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A Multiple-Parameterization Approach for Local Stabilization of Constrained Takagi-Sugeno Fuzzy Systems with Nonlinear Consequents

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Abstract

This paper addresses the local stabilization of constrained nonlinear systems with input saturation described by Takagi-Sugeno fuzzy models with nonlinear consequents. To reduce the design conservativeness, we propose a new delayed multiple-parameterization control approach based on a nonquadratic Lyapunov function with multiple delayed fuzzy summations. Both input saturation and state constraints are explicitly taken into account in the control design procedure. This multiple-parameterization condition is given in terms of linear matrix inequalities. Compared to existing results, the new approach offers a unified and concise control framework to design both non-delayed and delayed multidimensional nonlinear fuzzy controllers. Numerical examples are provided to demonstrate the effectiveness of the proposed multiple-parameterization approach in both reducing the design conservativeness and enlarging the estimation of the domain of attraction.

Keywords: Takagi-Sugeno fuzzy systems, input saturation, local stabilization, delayed control, nonquadratic Lyapunov functions.

1. Introduction

In practical control applications, input and state constraints frequently arise not only from physical and technical restrictions but also from safety reasons [1]. Input constraints emerge from actuator saturation [2] whereas state constraints can be related to both safety/comfort specifications and modeling validity region. If these constraints are not properly taken into account in the control design, the closed-loop performance may be seriously degraded, possibly leading to instability in extreme cases [3]. Thus, performing the control design taking into account these constraints is essential to ensure closed-loop stability and performance. For instance, this control issue has been

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addressed for linear [2] and linear parameter varying [4] systems with input saturation, nonlinear systems such as vibrating flexible string systems with dead-zone nonlinearity and input constraint [5–7], vibrating flexible riser systems with input saturation [8, 9], and nonlinear systems with input saturation described by Takagi-Sugeno fuzzy models [1, 3].

Using Takagi-Sugeno (TS) fuzzy modeling [10, 11], nonlinear systems can be represented by convex relations of simpler local models, which are usually valid within a given domain of validity [12]. A well-known method to obtain an *exact* TS fuzzy representation for nonlinear systems is the sector nonlinearity approach [12, Chapter 2]. However, it should be stressed that for this approach, the required number of fuzzy rules *exponentially* grows according to the number of system nonlinearities. This may lead to some major drawbacks on both the control design (conservativeness of results) and the real-time implementation (numerical complexity) of TS fuzzy model-based control technique. To avoid an excessive number of fuzzy rules while preserving the convexity property of classical TS fuzzy systems, a new class of TS fuzzy systems with local nonlinear models (N-TS) was proposed in [13, 14] for continuous-time nonlinear systems and employed in [15] for the discrete-time counterparts. As aforementioned, both TS and N-TS fuzzy representations are generally valid within a given subset of the state-space. Hence, taking into account these state constraints is necessary to ensure the fuzzy controller performance [16]. However, these constraints are usually neglected in the fuzzy control literature, especially for N-TS fuzzy models. Therefore, the first motivation of this work is (i) to perform the fuzzy control design taking into account both state and input constraints.

By means of the direct Lyapunov method, it is possible to obtain sufficient conditions to design fuzzy controllers for both TS and N-TS fuzzy systems. In addition, taking advantage of the convexity of these models, the stabilization conditions can be formulated in terms of linear matrix inequalities (LMIs). It is known that control approaches based on common quadratic Lyapunov functions and the parallel distributed compensation (PDC) control law [12] can lead to conservative design results, especially for complex nonlinear systems requiring a large number of fuzzy rules when using TS modeling [17]. Introducing LMI slack decision variables [12, 18, 19] and/or taking into account the information of the membership functions in continuous-time case [20] have been demonstrated as effective solutions to reduce the design conservativeness. Another control approach capable to progressively reduce conservativeness is based on asymptotically necessary and sufficient (ANS) conditions [21–23], which, however, may quickly lead to an excessive number of LMIs and intractable numerical complexity [19].

An alternative solution to reduce design conservativeness is based on the choice of different Lyapunov function candidates. In this context, various classes of Lyapunov functions have been proposed to deal with the conservativeness issue, such as piecewise Lyapunov functions [24], line integral Lyapunov functions [25], and fuzzy (or nonquadratic) Lyapunov functions [26–28]. The effectiveness of fuzzy Lyapunov functions for conservativeness reduction has been mainly demonstrated in the discrete-time case, especially through multiple-parameterization approach [29]. Nevertheless, the latter approach presents similar drawbacks as ANS conditions, *i.e.,* possibly leading to numerical intractability. More recently, delayed multiple-parameterization approach

[17, 30, 31] has been proven to be an efficient framework to provide less conservative design conditions for TS fuzzy systems regarding delayed fuzzy membership functions in fuzzy controllers and nonquadratic Lyapunov functions. With the delayed control approach, the design conservativeness can be reduced without increasing substantially the multiple-parameterization dimension, thus avoiding excessive numerical complexity. However, both multiple-parameterization and delayed approaches have not been proposed for the class of N-TS fuzzy systems. Then, the second motivation for this work is (ii) to derive the control design conditions based on delayed multiple-parameterization approach for N-TS fuzzy systems.

Since the state and the input constraints are explicitly taken into account in the proposed control design, the local stability analysis is considered for the closed-loop N-TS fuzzy systems. Within the local stabilization context, it is of particular interest to characterize the closed-loop domain of attraction (DoA) [32]. In the case of TS or N-TS fuzzy models, finding a DoA estimation is useful to provide the state-space region where the fuzzy controllers can guarantee the closed-loop asymptotic stability. Then, the third motivation of this paper is (iii) to determine an estimation of the domain of attraction, as large as possible, for the closed-loop N-TS fuzzy systems.

Based on the aforementioned discussion, this paper investigates the local stabilization of constrained discrete-time nonlinear systems with input saturation represented by N-TS fuzzy models. This control problem was recently addressed in [16] using a fuzzy dynamic output feedback controller whose design is based on a standard fuzzy Lyapunov function, *i.e.*, without delayed membership functions. The authors in [16] have tackled only the control problems (i) and (iii) mentioned above. However, solving these problems regarding delayed multiple-parameterization Lyapunov functions is more involved than the case of a standard fuzzy Lyapunov function, which is a particular case of the previous one. Therefore, our main contributions are summarized as follows:

- Propose a new multiple-parameterization framework for local stabilization of constrained N-TS fuzzy systems with input saturation;
- Provide a *unified* control approach to design both *non-delayed* and *delayed* nonlinear fuzzy controllers for the studied class of fuzzy systems;
- Estimate the DoA of the closed-loop fuzzy systems taking into account both state and input constraints.

Furthermore, numerical examples are appropriately given to illustrate the effectiveness of the proposed control approach on reducing the design conservativeness and enlarging the DoA estimation.

Notation. The symbol '$\star$' denotes matrix blocks deduced by symmetry. For a matrix X, $X^\top$ is the transpose matrix, $X \succ 0$ means that X is a positive definite matrix and $X_{(l)}$ is the lth row. $\mathrm{diag}(X_1, X_2)$ denotes a block-diagonal matrix composed of X_1 and X_2. For an integer $p > 1$, we denote $\mathcal{I}_p = \{1, \ldots, p\} \subset \mathbb{N}$. $\mathrm{co}\{\mathcal{S}\}$ denotes the convex hull of the set $\mathcal{S}$. Function arguments are omitted when their meaning is straightforward.

2. Takagi-Sugeno Fuzzy Systems with Nonlinear Consequents

This section describes the considered class of N-TS fuzzy models and their advantages for the control design of nonlinear systems.

2.1. System Description

Consider the following class of N-TS fuzzy systems [15]:

$$x_{k+1} = \sum_{i=1}^{r} h_i(z_k)(A_i x_k + G_i \varphi(x_k) + B_i \mathrm{sat}(u_k)) \tag{1}$$

where $x_k \in \mathbb{R}^{n_x}$ is the state vector, $u_k \in \mathbb{R}^{n_u}$ is the input vector, $z_k \in \mathbb{R}^{n_z}$ is the vector of measurable premise variables, and r is the number of fuzzy rules. The state-space matrices (A_i, B_i, G_i) of the ith local fuzzy rule are of appropriate dimensions. The normalized membership functions $h_i(z)$, $i \in \mathcal{I}_r$, verify the convex sum property, *i.e.*, $\sum_{i=1}^{r} h_i(z) = 1$, and $h_i(z) \geq 0,\ i \in \mathcal{I}_r$. Moreover, the control input is subject to a componentwise saturation map $\mathrm{sat}(\cdot) : \mathbb{R}^{n_u} \mapsto \mathbb{R}^{n_u}$ defined as

$$\mathrm{sat}(u_{(l)}) = \mathrm{sign}(u_{(l)}) \min(|u_{(l)}|, \bar{u}_{(l)}),$$

for all $l \in \mathcal{I}_{n_u}$, where $\bar{u}_{(l)} > 0$ is the saturation bound of the lth control input component. The following assumptions are considered for system (1).

Assumption 1. *The vector of nonlinearities $\varphi(x) \in \mathbb{R}^{n_\varphi}$ satisfies the sector-boundedness condition $\varphi_{(i)}(x) \in co\{0, E_i x\}$, $\forall i \in \mathcal{I}_{n_\varphi}$, which can be equivalently rewritten as follows [32]:*

$$\varphi(x)^\top \mathcal{L}^{-1} (Ex - \varphi(x)) \geq 0 \tag{2}$$

where the matrix $E = [E_1^\top, \ldots, E_{n_\varphi}^\top]^\top \in \mathbb{R}^{n_\varphi \times n_x}$ is given, and $\mathcal{L} \in \mathbb{R}^{n_\varphi \times n_\varphi}$ is a positive definite diagonal matrix.

Inequality (2) in Assumption 1 is especially useful to derive convex control design conditions for N-TS fuzzy systems.

Assumption 2. *The state trajectories of system* (1) *is constrained into the following polyhedral set:*

$$\mathcal{D}_x = \{x \in \mathbb{R}^{n_x} : S_{(j)} x \leq 1,\ j \in \mathcal{I}_{n_e}\}, \tag{3}$$

where the given matrix $S \in \mathbb{R}^{n_e \times n_x}$ characterizes the domain of validity $\mathcal{D}_x$.

The domain $\mathcal{D}_x$ represents the state-space region where trajectories are constrained to evolve due to both physical limitations and validity region of the N-TS fuzzy model (1). Taking this domain into account in the control design is essential to ensure both suitable closed-loop performance and stability [3, 16].

2.2. System Motivation

To highlight the interests of using N-TS fuzzy models, we consider an input-saturated nonlinear system constructed by N inverted pendulums series connected via linear springs. Using the Euler's discretization method, the system dynamics can be described by the following state-space representation:

$$x_{k+1(j)} = x_{k(j)} + Tx_{k(n)} \tag{4a}$$

$$\begin{aligned} x_{k+1(n)} &= x_{k(n)} + \frac{gT}{l_\kappa}\sin x_{k(j)} + \frac{T}{m_\kappa l_\kappa^2}\mathrm{sat}(u_{k(\kappa)}) \\ &+ (1-\delta_1^\kappa)\frac{k_{\kappa-1}a^2T}{m_\kappa l_\kappa^2}\Big(\sin x_{k(j-2)}\cos x_{k(j-2)} - \sin x_{k(j)}\cos x_{k(j)}\Big) \\ &+ (1-\delta_N^\kappa)\frac{k_\kappa a^2T}{m_\kappa l_\kappa^2}\Big(\sin x_{k(j+2)}\cos x_{k(j+2)} - \sin x_{k(j)}\cos x_{k(j)}\Big) \end{aligned} \tag{4b}$$

where $\kappa \in \mathcal{I}_N$, $j = 2\kappa - 1$, $n = 2\kappa$, and δ_p^q, $p, q \in \mathbb{Z}$, denotes the Kronecker delta, $x_{k(\kappa)}$ is the rod angle with respect to the vertical axis, m_κ is the mass, and l_κ is the rod length of the κth pendulum. $u_{k(\kappa)}$ is the torque applied to the base of the κth pendulum. T is the sample time, g is the gravitational acceleration, k_s, $s \in \mathcal{I}_{N-1}$, are the spring elastic constants, and a is their connection heights.

By the sector nonlinearity approach [12, Chapter 2], an *exact* TS fuzzy representation can be obtained for system (4) choosing the premise variables as $z_{k(\kappa)} = \sin x_{k(j)}$, and $z_{k(N+\kappa)} = \sin x_{k(j)}\cos x_{k(j)}$, for all $j = 2\kappa - 1$, $\kappa \in \mathcal{I}_N$. This leads to a classical TS fuzzy model with 2^{2N} fuzzy rules. However, note that $\sin x_{k(\kappa)}\cos x_{k(\kappa)} \in \mathrm{co}\{0, x_{k(\kappa)}\}$. Then, a N-TS fuzzy model can be constructed with N premise variables $z_{k(\kappa)} = \sin x_{k(j)}$ and N local nonlinearities $\varphi_\kappa(x_k) = \sin x_{k(j)}\cos x_{k(j)}$, for $j = 2\kappa - 1$, $\kappa \in \mathcal{I}_N$. In this case, the number of fuzzy rules is reduced to only 2^N. As a direct consequence, not only the numerical complexity for control design and implementation but also the conservativeness of the results can be drastically reduced when N-TS fuzzy models are used, see also [13, 15].

3. Problem Formulation

The considered control problem is formulated in this section. First, we provide technical definitions related to the theory of multisets to represent compactly multiple fuzzy summations with arbitrary delays. Then, the control problem is stated.

3.1. Technical Definitions

Hereafter, technical definitions related to the theory of multisets [33] and multiple fuzzy summations are presented. Similar notations on multisets of delays and multi-indexes can be found in [17, 22].

Definition 1 (Multisets, see [33])**.** *Let $D = \{d_1, d_2, \dots, d_n\}$ be a set. A multiset G_D over D is a cardinal-valued function $G_D : D \mapsto \mathbb{N}$ such that for $d \in$ Dom(G_D) implies the cardinal $\mathbf{1}_{G_D}(d) > 0$. The value $\mathbf{1}_{G_D}(d)$ is called multiplicity of d and represents the number of occurrences of d in G_D. A multiset G_D is denoted here by the* *set of pairs $G_D = \{\langle \mathbf{1}_{G_D}(d_1), d_1\rangle, \ldots, \langle \mathbf{1}_{G_D}(d_n), d_n\rangle\}$. If the multiplicity of a given element $d \in D$ is 1, it is simply denoted $\langle 1, d\rangle = d$.*

From the last definition, the cardinality of a multiset G_D is computed by $|G_D| = \sum_{d\in D} \mathbf{1}_{G_D}(d)$.

Definition 2 (Operations on multisets)**.** *Consider two multisets G_A and G_B. Useful operations on multisets are defined as follows:*

- *Union: $G_\cup = G_A \cup G_B = \{d \in G_\cup : \mathbf{1}_{G_\cup}(d) = \max\{\mathbf{1}_{G_A}(d), \mathbf{1}_{G_B}(d)\}\}$*
- *Intersection: $G_\cap = G_A \cap G_B = \{d \in G_\cap : \mathbf{1}_{G_\cap}(d) = \min\{\mathbf{1}_{G_A}(d), \mathbf{1}_{G_B}(d)\}\}$*
- *Sum: $G_\oplus = \{d \in G_\oplus : \mathbf{1}_{G_\oplus}(d) = \mathbf{1}_{G_A}(d) + \mathbf{1}_{G_B}(d)\}$.*

Definition 3 (Multiple fuzzy summation)**.** *Let*

$$\mathbb{P}_{G_0^P} = \sum_{i_1=1}^{r} \cdots \sum_{i_n=1}^{r} h_{i_1}(z_{k+d_1}) \ldots h_{i_n}(z_{k+d_n}) P_{(i_1,\ldots,i_n)} \tag{5}$$

be a multiple fuzzy summation of matrices $P_{(i_1,\ldots,i_n)}$. The multiset of delays involved in this summation is denoted by $G_0^P = \{d_1, \ldots, d_n\}$, for $d_i \in \mathbb{Z}_{\leq 0}$, $i \in \mathcal{I}_n$. If the summation is evaluated at the instant $k+\alpha$, the summation is denoted by $\mathbb{P}_{G_\alpha^P}$ and the multiset of delays is $G_\alpha^P = \{\alpha + d_1, \ldots, \alpha + d_n\}$.

The above notations and operations on multisets are illustrated in the following example.

Example 1. *Consider the 4-dimensional fuzzy summation*

$$\sum_{i_1=1}^{r} \sum_{i_2=1}^{r} \sum_{i_3=1}^{r} \sum_{i_4=1}^{r} h_{i_1}(z_k) h_{i_2}(z_k) h_{i_3}(z_k) h_{i_4}(z_{k-1}) P_{(i_1,i_2,i_3,i_4)}.$$

The multiset of delays associated to this fuzzy summation is $G_0^P = \{\langle 3, 0\rangle, -1\}$ and its cardinality is $|G_0^P| = 4$. If this fuzzy summation is evaluated at the instant $k+\alpha$, then the corresponding multiset of delays is $G_\alpha^P = \{\langle 3, \alpha\rangle, \alpha - 1\}$.

To illustrate the operations on multisets, we consider the multiset of delays $G_0^X = \{0, \langle 2, -1\rangle, -2\}$. The union of G_0^X with the multiset of delays G_0^P is $G_0^P \cup G_0^X = \{\langle 3, 0\rangle, \langle 2, -1\rangle, -2\}$; the intersection is $G_0^P \cap G_0^X = \{0, -1, -2\}$ and the sum is $G_0^P \oplus G_0^X = \{\langle 4, 0\rangle, \langle 3, -1\rangle, -2\}$.

Definition 4 (Index set and multi-index)**.** *The index set of a multiple fuzzy summation with multiset of delays G_0^P is the set of all indexes in the sum. It is denoted here by $\mathbb{I}_{G_0^P} = \{\mathbf{i} = (i_1, \ldots, i_{|G_0^P|}) : i_j \in \mathcal{I}_r,\ j \in \mathcal{I}_{|G_0^P|}\}$. An element $\mathbf{i} \in \mathbb{I}_{G_0^P}$ is called multi-index.*

Remark 1. *For the formulations presented in this paper, the following two subsets of $\mathbb{I}_{G_0^P}$ will be useful. The set of upper-triangle indexes, which is defined by $\mathbb{I}^+_{G_0^P} = \{\mathbf{i} \in \mathbb{I}_{G_0^P} : i_j \leq i_{j+1}, j \in \mathcal{I}_{|G_0^P|-1}\}$, and the set of multi-index permutations defined by $\mathcal{P}(\mathbf{i}) \subset \mathbb{I}_{G_0^P}$, for some $\mathbf{i} \in \mathbb{I}_{G_0^P}$. These subsets of $\mathbb{I}_{G_0^P}$ can also be found in [21, 22].*

Example 2. *Consider the index set* $\mathbb{I}_G = \{\mathbf{i} = (i_1, i_2, i_3) : i_j \in \mathcal{I}_2, j \in \mathcal{I}_3\}$*. The set of upper-triangle indexes is* $\mathbb{I}_G^+ = \{(1,1,1),(1,1,2),(1,2,2),(2,2,2)\}$*. In addition, consider the multi-index* $\mathbf{i} = (1,2,2) \in \mathbb{I}_G^+$*. The set of permutations is* $\mathcal{P}(\mathbf{i}) = \big\{(1,2,2),(2,1,2),(2,2,1)\big\}$*.*

Definition 5 (Projection of a multi-index)**.** *The projection of a multi-index* $\mathbf{i} \in \mathbb{I}_{G_A}$ *to the multiset* G_B*, denoted by* $\mathrm{pr}^{\mathbf{i}}_{G_B}$*, is the part of* $\mathbf{i}$ *corresponding to delays in* $G_A \cap G_B$*.*

It is important to note that the projection may be not unique. For instance, consider the multi-index $\mathbf{i} = (1,2,3,4) \in \mathbb{I}_{G_0^P}$, where $G_0^P = \{\langle 3, 0\rangle, -1\}$. The projection to the multiset of delays $G_0^C = \{0, -1\}$ can be $\mathrm{pr}^{\mathbf{i}}_{G_0^C} = (1,4)$, or $\mathrm{pr}^{\mathbf{i}}_{G_0^C} = (2,4)$, or $\mathrm{pr}^{\mathbf{i}}_{G_0^C} = (3,4)$.

Remark 2. *The multiple fuzzy summation* (5) *can be rewritten using the multi-index notation as follows:*

$$\mathbb{P}_{G_0^P} = \prod_{j=1}^{n} \sum_{i_j=1}^{r} h_{i_j}(z_{k+d_j}) P_{(i_1,\ldots,i_n)} = \sum_{\mathbf{i}_0^P \in \mathbb{I}_{G_0^P}} h_{\mathbf{i}_0^P} P_{\mathbf{i}_0^P},$$

with $\mathbb{I}_{G_0^P} = \{\mathbf{i}_0^P = (i_1, \ldots, i_n) : i_j \in \mathcal{I}_r,\ j \in \mathcal{I}_n\}$*. Similarly, the N-TS fuzzy system* (1) *can be rewritten in the form:*

$$x_{k+1} = \mathbb{A}_{G_0^A} x_k + \mathbb{G}_{G_0^G} \varphi(x_k) + \mathbb{B}_{G_0^B} \mathrm{sat}(u_k), \tag{6}$$

with $G_0^A = G_0^G = G_0^B = \{0\}$*.*

3.2. Problem Statement

For the stabilization of the N-TS fuzzy model (6), the following delayed nonlinear fuzzy control law is proposed:

$$u_k = \mathbb{F}_{G_0^F} \mathbb{H}^{-1}_{G_0^H} x_k + \mathbb{K}_{G_0^K} \mathbb{L}^{-1}_{G_0^L} \varphi(x_k), \tag{7}$$

where $\mathbb{F}_{G_0^F}$, $\mathbb{H}_{G_0^H}$, $\mathbb{K}_{G_0^K}$ are respectively the multiple fuzzy summations of matrices $F_{\mathbf{i}_0^F} \in \mathbb{R}^{n_u \times n_x}$, $\mathbf{i}_0^F \in \mathbb{I}_{G_0^F}$, $H_{\mathbf{i}_0^H} \in \mathbb{R}^{n_x \times n_x}$, $\mathbf{i}_0^H \in \mathbb{I}_{G_0^H}$, and $K_{\mathbf{i}_0^K} \in \mathbb{R}^{n_u \times n_\varphi}$, $\mathbf{i}_0^K \in \mathbb{I}_{G_0^K}$, and $\mathbb{L}_{G_0^L}$ is composed of diagonal matrices $L_{\mathbf{i}_0^L} \in \mathbb{R}^{n_\varphi \times n_\varphi}$, $\mathbf{i}_0^L \in \mathbb{I}_{G_0^L}$. For real-time implementation, the multisets of delays involved in (7) cannot be defined with positive delays, which corresponds to premise variables at future time samples, since it leads to a non-causal closed-loop dynamics.

Remark 3. *The proposed control law* (7) *generalizes other fuzzy controllers existing in the literature. For instance, the PDC control law can be obtained by selecting* $G_0^F = \{0\}$*,* $G_0^H = \{\emptyset\}$*, and* $\varphi(x_k) = 0$*, and the classical non-PDC law can be constructed with* $G_0^F = G_0^H = \{0\}$ *and* $\varphi(x_k) = 0$*. Moreover, the nonlinear non-PDC control law used in [15] for the stabilization of N-TS fuzzy models can also be obtained by selecting* $G_0^F = G_0^H = G_0^K = G_0^L = \{0\}$*.*

Substituting (7) into (6), the closed-loop dynamics is rewritten as

$$x_{k+1} = \mathbb{A}_{cl} x_k + \mathbb{G}_{cl}\varphi(x_k) - \mathbb{B}_{G_0^B}\psi(u_k) \tag{8}$$

where $\mathbb{A}_{cl} = \mathbb{A}_{G_0^A} + \mathbb{B}_{G_0^B}\mathbb{F}_{G_0^F}\mathbb{H}_{G_0^H}^{-1}$, $\mathbb{G}_{cl} = \mathbb{G}_{G_0^G} + \mathbb{B}_{G_0^B}\mathbb{K}_{G_0^K}\mathbb{L}_{G_0^L}^{-1}$, and $\psi(u_k) = u_k - \mathrm{sat}(u_k)$ is the dead-zone nonlinearity. The following lemma is useful to deal with this nonlinearity.

Lemma 1. *Consider the set*

$$\mathcal{D}_u = \Big\{ x_k \in \mathbb{R}^{n_x} : |\Pi| \le \bar{u}_{(l)}, \forall l \in \mathcal{I}_{n_u} \Big\}, \tag{9}$$

where

$$\Pi = \Big(\mathbb{F}_{G_0^F}\mathbb{H}_{G_0^H}^{-1} - \mathbb{W}_{G_0^W}\mathbb{H}_{G_0^H}^{-1} \Big)_{(l)} x_k + \Big(\mathbb{K}_{G_0^K}\mathbb{L}_{G_0^L}^{-1} - \mathbb{J}_{G_0^J}\mathbb{L}_{G_0^L}^{-1} \Big)_{(l)} \varphi(x_k),$$

and $\mathbb{W}_{G_0^W}$ and $\mathbb{J}_{G_0^J}$ are respectively the multiple fuzzy summations of matrices $W_{\mathbf{i}_0^W} \in \mathbb{R}^{n_u \times n_x}$, $\mathbf{i}_0^W \in \mathbb{I}_{G_0^W}$, and $J_{\mathbf{i}_0^J} \in \mathbb{R}^{n_u \times n_\varphi}$, $\mathbf{i}_0^J \in \mathbb{I}_{G_0^J}$. If $x_k \in \mathcal{D}_u$, then

$$\psi(u_k)^\top \mathbb{U}_{G_0^U}^{-1} \Big(\psi(u_k) - \mathbb{W}_{G_0^W}\mathbb{H}_{G_0^H}^{-1} x_k - \mathbb{J}_{G_0^J}\mathbb{L}_{G_0^L}^{-1}\varphi(x_k) \Big) \le 0 \tag{10}$$

for positive definite diagonal matrices $U_{\mathbf{i}_0^U} \in \mathbb{R}^{n_u \times n_u}$, $\mathbf{i}_0^U \in \mathbb{I}_{G_0^U}$, and u_k given in (7).

Proof. The proof follows similar steps as in [1, Lemma 1] and is omitted here for brevity. □

To study the local asymptotic stability of the closed-loop system (8), the following delayed nonquadratic Lyapunov function candidate is considered:

$$V(x_k) = x_k^\top \mathbb{P}_{G_0^P}^{-1} x_k \tag{11}$$

where $\mathbb{P}_{G_0^P} \in \mathbb{R}^{n_x \times n_x}$ is the multiple fuzzy summation of matrices $P_{\mathbf{i}_0^P} = P_{\mathbf{i}_0^P}^\top \succ 0$, $\mathbf{i}_0^P \in \mathbb{I}_{G_0^P}$.

Remark 4. *The level set associated to the function (11) is defined by*

$$\mathcal{L}_V = \Big\{ x_k \in \mathbb{R}^{n_x} : x_k^\top \mathbb{P}_{G_0^P}^{-1} x_k \le 1 \Big\}.$$

Moreover, if $\Delta V = V(x_{k+1}) - V(x_k) < 0$ holds along trajectories of system (8) for all $x_k \in \mathcal{L}_V \setminus \{0\}$, then (11) is said to be a Lyapunov function and $\mathcal{L}_V$ a contractively invariant set with respect to the closed-loop system (8). Note that $\mathcal{L}_V$ is a subset of the DoA of the closed-loop system [32].

In the light of the previous discussions, this work is concerned with proposing sufficient conditions to solve the following control problem.

Problem 1. *Consider the N-TS fuzzy model (6). Design a nonlinear fuzzy controller of the form (7) such that $\mathcal{L}_V \subset \mathcal{D}_x \cap \mathcal{D}_u$, as large as possible, is a contractively invariant set with respect to the closed-loop system (8).*

4. Multiple-Parameterization Stabilization Conditions for Constrained N-TS Fuzzy Systems

The following theorem provides sufficient conditions to guarantee that $\mathcal{L}_V \subset \mathcal{D}_x \cap \mathcal{D}_u$ is a contractively invariant set with respect to system (8).

Theorem 1. *Given $G_V = G_0^P \cup G_1^P \cup (G_0^A \oplus G_0^H) \cup (G_0^G \oplus G_0^L) \cup (G_0^B \oplus G_0^F) \cup (G_0^B \oplus G_0^K) \cup (G_0^B \oplus G_0^U)$. If there exist matrices $P_{\mathbf{i}_j^P} = P_{\mathbf{i}_j^P}^\top \succ 0$, $\mathbf{i}_j^P = \mathrm{pr}^{\mathbf{i}}_{G_j^P}$, $j \in \{0,1\}$, $H_{\mathbf{i}_0^H}$, $\mathbf{i}_0^H = \mathrm{pr}^{\mathbf{i}}_{G_0^H}$, $K_{\mathbf{i}_0^K}$, $\mathbf{i}_0^K = \mathrm{pr}^{\mathbf{i}}_{G_0^K}$, $F_{\mathbf{i}_0^F}$, $\mathbf{i}_0^F = \mathrm{pr}^{\mathbf{i}}_{G_j^F}$, $W_{\mathbf{i}_0^W}$, $\mathbf{i}_0^W = \mathrm{pr}^{\mathbf{i}}_{G_0^W}$, $J_{\mathbf{i}_0^J}$, $\mathbf{i}_0^J = \mathrm{pr}^{\mathbf{i}}_{G_0^J}$, and positive definite diagonal matrices $L_{\mathbf{i}_0^L}$, $\mathbf{i}_0^L = \mathrm{pr}^{\mathbf{i}}_{G_0^L}$, and $U_{\mathbf{i}_0^U}$, $\mathbf{i}_0^U = \mathrm{pr}^{\mathbf{i}}_{G_0^U}$, $\mathbf{i} \in \mathbb{I}_{G_V}$, such that (12), (13) and (14) are feasible. Then, the control law (7) guarantees that $\mathcal{L}_V \subset \mathcal{D}_x \cap \mathcal{D}_u$ is a contractively invariant set of the closed-loop system (8).*

$$\begin{bmatrix} 1 & S_{(j)}\mathbb{P}_{G_0^P} \\ \star & \mathbb{P}_{G_0^P} \end{bmatrix} \succeq 0, \quad \forall j \in \mathcal{I}_{n_e} \tag{12}$$

$$\begin{bmatrix} \mathbb{H}_{G_0^H} + \mathbb{H}_{G_0^H}^\top - \mathbb{P}_{G_0^P} & -\mathbb{H}_{G_0^H}^\top E^\top & \left(\mathbb{F}_{G_0^F} - \mathbb{W}_{G_0^W}\right)_{(l)}^\top \\ \star & 2\mathbb{L}_{G_0^L} & \left(\mathbb{K}_{G_0^K} - \mathbb{J}_{G_0^J}\right)_{(l)}^\top \\ \star & \star & \bar{u}_{(l)}^2 \end{bmatrix} \succ 0, \quad \forall l \in \mathcal{I}_{n_u} \tag{13}$$

$$\begin{bmatrix} -\mathbb{P}_{G_1^P} & \Phi_1 & \Phi_2 & -\mathbb{B}_{G_0^B}\mathbb{U}_{G_0^U} \\ \star & -\mathbb{H}_{G_0^H} - \mathbb{H}_{G_0^H}^\top + \mathbb{P}_{G_0^P} & \mathbb{H}_{G_0^H}^\top E^\top & \mathbb{W}_{G_0^W}^\top \\ \star & \star & -2\mathbb{L}_{G_0^L} & \mathbb{J}_{G_0^J}^\top \\ \star & \star & \star & -2\mathbb{U}_{G_0^U} \end{bmatrix} \prec 0 \tag{14}$$

where $\Phi_1 = \mathbb{A}_{G_0^A}\mathbb{H}_{G_0^H} + \mathbb{B}_{G_0^B}\mathbb{F}_{G_0^F}$, and $\Phi_2 = \mathbb{G}_{G_0^G}\mathbb{L}_{G_0^L} + \mathbb{B}_{G_0^B}\mathbb{K}_{G_0^K}$.

Proof. Multiplying (12) with $\mathrm{diag}\left(1, \mathbb{P}_{G_0^P}^{-1}\right)$ on the left and its transpose on the right, yields

$$\begin{bmatrix} 1 & S_{(j)} \\ \star & \mathbb{P}_{G_0^P}^{-1} \end{bmatrix} \succeq 0. \tag{15}$$

Then, pre- and post-multiplying (15) with $\begin{bmatrix} 1 & -x_k{}^\top \end{bmatrix}$ and its transpose, respectively, leads to

$$1 - x_k^\top S_{(j)}^\top - S_{(j)}x_k + x_k^\top \mathbb{P}_{G_0^P}^{-1} x_k \geq 0.$$

Since for all $x_k \in \mathcal{L}_V$, one has $x_k^\top \mathbb{P}_{G_0^P}^{-1} x_k \leq 1$, which implies that $S_{(j)}x_k \leq 1$. This proves the inclusion $\mathcal{L}_V \subseteq \mathcal{D}_x$.

Inequality (13) implies clearly that $\mathbb{H}_{G_0^H} + \mathbb{H}_{G_0^H}^\top \succ \mathbb{P}_{G_0^P} \succ 0$. This, in its turn, implies that $\mathbb{H}_{G_0^H}$ is regular. Since $\mathbb{H}_{G_0^H}^\top \mathbb{P}_{G_0^P}^{-1} \mathbb{H}_{G_0^H} \succeq \mathbb{H}_{G_0^H} + \mathbb{H}_{G_0^H}^\top - \mathbb{P}_{G_0^P}$, inequality

(13) implies that

$$\begin{bmatrix} \mathbb{H}_{G_0^H}^\top \mathbb{P}_{G_0^P}^{-1} \mathbb{H}_{G_0^H} & -\mathbb{H}_{G_0^H}^\top E^\top & \left(\mathbb{F}_{G_0^F} - \mathbb{W}_{G_0^W}\right)_{(l)}^\top \\ \star & 2\mathbb{L}_{G_0^L} & \left(\mathbb{K}_{G_0^K} - \mathbb{J}_{G_0^J}\right)_{(l)}^\top \\ \star & \star & \bar{u}_{(l)}^2 \end{bmatrix} \succ 0. \tag{16}$$

Then, multiplying (16) with diag $\left(\mathbb{H}_{G_0^H}^{-\top}, \mathbb{L}_{G_0^L}^{-1}, 1\right)$ on the left and its transpose on the right, followed by the well-known Schur complement lemma, we obtain

$$\begin{bmatrix} \mathbb{P}_{G_0^P}^{-1} & -E^\top \mathbb{L}_{G_0^L}^{-1} \\ \star & 2\mathbb{L}_{G_0^L}^{-1} \end{bmatrix} - \frac{1}{\bar{u}_{(l)}^2} \Sigma_{(l)}^\top \Sigma_{(l)} \succ 0, \quad \forall l \in \mathcal{I}_{n_u} \tag{17}$$

where $\Sigma = \begin{bmatrix} (\mathbb{F}_{G_0^F} - \mathbb{W}_{G_0^W})\mathbb{H}_{G_0^H}^{-1} & (\mathbb{K}_{G_0^K} - \mathbb{J}_{G_0^J})\mathbb{L}_{G_0^L}^{-1} \end{bmatrix}$. Following the same steps as in [1], we can conclude from (17) that $\mathcal{L}_V \subset \mathcal{D}_u$.

Using again the property $-\mathbb{H}_{G_0^H}^\top \mathbb{P}_{G_0^P}^{-1} \mathbb{H}_{G_0^H} \preceq \mathbb{P}_{G_0^P} - \mathbb{H}_{G_0^H} - \mathbb{H}_{G_0^H}^\top$, it follows from (14) that

$$\begin{bmatrix} -\mathbb{P}_{G_1^P} & \Phi_1 & \Phi_2 & -\mathbb{B}_{G_0^B}\mathbb{U}_{G_0^U} \\ \star & -\mathbb{H}_{G_0^H}^\top \mathbb{P}_{G_0^P}^{-1} \mathbb{H}_{G_0^H} & \mathbb{H}_{G_0^H}^\top E^\top & \mathbb{W}_{G_0^W}^\top \\ \star & \star & -2\mathbb{L}_{G_0^L} & \mathbb{J}_{G_0^J}^\top \\ \star & \star & \star & -2\mathbb{U}_{G_0^U} \end{bmatrix} \prec 0. \tag{18}$$

Then, multiplying (18) with diag $\left(\mathbb{P}_{G_1^P}^{-1}, \mathbb{H}_{G_0^H}^{-\top}, \mathbb{L}_{G_0^L}^{-1}, \mathbb{U}_{G_0^U}^{-1}\right)$ on the left and its transpose on the right, respectively, followed by the Schur complement lemma, the following inequality can be obtained:

$$\Psi^\top \mathbb{P}_{G_1^P}^{-1} \Psi + \begin{bmatrix} -\mathbb{P}_{G_0^P}^{-1} & E^\top \mathbb{L}_{G_0^L}^{-1} & \mathbb{H}_{G_0^H}^{-\top} \mathbb{W}_{G_0^W}^\top \mathbb{U}_{G_0^U}^{-1} \\ \star & -2\mathbb{L}_{G_0^L}^{-1} & \mathbb{L}_{G_0^L}^{-\top} \mathbb{J}_{G_0^J}^\top \mathbb{U}_{G_0^U}^{-1} \\ \star & \star & -2\mathbb{U}_{G_0^U}^{-1} \end{bmatrix} \prec 0 \tag{19}$$

where $\Psi = \begin{bmatrix} \mathbb{A}_{cl} & \mathbb{G}_{cl} & -\mathbb{B}_{G_0^B} \end{bmatrix}$.

Pre- and postmultiplying (19) by $\begin{bmatrix} x_k^\top & \varphi(x_k)^\top & \psi(u_k)^\top \end{bmatrix}$ and its transpose, it follows that

$$\begin{aligned} &\Delta V + 2\varphi(x_k)^\top \mathbb{L}_{G_0^L}^{-1} \left(E x_k - \varphi(x_k)\right) \\ &\quad - 2\psi^\top(u_k)\mathbb{U}_{G_0^U}^{-1} \left(\psi(u_k) - \mathbb{W}_{G_0^W} \mathbb{H}_{G_0^H}^{-1} x_k - \mathbb{J}_{G_0^J} \mathbb{L}_{G_0^L}^{-1} \varphi(x_k)\right) < 0. \end{aligned} \tag{20}$$

Since $\mathbb{L}_{G_0^L}^{-1} \succ 0$ and $\mathbb{U}_{G_0^U}^{-1} \succ 0$, by properties (2) and (10), it follows from (20) that $\Delta V < 0$, for $\forall x_k \in \mathcal{D}_x \cap \mathcal{D}_u$. This guarantees that the origin of the closed-loop system (8) is asymptotically stable and $\mathcal{L}_V \subset \mathcal{D}_x \cap \mathcal{D}_u$ is a contractively invariant set with respect to system (8). This completes the proof. □

Remark 5. *Without considering $\mathcal{H}_\infty$ performance in the design condition of [15, Corollary 12], this condition can be seen as a special case of* (14) *in Theorem 1 for N-TS fuzzy systems without input saturation and state constraints. Indeed, the design conditions in [15] can be recovered from Theorem 1 by choosing the multisets of delays as* $G_0^P = G_0^F = G_0^K = G_0^H = G_0^L = \{0\}$.

The proposed multiple-parameterization approach is based on the choice of multisets of delays in Theorem 1, *i.e.*, by increasing the fuzzy summation dimensions in both delayed controller (7) and Lyapunov function (11). In particular, Theorem 1 allows to design both *non-delayed* and *delayed* controllers with appropriate choices of the multisets of delays.

For example, conditions to design a *non-delayed* controller can be obtained with $G_0^P = \{\langle \mathbf{1}_{G_0^P}(0), 0\rangle\}$ and $G_0^F = G_0^H = G_0^K = G_0^L = G_0^J = G_0^W = G_0^U = \{\langle \mathbf{1}_{G_0^F}(0), 0\rangle\}$. If we consider $|G_0^P| = n_P$ and assume that $|G_0^F| = |G_0^P|$, the multiset of all delays in Theorem 1 is $G_V = \{0, \langle n_P, 0\rangle, \langle n_P, 1\rangle\}$. Hence, condition (14) corresponds to a $(2n_P+1)$-dimensional fuzzy summation. For illustrations, taking $n_P = 1$, condition (14) becomes

$$\sum_{i_1=1}^{r}\sum_{i_2=1}^{r}\sum_{i_3=1}^{r} h_{i_1}(z_k)h_{i_2}(z_k)h_{i_3}(z_{k+1})\Xi^* \prec 0 \tag{21}$$

where

$$\Xi^* = \begin{bmatrix} -P_{i_3} & A_{i_1}H_{i_2} + B_{i_1}F_{i_2} & G_{i_1}L_{i_2} + B_{i_1}K_{i_2} & -B_{i_1}U_{i_2} \\ \star & P_{i_2} - H_{i_2} - H_{i_2}^\top & H_{i_2}E^\top & W_{i_1}^\top \\ \star & \star & -2L_{i_2} & J_{i_2}^\top \\ \star & \star & \star & -2U_{i_2} \end{bmatrix}.$$

Moreover, the design conditions to obtain a *delayed* fuzzy controller can be obtained with $G_0^P = \{\langle \mathbf{1}_{G_0^P}(-1), -1\rangle\}$ and $G_0^F = G_0^H = G_0^K = G_0^L = G_0^J = G_0^W = G_0^U = \{\langle \mathbf{1}_{G_0^F}(0), 0\rangle, \langle \mathbf{1}_{G_0^F}(-1), -1\rangle\}$. In this case, with $|G_0^P| = n_P$ and $G_0^F = \{\langle n_P, 0\rangle, \langle n_P, -1\rangle\}$, the multiset G_V is $G_V = \{0, \langle n_P, 0\rangle, \langle n_P, -1\rangle\}$. For example, to obtain a 3-dimensional fuzzy summation based condition to design a *delayed* controller from (14), one can choose $G_0^P = \{-1\}$ and $G_0^F = \{0, -1\}$, which results in

$$\sum_{i_1=1}^{r}\sum_{i_2=1}^{r}\sum_{i_3=1}^{r} h_{i_1}(z_k)h_{i_2}(z_k)h_{i_3}(z_{k-1})\Xi \prec 0 \tag{22}$$

where

$$\Xi = \begin{bmatrix} -P_{i_2} & A_{i_1}H_{(i_2,i_3)} + B_{i_1}F_{(i_2,i_3)} & G_{i_1}L_{(i_2,i_3)} + B_{i_1}K_{(i_2,i_3)} & -B_{i_1}U_{(i_2,i_3)} \\ \star & P_{i_3} - H_{(i_2,i_3)} - H_{(i_2,i_3)}^\top & H_{(i_2,i_3)}E^\top & W_{(i_2,i_3)}^\top \\ \star & \star & -2L_{(i_2,i_3)} & J_{(i_2,i_3)}^\top \\ \star & \star & \star & -2U_{(i_2,i_3)} \end{bmatrix}.$$

It can be clearly observed in (21) and (22) that for a fixed number of fuzzy summations, design conditions based on both *delayed* fuzzy controllers and Lyapunov functions may lead to more degrees of freedom (thus, less conservativeness) than those for *non-delayed* ones. The extension to higher fuzzy summation dimensions is obvious from the above discussion.

Remark 6. *From practical aspects, there are two ways of reducing conservativeness with the proposed delayed multiple-parameterization approach. First, for the case of non-delayed fuzzy controllers, less conservative results can be obtained by increasing the multiple-parameterization dimension. Second, by introducing delayed membership functions in the fuzzy controller* (7) *and nonquadratic Lyapunov function* (11)*, conservativeness can be further reduced since more degrees of freedom can be introduced into the design conditions without increasing the dimension of the overall fuzzy summation dimension related to the multiset of delays G_V in Theorem 1.*

5. LMI-Based Design Conditions of Constrained N-TS Fuzzy Systems

The stabilization conditions in Theorem 1 are expressed in terms of (possibly delayed) membership functions. Hence, they cannot be solved directly by numerical solvers. For practical use, the following lemma provides a technical tool to derive a *finite* set of LMI-based design conditions from Theorem 1.

Lemma 2. *Consider a multiple fuzzy summation with multiset of delays $G = D_1 \oplus \cdots \oplus D_q$, where $D_k = \{\langle \mathbf{1}_{D_k}(d_k), d_k\rangle\}$ for all $k \in \mathcal{I}_q$. The following inequality*

$$\sum_{\mathbf{i}_1 \in \mathbb{I}_{D_1}} \cdots \sum_{\mathbf{i}_q \in \mathbb{I}_{D_q}} h_{\mathbf{i}_1}(z_{k+d_1}) \ldots h_{\mathbf{i}_q}(z_{k+d_q}) \Upsilon_{(\mathbf{i}_1,\ldots,\mathbf{i}_q)} \prec 0 \tag{23}$$

is verified if

$$\Theta = \sum_{\mathbf{i}_1 \in \mathcal{P}(\mathbf{j}_1)} \cdots \sum_{\mathbf{i}_q \in \mathcal{P}(\mathbf{j}_q)} \Upsilon_{(\mathbf{i}_1,\ldots,\mathbf{i}_q)} \prec 0, \quad \mathbf{j}_k \in \mathbb{I}^+_{D_k},\ k \in \mathcal{I}_q. \tag{24}$$

Proof. Condition (23) can be *equivalently* rewritten in the following form:

$$\sum_{\mathbf{j}_1 \in \mathbb{I}^+_{D_1}} \cdots \sum_{\mathbf{j}_q \in \mathbb{I}^+_{D_q}} \Theta \prec 0 \tag{25}$$

with Θ defined in (24). It is clear that inequality (24) is a sufficient condition to ensure (25). This concludes the proof. □

Remark 7. *Given a multidimensional fuzzy summation with multiset of delays $G = D_1 \oplus \cdots \oplus D_q$, with $D_k = \{\langle \mathbf{1}_{D_k}(d_k), d_k\rangle\}$, $k \in \mathcal{I}_q$, the number of LMIs to ensure its negativity obtained with Lemma 2 is $\prod_{k=1}^{q} |\mathbb{I}^+_{D_k}|$, where $|\mathbb{I}^+_{D_k}| = \frac{(r+|D_k|-1)!}{|D_k|!(r-1)!}$.*

The following example illustrates the application of Lemma 2 to obtain a finite set of LMI-based conditions from a multiple fuzzy summation condition.

Example 3. *Consider a 4-dimensional fuzzy summation in the form* (23) *with multiset of delays* $G = \{\langle 2,0\rangle, \langle 2,1\rangle\}$*. Then,* $G = D_1 \oplus D_2$*, with* $D_1 = \{\langle 2,0\rangle\}$*,* $D_2 = \{\langle 2,1\rangle\}$ *and* $|D_1| = \mathbf{1}_G(0) = |D_2| = \mathbf{1}_G(1) = 2$*. By Lemma 2, the negativeness of this fuzzy summation is guaranteed if the following set of LMIs hold:*

$$\begin{aligned}
&\Gamma_{(1,1,1,1)} \prec 0, \quad \Gamma_{(2,2,2,2)} \prec 0, \quad \Gamma_{(1,1,1,2)} + \Gamma_{(1,1,2,1)} \prec 0, \quad \Gamma_{(1,1,2,2)} \prec 0,\\
&\Gamma_{(1,2,1,1)} + \Gamma_{(2,1,1,1)} \prec 0, \quad \Gamma_{(1,2,1,2)} + \Gamma_{(2,1,1,2)} + \Gamma_{(1,2,2,1)} + \Gamma_{(2,1,2,1)} \prec 0,\\
&\Gamma_{(1,2,2,2)} + \Gamma_{(2,1,2,2)} \prec 0, \quad \Gamma_{(2,2,1,1)} \prec 0, \quad \Gamma_{(2,2,1,2)} + \Gamma_{(2,2,2,1)} \prec 0.
\end{aligned}$$

Remark 8. *Note that the explicit dependence of the set* $\mathcal{L}_V$*, obtained from Theorem 1, on the premise variables leads to difficulties in finding the largest contractively invariant set, especially in the case of delayed controllers [3]. A suitable alternative is to consider the following shape-independent subset of* $\mathcal{L}_V$*:*

$$\mathcal{E}_V = \bigcap_{\mathbf{i}_0^P \in \mathbb{I}_{G_0^P}} \mathcal{E}(P_{\mathbf{i}_0^P}^{-1}) \subset \mathcal{L}_V,$$

where $\mathcal{E}(P_{\mathbf{i}_0^P}^{-1}) = \left\{ x_k \in \mathbb{R}^{n_x} : x_k^\top P_{\mathbf{i}_0^P}^{-1} x_k \leq 1 \right\}$*. The maximization of* $\mathcal{L}_V$ *can be thus performed by maximizing* $\mathcal{E}_V$*.*

By Lemma 2, LMI-based sufficient conditions can be obtained to guarantee conditions (12), (13) and (14) in Theorem 1. Hence, the maximization of $\mathcal{E}_V$ can be reformulated as a convex optimization problem in the the following corollary.

Corollary 1. *Given* G_V *as in Theorem 1. If there exist matrices* $\bar{P} = \bar{P}^\top \succ 0$*,* $P_{\mathbf{i}_j^P} = P_{\mathbf{i}_j^P}^\top \succ 0$*,* $\mathbf{i}_j^P = \mathrm{pr}^{\mathbf{i}}_{G_j^P}$*,* $j \in \{0,1\}$*,* $H_{\mathbf{i}_0^H}$*,* $\mathbf{i}_0^H = \mathrm{pr}^{\mathbf{i}}_{G_0^H}$*,* $K_{\mathbf{i}_0^K}$*,* $\mathbf{i}_0^K = \mathrm{pr}^{\mathbf{i}}_{G_0^K}$*,* $F_{\mathbf{i}_j^F}$*,* $\mathbf{i}_j^F = \mathrm{pr}^{\mathbf{i}}_{G_j^F}$*,* $W_{\mathbf{i}_0^W}$*,* $\mathbf{i}_0^W = \mathrm{pr}^{\mathbf{i}}_{G_0^W}$*,* $J_{\mathbf{i}_0^J}$*,* $\mathbf{i}_0^J = \mathrm{pr}^{\mathbf{i}}_{G_0^J}$*, and diagonal matrices* $L_{\mathbf{i}_0^L} \succ 0$*,* $\mathbf{i}_0^L = \mathrm{pr}^{\mathbf{i}}_{G_0^L}$ *and* $U_{\mathbf{i}_0^U} \succ 0$*,* $\mathbf{i}_0^U = \mathrm{pr}^{\mathbf{i}}_{G_0^U}$*,* $\mathbf{i} \in \mathbb{I}_{G_V}$*, such that the optimization problem*

$$\begin{aligned}
&\max_{\bar{P}, P_{\mathbf{i}_0^P}, P_{\mathbf{i}_1^P}, H_{\mathbf{i}_0^H}, K_{\mathbf{i}_0^K}, F_{\mathbf{i}_j^F}, W_{\mathbf{i}_0^W}, J_{\mathbf{i}_0^J}, L_{\mathbf{i}_0^L}, U_{\mathbf{i}_0^U}} \quad \det\left(\bar{P}\right)^{1/n_x}\\
&\textit{subject to } (12), (13), (14) \textit{ and } (27)
\end{aligned} \tag{26}$$

with

$$P_{\mathbf{i}_0^P} \succeq \bar{P}, \ \mathbf{i}_0^P \in \mathbb{I}_{G_0^P}, \tag{27}$$

is feasible. Then, Problem 1 is solved with the guaranteed contractively invariant set $\mathcal{E}_V \subset \mathcal{L}_V$*.*

Proof. Note that inequality (27) directly implies the inclusion $\mathcal{E}(\bar{P}^{-1}) \subseteq \mathcal{E}_V$. The rest of the proof is a direct consequence of Theorem 1 and Remark 8. □

It should be highlighted that the index $\det(\bar{P})^{1/n_x}$ is proportional to the volume of the ellipsoid $\mathcal{E}(\bar{P}^{-1})$.

Remark 9. *The computational complexity of an LMI solver based on interior point methods can be estimated as being proportional to* $\log_{10}(N_d^3 N_l)$*, where* N_d *is the number of decision variables and* N_l *the number of LMI rows [1, 29]. For the optimization problem in Corollary 1 with LMI conditions derived using Lemma 2, these quantities can be computed as follows:*

$$\begin{aligned} N_d &= n_x\left(\frac{n_x+1}{2}\right)\left(1+r^{|G_0^P|}\right)+r^{|G_0^L|}n_\varphi+r^{|G_0^U|}n_u+ \\ &\qquad r^{|G_0^H|}n_x^2+n_xn_u\left(r^{|G_0^F|}+r^{|G_0^W|}\right)+n_un_\varphi\left(r^{|G_0^K|}+r^{|G_0^J|}\right), \\ N_l &= N_{l_1}+N_{l_2}+N_{l_3}+N_{l_4}, \end{aligned}$$

where $N_{l_1}=(1+n_x)n_e\prod_{k=1}^q|\mathbb{I}_{D_k}^+|$ *is the number of LMI rows in (12). In this case, the multisets* D_k*,* $k\in\mathcal{I}_q$*, are defined as in Lemma 2 for* $G=G_0^P$*;* $N_{l_2}=(n_x+n_\varphi+1)n_u\prod_{k=1}^q|\mathbb{I}_{\widehat{D}_k}^+|$*, with* $\widehat{D}_k$ *related to the multiset* $G=G_0^P\cup G_0^H\cup G_0^L\cup G_0^F\cup G_0^K$*, is the number of LMI rows in (13);* $N_{l_3}=(2n_x+n_\varphi+n_u)\prod_{k=1}^q|\mathbb{I}_{\bar{D}_k}^+|$*, with* $\bar{D}_k$ *related to* G_V *in Theorem 1, is the number of LMI rows in (14);* $N_{l_4}=n_xr^{|G_0^P|}$ *is the number of LMI rows in (27). Note that the LMI rows computation is based on the procedure described in Remark 7.*

6. Illustrative Examples

Three numerical examples are given in this section to demonstrate effectiveness of the proposed multiple-parameterization approach in reducing control design conservativeness and enlarging the estimation of DoA. The first two are academic examples, while the third example is physically motivated.

6.1. Example 1

Consider the following nonlinear input-saturated system:

$$\begin{aligned} \begin{bmatrix} x_{k+1(1)} \\ x_{k+1(2)} \end{bmatrix} &= \begin{bmatrix} x_{k(1)}-x_{k(1)}x_{k(2)} \\ -x_{k(1)}-0.5x_{k(2)} \end{bmatrix} \\ &\quad + \begin{bmatrix} 0.5x_{k(1)}^3 \\ 0.2x_{k(2)}\left(1.2+\sin x_{k(2)}\right) \end{bmatrix} + \begin{bmatrix} 5+x_{k(1)} \\ 2x_{k(1)} \end{bmatrix}\mathrm{sat}(u_k), \end{aligned} \tag{28}$$

where $x_{k(1)}\in[-b,b]$, $b>0$, $x_{k(2)}\in[-2,2]$ and $\bar{u}=0.7$. System (28) has two sector nonlinearities, namely $\varphi_{k(1)}=x_{k(1)}^3\in\mathrm{co}\{0,b^2x_{k(1)}\}$ and $\bar{\varphi}_{k(2)}=x_{k(2)}(1.2+\sin x_{k(2)})\in\mathrm{co}\{0.2x_{k(2)},2.2x_{k(2)}\}$. Applying a simple loop transformation [32] to $\bar{\varphi}_{k(2)}$, the following nonlinearity can be derived $\varphi_{k(2)}=x_{k(2)}(1.2+\sin x_{k(2)})-0.2x_{k(2)}\in\mathrm{co}\{0,2x_{k(2)}\}$. Considering $z_{k(1)}=x_{k(1)}$ as a premise variable, the nonlinear system (28) can be *exactly* represented by a 2-rule N-TS fuzzy model (1) with the membership functions $h_1(x_{k(1)})=\frac{x_{k(1)}+b}{2b}$, $h_2(x_{k(1)})=1-h_1(x_{k(1)})$, and local state-space matrices

$$A_1=\begin{bmatrix} 1 & -b \\ -1 & -0.46 \end{bmatrix},\ B_1=\begin{bmatrix} 5+b \\ 2b \end{bmatrix},\ A_2=\begin{bmatrix} 1 & b \\ -1 & -0.46 \end{bmatrix},\ B_2=\begin{bmatrix} 5-b \\ -2b \end{bmatrix}$$

and $G_1 = G_2 = \text{diag}(0.5, 0.2)$. The validity domain $\mathcal{D}_x$ of the nonlinear system (28) explicitly depends on the parameter b. Denote b^* as the maximal value of b for which the control design is feasible. This value is used to illustrate the design conservativeness of the proposed control approach according to the procedure discussed in Remark 6. Applying Corollary 1 with different choices of multisets of delays, the obtained values for b^* are given in Table 1. We assume that $G_0^F = G_0^H = G_0^K = G_0^L = G_0^J = G_0^W = G_0^U$ and $|G_0^P|$ is up to 3.

Note that taking into account both input and state constraints, the proposed 3-dimensional fuzzy summation based conditions to design non-delayed controllers are an extension of those given in [15]. These conditions can be extended to 6-dimensional fuzzy summation based conditions by setting $G_0^P = \{\langle 2, 0\rangle\}$ and $G_0^F = \{\langle 3, 0\rangle\}$, which corresponds to $G_V = \{\langle 0, 4\rangle, \langle 2, 1\rangle\}$ in Corollary 1. For this multiset of delays, (14) can be rewritten as a 6-dimensional summation similar to (21).

Moreover, a 3-dimensional delayed based condition can be obtained with $G_0^P = \{-1\}$ and $G_0^F = \{0, -1\}$, which results in $G_V = \{\langle 2, 0\rangle, -1\}$. To obtain 6-dimensional fuzzy summation based conditions, we consider $G_0^P = \{-1\}$ and $G_0^F = \{\langle 0, 4\rangle, -1\}$, which leads to $G_V = \{\langle 0, 5\rangle, -1\}$. Similarly, for $G_0^P = \{\langle 2, -1\rangle\}$ and $G_0^F = \{\langle 3, 0\rangle, \langle 2, -1\rangle\}$, one has $G_V = \{\langle 4, 0\rangle, \langle 2, -1\rangle\}$. Then, note that delayed conditions offer more flexibility to design since different multisets of delays can be chosen without increasing $|G_V|$.

Table 1: Parameter b^* obtained with Corollary 1 for different multisets of delays.

G_0^P	G_0^F	$\lvert G_V\rvert$	b^*
$\{0\}$ (extension of [15])	$\{0\}$	3	1.367
$\{-1\}$	$\{0, -1\}$	3	1.369
$\{\langle 2, 0\rangle\}$	$\{\langle 3, 0\rangle\}$	6	1.432
$\{-1\}$	$\{\langle 4, 0\rangle, -1\}$	6	1.437
$\{\langle 2, -1\rangle\}$	$\{\langle 3, 0\rangle, \langle 2, -1\rangle\}$	6	1.442
$\{\langle 3, 0\rangle\}$	$\{\langle 3, 0\rangle\}$	7	1.436
$\{-1\}$	$\{\langle 5, 0\rangle, -1\}$	7	1.438
$\{\langle 2, -1\rangle\}$	$\{\langle 4, 0\rangle, \langle 2, -1\rangle\}$	7	1.443
$\{\langle 3, -1\rangle\}$	$\{\langle 3, 0\rangle, \langle 3, -1\rangle\}$	7	1.445

From the results in Table 1, it is clear that increasing the fuzzy summation dimension leads to less design conservativeness. However, observe also that there is a kind of limit bound for which the conservativeness can be reduced since in comparison to the 6-dimensional based conditions, the results with $|G_V| = 7$ are just slightly improved.

To further illustrate the advantages of the proposed control approach, let us consider $b = 1.367$, the value for which all design conditions in Table 1 are feasible. As depicted in Figure 1 (a), the contractively invariant set $\mathcal{E}_V$ obtained with the 6-dimensional non-delayed design conditions encompasses the one provided by the 3-dimensional non-delayed conditions (*i.e.*, extension of [15]). Consider now $b = 1.389$, a value for which the 3-dimensional conditions are infeasible. Figure 1 (b) depicts the set $\mathcal{E}(P_{\mathbf{i}_0^P}^{-1})$ together with several closed-loop trajectories

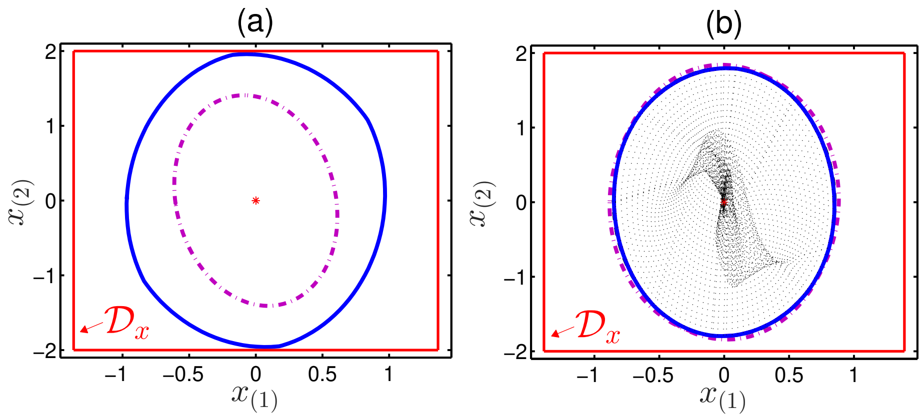

Figure 1: Illustrations on the contractively invariant sets. (a) $\mathcal{E}_V$ obtained from Corollary 1 using the non-delayed conditions with 3 (dash-dotted line) and 6 (solid line) fuzzy summations for $b = 1.367$. (b) $\mathcal{E}(P^{-1}_{\mathbf{i}_0^P})$ (dash-dotted line), $\mathcal{E}_V$ (solid line) and closed-loop trajectories obtained with 6 fuzzy summations and $G_0^P = \{\langle 2, -1\rangle\}$ for $b = 1.389$.

6.2. Example 2

Consider the discrete-time N-TS fuzzy model (1) described in [14, Example 1] with the following local matrices:

$$A_1 = \begin{bmatrix} 1+T & T & \frac{2}{\pi}\sin\frac{\pi}{2}\bar{T} & -0.1T \\ \bar{T} & 1-2\bar{T} & 0 & 0 \\ \bar{T} & a^2\bar{T} & 1-0.3\bar{T} & 0 \\ 0 & 0 & \frac{2}{\pi}\sin\frac{\pi}{2}T & 1-T \end{bmatrix}, \quad B_1 = \begin{bmatrix} (1+a^2)T \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 1+T & \bar{T} & \frac{2}{\pi}\sin\frac{\pi}{2}T & -0.1T \\ T & 1-2\bar{T} & 0 & 0 \\ T & 0 & 1-0.3\bar{T} & 0 \\ 0 & 0 & \frac{2}{\pi}\sin\frac{\pi}{2}T & 1-T \end{bmatrix}, \quad B_2 = \begin{bmatrix} \bar{T} \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad G_1 = G_2 = \begin{bmatrix} T \\ 0 \\ 0 \\ T \end{bmatrix}$$

where $x_{(1)} \in [-a, a]$ with $a > 0$, $x_{(3)} \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, the sampling time is $T = 0.5$s, the control input bound is $\bar{u} = 2$. The sector nonlinearity is chosen as $\varphi(x) = \sin x_{(3)} - (\frac{2}{\pi}\sin\frac{\pi}{2})x_{(3)} \in \text{co}\{0, (1 - (\frac{2}{\pi}\sin\frac{\pi}{2})x_{(3)}\}$. In addition, the normalized membership functions are defined by $h_1(x_1) = x_1^2/a^2$ and $h_2(x_1) = 1 - h_1(x_1)$. Similar to Example 6.1, it is evaluated the maximal a, denoted by a^*, for which a given control design condition is feasible. The results obtained with $G_0^F = G_0^H = G_0^K = G_0^L = G_0^J = G_0^W = G_0^U$ are summarized in Table 2. It can be confirmed that fuzzy summation based design conditions with higher dimensions can provide less conservative results.

Consider the case with $a = 7$, for which the design conditions based on 3-dimensional fuzzy summations are *infeasible*. The conservativeness reduction is evaluated in terms of the index $\det(\bar{P})^{1/n_x}$ obtained from Corollary 1. The values of this index for different multisets of delays are summarized in Table 3. Observe that the volume of the ellipsoid $\mathcal{E}(\bar{P}^{-1})$ can be increased with higher dimensions in the multiple-parameterization conditions.

Table 2: Parameter a^* obtained with Corollary 1 for different multisets of delays.

G_0^P	G_0^F	$\|G_V\|$	a^*
$\{0\}$ (extension of [15])	$\{0\}$	3	6.592
$\{-1\}$	$\{0,-1\}$	3	6.592
$\{\langle 2,0\rangle\}$	$\{\langle 2,0\rangle\}$	5	7.931
$\{-1\}$	$\{\langle 3,0\rangle,-1\}$	5	8.224
$\{\langle 2,0\rangle\}$	$\{\langle 3,0\rangle\}$	6	8.952
$\{-1\}$	$\{\langle 4,0\rangle,-1\}$	6	9.854

Table 3: Values of index $\det(\bar{P})^{1/n_x}$ obtained with different multisets of delays in Corollary 1 for $a = 7$.

G_0^P	G_0^F	$\|G_V\|$	$\det(\bar{P})^{1/n_x}$
$\{\langle 2,0\rangle\}$	$\{\langle 2,0\rangle\}$	5	0.3015
$\{-1\}$	$\{\langle 3,0\rangle,-1\}$	5	0.3864
$\{\langle 2,0\rangle\}$	$\{\langle 3,0\rangle\}$	6	0.3935
$\{-1\}$	$\{\langle 4,0\rangle,-1\}$	6	0.4285

6.3. Example 3 (Stabilization of Connected Inverted Pendulums)

Let us consider the stabilization problem of a constrained system composed of two inverted pendulums interconnected via a linear spring, *i.e.*, the nonlinear system (4) with $N = 2$. The system parameters are $T = 0.2$s, $g = 9.8$m/s^2, $l_1 = 0.5$m, $l_2 = 0.7$m, $m_1 = 0.25$kg, $m_2 = 0.35$kg, $k_1 = 60$N/m and $a = 0.4$m.

A 4-rule N-TS fuzzy model can be obtained applying the sector nonlinearity approach with $z_{(\kappa)} = \sin x_{(j)} \in [\frac{2}{\pi}, 1]$ and $\varphi_\kappa(x) = \sin x_{(j)} \cos x_{(j)} \in \text{co}\{0, x_{(j)}\}$, $j = 2\kappa - 1$, $\kappa \in \mathcal{I}_2$. Note however that 16 fuzzy rules are required to derive an exact representation of this nonlinear system in the classical TS fuzzy form. The membership functions are $h_1(z_k) = w_0^1(z_{(1)})w_0^2(z_{(2)})$, $h_2(z) = w_0^1(z_{(1)})w_1^2(z_{(2)})$, $h_3(z) = w_1^1(z_{(1)})w_0^2(z_{(2)})$, $h_4(z) = w_1^1(z_{(1)})w_1^2(z_{(2)})$, where

$$w_0^\kappa(z_{(\kappa)}) = \begin{cases} 1, & z_{(\kappa)} = 0, \\ \dfrac{z_{(\kappa)} - \frac{2}{\pi}\text{asin} z_{(\kappa)}}{\left(1 - \frac{2}{\pi}\right)\text{asin} z_{(\kappa)}}, & z_{(\kappa)} \neq 0, \end{cases}$$

and $w_1^\kappa(z_{(\kappa)}) = 1 - w_0^\kappa(z_{(\kappa)})$, $\kappa \in \mathcal{I}_2$. The local state-space matrices are given as follows:

$$A_i = \begin{bmatrix} 1 & \bar{T} & 0 & 0 \\ \dfrac{gT}{l_1} b_j & 1 & 0 & 0 \\ 0 & 0 & 1 & \bar{T} \\ 0 & 0 & \dfrac{gT}{l_2} d_l & 1 \end{bmatrix}, \qquad B_i = \begin{bmatrix} 0 & 0 \\ \dfrac{T}{m_1 l_1^2} & 0 \\ 0 & 0 \\ 0 & \dfrac{T}{m_2 l_2^2} \end{bmatrix},$$

$$G_i = \begin{bmatrix} 0 & -\dfrac{k_1 a^2 \bar{T}}{m_1 l_1^2} & 0 & \frac{k_1 a^2 T}{m_2 l_2^2} \\ 0 & \dfrac{k_1 a^2 \bar{T}}{m_1 l_1^2} & 0 & -\frac{k_1 a^2 T}{m_2 l_2^2} \end{bmatrix}^\top, \qquad i = l + 2(j-1),$$

for $j, l \in \mathcal{I}_2$, $b_1 = d_1 = 1$ and $b_2 = d_2 = \frac{2}{\pi}$. Moreover, the input and state constraints are respectively given as $\bar{u} = \begin{bmatrix} 8 & 6 \end{bmatrix}^\top$ and $|x_{(1)}| \leq \frac{\pi}{3}$, $|x_{(3)}| \leq \frac{\pi}{3}$. For the control design, the 3-dimensional delayed conditions with multisets of delays used in Examples 6.1 and 6.2 are considered. The number of decision variables involved in the solution is $N_d = 754$ and the number of LMI rows is $N_l = 800$. Hence, the numerical complexity is estimated as $\log_{10}(N_d^3 N_l) = 11.535$.

The closed-loop behavior of the nonlinear system corresponding to the initial condition $x_0 = \begin{bmatrix} \frac{\pi}{6} & 0 & -\frac{\pi}{4} & 0 \end{bmatrix}^\top$ is depicted in Figure 2. Note that although the input signal u_k exceeds its amplitude bound $\bar{u}$, the control signal actually applied to the system is $\mathrm{sat}(u_k)$, which is always bounded by $\bar{u}_{(l)}$. Observe also that despite the saturation of both actuators at the beginning of the simulation, the proposed nonlinear controller can guarantee an asymptotic stability of the closed-loop system. This illustrates the effectiveness of the proposed control approach in dealing with complex constrained nonlinear systems in real-world applications.

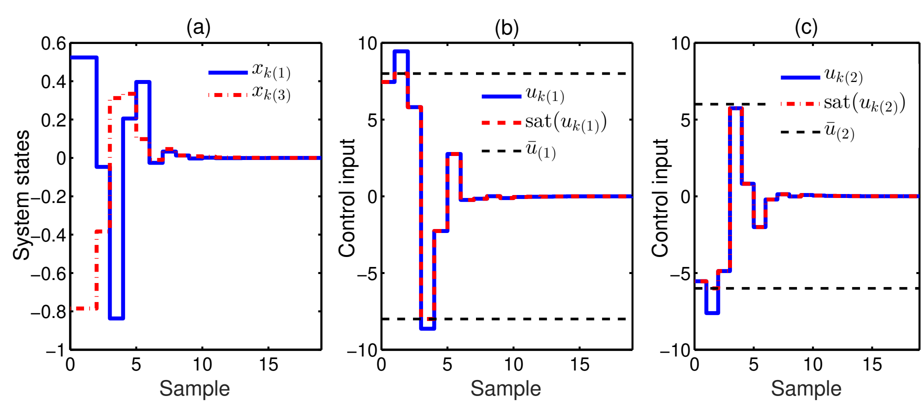

Figure 2: Closed-loop behaviors of the 2 connected inverted pendulums system. (a) Position trajectories. (b) and (c) Control input.

7. Concluding Remarks

This work has proposed a multiple-parameterization approach for stabilization of N-TS fuzzy systems with input saturation subject to state constraints. This control approach offers the possibility to design both non-delayed and delayed nonlinear fuzzy controllers. In addition, a procedure to obtain convex design conditions in terms of LMIs was proposed. Several numerical tests were performed to illustrate the advantages of the proposed control approach compared to existing fuzzy summation based results. Future research directions are related to the development of fuzzy output feedback control designs based on the proposed delayed multiple-parameterization approach and switching mechanisms [30] to further reduce the design conservativeness as well as the online computational burden.

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