T-norms and t-conorms on a family of lattices

C. Bejines[a,*]

[a] *Department of Cybernetics, Faculty of Electrical Engineering, Czech Technical University in Prague, Prague, Czech Republic*

Abstract

This paper provides a complete classification of all t-norms on a family of lattices in terms of t-norms on discrete chains. Moreover, the cardinal of some classes on discrete chains is computed. Therefore, the number of t-norms on the family of lattices is obtained. Also, new results involving Archimedean and divisible t-norms are presented. Finally, we bring out dual results for t-conorms.

Keywords: T-norm, t-conorm, lattice, Archimedean, divisible, 1-Lipschitz.

1. Introduction

T-norms and t-conorms are basic tools in the framework of Fuzzy Logic. They extend the conjunction and disjunction of classical sets and are suitable to define fuzzy algebraic structures (for instance, see [1, 32]).

Firstly, they were defined on the interval $[0, 1]$, but the need to work with incomparable elements is required in many contexts, so a more general algebraic structure is fundamental (see [19]). Consequently, they have been studied on bounded lattices (for instance, see [13, 15]).

One object of interest is to find which operators defined on bounded lattices are t-norms (or t-conorms), or at least to estimate the number of them. If the bounded lattice is not finite, the number of them is infinite. Also, finite lattices are used in the practice. For these reasons, we have focused on a family of finite lattices in the paper.

*Corresponding author

Email address: c.bejines@alumni.unav.es (C. Bejines)

A pioneering article in this context was written by De Baets and Mesiar in 1999. They provided the number of t-norms on discrete chains up to a length of fourteen by computational methods (see [12]). Also, Bartušek and Navara studied conjunctions on finite chains using a computer program generating all t-norms (see [4]), and Petrík focused on the building of finite negative totally ordered monoids, in particular, discrete t-norms (see [30, 31]). Recently, the number of t-norms on other families of lattices is found in [5].

In general, the study of obtaining t-norms and t-conorms on bounded lattices is a current topic (for instance, see [3, 10, 17]). There are different lines of research focused on obtaining and building new t-norms. In 2005, Zhang provided a method to obtain t-norms through monotone functions on bounded partially ordered sets (see [38]). In particular, his work can be applied to finite lattices. In the studies of Palmeira et al., methods and techniques are presented to extend t-norms (and t-conorms) from sublattices to lattices that contain them (see [28, 29]).

Another way to obtain new t-norms from ones already given is through the ordinal sum. However, the ordinal sum of t-norms on a finite lattice does not have to be a t-norm. In 2008, Saminger et al. studied the ordinal sum of t-norms and the horizontal sum of bounded lattices (see [35]). In 2012, Medina determined several necessary and sufficient conditions so that the ordinal sum of two t-norms is also a t-norm (see [26]). In 2018, it is provided a method to construct ordinal sums of t-norms (and t-conorms) on any bounded lattice (see [9]). Recently, new constructions of ordinal sums of t-norms (and t-conorms) on bounded lattices have been presented (see [11, 14, 16]).

It is worth noting that t-norms and t-conorms are not the only operators that are being investigated. Uninorms were recently defined on bounded lattices (see [22]) and characterized by means of t-norms and t-conorms (see [8, 21]). More generally, aggregation functions and t-operators can be defined on bounded lattices (see [7, 23]). All these operations are generalizations of the two concepts that we study in-depth in this paper: the t-norm and the t-conorm.

In this article, we describe each and every one of the t-norms defined on a family of finite lattices in terms of the t-norms defined on the discrete chains. Some of them can be obtained using the ordinal sum, but most are not. The lattices (see Figure 1) resemble the horizontal sum of discrete chains (see [33]). Our family of lattices has the same elements as a horizontal sum of discrete chains, but the imposition of a new connection between two elements makes our lattice more complex to study.

After the introduction, we present the basic concepts to understand the article. Auxiliary information about t-norms on discrete chains is exposed in Section 3. More precisely, we compute the number of some classes of t-norms that are used in the following section. Section 4 is the main one and each t-norm is completely described. Results are separated according to the properties of the t-norms. A significant class of t-norms is the class of Archimedean t-norms (see for instance [34, 37]). Section 5 determines what are the Archimedean t-norms. Another relevant class is the set of divisible t-norms. This property is the proper equivalent to the continuity of t-norms on the interval $[0,1]$ in the framework of discrete t-norms (see [24]). When the lattice is a discrete chain, these t-norms are exactly the 1-Lipschitz t-norms and they are well-known (see [25]). Also, t-norms generate partial orders, and whenever a t-norm is divisible, the corresponding partial order is a lattice (see [2]). More details about divisible t-norms can be found in [18]. Section 6 is devoted to describing them. Finally, Section 7 highlights the importance of the duality between t-norms and t-conorms. Each result written in the previous parts has a dual result for t-conorms.

2. Preliminaries

We introduce the notions that will be used throughout the article.

Definition 2.1 ([6]). *Let $(L,\leq)$ be a lattice. We say that L is a bounded lattice if there are $0,1\in L$ such that $0\leq x$ and $x\leq 1$ for each $x\in L$.*

Notice that each finite lattice is a bounded lattice.

The first definition of the t-norm was given by Menger in [27]. The following two notions are the generalization of t-norms and t-conorms from the interval $[0,1]$ to a bounded lattice L.

Definition 2.2 ([13])**.** *Let $(L,\leq,0,1)$ be a bounded lattice. A function $T : L\times L \longrightarrow L$ is called a triangular norm (a t-norm) on L if it satisfies the following axioms:*

1. $T(x,y)=T(y,x)$ *for all* $x,y\in L$.
2. $T(x,1)=x$ *for all* $x\in L$.
3. *If* $x\leq y$, *then* $T(x,z)\leq T(y,z)$ *for all* $x,y,z\in L$.
4. $T(x,T(y,z))=T(T(x,y),z)$ *for all* $x,y,z\in L$.

Definition 2.3 ([13])**.** *Let $(L,\leq,0,1)$ be a bounded lattice. A function $S : L\times L \longrightarrow L$ is called a triangular conorm (a t-conorm) on L if it satisfies the following axioms:*

1. $S(x,y)=S(y,x)$ *for all* $x,y\in L$.
2. $S(x,0)=x$ *for all* $x\in L$.
3. *If* $x\leq y$, *then* $S(x,z)\leq S(y,z)$ *for all* $x,y,z\in L$.
4. $S(x,S(y,z))=S(S(x,y),z)$ *for all* $x,y,z\in L$.

Computationally, in [12] De Baets and Mesiar calculated the number of t-norms on discrete chains. We show that data in Table 1. Due to the duality between t-norms and t-conorms, this table provides the same information about the number of t-conorms because a discrete chain is a self-dual lattice (more information in Section 7).

3. Auxiliary information about t-norms on chains

To illustrate the results of the paper, in this section we provide the number of t-norms defined on a chain that are used in the main section.

Given a discrete chain

$$C=\{0,\alpha_1,\alpha_2,\ldots,\alpha_n,1\}$$

n	P_n	n	P_n
1	2	7	13775
2	6	8	86417
3	22	9	590489
4	94	10	4446029
5	451	11	37869449
6	2386	12	382549464

Table 1: Numbers of t-norms on discrete chains with $n+2$ elements, that is, n non-trivial elements.

satisfying $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_n < 1$, we present the number of the following families of t-norms on C.

1. T-norms T on the chain C satisfying $T(\alpha_1, \alpha_1) = 0$, that is, t-norms on a discrete chain satisfying that the smallest non-trivial element is not idempotent. The number of them is denoted by A_n. See Column 2 in Table 2.
2. T-norms T on the chain C satisfying $T(\alpha_1, \alpha_1) = \alpha_1$, that is, t-norms on a discrete chain satisfying that the smallest non-trivial element is idempotent. The number of them is denoted by A'_n. See Column 3 in Table 2.
3. T-norms T on the chain C satisfying $T(\alpha_n, \alpha_n) < \alpha_n$, that is, t-norms on a chain satisfying that the biggest non-trivial element is not idempotent. The number of them is denoted by B_n. See Column 4 in Table 2.
4. T-norms T on the chain C satisfying $T(\alpha_n, \alpha_n) = \alpha_n$, that is, t-norms on a chain satisfying that the biggest non-trivial element is idempotent. The number of them is denoted by B'_n. See Column 5 in Table 2.
5. Fixed a $k \in \{1, 2, ..., n\}$, t-norms T_k on the chain C satisfying $T_k(\alpha_1, \alpha_n) = 0$ and $T_k(\alpha_n, \alpha_n) \leq \alpha_k$. See Table 3.

Obviously, for each $n \in \mathbb{N}$, we have that $A_n + A'_n = P_n$ (see Columns 2 and 3) and $B_n + B'_n = P_n$ (see Columns 4 and 5). Moreover, $A'_n = P_{n-1}$.

n	A_n	A'_n	B_n	B'_n
1	1	1	1	1
2	4	2	3	3
3	16	6	11	11
4	78	22	46	48
5	357	94	215	236
6	1935	451	1108	1278
7	11389	2386	6273	7502
8	72642	13775	39114	47303
9	504072	86417	271604	318885

Table 2: Auxiliary information about the number of some families of t-norms on a discrete chain.

4. Main results

We study the t-norms on the lattice described in Figure 1, where p, q are natural numbers.

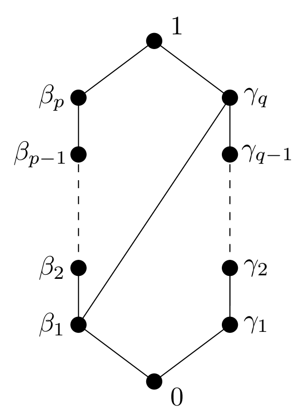

Figure 1: The lattice L.

Throughout the paper, C_β and C_γ denote the following discrete chains with

$k \setminus n$	1	2	3	4	5	6	7	8	9
1	1	2	4	8	16	32	64	128	256
2		3	7	18	51	158	526	1844	6691
3			10	28	89	323	1358	6581	35912
4				40	127	481	2156	11593	75332
5					181	643	2869	15652	105115
6						914	3635	19127	126985
7							5118	23093	145198
8								31842	167468
9									222733

Table 3: For each k and n, the number of t-norms T_k on a discrete chain with $n+2$ elements satisfying that $T_k(\alpha_1, \alpha_n) = 0$ and $T_k(\alpha_n, \alpha_n) \leq \alpha_k$.

the induced order:

$$C_\beta = \{0, \beta_1, \ldots, \beta_p, 1\}$$
$$C_\gamma = \{0, \gamma_1, \ldots, \gamma_q, 1\}$$

Therefore, C_β and C_γ are sublattices of L (see Figure 1).

Lemma 4.1. *If T is a t-norm on L, then $T(\beta_i, \gamma_j) = 0$ for each $i \in \{1, 2, ..., p\}$ and $j \in \{1, 2, ..., q-1\}$.*

Proof. $T(\beta_i, \gamma_j) \leq \beta_i \wedge \gamma_j = 0$ for each $i \in \{1, 2, ..., p\}$ and $j \in \{1, 2, ..., q-1\}$. □

Now, we present each t-norm on L. We have divided the propositions according to the values of $T(\beta_1, \beta_1)$ and $T(\gamma_q, \gamma_q)$.

Proposition 4.2. *Each t-norm T on L satisfying $T(\beta_1, \beta_1) = \beta_1$ and $T(\gamma_q, \gamma_q) =$*

β_1 *is expressed as follows:*

$$T(x,y) = \begin{cases} T'(x,y) & \text{if} \quad x, y \in C_\beta, \\ x \wedge y & \text{if} \quad x = 1 \text{ or } y = 1, \\ \beta_1 & \text{if} \quad (x,y) \in \{(\beta_i, \gamma_q), (\gamma_q, \beta_i) \mid i \in \{1, ..., p\}\}, \\ \beta_1 & \text{if} \quad x = y = \gamma_q, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

where T' is a t-norm on C_β satisfying $T'(\beta_1, \beta_1) = \beta_1$.

Proof. Firstly, we prove that each t-norm T on L satisfying $T(\beta_1, \beta_1) = \beta_1$ and $T(\gamma_q, \gamma_q) = \beta_1$ has this expression. Finally, we will prove that each binary operator T on L which has this expression is a t-norm satisfying the mentioned conditions.

If T is a t-norm on L, then $T|_{C_\beta^2}$ is a t-norm on C_β. Moreover, since $T(\beta_1, \beta_1) = \beta_1$, we have that $T|_{C_\beta^2}(\beta_1, \beta_1) = \beta_1$. Put $T' = T|_{C_\beta^2}$.

If $x = 1$ or $y = 1$, then $T(x,y) = x \wedge y$ because T is a t-norm. Without loss of generality due to commutativity, take $x = \beta_i$ for some $i \in \{1, ..., p\}$ and $y = \gamma_q$. By monotonicity,

$$\beta_1 = T(\beta_1, \beta_1) \leq T(x,y) \leq \beta_i \wedge \gamma_q = \beta_1.$$

Under our hypothesis, $T(\gamma_q, \gamma_q) = \beta_1$.

In the last case, we have that $T(\gamma_i, \gamma_q) \leq T(\gamma_q, \gamma_q) = \beta_1$ for each $i \in \{1, ..., q-1\}$. This implies that $T(\gamma_i, \gamma_q) \leq \gamma_i \wedge \beta_1 = 0$. Consequently, $T(\gamma_i, \gamma_j) \leq T(\gamma_i, \gamma_q) = 0$ for each $i, j \in \{1, ..., q-1\}$. Lemma 4.1 provides the rest of values.

Now, consider a binary operator T expressed by formula (1). The element 1 is the neutral element by definition of T and commutativity is straightforward. To prove the monotonicity, let us take $x, y, z \in L \setminus \{1\}$ such that $x \leq y$ (if one of them is equal to 1, the condition of monotonicity is trivial). If all of them belong to C_β, then $T(x,z) = T'(x,z) \leq T'(y,z) = T(y,z)$. If all of them belong to C_γ, then monotonicity is clear due to the definition of T. Otherwise:

1. If $x \in C_\beta$, necessarily $y \in C_\beta$. Hence, $z = \gamma_i$ for some $i \in \{1, ..., q\}$.

When $z = \gamma_q$, then $T(x,z) = \beta_1 = T(y,z)$ and when $z \neq \gamma_q$, then $T(x,z) = 0 \leq T(y,z)$.

2. If $x \in C_\gamma$, then $y \in C_\gamma$ and $z \in C_\beta$. If $x = \gamma_q$, then $x = y$ and the condition of monotonicity is trivial. If $x \neq \gamma_q$, then $T(x,z) = 0 \leq T(y,z)$.

Finally, let us check that T is associative. Take $x, y, z \in L \setminus \{0,1\}$ (whenever one of them is equal to 0 or 1, associativity is trivial).

If $x, y, z \in C_\beta$, then $T(x, T(y,z)) = T'(x, T'(y,z)) = T'(T'(x,y), z) = T(T(x,y), z)$.

If one of them is equal to γ_i for some $i < q$, then $T(x, T(y,z)) = 0 = T(T(x,y), z)$ by definition of T. Therefore, the last case must consider one of them equal to γ_q. It is easy to check by definition of T that $T(T(x,y), z) = \beta_1 = T(x, T(y,z))$. We conclude that T is a t-norm on L. Besides, $T(\beta_1, \beta_1) = \beta_1$ and $T(\gamma_q, \gamma_q) = \beta_1$. □

Proposition 4.3. *Each t-norm T on L satisfying $T(\beta_1, \beta_1) = \beta_1$ and $T(\gamma_q, \gamma_q) = \gamma_q$ is expressed as follows:*

$$T(x,y) = \begin{cases} T_1(x,y) & \text{if} \quad x, y \in C_\beta, \\ T_2(x,y) & \text{if} \quad x, y \in C_\gamma, \\ \beta_1 & \text{if} \quad (x,y) \in \{(\beta_i, \gamma_q), (\gamma_q, \beta_i) \mid i \in \{1, ..., p\}\}, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

where T_1 is a t-norm on C_β satisfying $T_1(\beta_1, \beta_1) = \beta_1$ and T_2 is a t-norm on C_γ satisfying $T_2(\gamma_q, \gamma_q) = \gamma_q$.

Proof. Firstly, we prove that each t-norm T on L satisfying $T(\beta_1, \beta_1) = \beta_1$ and $T(\gamma_q, \gamma_q) = \gamma_q$ has this expression. Finally, we will prove that each binary operator T on L which has this expression is a t-norm satisfying the mentioned conditions.

If T is a t-norm on L, then $T|_{C_\beta^2}$ and $T|_{C_\gamma^2}$ are t-norms on C_β and C_γ respectively. Notice that we can guarantee that $T|_{C_\gamma^2}$ is a t-norm because $T(\gamma_q, \gamma_q) \in C_\gamma$, and therefore, it is well-defined. Moreover, since $T(\beta_1, \beta_1) = \beta_1$ and $T(\gamma_q, \gamma_q) = \gamma_q$, we can set $T_1 = T|_{C_\beta^2}$ and $T_2 = T|_{C_\gamma^2}$.

Without loss of generality due to commutativity, take $x = \beta_i$ for some $i \in \{1, ..., p\}$ and $y = \gamma_q$. By monotonicity,

$$\beta_1 = T(\beta_1, \beta_1) \leq T(x, y) \leq \beta_i \wedge \gamma_q = \beta_1.$$

The last case is Lemma 4.1.

Now, consider a binary operator T expressed by formula (2). The element 1 is the neutral element because T_1 and T_2 are t-norms, and commutativity is straightforward for the same reason. To prove the monotonicity, let us take $x, y, z \in L \setminus \{1\}$ such that $x \leq y$ (if one of them is equal to 1, the condition of monotonicity is trivial). If all of them belong to C_β, then $T(x, z) = T_1(x, z) \leq T_1(y, z) = T(y, z)$. Analogously, if all of them belong to C_γ. Otherwise:

1. If $x \in C_\beta$, necessarily $y \in C_\beta$. Hence, $z = \gamma_i$ for some $i \in \{1, ..., q\}$. When $z = \gamma_q$, then $T(x, z) = \beta_1 = T(y, z)$ and when $z \neq \gamma_q$, then $T(x, z) = 0 \leq T(y, z)$.
2. If $x \in C_\gamma$, the procedure is analogous.

Finally, let us check that T is associative. Take $x, y, z \in L \setminus \{0, 1\}$ (whenever one of them is equal to 0 or 1, associativity is trivial). If $x, y, z \in C_\beta$, then $T(x, T(y, z)) = T_1(x, T_1(y, z)) = T_1(T_1(x, y), z) = T(T(x, y), z)$. Analogously, if $x, y, z \in C_\gamma$, then $T(x, T(y, z)) = T_2(x, T_2(y, z)) = T_2(T_2(x, y), z) = T(T(x, y), z)$.

Therefore, let us suppose that some of them (but not all) belong to C_β and the rest belongs to C_γ. If one of them is equal to γ_j for some $j < q$, then $T(x, T(y, z)) = 0 = T(T(x, y), z)$ by definition of T. Otherwise, we have two possibilities: exactly one or two elements are equal to γ_q. In any case, we have easily that $T(T(x, y), z) = \beta_1 = T(x, T(y, z))$. We conclude that T is a t-norm on L satisfying $T(\beta_1, \beta_1) = \beta_1$ and $T(\gamma_q, \gamma_q) = \gamma_q$. □

The following result involves a class of t-norms on chains depending on a value $k \in \{1, 2, ..., p\}$.

Proposition 4.4. *Each t-norm T on L satisfying $T(\beta_1, \beta_1) = 0$ and $T(\gamma_q, \gamma_q) = \beta_1$ is expressed as follows: There is $k \in \{1, 2, ..., p\}$ such that*

$$T(x,y) = \begin{cases} T_k(x,y) & \text{if} \quad x, y \in C_\beta, \\ x \wedge y & \text{if} \quad x = 1 \text{ or } y = 1, \\ \beta_1 & \text{if} \quad (x,y) \in \{(\beta_i, \gamma_q), (\gamma_q, \beta_i) \mid i > k\}, \\ \beta_1 & \text{if} \quad x = y = \gamma_q, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

where T_k is a t-norm on C_β satisfying $T_k(\beta_1, \beta_p) = 0$ and $T_k(\beta_p, \beta_p) \leq \beta_k$.

Proof. Firstly, we prove that each t-norm T on L satisfying $T(\beta_1, \beta_1) = 0$ and $T(\gamma_q, \gamma_q) = \beta_1$ has this expression. Finally, we will prove that each binary operator T on L which has this expression is a t-norm satisfying the mentioned conditions.

Suppose that T is a t-norm on L. First of all, notice that $T(\beta_1, \gamma_q) = 0$ because if $T(\beta_1, \gamma_q) = \beta_1$, on the one hand

$$T(T(\beta_1, \gamma_q), \gamma_q) = T(\beta_1, \gamma_q) = \beta_1,$$

and on the other hand

$$T(\beta_1, T(\gamma_q, \gamma_q)) = T(\beta_1, \beta_1) = 0,$$

which is a contradiction.

$$\beta_1 = T(\beta_1, \gamma_q) = T(\beta_1, T(\gamma_q, \gamma_q)) = T(T(\beta_1, \gamma_q), \gamma_q) = T(\beta_1, \beta_1) = 0,$$

that is a contradiction. Taking this into account, let us check that $T(\beta_1, \beta_p) = 0$. By associativity, we have

$$T(\beta_1, \beta_p) = T(T(\gamma_q, \gamma_q), \beta_p) = T(\gamma_q, T(\gamma_q, \beta_p)) \leq T(\gamma_q, \beta_1) = 0.$$

Now, take $k = \max\{i \in \{1, 2, ..., p\} \mid T(\beta_i, \gamma_q) = 0\}$. Clearly, the set is not empty because $T(\beta_1, \gamma_q) = 0$. Let us prove that $T(\beta_p, \beta_p) \leq \beta_k$. By contradiction, suppose that $T(\beta_p, \beta_p) > \beta_k$, by associativity and monotonicity, we have the following contradiction

$$\beta_1 = T(\gamma_q, T(\beta_p, \beta_p)) = T(T(\gamma_q, \beta_p), \beta_p) \leq T(\beta_1, \beta_p) = 0.$$

Therefore, putting $T_k = T|_{C_\beta^2}$, we conclude that T_k is a t-norm on C_β satisfying $T_k(\beta_1, \beta_p) = 0$ and $T_k(\beta_p, \beta_p) \leq \beta_k$.

Moreover, by definition of k, $T(x, y) = \beta_1$ for each pair $(x, y) \in \{(\beta_i, \gamma_q), (\gamma_q, \beta_i)\}$ with $i > k$. Finally, taking into account the definition of k, we have that $T(\beta_i, \gamma_q) = 0$ whenever $i \leq k$ and using Lemma 4.1, we have that $T(\beta_i, \gamma_j) = 0$ for each $i \in \{1, 2, ..., p\}$ and $j \in \{1, 2, ..., q-1\}$.

Now, let us consider a binary operator T expressed by formula (3) and let us prove that it is a t-norm. Monotonicity, commutativity and the fact that the element 1 is the neutral element are straightforward by definition of T.

To prove the associative property, take $x, y, z \in L \setminus \{0, 1\}$ (whenever one of them is equal to 0 or 1, associativity is trivial).

If one of them belongs to the set $\{\beta_1, \gamma_1, \gamma_2, ..., \gamma_{q-1}\}$, then $T(T(x, y), z) = 0 = T(x, T(y, z))$ by definition of T. Hence, let us suppose that $x, y, z \in \{\beta_2, \beta_3, ..., \beta_p, \gamma_q\}$. If none of them is equal to γ_q, then

$$T(x, T(y, z)) = T_k(x, T_k(y, z)) = T_k(T_k(x, y), z) = T(T(x, y), z).$$

Therefore, we can consider that $\gamma_q \in \{x, y, z\}$ in the rest of the proof. If all of them are equal to γ_q, it is trivial. If exactly two of the elements are equal to γ_q, using that $T(\gamma_q, \gamma_q) = \beta_1$ and the fact that $T(\beta_1, \beta_i) = 0$ for each $i \in \{1, 2, ..., p\}$, we obtain that $T(x, (T(y, z)) = 0 = T(T(x, y), z)$. If only one of the elements is equal to γ_q, we have the following three cases:

1. $x = \gamma_q$. We have that

$$T(x, T(y, z)) \leq T(\gamma_q, T(\beta_p, \beta_p)) \leq T(\gamma_q, \beta_k) = 0$$

and

$$T(T(x, y), z) \leq T(T(\gamma_q, \beta_p), \beta_p) \leq T(\beta_1, \beta_p) = 0.$$

2. $y = \gamma_q$. We have that

$$T(x, T(y, z)) \leq T(\beta_p, \beta_1) = 0$$

and

$$T(T(x, y), z) \leq T(\beta_1, \beta_p) = 0.$$

3. $z = \gamma_q$. It is similar to the first case.

We conclude that T is a t-norm on L. In addition, $T(\beta_1, \beta_1) \leq T(\beta_1, \beta_p) = 0$ and $T(\gamma_q, \gamma_q) = \beta_1$. □

Proposition 4.5. *Each t-norm T on L satisfying $T(\beta_1, \beta_1) = 0$ and $T(\gamma_q, \gamma_q) \in C_\gamma \setminus \{\gamma_q\}$ is expressed as one of the following two formulas:*

$$T(x, y) = \begin{cases} T_1(x, y) & \text{if} \quad x, y \in C_\beta, \\ T_2(x, y) & \text{if} \quad x, y \in C_\gamma, \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

where T_1 is a t-norm on C_β satisfying $T_1(\beta_1, \beta_1) = 0$ and T_2 is a t-norm on C_γ satisfying $T_2(\gamma_q, \gamma_q) < \gamma_q$, or, there is $k \in \{2, 3, ..., p\}$ such that

$$T(x, y) = \begin{cases} T_1(x, y) & \text{if} \quad x, y \in C_\beta, \\ T_2(x, y) & \text{if} \quad x, y \in C_\gamma, \\ \beta_1 & \text{if} \quad (x, y) \in \{(\beta_i, \gamma_q), (\gamma_q, \beta_i) \mid i \geq k\}, \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

where T_2 is a t-norm on C_γ satisfying $T_2(\gamma_q, \gamma_q) < \gamma_q$ and T_1 is a t-norm on C_β satisfying

1. *$T_1(\beta_i, \beta_1) = 0$ for $i < k$.*
2. *If $T_1(\beta_k, \beta_k) = \beta_k$, then*

$$T_1(\beta_1, \beta_p) = T_1(\beta_1, \beta_k) = \beta_1.$$

3. *If $T_1(\beta_k, \beta_k) < \beta_k$, then*

$$T_1(\beta_1, \beta_p) = T_1(\beta_1, \beta_k) = 0$$

and

$$T(\beta_p, \beta_p) \leq \beta_{k-1}.$$

Proof. Firstly, we prove that each t-norm T on L satisfying $T(\beta_1, \beta_1) = 0$ and $T(\gamma_q, \gamma_q) \in C_\gamma \setminus \{\gamma_q\}$ has one of these expressions. Finally, we will prove that each binary operator T on L which has one of these expressions is a t-norm satisfying the mentioned conditions.

Suppose that T is a t-norm on L. We consider two possibilities:

1. Case $T(\beta_p, \gamma_q) = 0$. By monotonicity, $T(\beta_i, \gamma_q) = 0$ for all $i \in \{1, 2, ..., p\}$. Since T is a t-norm satisfying $T(\gamma_q, \gamma_q) \in C_\gamma \setminus \{\gamma_q\}$, the restriction $T_2 := T|_{C_\gamma}$ is a t-norm on C_γ satisfying the same condition. Also, the restriction $T_1 = T|_{C_\beta}$ is a t-norm on C_β satisfying $T_1(\beta_1, \beta_1) = 0$. Therefore, T is given by formula (4).

2. Case $T(\beta_p, \gamma_q) = \beta_1$. We are going to check that T is given by formula (5). Take

$$k = \min\{j \mid T(\beta_j, \gamma_q) = \beta_1\}.$$

By monotonicity $T(\beta_1, \gamma_q) \leq T(\gamma_q, \gamma_q) \leq \gamma_{q-1}$, hence $T(\beta_1, \gamma_q) \leq \beta_1 \wedge \gamma_{q-1} = 0$. Since $T(\beta_1, \gamma_q) = 0$, we have that $k \in \{2, 3, ..., p\}$. By monotonicity and commutativity, $T(\beta_i, \gamma_q) = T(\gamma_q, \beta_i) = \beta_i$ for all $i \geq k$. Analogously to the previous case, we have that $T_1 := T|_{C_\beta}$ is a t-norm on C_β satisfying $T_1(\beta_1, \beta_1) = 0$ and $T_2 := T|_{C_\gamma}$ is a t-norm on C_γ satisfying $T_2(\gamma_q, \gamma_q) \in C_\gamma \setminus \{\gamma_q\}$. If $i < k$, we have

$$T(\beta_1, \beta_i) = T(\beta_1, \beta_i) = T(T(\gamma_q, \beta_k), \beta_i) = T(\gamma_q, T(\beta_k, \beta_i)) \leq T(\gamma_q, \beta_i) = 0.$$

Now, we will check that $T_1(\beta_k, \beta_1) = T_1(\beta_p, \beta_1)$:

$$T(\beta_k, \beta_1) = T(\beta_k, T(\gamma_q, \beta_p)) = T(T(\beta_k, \gamma_q), \beta_p) = T(\beta_1, \beta_p) = T(\beta_p, \beta_1).$$

If $T_1(\beta_k, \beta_k) = \beta_k$, we have that

$$T(\beta_1, \beta_p) = T(\beta_1, \beta_k) = T(T(\gamma_q, \beta_k), \beta_k) = T(\gamma_q, T(\beta_k, \beta_k)) = T(\gamma_q, \beta_k) = \beta_1.$$

If $T_1(\beta_k, \beta_k) < \beta_k$, we have that

$$T(\beta_1, \beta_p) = T(\beta_1, \beta_k) = T(T(\gamma_q, \beta_k), \beta_k) = T(\gamma_q, T(\beta_k, \beta_k)) \leq T(\gamma_q, \beta_{k-1}) = 0$$

and

$$0 = T(\beta_1, \beta_p) = T(T(\gamma_q, \beta_p), \beta_p) = T(\gamma_q, T(\beta_p, \beta_p))$$

hence, $T(\beta_p, \beta_p) < \beta_k$. Therefore, $T_1 = T|_{C_\beta}$ satisfies the required properties.

Now, let us consider a binary operator T expressed by formula (4) or (5) and let us prove that it is a t-norm. If T is given by the first one, axioms of t-norms are trivially fullfilled.

We study when T is given by formula (5). Monotonicity, commutativity and the fact that the element 1 is the neutral element are straightforward by definition of T taking into account that T_1 and T_2 are t-norms.

To prove the associative property, take $x, y, z \in L \setminus \{0, 1\}$ (whenever one of them is equal to 0 or 1, associativity is trivial).

If $x, y, z \in C_\beta$ or $x, y, z \in C_\gamma$, then $T(T(x, y), z) = T(x, T(y, z))$ because T_1 and T_2 are t-norms on C_β and C_γ respectively.

Therefore, in the rest of the proof let us suppose that some of them (but not all) belong to C_β and the rest belongs to C_γ. If one of them is equal to γ_j for some $j < q$, then $T(x, T(y, z)) = 0 = T(T(x, y), z)$ using monotonicity. Otherwise, at least one of them is equal to γ_q. If exactly two of them are equal to γ_q, it is easy to check that $T(x, T(y, z)) = 0 = T(T(x, y), z)$ using monotonicity and the fact $T(\beta_1, \gamma_q) = 0$. If only one of the elements is equal to γ_q, we have three cases:

1. $x = \gamma_q$, $y = \beta_i$ and $z = \beta_j$.

   If $i, j < k$, then
   $$T(T(\gamma_q, \beta_i), \beta_j) = T(0, \beta_j) = 0$$
   and
   $$T(\gamma_q, T(\beta_i, \beta_j)) \leq T(\gamma_q, \beta_i \wedge \beta_j) = 0.$$
   If $i < k$ and $j \geq k$, then
   $$T(T(\gamma_q, \beta_i), \beta_j) = T(0, \beta_j) = 0$$
   and
   $$T(\gamma_q, T(\beta_i, \beta_j)) \leq T(\gamma_q, \beta_i) = 0.$$
   If $i \geq k$ and $j < k$, then
   $$T(T(\gamma_q, \beta_i), \beta_j) = T(\beta_1, \beta_j) = 0$$

and

$$T(\gamma_q, T(\beta_i, \beta_j)) \leq T(\gamma_q, \beta_j) = 0.$$

If $i, j \geq k$, then we cosider the following two cases:

A) $T(\beta_k, \beta_k) = \beta_k$.

$$T(T(\gamma_q, \beta_i), \beta_j) = T(\beta_1, \beta_j) = \beta_1$$

and

$$T(\gamma_q, T(\beta_i, \beta_j)) \geq T(\gamma_q, T(\beta_k, \beta_k)) = T(\gamma_q, \beta_k) = \beta_1.$$

B) $T(\beta_k, \beta_k) < \beta_k$.

$$T(T(\gamma_q, \beta_i), \beta_j) = T(\beta_1, \beta_j) = 0$$

and

$$T(\gamma_q, T(\beta_i, \beta_j)) \leq T(\gamma_q, T(\beta_p, \beta_p)) \leq T(\gamma_q, \beta_{k-1}) = 0.$$

2. $x = \beta_i$, $y = \gamma_q$ and $z = \beta_j$. If $i < k$ or $j < k$, it is easy to check that

$$T(\beta_i, T(\gamma_q, \beta_j)) = 0 = T(T(\beta_i, \gamma_q), \beta_j).$$

Otherwise, let us assume that $i, j \geq k$. Then,

$$T(\beta_i, T(\gamma_q, \beta_j)) = T(\beta_i, \beta_1) = T(\beta_p, \beta_1)$$

and

$$T(T(\beta_i, \gamma_q), \beta_j) = T(\beta_1, \beta_j) = T(\beta_1, \beta_p).$$

3. $x = \beta_i$, $y = \beta_j$ and $z = \gamma_q$. It is similar to the first case.

We conclude that T is a t-norm on L. Moreover, $T(\beta_1, \beta_1) \leq T(\beta_1, \beta_p) = 0$ and $T(\gamma_q, \gamma_q) = T'(\gamma_q, \gamma_q) \in C_\gamma \setminus \{\gamma_q\}$. □

Proposition 4.6. *Each t-norm T on L satisfying $T(\beta_1, \beta_1) = 0$, $T(\gamma_q, \gamma_q) = \gamma_q$, and $T(\beta_1, \gamma_q) = \beta_1$ is expressed as follows:*

$$T(x, y) = \begin{cases} T'(x, y) & \text{if} \quad x, y \in C_\gamma, \\ x \wedge y & \text{if} \quad x = 1 \text{ or } y = 1, \\ \beta_1 & \text{if} \quad (x, y) \in \{(\beta_i, \gamma_q), (\gamma_q, \beta_i) \mid i \geq 1\}, \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

where T' is a t-norm on C_γ satisfying $T'(\gamma_q, \gamma_q) = \gamma_q$.

Proof. Firstly, we prove that each t-norm T on L satisfying $T(\beta_1, \beta_1) = 0$, $T(\gamma_q, \gamma_q) = \gamma_q$ and $T(\beta_1, \gamma_q) = \beta_1$ has this expression. Finally, we will prove that each binary operator T on L which has this expression is a t-norm satisfying the mentioned conditions.

If T is a t-norm on L that satisfies the hypotheses, we have that

$$\beta_1 = T(\beta_1, \gamma_q) \leq T(\beta_i, \gamma_q) \leq \beta_2 \wedge \gamma_q \leq \beta_1$$

for each $i \geq 1$. Moreover, $T' = T|_{C_\gamma^2}$ is a t-norm on C_γ satisfying $T'(\gamma_q, \gamma_q) = \gamma_q$. Now, we are going to prove that $T(\beta_i, \beta_j) = 0$ for each $i, j \in \{1, 2, ..., p\}$. By hypothesis, $T(\beta_1, \beta_1) = 0$. Firstly, by contradiction, let us suppose that $T(\beta_1, \beta_i) = \beta_1$ for some $i \in \{1, 2, ..., p\}$. Hence, $\beta_1 = T(\gamma_q, \beta_i)$ by monotonicity. Therefore,

$$0 = T(\beta_1, \beta_1) = T(T(\gamma_q, \beta_i), \beta_1) = T(\gamma_q, T(\beta_i, \beta_1)) = T(\gamma_q, \beta_1) = \beta_1,$$

that is a contradiction. We conclude that $T(\beta_1, \beta_i) = 0$ for each $i \in \{1, 2, ..., p\}$. Secondly, let us check that $T(\beta_i, \beta_j) = 0$ for each $i, j \in \{1, 2, ..., p\}$. By contradiction, assume that $T(\beta_i, \beta_j) \geq \beta_1$ for some $i, j \in \{1, 2, ..., p\}$. Then, taking into account that $\beta_1 = T(\gamma_q, \beta_j)$ as a consequence of the hypothesis due to monotonicity, we have that

$$0 = T(\beta_1, \beta_i) = T(T(\gamma_q, \beta_j), \beta_i) = T(\gamma_q, T(\beta_j, \beta_i)) \leq T(\gamma_q, \beta_1) = \beta_1,$$

that is a contradiction. Hence $T(\beta_i, \beta_j) = 0$ for each $i, j \in \{1, 2, ..., p\}$. The rest of values are obtained from Lemma 4.1.

Now, let us consider a binary operator T expressed by formula (6). Monotonicity, commutativity and the fact that the element 1 is the neutral element are straightforward by definition of T taking into account that T' is a t-norm on C_γ. Let us check that T is associative. Take $x, y, z \in L \setminus \{0, 1\}$ (whenever one of them is equal to 0 or 1, associativity is trivial).

If all of them belong to C_γ, then

$$T(x, T(y, z)) = T'(x, T'(y, z)) = T'(T'(x, y), z) = T(T(x, y), z).$$

If all of them belong to C_β, then

$$T(x, T(y, z)) = 0 = T(T(x, y), z).$$

Therefore, let us suppose that some of them (but not everyone) belong to C_β and the rest belongs to C_γ. If one of them is equal to γ_j for some $j < q$, then $T(x, T(y, z)) = 0 = T(T(x, y), z)$ by definition of T. Otherwise, the elements that belong to C_γ are equal to γ_q. Since $T(\gamma_q, \beta_i) = \beta_1$ and $T(\beta_i, \beta_j) = 0$ for each $i, j \in \{1, 2, ..., p\}$, we have that if exactly two of them are equal to γ_q, then

$$T(x, T(y, z)) = \beta_1 = T(T(x, y), z)$$

and if exactly one of them is equal to γ_q, then

$$T(x, T(y, z)) = 0 = T(T(x, y), z).$$

Therefore, T is a t-norm on L and it satisfies $T(\beta_1, \beta_1) = 0$, $T(\gamma_q, \gamma_q) = T'(\gamma_q, \gamma_q) = \gamma_q$, and $T(\beta_1, \gamma_q) = \beta_1$. □

Proposition 4.7. *Each t-norm T on L satisfying $T(\gamma_q, \gamma_q) = \gamma_q$ and $T(\beta_1, \gamma_q) = 0$ is expressed as follows:*

$$T(x, y) = \begin{cases} T_1(x, y) & \text{if} \quad x, y \in C_\beta, \\ T_2(x, y) & \text{if} \quad x, y \in C_\gamma, \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

where T_1 is a t-norm on C_β satisfying $T_1(\beta_1, \beta_1) = 0$, and T_2 is a t-norm on C_γ satisfying $T_2(\gamma_q, \gamma_q) = \gamma_q$.

Proof. Firstly, we prove that each t-norm T on L satisfying $T(\beta_1, \beta_1) = 0$, $T(\gamma_q, \gamma_q) = \gamma_q$ and $T(\beta_1, \gamma_q) = 0$ has this expression. Finally, we will prove that each binary operator T on L which has this expression is a t-norm satisfying the mentioned conditions.

If T is a t-norm on L, then $T|_{C_\beta^2}$ and $T|_{C_\gamma^2}$ are t-norms on C_β and C_γ respectively. Notice that we can guarantee that $T|_{C_\gamma^2}$ is well-defined because $T(\gamma_q, \gamma_q) = \gamma_q \in C_\gamma$. Moreover, since $T(\beta_1, \beta_1) = 0$ and $T(\gamma_q, \gamma_q) = \gamma_q$, we can put $T_1 = T|_{C_\beta^2}$ and $T_2 = T|_{C_\gamma^2}$. Now, let us check that $T(\beta_i, \gamma_q) = 0$ for each $i \in \{1, 2, ..., p\}$. By monotonicity and associativity, we have

$$0 = T(\beta_1, \gamma_q) \geq T(T(\beta_i, \gamma_q), \gamma_q) = T(\beta_i, T(\gamma_q, \gamma_q)) = T(\beta_i, \gamma_q) \geq 0.$$

The rest of values are obtained from Lemma 4.1.

Now, let us consider a binary operator T expressed by formula (7). Monotonicity, commutativity and the fact that the element 1 is the neutral element are straightforward by definition of T taking into account that T_1 and T_2 are t-norms on C_β and C_γ. Let us check that T is associative. Take $x, y, z \in L \setminus \{0, 1\}$ (whenever one of them is equal to 0 or 1, associativity is trivial). If $x, y, z \in C_\beta$ or $x, y, z \in C_\gamma$, then $T(x, T(y, z)) = T(T(x, y), z)$ because T_1 and T_2 are t-norms on C_β and C_γ respectively. Otherwise, by definition of T, we have that $T(x, T(y, z)) = 0 = T(T(x, y), z)$. Therefore, T is a t-norm on L. In addition, $T(\beta_1, \beta_1) = T_1(\beta_1, \beta_1) = 0$, $T(\gamma_q, \gamma_q) = T_2(\gamma_q, \gamma_q) = \gamma_q$, and $T(\beta_1, \gamma_q) = 0$. □

After studying each case, we propose a theorem that compiles the previous results.

Theorem 4.8. *If T is a t-norm on L, then T is expressed by formula* (1), (2), (3), (4), (5), (6) *or* (7).

Proof. We have that $T(\beta_1, \beta_1) \leq \beta_1$, that is, two possible values.

1. If $T(\beta_1, \beta_1) = \beta_1$, by monotonicity $\beta_1 \leq T(\gamma_q, \gamma_q) \leq \gamma_q$. Hence, $T(\gamma_q, \gamma_q) \in \{\beta_1, \gamma_q\}$. This implies that T is expressed by formula (1) or (2) using Proposition 4.2 and 4.3 respectively.
2. If $T(\beta_1, \beta_1) = 0$, we consider several cases: If $T(\gamma_q, \gamma_q) = \beta_1$, by Proposition 4.4, T is expressed by formula (3). If $T(\gamma_q, \gamma_q) \in C_\gamma \setminus \{\gamma_q\}$, by Proposition 4.5, T is expressed by formula (4) or (5). If $T(\gamma_q, \gamma_q) = \gamma_q$, we must take into account the two possible values of $T(\beta_1, \gamma_q)$, that is, $T(\beta_1, \gamma_q) \in \{0, \beta_1\}$. By Proposition 4.6 and 4.7, T is expressed by formula (6) or (7), respectively.

□

5. Archimedean t-norms

In this section, we enunciate the results that involve the Archimedean t-norms on L.

Let us consider the following notation (it can be found in [24]). Given a bounded lattice L, a t-norm T and $x \in L$, the term $x_T^{(n)}$ denotes the following

$$x_T^{(1)} = x \text{ and } x_T^{(n)} = T(x_T^{(n-1)}, x) \text{ for } n \geq 2.$$

Definition 5.1 ([36])**.** *Let L be a bounded lattice and $T : L \times L \longrightarrow L$ a t-norm on L. The t-norm T is called Archimedean if for each $x, y \in L \setminus \{0, 1\}$ there is $n \in \mathbb{N}$ such that $x_T^{(n)} < y$.*

We will use the following characterization for Archimedean t-norms on finite lattices.

Proposition 5.2 ([5])**.** *Let L be a finite lattice and $T : L \times L \longrightarrow L$ a t-norm. The following facts are equivalent.*

(1) *T is an Archimedean t-norm.*

(2) *T has only two idempotents: 0 and 1.*

In [26], Medina defined subidempotence when the t-norm satisfies condition (2). His aim was to characterize the ordinal sum of t-norms. Subidempotence plays a significant role in his results. Under the hypotheses of Proposition 5.2, the set of subidempotent t-norms and the set of Archimedean t-norms are equal.

In Table 4, we present the Archimedean t-norms on a discrete chain satisfying the same condition of Table 3.

The condition $T'_n(\alpha_n, \alpha_n) \leq \alpha_n$ is redundant. In addition, Archimedean property implies that $T'_n(\alpha_1, \alpha_n) = 0$ as the following result shows. Therefore, the diagonal of Table 4 provides the number of Archimedean t-norms on a discrete chain with $n + 2$ elements.

Proposition 5.3. *If T is an Archimedean t-norm on a chain*

$$C = \{0 < \alpha_1 < \alpha_2 < \cdots < \alpha_n < 1\},$$

$k \setminus n$	1	2	3	4	5	6	7	8	9
1	1	2	4	8	16	32	64	128	256
2		2	6	17	50	157	525	1843	6690
3			6	22	80	309	1335	6541	35839
4				22	95	419	2024	11279	74493
5					95	471	2467	14559	101633
6						471	2670	16508	118239
7							2670	17387	127559
8								17387	131753
9									131753

Table 4: For each k and n, the number of Archimedean t-norms T'_k on a discrete chain with $n+2$ elements satisfying that $T'_k(\alpha_1, \alpha_n) = 0$ and $T'_k(\alpha_n, \alpha_n) \leq \alpha_k$.

then $T(\alpha_1, \alpha_n) = 0$.

Proof. By contradiction, let us suppose that $T(\alpha_1, \alpha_n) = \alpha_1$. We have that

$$\alpha_1 = T(\alpha_1, \alpha_n) = T(T(\alpha_1, \alpha_n), \alpha_n) = T(\alpha_1, T(\alpha_n, \alpha_n)) \leq T(\alpha_1, \alpha_{n-1}) \leq \alpha_1$$

Hence $T(\alpha_1, \alpha_{n-1}) = \alpha_1$, again

$$\alpha_1 = T(\alpha_1, \alpha_{n-1}) = T(T(\alpha_1, \alpha_{n-1}), \alpha_{n-1}) = T(\alpha_1, T(\alpha_{n-1}, \alpha_{n-1})) \leq T(\alpha_1, \alpha_{n-2}) \leq \alpha_1$$

Recursively, in a finite number of steps, we obtain that $T(\alpha_1, \alpha_1) = \alpha_1$, that is, a contradiction. □

Now, we present all Archimedean t-norms on the lattice L (see Figure 1).

Corollary 5.4. *If T is an Archimedean t-norm on L, we have three possible cases:*

1. *There is* $k \in \{1, 2, 3, ..., p\}$ *such that*

$$T(x,y) = \begin{cases} T_k(x,y) & \text{if} \quad x, y \in C_\beta, \\ x \wedge y & \text{if} \quad x = 1 \text{ or } y = 1, \\ \beta_1 & \text{if} \quad (x,y) \in \{(\beta_i, \gamma_q), (\gamma_q, \beta_i) \mid i > k\}, \\ \beta_1 & \text{if} \quad x = y = \gamma_q, \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

where T_k *is an Archimedean t-norm on* C_β *satisfying* $T_k(\beta_p, \beta_p) \leq \beta_k$*, and* T' *is an Archimedean t-norm on* C_γ*.*

2. $$T(x,y) = \begin{cases} T_1(x,y) & \text{if} \quad x, y \in C_\beta, \\ T_2(x,y) & \text{if} \quad x, y \in C_\gamma, \\ 0 & \text{otherwise.} \end{cases} \tag{9}$$

where T_1 *is an Archimedean t-norm on* C_β *and* T_2 *is an Archimedean t-norm on* C_γ*.*

3. *There is* $k \in \{2, 3, ..., p\}$ *such that*

$$T(x,y) = \begin{cases} T_1(x,y) & \text{if} \quad x, y \in C_\beta, \\ T_2(x,y) & \text{if} \quad x, y \in C_\gamma, \\ \beta_1 & \text{if} \quad (x,y) \in \{(\beta_i, \gamma_q), (\gamma_q, \beta_i) \mid i \geq k\}, \\ 0 & \text{otherwise.} \end{cases} \tag{10}$$

where T_1 *is an Archimedean t-norm on* C_β *satisfying that* $T_1(\beta_p, \beta_p) < \beta_k$*, and* T_2 *is an Archimedean t-norm on* C_γ*.*

Proof. It is a consequence of Theorem 4.8, Proposition 5.2, and Proposition 5.3. □

6. Divisible t-norms

In this section we focus on the divisible t-norms defined on the lattice of Figure 1. We study each t-norm presented in Section 4 and determine when they are divisible t-norms. It is important to point out that, sometimes, the existence of divisible t-norms depends on the values of p and q. Let us start recalling the notion.

Definition 6.1 ([20]). *Let L be a bounded lattice and $T : L \times L \longrightarrow L$ a t-norm on L. The t-norm T is called divisible if for all $x, y \in L$ satisfying $x \leq y$, there is $z \in L$ such that $T(y, z) = x$.*

When the lattice is a discrete chain, a t-norm is 1-Lipschitz if and only if it is divisible (see [12]).

Under the premise that the lattice is a discrete chain, their divisible t-norms (or 1-Lipschitz t-norms) are completely described in [25]. The following result will help us to know the number of them.

Lemma 6.2 ([25]). *Let C be the chain $\{0, \alpha_1, \alpha_2, ..., \alpha_n, 1\}$. Then there is a bijection between the set of all divisible t-norms on C and the power set $\wp(C \setminus \{0, 1\})$. Thus there are 2^n divisible t-norms on C.*

In fact, a divisible t-norm on a discrete chain is determined by its non trivial idempotent elements.

Remark 6.3. *The only divisible and Archimedean t-norm on a discrete chain $C = \{0, \alpha_1, \alpha_2, ..., \alpha_n, 1\}$ is the Lukasiewicz t-norm T_L (see [25]), whose values $T_L(\alpha_i, \alpha_j)$ are the following*

$$T_L(\alpha_i, \alpha_j) = \begin{cases} \alpha_{i+j-n-1} & \text{if} \quad i + j - n - 1 > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Now, we present two lemmas which will be used in some propositions.

Lemma 6.4. *If T is a divisible t-norm on L and $\omega \in L$ is an idempotent element of T, we have that*

1. *If $x \leq \omega$, $T(\omega, x) = x$.*
2. *If $\omega \leq x$, $T(\omega, x) = \omega$.*

Proof. If $x \leq \omega$, by divisibility, there is $z \in L$ such that $T(\omega, z) = x$. By associativity,

$$T(\omega, x) = T(\omega, T(\omega, z)) = T(T(\omega, \omega), z) = T(\omega, z) = x.$$

The second case is clear using the monotonicity of T. □

Lemma 6.5. *If T is a divisible t-norm on L and $T(\gamma_q, \gamma_q) \in C_\gamma$, then $T|_{C_\gamma^2}$ is a divisible t-norm on C_γ.*

Proof. The restriction is well-defined because $T(\gamma_q, \gamma_q) \in C_\gamma$. Take $x, y \in C_\gamma$ satisfying $x \leq y$. If $x = 0$ or $x = 1$, divisibility is clear. Otherwise, $x = \gamma_j$ for some $j \in \{1, 2, ..., q\}$. Since T is a divisible t-norm, there is $z \in L$ such that $T(y, z) = x$. This implies that $\gamma_j \leq z$, hence $z \in C_\gamma$. □

Unlike the case of C_γ, if a t-norm T is divisible on L, it is not possible to ensure that the restriction on the chain C_β is also a divisible t-norm (due to the structure of the lattice). Therefore, each case must be analyzed independently. Below, based on formulas (1)-(7), we present the divisible t-norms on L.

Proposition 6.6. *If $q \geq 2$, there are no divisible t-norms on L satisfying formula* (1)*.*

Proof. By contradiction, let us suppose that T is a divisible t-norm according to formula (1). Since $\gamma_1 \leq \gamma_q$, there is $z \in L$ such that $T(z, \gamma_q) = \gamma_1$. However, by definition of T, $T(z, \gamma_q) \in \{0, \beta_1, \gamma_q\}$, and $q \geq 2$. □

Proposition 6.7. *If $q = 1$ and T is a divisible t-norm on L satisfying $T(\beta_1, \beta_1) = \beta_1$ and $T(\gamma_1, \gamma_1) = \beta_1$, then T is expressed as follows:*

$$T(x, y) = \begin{cases} T'(x, y) & \text{if} \quad x, y \in C_\beta, \\ x \wedge y & \text{if} \quad x = 1 \text{ or } y = 1, \\ \beta_1 & \text{if} \quad (x, y) \in \{(\beta_i, \gamma_1), (\gamma_1, \beta_i) \mid i \in \{1, ..., p\}\}, \\ \beta_1 & \text{if} \quad x = y = \gamma_1, \\ 0 & \text{otherwise.} \end{cases} \tag{11}$$

where T' is a divisible t-norm on C_β satisfying $T'(\beta_1, \beta_1) = \beta_1$.

Proof. Firstly, we prove that each divisible t-norm T on L satisfying $T(\beta_1, \beta_1) = \beta_1$ and $T(\gamma_1, \gamma_1) = \beta_1$ has this expression. Finally, we will prove that each binary operator T on L which has this expression is a divisible t-norm satisfying the mentioned conditions.

If T is a divisible t-norm on L, by Proposition 4.2, T must be like formula (1). Let us prove that $T' = T|_{C_\beta^2}$ is divisible. Take $x, y \in C_\beta \setminus \{0,1\}$ satisfying $x \leq y$. If $x = \beta_1$, by monotonicity of T, taking $z = \beta_1$ we have that $\beta_1 = T(\beta_1, \beta_1) \leq T(y, z) \leq \beta_1$. If $x > \beta_1$, since T is divisible, there is $z \in L$ such that $T(y, z) = x$. This implies that $x \leq z$, hence $x \in C_\beta$. Therefore, T' is divisible.

Conversely, take the binary operator T of the formula. We know that T is a t-norm due to Proposition 4.2. Now, let us prove that T is divisible. Take $x, y \in L$ such that $x \leq y$. If $x = \beta_1$, put $z = \beta_1$. If $x > \beta_1$, divisibility of T' provides an element $z \in C_\beta \subseteq L$ such that $T(z, y) = x$. If $x \notin C_\beta$, then $x = \gamma_1$. Here, divisibility is trivial because $y = x$ or $y = 1$. □

Proposition 6.8. *Each divisible t-norm T on L satisfying $T(\beta_1, \beta_1) = \beta_1$ and $T(\gamma_q, \gamma_q) = \gamma_q$ is expressed as follows:*

$$T(x,y) = \begin{cases} T_1(x,y) & \text{if} \quad x, y \in C_\beta, \\ T_2(x,y) & \text{if} \quad x, y \in C_\gamma, \\ \beta_1 & \text{if} \quad (x,y) \in \{(\beta_i, \gamma_q), (\gamma_q, \beta_i) \mid i \in \{1, ..., p\}\}, \\ 0 & \text{otherwise.} \end{cases} \tag{12}$$

where T_1 is a divisible t-norm on C_β satisfying $T_1(\beta_1, \beta_1) = \beta_1$ and T_2 is a divisible t-norm on C_γ satisfying $T_2(\gamma_q, \gamma_q) = \gamma_q$.

Proof. Firstly, we prove that each divisible t-norm T on L satisfying $T(\beta_1, \beta_1) = \beta_1$ and $T(\gamma_q, \gamma_q) = \gamma_q$ has this expression. Finally, we will prove that each binary operator T on L which has this expression is a divisible t-norm satisfying the mentioned conditions.

If T is a divisible t-norm on L, by Proposition 4.3, T must be like formula (2). By Lemma 6.5, $T_2 = T|_{C_\gamma^2}$ is divisible on C_γ. Now, let us prove that $T_1 = T|_{C_\beta^2}$ is divisible. Take $x, y \in C_\beta \setminus \{0,1\}$. If $x = \beta_1$, by monotonicity of T, taking $z = \beta_1$ we have that $\beta_1 = T(\beta_1, \beta_1) \leq T(y, z) \leq \beta_1$. If $x > \beta_1$, since T is divisible, there is $z \in L$ such that $T(y, z) = x$. This implies that $x \leq z$, hence $x \in C_\beta$. Therefore, T_1 is divisible on C_β.

Conversely, take the binary operator T of the formula. We know that T is a t-norm due to Proposition 4.3. Now, let us prove that T is divisible. Take $x, y \in L$ such that $x \leq y$, we enumerate all the possibilities:

1. If $x, y \in C_\beta$, there is $z \in C_\beta$ such that $T(z, y) = x$ because T_1 is divisible on C_β.
2. If $x \in C_\beta$ and $y \in C_\gamma$, necessarily $x = \beta_1$ and $y = \gamma_q$. Putting $z = \beta_1$ we have that $T(y, z) = x$.
3. If $x \in C_\gamma$, then $y \in C_\gamma$. Since T_2 is divisible on C_γ, there is $z \in C_\gamma$ such that $T(z, y) = x$.

□

Proposition 6.9. *If $q \geq 2$, there are no divisible t-norms on L satisfying formula* (3).

Proof. It is similar to Proposition 6.6. □

Now, we study formula (3) whenever $q = 1$. If $p = 1$, L is a discrete chain and its divisible t-norms are well-known. The following result shows the case $p \geq 2$.

Proposition 6.10. *If $q = 1$ and $p \geq 2$, each divisible t-norm T on L satisfying $T(\beta_1, \beta_1) = 0$ and $T(\gamma_1, \gamma_1) = \beta_1$ is expressed as:*

$$T(x, y) = \begin{cases} T_L(x, y) & \text{if} \quad x, y \in C_\beta, \\ x \wedge y & \text{if} \quad x = 1 \text{ or } y = 1, \\ \beta_1 & \text{if} \quad x = y = \gamma_1, \\ 0 & \text{otherwise.} \end{cases} \tag{13}$$

or

$$T(x, y) = \begin{cases} T_L(x, y) & \text{if} \quad x, y \in C_\beta, \\ x \wedge y & \text{if} \quad x = 1 \text{ or } y = 1, \\ \beta_1 & \text{if} \quad (x, y) \in \{(\beta_p, \gamma_1), (\gamma_1, \beta_p)\}, \\ \beta_1 & \text{if} \quad x = y = \gamma_1, \\ 0 & \text{otherwise.} \end{cases} \tag{14}$$

where T_L is the Łukasiewicz t-norm (see Remark 6.3*).*

Proof. Firstly, we prove that each divisible t-norm T on L satisfying $T(\beta_1, \beta_1) = 0$ and $T(\gamma_1, \gamma_1) = \beta_1$ has one of these expressions. Finally, we will prove that each binary operator T on L which has one of these expressions, is a divisible t-norm satisfying the mentioned conditions.

By Proposition 4.4, T is expressed by formula (3) for some $k \in \{1, 2, ..., p\}$. Let us check that $k \leq p - 2$ is not possible under the hypothesis of divisibility. For each $z \leq \beta_p$, we have that

$$T_k(z, \beta_p) \leq T_k(\beta_p, \beta_p) \leq \beta_{p-2} < \beta_{p-1}.$$

Hence, there is not $z \in L$ such that $T(z, \beta_p) = \beta_{p-1}$, that is, divisibility fails. Therefore, $k \geq p - 1$.

Now, taking into account that T is a divisible t-norm, let us study the two values of k:

1. Case $k = p$. Take $x, y \in C_\beta \setminus \{0, 1\}$ satisfying $x \leq y$. Since T is divisible, there is $z \in L$ such that $T(y, z) = x$. If $x > \beta_1$, then $z \in C_\beta$. If $x = \beta_1$, then $z \in C_\beta \cup \{\gamma_1\}$. But $T(\gamma_1, y) = 0$ by definition of T whenever $y = \beta_i$ for each i. Hence, $z \in C_\beta$ and $T|_{C_\beta^2}$ is a divisible t-norm on C_β. According to Lemma 6.4 and the fact that $T(\beta_1, \beta_p) = 0$, we have that $T|_{C_\beta^2}$ has no non-trivial idempotent elements. Therefore, $T|_{C_\beta^2} = T_L$ (check Remark 6.3), and formula (13) is obtained.

2. Case $k = p - 1$.
   If $p = 2$, then $T(\beta_2, \beta_2) \leq \beta_1$. Divisibility provides $z \in L$ such that $T(\beta_2, z) = \beta_1$. Since $T(\beta_1, \beta_2) = 0$, $z = \beta_2$. Therefore, the values of $T|_{C_\beta^2}$ are completely determined and it is equal to the Łukasiewicz t-norm T_L.
   If $p \geq 3$, since $T(\beta_1, \beta_p) = 0$ and T is divisible, Lemma 6.4 states that the elements $\beta_1, \beta_2, ..., \beta_p$ are not idempotents. Let us prove that $T(\beta_i, \beta_j) < \beta_i \wedge \beta_j$ for each $i, j \in \{1, 2, ..., p\}$. Without loss of generality, suppose $\beta_i \leq \beta_j$. Take

$$t = \min\{m \in \{1, 2, ..., p\} \mid T(\beta_m, \beta_i) = \beta_i\}.$$

The set is non-empty and $i < t \leq j$. By associativity,

$$\beta_i = T(\beta_i, \beta_t) = T(T(\beta_i, \beta_t), \beta_t) = T(\beta_i, T(\beta_t, \beta_t))$$

this is a contradiction by definition of t and the fact that β_t is not idempotent. Hence, we can conclude that $T(\beta_i, \beta_j) < \beta_i$, equivalently, $T(\beta_i, \beta_j) < \beta_i \wedge \beta_j$ for each $i, j \in \{1, 2, ..., p\}$.

Given $\beta_i \leq \beta_j$, divisibility of T provides an element $z \in L$ such that $T(\beta_j, z) = \beta_i$. By monotonicity of T and the values of $T(\beta_j, \gamma_1)$, notice that we can guarantee that $z \in C_\beta$ for each case, except for $i = 1$ and $j = p$ simultaneously. In that case, let us check that $T(\beta_p, \beta_2) = \beta_1$. We list the following steps:

(a) $T(\beta_p, \beta_i) = \beta_{i-1}$ for $i > 2$: Since $\beta_{i-1} \leq \beta_p$ and $i - 1 \neq 1$, there is $z \in C_\beta$ such that $T(\beta_p, z) = \beta_{i-1}$. For $i = p$, we have that $z \geq \beta_{p-1}$, that is, $T(\beta_p, z) = \beta_{p-1}$. Using that $T(\beta_p, z) < \beta_p \wedge z$, we have that $z > \beta_{p-1}$, that is, $z = \beta_p$. Hence, $T(\beta_p, \beta_p) = \beta_{p-1}$. The rest of values are obtained recursively. Taking into account that $T(\beta_p, \beta_1) = 0$, $T(\beta_p, \beta_i)$ is completely determined except for $i = 2$.

(b) $T(\beta_{p-1}, \beta_3) = \beta_1$: A process similar to the previous one provides that $T(\beta_{p-1}, \beta_i) = \beta_{i-2}$. In particular, for $i = 3$ we have $T(\beta_{p-1}, \beta_3) = \beta_1$. Notice that $i = 3$ can be considered because $p \geq 3$.

(c) $T(\beta_p, \beta_2) = \beta_1$: We know that $T(\beta_p, \beta_2) < \beta_p \wedge \beta_2 = \beta_2$. If $T(\beta_p, \beta_2) = 0$, then

$$0 = T(\beta_p, \beta_2) = T(\beta_p, T(\beta_p, \beta_3)) = T(T(\beta_p, \beta_p), \beta_3) = T(\beta_{p-1}, \beta_3) = \beta_1.$$

A contradiction, hence $T(\beta_p, \beta_2) = \beta_1$. This implies that $T|_{C_\beta^2}$ is a divisible t-norm on the discrete chain C_β. Since $T|_{C_\beta^2}$ has no non-trivial idempotents, $T|_{C_\beta^2} = T_L$ (check Remark 6.3), and formula (14) is obtained.

Conversely, if T is expressed by formula (13) or formula (14), it is easy to prove that each case is a divisible t-norm, bearing in mind that T_L is a divisible t-norm. □

Proposition 6.11. *Each divisible t-norm T on L satisfying $T(\beta_1, \beta_1) = 0$ and $T(\gamma_q, \gamma_q) \in C_\gamma \setminus \{\gamma_q\}$ is expressed as follows: There is $k \in \{2, 3, ..., p\}$ such that*

$$T(x,y) = \begin{cases} T_1(x,y) & \text{if} \quad x, y \in C_\beta, \\ T_2(x,y) & \text{if} \quad x, y \in C_\gamma, \\ \beta_1 & \text{if} \quad (x,y) \in \{(\beta_i, \gamma_q), (\gamma_q, \beta_i) \mid i \geq k\}, \\ 0 & \text{otherwise.} \end{cases} \tag{15}$$

where T_2 is a divisible t-norm on C_γ satisfying $T_2(\gamma_q, \gamma_q) < \gamma_q$ and T_1 is a t-norm on C_β satisfying that

1. *$T_1(\beta_i, \beta_1) = 0$ for $i < k$.*
2. *If $T_1(\beta_k, \beta_k) = \beta_k$, then*

$$T_1(\beta_1, \beta_p) = T_1(\beta_1, \beta_k) = \beta_1.$$

3. *If $T_1(\beta_k, \beta_k) < \beta_k$, then*

$$T_1(\beta_1, \beta_p) = T_1(\beta_1, \beta_k) = 0.$$

and

$$T(\beta_p, \beta_p) \leq \beta_{k-1}.$$

4. *For $\beta_i \leq \beta_j$, with $i \geq 2$, there is $z \in C_\beta$ such that $T_1(\beta_j, z) = \beta_i$.*
5. *For $\beta_1 \leq \beta_i$, with $i < k$, there is $z \in C_\beta$ such that $T_1(\beta_i, z) = \beta_1$.*

Proof. Firstly, we prove that each divisible t-norm T on L satisfying $T(\beta_1, \beta_1) = 0$ and $T(\gamma_q, \gamma_q) \in C_\gamma \setminus \{\gamma_q\}$ has this expression. Finally, we will prove that each binary operator T on L which has this expression, is a divisible t-norm satisfying the mentioned conditions.

By Proposition 4.5, T is expressed as formula (4) or there is $k \in \{2, ..., p\}$ such that T is expressed as (5). Let us check that the first one is not a divisible t-norm: We have that $\beta_1 \leq \gamma_q$, but $T(\gamma_q, x) \neq \beta_1$ for all $x \in L$. Therefore, T must be expressed as the second formula. Therefore, let us prove that T_2 is divisible and the items 4 and 5 of T_1.

- T_2 is divisible: Take $x, y \in C_\gamma$ satisfying $x \leq y$. Since T is divisible, there $z \in L$ such that $T(y, z) = x$. Then, $x \leq z$. Taking into account the configuration of L, $z \in C_\gamma$.

- Item 4: Take $\beta_i \leq \beta_j$, with $i \geq 2$. Since T is divisible, there is $z \in L$ such that $T_1(\beta_j, z) = \beta_i$. Then, $\beta_i \leq z$. Taking into account that $i \geq 2$ and the configuration of L, $z \in C_\beta$.

- Item 5: $\beta_1 \leq \beta_i$, with $i < k$. Since T is divisible, there is $z \in L$ such that $T_1(\beta_i, z) = \beta_1$. Then, $\beta_1 \leq z$. This implies that $z \in C_\beta$ or $z = \gamma_q$. However, $z = \gamma_q$ is not possible because $T(\gamma_q, \beta_i) = 0$ whenever $i < k$.

We conclude that T is expressed by formula (15).

Conversely, if a binary operator T is expressed by formula (15), then T is a t-norm because it is a particular case of formula (5). Let us prove that T is divisible. Take $x, y \in L$ such that $x \leq y$.

1. If $x \in C_\gamma$, then $y \in C_\gamma$. By divisibility of T_2 on C_γ, there is $z \in C_\gamma$ such that $T(y, z) = x$.
2. If $x \in C_\beta \setminus \{\beta_1\}$, then $y \in C_\beta$. By item 4, there is $z \in C_\beta$ such that $T(y, z) = x$.
3. If $x = \beta_1$ and $y = \beta_i$ with $i < k$, then by item 5, there is $z \in C_\beta$ such that $T(y, z) = x$.
4. If $x = \beta_1$ and $y = \beta_i$ with $i \geq k$, then $T(y, \gamma_q) = x$.
5. If $x = \beta_1$ and $y = \gamma_q$, then $T(y, \beta_k) = x$.

Hence T is a divisible t-norm on L satisfying required conditions. □

Proposition 6.12. *If $p \leq 2$, each divisible t-norm T on L satisfying $T(\beta_1, \beta_1) = 0$, $T(\gamma_q, \gamma_q) = \gamma_q$, and $T(\beta_1, \gamma_q) = \beta_1$ is expressed as follows:*

$$T(x, y) = \begin{cases} T'(x, y) & \text{if} \quad x, y \in C_\gamma, \\ x \wedge y & \text{if} \quad x = 1 \text{ or } y = 1, \\ \beta_1 & \text{if} \quad (x, y) \in \{(\beta_i, \gamma_q), (\gamma_q, \beta_i) \mid i \geq 1\}, \\ 0 & \text{otherwise.} \end{cases} \tag{16}$$

where T' is a divisible t-norm on C_γ satisfying $T'(\gamma_q, \gamma_q) = \gamma_q$.

Proof. Firstly, we prove that each divisible t-norm T on L satisfying $T(\beta_1, \beta_1) = 0$, $T(\gamma_q, \gamma_q) = \gamma_q$, and $T(\beta_1, \gamma_q) = \beta_1$ has this expression. Finally, we will prove that each binary operator T on L which has this expression, is a divisible t-norm satisfying the mentioned conditions.

By Proposition 4.6, T is expressed by formula (6). By Lemma 6.5, $T' = T|_{C_\gamma^2}$ is a divisible t-norm on C_γ. Moreover, $T'(\gamma_q, \gamma_q) = T(\gamma_q, \gamma_q) = \gamma_q$.

Conversely, take a binary operator T from formula (16). We know that T is a t-norm because it is a particular case of formula (6). Finally, let us prove that T is divisible, taking $x, y \in L$ such that $x \leq y$.

1. If $x \in C_\gamma$, then $y \in C_\gamma$. Since T' is divisible on C_γ, there is $z \in C_\gamma$ such that $T(y, z) = x$.
2. If $x = \beta_1$, then $y \in \{\beta_1, \beta_p\}$ (we admit $p = 1$). We have that $T(\gamma_q, \beta_i) = \beta_1$ for each $i = 1, 2$.
3. If $x = \beta_2$ (only in the case $p = 2$), then $y = \beta_2$ or $y = 1$. Putting $z = 1$ and $z = \beta_2$ in each case respectively to obtain $T(y, z) = x$.

Therefore, T is a divisible t-norm on L satisfying $T(\gamma_q, \gamma_q) = T'(\gamma_q, \gamma_q) = \gamma_q$. □

Proposition 6.13. *If $p \geq 3$, there are no divisible t-norms on L satisfying formula* (6).

Proof. By contradiction, let us suppose that T satisfies formula (6) and it is a divisible t-norm. Since $\beta_2 \leq \beta_3$, then there is $z \in L$ such that $T(\beta_3, z) = \beta_2$. This implies that $z \geq \beta_2$, hence $\beta_2 \leq z \leq \beta_p$ ($z = 1$ is trivially imposible). However, $T(\beta_2, \beta_i) = 0$ for each $1 \leq i \leq p$. □

Proposition 6.14. *There are no divisible t-norms on L satisfying formula* (7).

Proof. If T is a divisible t-norm on L satisfying formula (7), then there is $z \in L$ such that $T(z, \gamma_q) = \beta_1$. However, by definition of T, that is not possible. □

7. Results about t-conorms

We recall that given a finite lattice $(L, \leq_L)$, the dual lattice $(L', \leq_{L'})$ is formed by the same elements and

$$x \leq_{L'} y \iff y \leq_L x.$$

Therefore, $0_L = 1_{L'}$ and $1_L = 0_{L'}$.

Discrete chains are self-dual lattices. Lattices of the family of Figure 1 are self-dual if and only if $p = q$. However, their dual lattices belong to the same class, so we may apply the results interchanging the roles of p and q.

Given a t-norm T on a lattice, the following result provides a method to build a t-conorm S defined on the dual lattice. Of course, an analogous induction from t-conorms to t-norms on the dual lattice can be obtained.

Proposition 7.1. *Let $(L, \leq_L)$ a finite lattice. If $T : L \times L \longrightarrow L$ is a t-norm, then $S : L' \times L' \longrightarrow L'$ defined by $S(x, y) = T(x, y)$ is a t-conorm on the dual lattice $(L', \leq_{L'})$.*

Proof. Clearly, S is well-defined because L' and L have the same elements.

1. $S(x, y) = T(x, y) = T(y, x) = S(y, x)$ for each $x, y \in L'$.
2. $S(x, 0_{L'}) = T(x, 0_{L'}) = T(x, 1_L) = x$ for each $x \in L'$.
3. Suppose that $x \leq_{L'} y$. Equivalently, $y \leq_L x$. We have that

$$S(x, z) = T(x, z) \geq_L T(y, z) = S(y, z)$$

   that is, $S(x, z) \leq_{L'} S(y, z)$.
4. $S(x, S(y, z)) = T(x, T(y, z)) = T(T(x, y), z)) = S(S(x, y), z)$ for each $x, y, z \in L'$.

□

Due to this proposition, we can transfer the results described for t-norms in the previous sections to t-conorms taking into account that the dual of each lattice is other lattice of the same family. For instance, if $p = 5$ and $q = 11$, its dual is the lattice that takes $p = 11$ and $q = 5$. Hence, we have provided a complete description of t-conorms defined on the lattices described in Figure 1. Similarly, for Archimedean and divisible t-conorms.

Conclusion

We have provided a complete classification and description of the t-norms defined on a family of lattices in terms of t-norms on discrete chains. Complementarily, the number of t-norms required on discrete chains is obtained by computational methods. We have done the same for Archimedean and divisible t-norms. They extend the results given by other authors in discrete chains (see [12, 25]). The last part has highlighted that each result can be transferred to t-conorms. We hope that similar arguments could be applied to some more general lattices, too.

Acknowledgements

I wish to acknowledge Mirko Navara for his feedback and suggestions that have helped to present the results. Also, I am grateful to the anonymous referees for their comments, which led to several improvements on the originally submitted version of this paper, including an important correction for Proposition 4.5. This research has been supported by the Ministry of Education of the Czech Republic under Project GACR 19-09967S.

References

[1] Anthony, J., & Sherwood, H. (1979). Fuzzy groups redefined. *Journal of Mathematical Analysis and Applications*, *69*, 124–130.

[2] Aşıcı, E., & Karaçal, F. (2014). On the t-partial order and properties. *Information Sciences*, *267*, 323–333.

[3] Aşıcı, E., & Mesiar, R. (2020). New constructions of triangular norms and triangular conorms on an arbitrary bounded lattice. *International Journal of General Systems*, *49*, 143–160.

[4] Bartušek, T., & Navara, M. (2002). Program for generating fuzzy logical operations and its use in mathematical proofs. *Kybernetika*, *38*, 235–244.

[5] Bejines, C., Bruteničová, M., Chasco, M., Elorza, J., & Janiš, V. (2021). The number of t-norms on some special lattices. *Fuzzy Sets and Systems*, *408*, 26–43.

[6] Birkhoff, G. (1940). *Lattice theory* volume 25. American Mathematical Soc.

[7] Bodjanova, S., & Kalina, M. (2018). Nullnorms and t-operators on bounded lattices: Coincidence and differences. In J. Medina, M. Ojeda-Aciego, J. L. Verdegay, D. A. Pelta, I. P. Cabrera, B. Bouchon-Meunier, & R. R. Yager (Eds.), *Information Processing and Management of Uncertainty in Knowledge-Based Systems. Theory and Foundations* (pp. 160–170). Cham: Springer International Publishing.

[8] Çaylı, G. D. (2018). A characterization of uninorms on bounded lattices by means of triangular norms and triangular conorms. *International Journal of General Systems*, *47*, 772–793.

[9] Çaylı, G. D. (2018). On a new class of t-norms and t-conorms on bounded lattices. *Fuzzy Sets and Systems*, *332*, 129–143.

[10] Çaylı, G. D. (2019). Some methods to obtain t-norms and t-conorms on bounded lattices. *Kybernetika*, *55*, 273–294.

[11] Dan, Y., Hu, B. Q., & Qiao, J. (2020). New construction of t-norms and t-conorms on bounded lattices. *Fuzzy Sets and Systems*, *395*, 40–70.

[12] De Baets, B., & Mesiar, R. (1999). Triangular norms on product lattices. *Fuzzy Sets and Systems*, *104*, 61–75.

[13] De Cooman, G., & Kerre, E. E. (1994). Order norms on bounded partially ordered sets. *J. Fuzzy Math*, *2*, 281–310.

[14] Dvořák, A., & Holčapek, M. (2020). New construction of an ordinal sum of t-norms and t-conorms on bounded lattices. *Information Sciences*, *515*, 116–131.

[15] El-Zekey, M. (2020). Lattice-based sum of t-norms on bounded lattices. *Fuzzy Sets and Systems*, *386*, 60–76.

[16] Ertuğrul, Ü., Karaçal, F., & Mesiar, R. (2015). Modified ordinal sums of triangular norms and triangular conorms on bounded lattices. *International Journal of Intelligent Systems*, *30*, 807–817.

[17] Ertuğrul, Ü., Kesicioğlu, M. N., & Karaçal, F. (2019). Some new construction methods for t-norms on bounded lattices. *International Journal of General Systems*, *48*, 775–791.

[18] Godo, L., & Sierra, C. (1988). A new approach to connective generation in the framework of expert systems using fuzzy logic. In *[1988] Proceedings. The Eighteenth International Symposium on Multiple-Valued Logic* (pp. 157–162). IEEE.

[19] Goguen, J. A. (1967). L-fuzzy sets. *Journal of mathematical analysis and applications*, *18*, 145–174.

[20] Höhle, U., & Klement, E. P. (2012). *Non-classical logics and their applications to fuzzy subsets: a handbook of the mathematical foundations of fuzzy set theory* volume 32. Springer Science & Business Media.

[21] Karaçal, F., Ertuğrul, Ü., & Mesiar, R. (2017). Characterization of uninorms on bounded lattices. *Fuzzy Sets and Systems*, *308*, 54–71.

[22] Karaçal, F., & Mesiar, R. (2015). Uninorms on bounded lattices. *Fuzzy Sets and Systems*, *261*, 33–43.

[23] Karaçal, F., & Mesiar, R. (2017). Aggregation functions on bounded lattices. *International Journal of General Systems*, *46*, 37–51. doi:10.1080/03081079.2017.1291634.

[24] Klement, E. P., Mesiar, R., & Pap, E. (2000). *Triangular Norms*. Dordrecht: Springer Netherlands.

[25] Mayor, G., & Torrens, J. (2005). Triangular norms on discrete settings. In *Logical, Algebraic, Analytic and Probabilistic Aspects of Triangular Norms* (pp. 189–230). Elsevier.

[26] Medina, J. (2012). Characterizing when an ordinal sum of t-norms is a t-norm on bounded lattices. *Fuzzy Sets and Systems*, *202*, 75–88.

[27] Menger, K. (1942). Statistical metrics. *Proceedings of the National Academy of Sciences of the United States of America*, *28*, 535–537.

[28] Palmeira, E., & Bedregal, B. (2012). Extension of fuzzy logic operators defined on bounded lattices via retractions. *Computers & Mathematics with Applications*, *63*, 1026–1038.

[29] Palmeira, E., Bedregal, B., Mesiar, R., & Fernandez, J. (2014). A new way to extend t-norms, t-conorms and negations. *Fuzzy Sets and Systems*, *240*, 1–21.

[30] Petrík, M., & Vetterlein, T. (2017). Rees coextensions of finite, negative tomonoids. *Journal of Logic and Computation*, *27*, 337–356.

[31] Petrík, M., & Vetterlein, T. (2019). Rees coextensions of finite tomonoids and free pomonoids. In *Semigroup Forum* (pp. 345–367). Springer volume 99.

[32] Recasens, J. (2011). *Indistinguishability Operators - Modelling Fuzzy Equalities and Fuzzy Equivalence Relations* volume 260 of *Studies in Fuzziness and Soft Computing*. Springer.

[33] Saminger-Platz, S. (2006). On ordinal sums of triangular norms on bounded lattices. *Fuzzy Sets and Systems*, *157*, 1403–1416.

[34] Saminger-Platz, S. (2009). The dominance relation in some families of continuous archimedean t-norms and copulas. *Fuzzy Sets and Systems*, *160*, 2017–2031.

[35] Saminger-Platz, S., Klement, E. P., & Mesiar, R. (2008). On extensions of triangular norms on bounded lattices. *Indagationes Mathematicae*, *19*, 135–150.

[36] Schweizer, B., & Sklar, A. (2011). *Probabilistic metric spaces*. Courier Corporation.

[37] Xia, M., Xu, Z., & Zhu, B. (2012). Some issues on intuitionistic fuzzy aggregation operators based on archimedean t-conorm and t-norm. *Knowledge-Based Systems*, *31*, 78–88.

[38] Zhang, D. (2005). Triangular norms on partially ordered sets. *Fuzzy Sets and Systems*, *153*, 195–209.