korobov class $E_{\alpha,d}$ given by all functions with Fourier coefficients $\hat{f}(h)$ such that

$$|\hat{f}(h)| \leq \prod_{i=1}^{d} (\max(1, |h_i|))^{-\alpha},$$

where α might be large. The authors of [21] proved that the optimal worst case error for this class is 1 for $n < 2^d$. This holds for the whole Korobov class and also for the subset of trigonometric polynomials of degree 1 (which is larger than the unit ball considered in Corollary 4). For these polynomials the error is zero for $n = 2^d$, since a product rule with 2^d evaluations is exact.

As a by-product the authors of [21] also obtain the fact that exactly 2^d function values are needed to obtain an exact quadrature formula for this space of polynomials. This last property also follows from Corollary 4 and is of course independent of the used norm.

3.2 Gaussian integration for polynomials of degree 2

Let F_1 be the space of polynomials on $\mathbb{R}$ with degree at most 2, equipped with the scalar product

$$\langle f, g \rangle = f(0)g(0) + \frac{1}{2} f'(0)g'(0) + \frac{1}{4} \int_{\mathbb{R}} f''(x)g''(x)\, \mathrm{d}\mu_1(x),$$

where μ_1 is the standard Gaussian measure on $\mathbb{R}$. We consider the integration problem

$$S_1 \colon F_1 \to \mathbb{R}, \quad S_1(f) = \int_{\mathbb{R}} f(x)\, \mathrm{d}\mu_1(x).$$

The tensor product problem for $d \in \mathbb{N}$ is given by the functional

$$\mathbf{S}_d \colon F_d \to \mathbb{R}, \quad \mathbf{S}_d(f) = \int_{\mathbb{R}^d} f(x)\, \mathrm{d}\mu_d(x),$$

on the tensor product space $\mathbf{F}_d$, which consists of all d-variate polynomials of degree 2 or less in every variable. Here, μ_d is the standard Gaussian measure on $\mathbb{R}^d$. By Theorem 3, this problem suffers from the curse of dimensionality. To see this, it is enough to choose $e_1(x) = 1$ and $e_2(x) = x$. We observe that the functions $e_1^2 = 1$, $e_2^2 = x^2$ and $\sqrt{2}e_1e_2 = \sqrt{2}x$ are orthonormal in F_1 and that $S_1(e_i^2) = 1$ and $S_1(e_1e_2) = 0$. For this, note that $S_1(x^j)$ is the jth moment of a standard Gaussian variable. In particular, $e(0, S_1) = \sqrt{2}$. Thus, the functional $S_1' = 2^{-1/2}S_1$ satisfies the conditions of Theorem 3 and an application of the theorem immediately yields the following.

Corollary 5. *Take the RKHS F_1 on $\mathbb{R}$ which is generated by the orthonormal system 1, x^2 and $\sqrt{2}x$. Then the problem $\mathbf{S}_d(f) = \int_{\mathbb{R}^d} f(x)\, d\mu_d(x)$ of Gaussian integration on the tensor product space $\mathbf{F}_d$ satisfies*

$$\frac{e(n, \mathbf{S}_d)^2}{e(0, \mathbf{S}_d)^2} \geq 1 - n\, 2^{-d}.$$

In particular, we obtain the curse of dimensionality and the fact that exactly $n = 2^d$ function values are needed for exact integration.

For the last statement it is enough to consider product Gauss formulas with $n = 2^d$ function values that are exact for all polynomials from $\mathbf{F}_d$.

3.3 Integration for polynomials of degree 2 on $[-\frac{1}{2}, \frac{1}{2}]$

Let F_1 be the space of polynomials on $\mathbb{R}$ with degree at most 2, defined on an interval of unit length. For convenience and symmetry we take the interval $[-1/2, 1/2]$. The univariate problem is given by $S_1(f) = \int_{-1/2}^{1/2} f(x)\,\mathrm{d}x$ and again we want to apply Theorem 3. For our construction we need $S_1(e_i^2) = \frac{1}{2}\sqrt{2}$ and $S_1(e_1e_2) = 0$. This is achieved with the choice $e_1 = 2^{-1/4}$ and $e_2(x) = 72^{1/4}x$. If we apply Theorem 3 then we obtain the following.

Corollary 6. *Take the RKHS F_1 on $I = [-1/2, 1/2]$ which is generated by the orthonormal system $\frac{1}{2}\sqrt{2}$, $\sqrt{72}x^2$ and $\sqrt{12}x$. Then the integration problem $\mathbf{S}_d(f) = \int_{I^d} f(x)\,\mathrm{d}x$ on $\mathbf{F}_d$ satisfies*

$$e(n, \mathbf{S}_d)^2 \geq 1 - n\, 2^{-d}.$$

In particular, we obtain the curse of dimensionality and the fact that exactly $n = 2^d$ function values are needed for the exact integration.

For the last statement it is again enough to consider product Gauss formulas with $n = 2^d$ function values that are exact for all polynomials from $\mathbf{F}_d$.

Observe that we are forced by our approach to take this norm on F_1 and on the tensor product space. We admit that the norm on F_1 is not a standard Sobolev norm, actually it looks rather artificial. The norm in F_1 is a weighted ℓ_2-norm of Taylor coefficients. For $d = 1$ and $f(x) = ax^2 + bx + c$ we obtain the norm

$$\|f\|^2 = \frac{a^2}{72} + \frac{b^2}{12} + 2c^2.$$

It can also be written in the form

$$\begin{aligned}\|f\|^2 &= 2f(0)^2 + \frac{1}{12}f'(0)^2 + \frac{1}{288}f''(0)^2 \\ &= 2f(0)^2 + \frac{1}{12}f'(0)^2 + \frac{1}{288}\int_{-1/2}^{1/2} f''(x)^2 \,\mathrm{d}x\end{aligned}$$

and so looks at least a little like a Sobolev norm. By our approach, we are not free to choose the norm and obtain lower bounds only for very specific norms. For the given norm we obtain

$$\Gamma = \sup_{x\in I} K_1(x,x)^{1/2} = 8^{1/2}$$

and Γ^d is the norm of the embedding of $\mathbf{F}_d$ into the space of continuous functions with the sup norm. Hence functions in the unit ball of $\mathbf{F}_d$ may take large values if d is large, but the integral is bounded by one.

3.4 Integration of functions with zero boundary conditions

As another application of Theorem 3, we consider the integration of smooth functions with zero on the boundary. For that sake, let $e_1(x) = 2^{1/4}\sin(\pi x)$ and $e_2(x) = 2^{1/4}\sin(2\pi x)$ for $x \in [0,1]$. Further, let F_1 be a three-dimensional space of functions on $[0,1]$, such that the system

$$\begin{aligned}e_1^2(x) &= \sqrt{2}\sin^2(\pi x), \\ e_2^2(x) &= \sqrt{2}\sin^2(2\pi x), \\ \sqrt{2}e_1(x)e_2(x) &= 2\sin(\pi x)\sin(2\pi x)\end{aligned} \tag{2}$$

forms an orthonormal basis of F_1. The norm on F_1 is uniquely determined and it can be expressed for example by

$$\|f\|^2 = \frac{1}{3} f(1/2)^2 + \frac{1}{12\pi^2} \int_0^1 (1 + 4\sin^2(2\pi x)) f'(x)^2 dx$$

or

$$\|f\|^2 = \frac{1}{2} f(1/2)^2 + \frac{1}{16\pi^2} f'(1/2)^2 + \frac{1}{128\pi^4} [f''(1/4) + f''(3/4)]^2,$$

which coincide on F_1. We consider the integration problem on F_1 defined by

$$S_1 : F_1 \to \mathbb{R}, \quad S_1(f) = \int_0^1 f(x)\,\mathrm{d}x$$

and its tensor product version $\mathbf{S}_d$ on $\mathbf{F}_d$. We observe that $S_1(e_1^2) = S_1(e_2^2) = \sqrt{2}/2$ and $S_1(e_1 e_2) = 0$.

Corollary 7. *Let F_1 be a three-dimensional RKHS on $[0,1]$ such that the functions in (2) form its orthonormal basis. Then the integration problem $\mathbf{S}_d(f) = \int_{[0,1]^d} f(x)\,\mathrm{d}x$ satisfies*

$$e(n, \mathbf{S}_d)^2 \geq 1 - n\, 2^{-d},$$

i.e. it suffers from the curse of dimensionality.

Remark 3. Let us observe that every $f \in F_1$ satisfies $f(0) = f(1) = f'(0) = f'(1) = 0$. This means that the functions from F_d and all their partial derivatives of order at most one in any of the variables vanish on the boundary of the unit cube.

3.5 Hilbert spaces with decomposable kernels

Another known method to prove lower bounds for tensor product functionals works for so called decomposable kernels and slight modifications, see [16, Chapter 11]. There is some intersection where our method and the decomposable kernel method both work.

Let F_1 be a RKHS on $D_1 \subset \mathbb{R}$ with reproducing kernel K_1. The kernel K_1 is called decomposable if there exists $a^* \in \mathbb{R}$ such that the sets

$$D_{(1)} = \{x \in D_1 \mid x \leq a^*\} \quad \text{and} \quad D_{(2)} = \{x \in D_1 \mid x \geq a^*\}$$

are nonempty and $K_1(x,y) = 0$ if $(x,y) \in D_{(1)} \times D_{(2)}$ or $(x,y) \in D_{(2)} \times D_{(1)}$. If K_1 is decomposable, then F_1 is an orthonormal sum of $F_{(1)}$ and $F_{(2)}$ consisting of the functions in F_1 with support in $D_{(1)}$ and $D_{(2)}$, respectively.

Choosing now arbitrary suitably scaled functions e_1 with support in $D_{(1)}$ and e_2 with support in $D_{(2)}$ such that $e_1^2 \in F_{(1)}$ and $e_2^2 \in F_{(2)}$, we automatically have that e_1^2 and e_2^2 are orthonormal in F_1 and $e_1 e_2 = 0$. The proof of Theorem 3 is easily adapted to this case and gives the next corollary.

Corollary 8. *Let F_1 be a RKHS on $D_1 \subset \mathbb{R}$ with decomposable reproducing kernel. Let e_1 and e_2 be as above and let $h_1 = \frac{1}{2}\sqrt{2}(e_1^2 + e_2^2)$. Then the tensor product problem $\mathbf{S}_d = \langle \cdot, \mathbf{h}_d \rangle$ satisfies*

$$e(n, \mathbf{S}_d)^2 \geq 1 - n\, 2^{-d}.$$

In particular, it suffers from the curse of dimensionality.

One particular example, where this corollary is applicable, is the centered L_2-discrepancy. Here F_1 consists of absolutely continuous functions f on $[0,1]$ with $f(1/2) = 0$ and $f' \in L_2[0,1]$. The norm of f in F_1 is the L_2-norm of f'. The kernel of F_1 is given by $K_1(x,y) = \big(|x-1/2| + |y-1/2| - |x-y|\big)/2$ and is decomposable with respect to $D_{(1)} = [0,1/2]$ and $D_{(2)} = [1/2,1]$. The normalized representer of the integration problem is $h_1(x) = \big(|x-1/2| - |x-1/2|^2\big)/2$. Then e_1^2 is the normalized restriction of h_1 to the interval $[0,1/2]$, similarly, e_2^2 is the normalized restriction of h_1 to the interval $[1/2,1]$. Since h_1 is nonnegative, such functions e_1 and e_2 exist.

Corollary 8 is a special case (for $\alpha = 1/2$) of [16, Theorem 11.8]. As such, it will not give any new results. Nevertheless, it seems appropriate to note the connection. It would be interesting to know if the full strength of [16, Theorem 11.8] can be obtained via this approach or the variants described in the next section.

3.6 Exact Integration

In all the examples from above, we obtained that

$$e(2^d - 1, \mathbf{S}_d) > 0,$$

so that it is not possible to compute the integral exactly with less than 2^d function values. One may ask whether this is the case for all nontrivial tensor product problems. Here a problem is called trivial if $e(1, S_1) = 0$. Then we have also $e(1, \mathbf{S}_d) = 0$ for all d. In general, the answer is "no", examples with $e(d, \mathbf{S}_d) > 0$ but $e(d+1, \mathbf{S}_d) = 0$ can be found in [16, Section 11.3] which is based on [12]. However, we obtain the following criterion under which the answer to the above question is "yes".

Corollary 9. *If there are functions e_1 and e_2 such that $e_1^2, e_2^2, e_1e_2 \in F_1$ are linearly independent with $S_1(e_i^2) \neq 0$ and $S_1(e_1e_2) = 0$, then*

$$e(2^d - 1, \mathbf{S}_d) > 0.$$

This follows from Theorem 3 since the statement on exact integration does not depend on the norm (resp. scalar product) of F_1. We may simply apply the theorem to the 3-dimensional space which is defined by the orthonormal basis $\tilde{e}_1^2, \tilde{e}_2^2$ and $\sqrt{2}\tilde{e}_1\tilde{e}_2$, where $\tilde{e}_i := c_i e_i$ with $c_i \in \mathbb{R}$ such that $S_1(\tilde{e}_i^2) = \sqrt{2}/2$.

4 Non-homogeneous tensor products

We now turn to tensor products whose factors F_i and h_i may be different for each $i \leq d$. We start with the following generalization of Theorem 3, which involves an additional parameter α_i.

Theorem 10. *For all $i \leq d$, let F_i be a RKHS and let S_i be a bounded linear functional on F_i with unit norm and nonnegative representer h_i. Assume that there are functions f_i and g_i in F_i and a number $\alpha_i \in (0, 1]$ such that (h_i, f_i, g_i) is orthonormal in F_i and $\alpha_i h_i = \sqrt{f_i^2 + g_i^2}$. Then the tensor product problem $\mathbf{S}_d = S_1 \otimes \ldots \otimes S_d$ satisfies for all $n \in \mathbb{N}$ that*

$$e(n, \mathbf{S}_d)^2 \geq 1 - n \prod_{i=1}^{d} (1 + \alpha_i^2)^{-1}.$$

Proof. Let D_i be the domain of the space F_i. Without loss of generality, we may assume that (h_i, f_i, g_i) is an orthonormal basis of F_i. In this case, the reproducing kernel of F_i is given by

$$K_i : D_i \times D_i \to \mathbb{R}, \quad K_i(x, y) = h_i(x)h_i(y) + f_i(x)f_i(y) + g_i(x)g_i(y).$$

Let us consider the functions

$$a_i = 2^{-1/4}\sqrt{\alpha_i h_i + f_i}, \qquad b_i = 2^{-1/4}\operatorname{sgn}(g_i)\sqrt{\alpha_i h_i - f_i}$$

on the domain D_i of F_i. These functions are well defined since $\alpha_i h_i \geq |f_i|$ and linearly independent since h and f are linearly independent. The function

$$M_i : D_i \times D_i \to \mathbb{R}, \quad M_i(x,y) = a_i(x)a_i(y) + b_i(x)b_i(y)$$

is a reproducing kernel on D_i and its diagonal is $\sqrt{2}\alpha_i h_i$. A simple computation shows for all $x, y \in D_i$ that

$$K_i(x,y) = M_i^2(x,y) + (1-\alpha_i^2)h_i(x)h_i(y).$$

Let now $\mathbf{K}_d$ be the reproducing kernel of the product space $\mathbf{F}_d = F_1 \otimes \ldots \otimes F_d$ with domain $\mathbf{D}_d = D_1 \times \ldots \times D_d$ and let $x_1, \ldots, x_n \in \mathbf{D}_d$. We have

$$\mathbf{K}_d(x_j, x_k) = \prod_{i=1}^d K_i(x_{j,i}, x_{k,i}) = \sum_{A \subset \{1,\ldots,d\}} \mathbf{K}_d^A(x_j, x_k),$$

where

$$\mathbf{K}_d^A(x_j, x_k) = \prod_{i \in A} M_i^2(x_{j,i}, x_{k,i}) \prod_{i \notin A} (1-\alpha_i^2) h_i(x_{j,i}) h_i(x_{k,i}).$$

The application of Proposition 2 yields

$$\left(\prod_{i \in A} M_i^2(x_{j,i}, x_{k,i}) \right)_{j,k=1}^n \succeq \frac{1}{n} \left(\prod_{i \in A} 2\alpha_i^2 h_i(x_{j,i}) h_i(x_{k,i}) \right)_{j,k=1}^n$$

and hence

$$\left(\mathbf{K}_d^A(x_j, x_k) \right)_{j,k=1}^n \succeq \frac{1}{n} \prod_{i \in A} 2\alpha_i^2 \prod_{i \notin A} (1-\alpha_i^2) \left(\mathbf{h}_d(x_j) \mathbf{h}_d(x_k) \right)_{j,k=1}^n,$$

where $\mathbf{h}_d = h_1 \otimes \ldots \otimes h_d$ is the representer of the product functional $\mathbf{S}_d$. Summing over all subsets A, we arrive at

$$\begin{aligned} \left(\mathbf{K}_d(x_j, x_k) \right)_{j,k=1}^n &= \sum_{A \subset \{1,\ldots,d\}} \left(\mathbf{K}_d^A(x_j, x_k) \right)_{j,k=1}^n \\ &\succeq \frac{1}{n} \sum_{A \subset \{1,\ldots,d\}} \prod_{i \in A} 2\alpha_i^2 \prod_{i \notin A} (1-\alpha_i^2) \left(\mathbf{h}_d(x_j) \mathbf{h}_d(x_k) \right)_{j,k=1}^n \\ &= \frac{1}{n} \prod_{i=1}^d (1+\alpha_i^2) \left(\mathbf{h}_d(x_j) \mathbf{h}_d(x_k) \right)_{j,k=1}^n. \end{aligned}$$

Now the statement follows by Proposition 1. □

As applications of this result, we consider spaces of trigonometric polynomials, Korobov spaces with increasing smoothness and Korobov spaces with product weights.

4.1 Trigonometric polynomials

The most prominent special case of Theorem 10 is the case of trigonometric polynomials of order at most one, i.e.,

$$h_i(x) = 1, \quad f_i(x) = \alpha_i \cos(2\pi x), \quad g_i(x) = \alpha_i \sin(2\pi x), \quad x \in [0,1], \tag{3}$$

which leads to the following result.

Corollary 11. *For all $1 \le i \le d$, let $\alpha_i \in (0,1]$ and let F_i be a RKHS on $[0,1]$ such that (h_i, f_i, g_i) defined in (3) are orthonormal in F_i. Then the integration problem $\mathbf{S}_d(f) = \int_{[0,1]^d} f(x)\mathrm{d}x$ satisfies on $\mathbf{F}_d = F_1 \otimes \cdots \otimes F_d$*

$$e(n, \mathbf{S}_d)^2 \ge 1 - n \prod_{i=1}^{d} (1 + \alpha_i^2)^{-1}.$$

Corollary 11 can be used to prove lower bounds for numerical integration on spaces with varying smoothness. Such classes were studied in [10, 19] for the approximation problem and upper bounds for numerical integration were provided in [16, Section 10.7.4]. We first recall the notation.

For a non-decreasing sequence of positive integers $r = (r_i)_{i=1}^{\infty}$ we consider the spaces H_{1,r_i} of 1-periodic real valued functions f defined on $[0,1]$ such that $f^{(r_i-1)}$ is absolutely continuous and $f^{(r_i)}$ belongs to $L_2([0,1])$. The norm on H_{1,r_i} is given by

$$\|f\|_{H_{1,r_i}}^2 = \left| \int_0^1 f(x)\mathrm{d}x \right|^2 + \int_0^1 |f^{(r_i)}(x)|^2 \mathrm{d}x.$$

The Korobov space of varying smoothness is then defined by

$$\mathbf{F}_d = H_{1,r_1} \otimes \cdots \otimes H_{1,r_d}.$$

If we set $\alpha_i = \sqrt{2} \cdot (2\pi)^{-r_i}$, then (h_i, f_i, g_i) from (3) form an orthonormal system in H_{1,r_i} and we denote their span in H_{1,r_i} by $\widetilde{H}_{1,r_i}$. We will prove lower bounds for $\mathbf{F}_d$ by actually considering only the 3^d-dimensional space

$$\widetilde{\mathbf{F}}_d = \widetilde{H}_{1,r_1} \otimes \cdots \otimes \widetilde{H}_{1,r_d}.$$

We consider the integration problem

$$\mathbf{S}_d(f) = \int_{[0,1]^d} f(x)\mathrm{d}x, \quad f \in \widetilde{\mathbf{F}}_d.$$

We call the problem *polynomially tractable* if there are positive constants $C, p, q > 0$ such that

$$n(\varepsilon, \mathbf{S}_d) \le C d^p \varepsilon^{-q}$$

for all $\varepsilon > 0$ and $d \in \mathbb{N}$. We call it *strongly polynomially tractable* if we can choose $p = 0$ in this estimate. Moreover, the problem is called *weakly tractable* if

$$\lim_{\varepsilon^{-1}+d\to\infty} \frac{\ln n(\varepsilon, \mathbf{S}_d)}{\varepsilon^{-1}+d} = 0.$$

It was observed in [16, Section 10.7.4] (see also Corollary 10.5 there), that

- if $L^{\sup} := \limsup_{i\to\infty} \frac{\ln(i)}{r_i} < 2\ln(2\pi)$, then integration on $\mathbf{F}_d$ is strongly polynomially tractable;
- if $L^{\sup} < +\infty$, then integration on $\mathbf{F}_d$ is weakly tractable.

We complement this by showing lower bounds for numerical integration on $\widetilde{\mathbf{F}}_d$ (which of course also apply to the larger space $\mathbf{F}_d$). By Corollary 11, we obtain the estimate

$$e(n, \mathbf{S}_d)^2 \ge 1 - n \prod_{i=1}^{d} (1+\alpha_i^2)^{-1} = 1 - n \prod_{i=1}^{d} (1 + 2\cdot(2\pi)^{-2r_i})^{-1}. \tag{4}$$

Corollary 12. *For $d \ge 2$, let $\mathbf{F}_d$ be the Korobov space of varying smoothness on $[0,1]^d$ given by the sequence $r = (r_i)_{i=1}^\infty$ and let $\widetilde{\mathbf{F}}_d$ be its 3^d-dimensional subspace of trigonometric polynomials of order at most one in each variable.*

(i) If $r = (r_i)_{i=1}^\infty$ is bounded, then numerical integration on $\widetilde{\mathbf{F}}_d$ (and hence also on $\mathbf{F}_d$) suffers from the curse of dimension.

(ii) If $L^{\inf} := \liminf_{i\to\infty} \frac{\ln i}{r_i} = \infty$, then numerical integration on $\widetilde{\mathbf{F}}_d$ (and hence also on $\mathbf{F}_d$) satisfies for any $\varepsilon, \beta > 0$ that

$$n(\varepsilon, \mathbf{S}_d) \ \ge\ c_{\varepsilon,\beta} \exp\left(d^{1-\beta}\right).$$

(iii) If $L^{\inf} > 2\ln(2\pi)$, then numerical integration on $\widetilde{\mathbf{F}}_d$ (and hence also on $\mathbf{F}_d$) is not polynomially tractable.

Proof. The proof is a direct consequence of (4). If $r_i \le R < \infty$ for all $i \in \mathbb{N}$, then

$$e(n, \mathbf{S}_d)^2 \ge 1 - n(1 + 2 \cdot (2\pi)^{-2R})^{-d}.$$

This implies $n(\varepsilon, \mathbf{S}_d) \ge (1-\varepsilon^2)(1+2\cdot(2\pi)^{-2R})^d$ and finishes the proof of (i).

To prove (ii) and (iii), we observe that there is some $0 < \beta < 1$ and $i_0 \in \mathbb{N}$ such that $2r_i \ln(2\pi) \le \beta \ln(i)$ for $i \ge i_0$. In the case of (ii) we can even find such $i_0 = i_0(\beta)$ for any $0 < \beta < 1$. Consequently, for d large enough,

$$\begin{aligned} n(\varepsilon, \mathbf{S}_d) &\ge (1-\varepsilon^2) \prod_{i=1}^{d} (1 + 2(2\pi)^{-2r_i}) \ge (1-\varepsilon^2) \prod_{i=i_0}^{d} (1 + 2i^{-\beta}) \\ &\ge (1-\varepsilon^2) \prod_{i=i_0}^{d} \exp(i^{-\beta}) = (1-\varepsilon^2) \exp\Big(\sum_{i=i_0}^{d} i^{-\beta}\Big) \\ &\ge (1-\varepsilon^2) \exp(c_\beta d^{1-\beta}), \end{aligned}$$

which shows both (ii) and (iii). $\square$

4.2 Korobov spaces with product weights

In a quite similar manner, Corollary 11 can be used to re-prove the lower bounds for numerical integration on Korobov spaces with product weights, see [3] or [16, Section 16.8]. Again, we first recall the necessary notation, see [15, Appendix A] for details. For a real parameter $s > 1/2$, we define

$$\varrho_{1,s,\gamma}(h) = \begin{cases} 1, & h = 0, \\ \dfrac{|2\pi h|^{2s}}{\gamma}, & h \in \mathbb{Z} \setminus \{0\}. \end{cases}$$

The space $H_{1,s,\gamma}$ of square-integrable functions on $[0,1]$ is defined by the norm

$$\|f\|^2_{H_{1,s,\gamma}} = \sum_{h \in \mathbb{Z}} \varrho_{1,s,\gamma}(h) |\hat{f}(h)|^2,$$

where

$$\hat{f}(h) = \int_0^1 \exp(-2\pi \mathrm{i}\, hx) f(x) \mathrm{d}x, \quad h \in \mathbb{Z}$$

are the Fourier coefficients of f and $\mathrm{i} = \sqrt{-1}$ is the imaginary unit.

If $\gamma = (\gamma_{d,j})_{d\in\mathbb{N},1\le j\le d}$ is a sequence of positive weights, the weighted Korobov space (with product weights γ) $H_{d,s,\gamma}$ is defined as the tensor product

$$H_{d,s,\gamma} = H_{1,s,\gamma_{d,1}} \otimes \cdots \otimes H_{1,s,\gamma_{d,d}}.$$

If $\alpha_{d,j} = \sqrt{2\gamma_{d,j}} \cdot (2\pi)^{-s}$, the functions $(1, \alpha_{d,j}\cos(2\pi x), \alpha_{d,j}\sin(2\pi x))$ are orthonormal in $H_{1,s,\gamma_{d,j}}$. We denote their linear span in $H_{1,s,\gamma_{d,j}}$ by $\widetilde{H}_{1,s,\gamma_{d,j}}$ and

$$\widetilde{H}_{d,s,\gamma} = \widetilde{H}_{1,s,\gamma_{d,1}} \otimes \cdots \otimes \widetilde{H}_{1,s,\gamma_{d,d}}.$$

Using Corollary 11, we can re-prove (in a rather straightforward way) the lower bounds of Theorem 16.16 in [16]. Moreover, we show that the same lower bounds apply also to the much smaller subspaces $\widetilde{H}_{d,s,\gamma}$.

Proposition 13. *Let $\mathbf{S}_d(f) = \int_{[0,1]^d} f(x)\mathrm{d}x$ denote the multivariate integration problem defined over the sequence of Korobov spaces $H_{d,s,\gamma}$, where $s > 1/2$ and $\gamma = (\gamma_{d,j})_{d\in\mathbb{N},1\le j\le d}$ is a bounded sequence. Let $\widetilde{H}_{d,s,\gamma}$ be the 3^d-dimensional subspaces of trigonometric polynomials of degree at most one in each variable in $H_{d,s,\gamma}$.*

(i) If $(\mathbf{S}_d)$ is strongly polynomially tractable on $\widetilde{H}_{d,s,\gamma}$, then

$$\sup_{d\in\mathbb{N}} \sum_{j=1}^{d} \gamma_{d,j} < \infty.$$

(ii) If $(\mathbf{S}_d)$ is polynomially tractable on $\widetilde{H}_{d,s,\gamma}$, then

$$\limsup_{d\to\infty} \frac{\sum_{j=1}^{d} \gamma_{d,j}}{\ln(d+1)} < \infty.$$

(iii) If $(\mathbf{S}_d)$ is weakly tractable on $\widetilde{H}_{d,s,\gamma}$, then

$$\lim_{d\to\infty} \frac{1}{d} \sum_{j=1}^{d} \gamma_{d,j} = 0.$$

Proof. We put $\alpha_{d,j} = \sqrt{2\gamma_{d,j}} \cdot (2\pi)^{-s}$ and obtain by Corollary 11

$$n(\varepsilon, \mathbf{S}_d) \ge (1-\varepsilon^2) \prod_{j=1}^{d} (1+\alpha_{d,j}^2) = (1-\varepsilon^2) \prod_{j=1}^{d} \Big(1 + 2\gamma_{d,j}(2\pi)^{-2s}\Big).$$

If $(\mathbf{S}_d)$ is strongly polynomially tractable, we observe from

$$n(\varepsilon, \mathbf{S}_d) \geq (1-\varepsilon^2) \cdot 2 \cdot (2\pi)^{-2s} \sum_{j=1}^{d} \gamma_{d,j}$$

that $\sum_{j=1}^{d} \gamma_{d,j}$ must be uniformly bounded in $d \in \mathbb{N}$.

If $(\mathbf{S}_d)$ is polynomially tractable or weakly tractable, we use the boundedness of γ to estimate

$$\ln n(\varepsilon, \mathbf{S}_d) \geq \ln(1-\varepsilon^2) + \sum_{j=1}^{d} \ln\Big(1 + 2\gamma_{d,j}(2\pi)^{-2s}\Big) \geq \ln(1-\varepsilon^2) + C \sum_{j=1}^{d} \gamma_{d,j}.$$

This estimate proves both (ii) and (iii). □

5 New variants of Schur's Theorem

In this section we present several variants of the uniform lower bound for the Schur product obtained in [22] and several consequences for the tractability of numerical integration.

5.1 Modifications of Schur's Theorem

The first generalization of Proposition 2 deals with matrices with reduced rank. This was first observed in [8].

Theorem 14 ([8])**.** *Let $M \in \mathbb{R}^{n \times n}$ be a positive semi-definite matrix with rank r. Then*

$$M \circ M \succeq \frac{1}{r} (\mathrm{diag} M)(\mathrm{diag} M)^T.$$

Proof. If M has also ones on the diagonal, then we may use the truncated singular value decomposition of M and observe that M is a Gram matrix of a set of n vectors in $\mathbb{S}^{r-1}$. The rest of the proof then follows in the same way as in Proposition 2. Alternatively, one may follow [22] but write $M = AA^T$, where $A \in \mathbb{R}^{n \times r}$. □

The next version deals with the Schur product of two possibly different matrices $M \neq N$. In this sense, it addresses a problem left open in [22].

Theorem 15. *Let $M, N \in \mathbb{R}^{n\times n}$ be positive semi-definite matrices with $M = AA^T$ and $N = BB^T$ with $A, B \in \mathbb{R}^{n\times D}$ and $D \geq \max(\operatorname{rank}(M), \operatorname{rank}(N))$. Then, for every $c \in \mathbb{R}^n$,*

$$\sum_{j,k=1}^{n} c_j c_k M_{j,k} N_{j,k} \geq \frac{1}{D}\Big(\sum_{j=1}^{n} c_j \langle A^j, B^j\rangle\Big)^2, \tag{5}$$

where A^j, B^j are the rows of A and B, respectively.

Proof. The proof is again similar to [22]. We write

$$\begin{aligned}
\sum_{j,k=1}^{n} c_j c_k M_{j,k} N_{j,k} &= \sum_{j,k=1}^{n} c_j c_k \sum_{l=1}^{D} A_{j,l} A_{k,l} \sum_{m=1}^{D} B_{j,m} B_{k,m} \\
&= \sum_{l,m=1}^{D} \Big(\sum_{j=1}^{n} c_j A_{j,l} B_{j,m}\Big)^2 \geq \sum_{l=1}^{D} \Big(\sum_{j=1}^{n} c_j A_{j,l} B_{j,l}\Big)^2 \\
&\geq \frac{1}{D}\Big(\sum_{j=1}^{n} c_j \sum_{l=1}^{D} A_{j,l} B_{j,l}\Big)^2.
\end{aligned}$$

□

Remark 4. Using $(AB^T)_{j,j} = \langle A^j, B^j\rangle$, the estimate (5) can be written as

$$M \circ N \succeq \frac{1}{D}(\operatorname{diag}(AB^T))(\operatorname{diag}(AB^T))^T.$$

The last generalization of Schur's Theorem, that, in a sense, combines Theorem 14 and Theorem 15, is the one we shall use later on.

Theorem 16. *Let $M \in \mathbb{R}^{n\times n}$ be a positive semi-definite matrix with rank r. Let $M = AA^T = BB^T$ with $A, B \in \mathbb{R}^{n\times D}$ for some $D \geq r$. Then, for every $c \in \mathbb{R}^n$,*

$$\sum_{j,k=1}^{n} c_j c_k M_{j,k}^2 \geq \frac{1}{2r}\Big(\sum_{j=1}^{n} c_j \langle A^j, B^j\rangle\Big)^2, \tag{6}$$

where $A^j, B^j \in \mathbb{R}^D$ are the rows of A and B, respectively.

Proof. We show that there exist two matrices $G, H \in \mathbb{R}^{n\times 2r}$ with rows denoted by G^j and H^j, respectively, such that $M = GG^T = HH^T$ and

$\langle G^j, H^j \rangle = \langle A^j, B^j \rangle$ for every $j = 1, \ldots, n$. The proof then follows by an application of Theorem 15 with $M = N$ and $2r$ instead of r.

Using the singular value decomposition theorem, we can write $A = U\Sigma V^T$ and $B = U\Sigma W^T$, where $U \in \mathbb{R}^{n\times r}, \Sigma \in \mathbb{R}^{r\times r}$ and $V, W \in \mathbb{R}^{D\times r}$. Here, U, V and W have orthonormal columns and Σ is a diagonal matrix. Furthermore,

$$\langle A^j, B^j \rangle = (AB^T)_{j,j} = e_j^T (U\Sigma V^T)(W\Sigma U^T) e_j = \varepsilon_j^T V^T W \varepsilon_j,$$

where $\varepsilon_j = \Sigma U^T e_j \in \mathbb{R}^r$ and $(e_j)_{j=1}^n$ is the canonical basis of $\mathbb{R}^n$. In the same way, we are looking for $G = U\Sigma X^T$ and $H = U\Sigma Z^T$ with matrices $X, Z \in \mathbb{R}^{2r\times r}$ with orthonormal columns and

$$\langle G^j, H^j \rangle = \varepsilon_j^T X^T Z \varepsilon_j = \varepsilon_j^T V^T W \varepsilon_j = \langle A^j, B^j \rangle, \quad j = 1, \ldots, n. \tag{7}$$

The matrix $V^T W$ is formed by the scalar products of the column vectors of V and W, respectively. Using an orthogonal projection onto their common linear span (which has dimension at most $2r$), we can find $X, Z \in \mathbb{R}^{2r\times r}$ such that $X^T Z = V^T W$, which is even stronger than (7). $\square$

5.2 Applications to numerical integration

Theorem 16 allows us to extend Theorem 3 to a larger class of tensor product problems with $e(0, \mathbf{S}_d) = \|\mathbf{h}_d\| = 1$.

Theorem 17. *Let M be a reproducing kernel on a set D and let $K = M^2$. Denote by $H(M)$ and $H(K)$ the Hilbert spaces with reproducing kernel M and K, respectively. Let $(b_\ell)_{\ell\in I}$ and $(\tilde{b}_\ell)_{\ell\in I}$ be two orthonormal bases of $H(M)$ and*

$$g = \sum_{\ell\in I} b_\ell \tilde{b}_\ell \in H(K).$$

We consider the normalized problem $S = \langle \cdot, h \rangle$ with $h = g/\|g\|$ on $H(K)$. Then

$$e(n, S)^2 \geq 1 - \frac{2n}{\|g\|^2}.$$

Proof. For any $x, y \in D$, we have

$$M(x, y) = \sum_{\ell\in I} b_\ell(x) b_\ell(y) = \sum_{\ell\in I} \tilde{b}_\ell(x) \tilde{b}_\ell(y).$$

Let $x_1, \dots, x_n \in D$ and let $M = (M(x_j, x_k))_{j,k \le n}$. Then

$$M = BB^T = \tilde{B}\tilde{B}^T,$$

where $B = (b_\ell(x_j))_{j \le n, \ell \in I}$ and $\tilde{B} = (\tilde{b}_\ell(x_j))_{j \le n, \ell \in I}$. Theorem 16 yields that

$$\sum_{j,k=1}^n c_j c_k M_{j,k}^2 \ge \frac{1}{2n} \Big(\sum_{j=1}^n c_j (B\tilde{B}^T)_{jj} \Big)^2 = \frac{\|g\|^2}{2n} \Big(\sum_{j=1}^n c_j h(x_j) \Big)^2$$

and thus the desired lower bound follows from Proposition 1. $\square$

Let us observe that Theorem 3 is obtained by considering particular orthonormal bases of $H(M_d)$. Namely, we take

$$b_\ell(x) = \prod_{i=1}^d e_{\ell_i}(x_i) \quad \text{for} \quad \ell \in \{1,2\}^d \tag{8}$$

and $\tilde{b}_\ell = b_\ell$, $\ell \in \{1,2\}^d$. Then we have

$$g(x) = \prod_{i=1}^d \left(e_1(x_i)^2 + e_2(x_i)^2 \right).$$

and we obtain Theorem 3 (up to a factor 2).

Another interesting choice of $(b_\ell)_{\ell \in I}$ and $(\tilde{b}_\ell)_{\ell \in I}$ of $H(M_d)$ is the following. We take again b_ℓ defined by (8) and

$$\tilde{b}_\ell(x) = \prod_{i=1}^d \tilde{e}_{\ell_i}^{(i)}(x_i) \quad \text{for} \quad \ell \in \{1,2\}^d \tag{9}$$

where

$$\begin{pmatrix} \tilde{e}_1^{(i)} \\ \tilde{e}_2^{(i)} \end{pmatrix} = U_i \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$$

and $U_i \in \mathbb{R}^{2 \times 2}$ is an orthogonal matrix.

If U_i is the identity matrix, we obtain $\tilde{e}_1^{(i)} = e_1$, $\tilde{e}_2^{(i)} = e_2$ and $\tilde{e}_1^{(i)} \cdot e_1 + \tilde{e}_2^{(i)} \cdot e_2 = e_1^2 + e_2^2$. If, on the other hand, we choose

$$U_i = \begin{pmatrix} \cos\varphi_i & \sin\varphi_i \\ \sin\varphi_i & -\cos\varphi_i \end{pmatrix}, \quad \varphi_i \in [0, 2\pi]$$

being a reflection across a line with angle $\varphi_i/2$, we obtain

$$\tilde{e}_1^{(i)} \cdot e_1 + \tilde{e}_2^{(i)} \cdot e_2 = \cos\varphi_i \cdot (e_1^2 - e_2^2) + 2\sin\varphi_i \cdot e_1 e_2.$$

Of course, we can mix these two examples by taking a different choice of U_i for each dimension $i \le d$, which leads to the following result.

Corollary 18. *Let F_1 be a RKHS on D_1. Assume that there are functions e_1 and e_2 on D_1 such that e_1^2, e_2^2 and $\sqrt{2}e_1e_2$ are orthonormal in F_1. Let*

$$\mathbf{h}_d(x) = \prod_{i=1}^d h_i(x_i),$$

where $h_i \in \operatorname{span}\{e_1^2 + e_2^2\} \cup \operatorname{span}\{e_1^2 - e_2^2, e_1e_2\}$ has unit norm $\|h_i\| = 1$. Then the tensor product problem $\mathbf{S}_d = \langle \cdot, \mathbf{h}_d \rangle$ satisfies

$$e(n, \mathbf{S}_d)^2 \ge 1 - n\, 2^{-d+1}.$$

In particular, it suffers from the curse of dimensionality.

For the next result we take again the space of trigonometric polynomials of degree 1, see Corollary 4.

Corollary 19. *Let F_1 be the RKHS on $[0,1]$ with the orthonormal system 1, $\cos(2\pi x)$ and $\sin(2\pi x)$. Let $d \ge 2$ and let $\{\varphi_i\}_{i=1}^\infty \subset [0, 2\pi]$ be a bounded sequence. Let*

$$\mathbf{h}_d(x) = \prod_{i=1}^d h_i(x_i),$$

where

$$h_i(x_i) = \cos\varphi_i \cdot \cos(2\pi x_i) + \sin\varphi_i \cdot \sin(2\pi x_i) = \cos(2\pi x_i - \varphi_i) \tag{10}$$

or $h_i = 1$. Then the corresponding problem $\mathbf{S}_d = \langle \cdot, \mathbf{h}_d \rangle$ satisfies

$$e(n, \mathbf{S}_d)^2 \ge 1 - n\, 2^{-d+1}$$

and the problem suffers from the curse of dimensionality.

Remark 5. Let us reformulate Corollary 19 as an integration problem. As in Section 3.1, we denote again $e_1(x) = 2^{1/4}\cos(\pi x)$ and $e_2 = 2^{1/4}\sin(\pi x)$ on $[0,1]$. Let $\varphi \in [0, 2\pi]$ and let $h(x) = \cos(2\pi x - \varphi), x \in [0,1]$, cf. (10). Then

$$h(x) = \cos\varphi \cdot \frac{e_1^2(x) - e_2^2(x)}{\sqrt{2}} + \sin\varphi \cdot \sqrt{2} e_1(x) e_2(x).$$

Consequently, if we define $S(f) = \langle f, h\rangle$ for $f \in F_1$, it satisfies

$$S(e_1^2) = \frac{\cos\varphi}{\sqrt{2}}, \quad S(e_2^2) = -\frac{\cos\varphi}{\sqrt{2}}, \quad S(\sqrt{2} e_1 e_2) = \sin\varphi$$

and we obtain

$$S(f) = 2\int_0^1 f(x)\cos(2\pi x - \varphi)\,\mathrm{d}x, \quad f \in F_1.$$

Similarly, if we denote in Corollary 19 by $I \subset \{1, \ldots, d\}$ those indices, for which h_i is given by (10), then

$$\mathbf{S}_d(f) = \langle f, \mathbf{h}_d\rangle = \int_{[0,1]^d} f(x) \prod_{i\in I} [2\cos(2\pi x_i - \varphi_i)]\,\mathrm{d}x.$$

We finish this section by a couple of open problems.

Open Problem 1. *We conjecture that Theorem 16 holds true with $\frac{1}{r}$ instead of $\frac{1}{2r}$ in* (6)*, see also [8, Theorem 1.9]. This would allow to improve the error bound in Corollary 18 and Corollary 19 to*

$$e(n, \mathbf{S}_d)^2 \geq 1 - n\,2^{-d}.$$

Open Problem 2. *Corollary 18 shows the curse for all problems $\mathbf{S}_d = \langle \cdot, \mathbf{h}_d\rangle$, where $\mathbf{h}_d = \bigotimes_{i=1}^d h_i$ is a tensor product with all components h_i being unit norm functions from either the span of $e_1^2 + e_2^2$ or from the span of $e_1^2 - e_2^2$ and $e_1 e_2$. Is the same true if we allow arbitrary $h_i \in F_1$?*

6 Randomly chosen sample points

We continue our analysis of high dimensional integration problems. In the previous sections, we mainly studied the quality of optimal sample points.

Optimal sample points are usually hard to find. In this section, we switch our point of view and ask for the quality of random point sets $\mathcal{X}_n = \{x_1, \ldots, x_n\}$, where the points $x_1, \ldots, x_n$ are independent and identically distributed in the domain of integration. With this, we continue the studies from [4, 5, 9] on the quality of random information. Like for optimal points, one can ask: How many random points do we need to solve the integration problem up to the error $\varepsilon > 0$? Does this number depend exponentially on d, i.e., do we have the curse for random information? We can use Proposition 1 to obtain the following result for Lebesgue integration on the unit cube (where the interval $[0,1]$ may clearly be replaced by any other interval of length 1).

Theorem 20. *Let F_1 be a RKHS on $[0,1]$ with (point-wise) non-negative and measurable kernel K_1 such that $1 \in F_1$ is the representer of the integral. We define*

$$\kappa = \operatorname*{ess\,sup}_{x \in [0,1]} K_1(x,x).$$

Let $\mathcal{X}_n$ be a set of n independent and uniformly distributed points in $[0,1]^d$. If $\kappa > 1$, then for any $\delta > 0$ there are constants $c > 0$ and $a > 1$ such that

$$\mathbb{E}\left[e(\mathcal{X}_n, \mathbf{S}_d)^2\right] \geq 1 - \delta$$

for all $n, d \in \mathbb{N}$ with $n \leq ca^d$. If $\kappa = 1$, then $e(\mathcal{X}_n, \mathbf{S}_d) = 0$ holds almost surely for all $n, d \in \mathbb{N}$.

This means that the tensor product problem for random information is either trivial or suffers from the curse of dimensionality. Note that the case $\kappa < 1$ does not occur. Moreover, if the kernel is continuous, the condition $\kappa > 1$ is equivalent to the claim that F_1 is at least two-dimensional. This will become apparent in the proof.

Proof. First note that the initial error is given by

$$e(0, S_1)^2 = \|1\|^2 = \langle 1, 1 \rangle = \int_0^1 1 \,\mathrm{d}x = 1,$$

so that the problem is properly normalized. Moreover, since $K_1(x, \cdot)$ can be written as the sum of the orthogonal functions 1 and $K_1(x, \cdot) - 1$, we have $K_1(x,x) = \|K_1(x, \cdot)\|^2 \geq \|1\|^2 = 1$ for all $x \in [0,1]$. Thus, $\kappa = 1$ means that $K_1(x,x) = 1$ almost everywhere. If $\kappa > 1$, then there is some $c_1 > 1$ such that the set of all $x \in [0,1]$ with $K_1(x,x) \geq c_1$ has Lebesgue-measure $p > 0$. We consider independent and uniformly distributed points $x_1, x_2, \ldots \in [0,1]^d$.

Let $\kappa = 1$. Then we have almost surely that $\mathbf{K}_d(x_1, x_1) = 1$ and the one-dimensional matrix $\mathbf{K}_d(x_1, x_1) - \alpha$ is positive semi-definite if, and only if, $\alpha \leq 1$. Proposition 1 yields that $e(x_1, \mathbf{S}_d) = 0$ as claimed.

Let now $\kappa > 1$. For all $i \in \mathbb{N}$, the number of coordinates of x_i with $K_1(x_{i,k}, x_{i,k}) \geq c_1$ is distributed according to the binomial distribution $B(d,p)$. Thus, the number of such coordinates is typically pd and greater than $pd/2$ with high probability. Namely, we have $\mathbf{K}_d(x_i, x_i) \geq c_1^{pd/2}$ with probability at least $1 - \exp(-p^2 d/2)$. We put $c_2 = c_1^{p/2}$ such that $c_1^{pd/2} = c_2^d$. For different indices $i, j \in \mathbb{N}$, Fubini's theorem yields that

$$\mathbb{E}\, \mathbf{K}_d(x_i, x_j) = \mathbb{E}\, \langle \mathbf{K}_d(x_i, \cdot), 1 \rangle = 1.$$

Using Markov's inequality and the non-negativity of $\mathbf{K}_d$, this implies that $\mathbf{K}_d(x_i, x_j) \leq c_2^d/(2n)$ with probability at least $1 - 2n/c_2^d$. By a union bound, all these inequalities hold simultaneously for all $i, j \leq n$ with probability at least $1 - n^3/c_2^d - n \exp(-p^2 d/2)$. In this case, we have

$$\begin{aligned}
&\left| \mathbf{K}_d(x_i, x_i) - \frac{c_2^d}{2n} \right| - \sum_{j \neq i} \left| \mathbf{K}_d(x_i, x_j) - \frac{c_2^d}{2n} \right| \\
&\qquad \geq \mathbf{K}_d(x_i, x_i) - \frac{c_2^d}{2n} - \sum_{j \neq i} \left(\mathbf{K}_d(x_i, x_j) + \frac{c_2^d}{2n} \right) > c_2^d - \frac{c_2^d}{2} - \frac{c_2^d}{2} = 0
\end{aligned}$$

for all $i \leq n$. Therefore, the matrix $(\mathbf{K}_d(x_i, x_j) - c_2^d/(2n))_{i,j \leq n}$ is diagonally dominant and hence positive definite by Gershgorin circle theorem [7, Theorem 6.1.1]. Proposition 1 implies that

$$e(\mathcal{X}_n, \mathbf{S}_d)^2 \geq 1 - \frac{2n}{c_2^d}$$

with the stated probability. This yields the assertion. □

Let us now consider two examples. The first example shows that random information may be much worse than optimal information. We consider the space F_1 of affine linear functions on $[0,1]$ with scalar product

$$\langle f, g \rangle = \langle f, g \rangle_2 + \langle f', g' \rangle_2.$$

The kernel of this space is given by

$$K_1(x,y) = 1 + \frac{12}{13}\left(x - \frac{1}{2}\right)\left(y - \frac{1}{2}\right).$$

The tensor product space $\mathbf{F}_d$ consists of all functions $f\colon [0,1]^d \to \mathbb{R}$ which are affine linear in each variable. This space satisfies the conditions of Theorem 20. On the other hand, the integral of any such function is given by its function value at the center of the cube. This means that the integration problem on $\mathbf{F}_d$ is trivial for optimal information, but suffers from the curse of dimensionality if we only have random information.

As another example, let us consider the integration problem on the space F_1 of polynomials with degree at most 2 with scalar product

$$\langle f, g\rangle = \langle f, g\rangle_2 + \langle f', g'\rangle_2 + \langle f'', g''\rangle_2.$$

The tensor product space $\mathbf{F}_d$ consists of polynomials of degree 2 or less in every variable. It was raised as an open problem in [16, Open Problem 44] whether the integration problem on the tensor product space $\mathbf{F}_d$ suffers from the curse of dimensionality. For optimal point sets, this question remains unsolved. For random point sets, Theorem 20 yields the curse of dimensionality. The assumptions are readily verified with the kernel K_1 given in [16]. In fact, the reproducing kernel even satisfies the condition

$$\inf_{x\in[0,1]} K_1(x,x) > 1. \tag{11}$$

One may ask whether this condition is already enough to obtain the curse of dimensionality for optimal information.

Open Problem 3. *Let F_1 be a RKHS of functions on $[0,1]$ with non-negative kernel K_1 satisfying (11) such that $1 \in F_1$ is the representer of the integral. Prove or disprove that Lebesgue integration on the tensor product space $\mathbf{F}_d$ with optimal information suffers from the curse of dimensionality.*

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