

Categorical foundations of variety-based bornology[☆]

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Abstract

Following the concept of topological theory of S. E. Rodabaugh, this paper introduces a new approach to (lattice-valued) bornology, which is based in bornological theories, and which is called variety-based bornology. In particular, motivated by the notion of topological system of S. Vickers, we introduce the concept of variety-based bornological system, and show that the category of variety-based bornological spaces is isomorphic to a full reflective subcategory of the category of variety-based bornological systems.

Keywords: bornological space, bornological system, bornological theory, ideal, powerset theory, reflective subcategory, system spatialization procedure, topological category, variety of algebras

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1. Introduction

The theory of bornological spaces (which takes its origin in the axiomatisation of the notion of boundedness of S.-T. Hu [20, 21]) has already found numerous applications in different branches of mathematics. For example, the main ideas of modern functional analysis are those of locally convex topology and convex bornology [18]. Additionally (as a link to physics), one could mention the notion of Hausdorff dimension in convex bornological spaces, motivated by the study of the complexity of strange attractors [6].

In 2011, M. Abel and A. Šostak [1] introduced the concept of lattice-valued (but fixed-basis, in the sense of [19]) bornological space, making thereby the first steps towards the theory of lattice-valued bornology. In a series of papers [28, 26, 27], the present authors made further steps in this direction, taking their inspiration in the well-developed theory of lattice-valued topology. In particular, we presented a variable-basis analogue (in the sense of [31]) of the concept of M. Abel and A. Šostak as well as introduced lattice-valued variable-basis bornological vector spaces (following the pattern of the fuzzy topological vector spaces of [23, 24]); found the necessary and sufficient condition for the category of lattice-valued bornological spaces to be topological (we notice that all the currently used categories for lattice-valued topology are topological [32]); showed that the category of strict (in the sense of [1]) fixed-basis lattice-valued bornological spaces is a quasitopos (we notice again that the categories for lattice-valued topology fail to have this convenient property); and also provided a bornological analogue of the notion of topological system of S. Vickers [37] (introduced as a common setting for both point-set and point-free topology), starting thus the theory of point-free bornology.

In 2012, S. Solovyov [35] presented a new framework for doing lattice-valued topology, namely, variety-based topology. This new setting was induced by an attempt of S. E. Rodabaugh [32], to provide a common framework for the existing categories for lattice-valued topology through the concept of powerset theory (we notice that there also exists the notion of topological theory of O. Wyler [38, 39], which is discussed in [2], and

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which is compared with the powerset theories of S. E. Rodabaugh in [32]). Unlike S. E. Rodabaugh, however, who tried to find the basic algebraic structure for lattice-valued topology, arriving thus at the new concept of semi-quantale, S. Solovyov decided to allow for an arbitrary algebraic structure, thus arriving at varieties of algebras. Briefly speaking, noticing that most of the currently popular approaches to lattice-valued topology are based in powersets of the form L^X , where X is a set and L is a complete lattice (possibly, with some additional axioms and/or algebraic operations), S. Solovyov decided to base his topology in powersets of the form A^X , in which the lattice L is replaced with an algebra A from an arbitrary variety (in the extended sense, i.e., allowing for a class of not necessarily finitary operations as in, e.g., [7, 30]). It appeared that such a general variety-based approach is still capable of producing some convenient results, e.g., all the categories of variety-based topological spaces are topological, and, moreover, one has a good concept of topological system, the category of which contains the category of variety-based topological spaces as a full (regular mono)-coreflective subcategory. Even more, variety-based approach to systems incorporates (apart from the lattice-valued extension of topological systems of J. T. Denniston, A. Melton and S. E. Rodabaugh [10, 11]), additionally, state property systems of D. Aerts [3, 4, 5] (which serve as the basic mathematical structure in the Geneva-Brussels approach to foundations of physics) as well as Chu spaces of [29] (which eventually are nothing else than many-valued formal contexts of Formal Concept Analysis [12]; we notice, however, that Chu spaces were introduced in a more general form by P.-H. Chu in [8]).

The purpose of this paper is to provide a variety-based setting for lattice-valued bornology. In particular, we introduce both variety-based bornological spaces and systems, and show that the category of the former is isomorphic to a full reflective (unlike coreflective in the topological case) subcategory of the latter. Both spaces and systems rely on powerset theories, which, however, are radically different from those of topology. More precisely, one of the main difficulties for introducing a variety-based approach to bornology lies in the fact that while classical topologies are just subframes [22] of powersets, bornologies are lattice ideals of powersets. Thus, while the switch from subframes to subalgebras is immediate, one needs a good concept of ideal in universal algebras, to switch from lattice ideals to algebra ideals. In this paper, we rely on the concept of algebra ideal of [15] (which eventually goes back to an earlier paper of A. Ursini). We also notice that motivated by, e.g., [36], we distinguish between powerset theories and bornological theories. The main reason for this in the topological setting of [36] is the fact that while powersets are Boolean algebras, topologies are just subframes, i.e., when dealing with topologies, one “forgets” a part of the available algebraic structure.

If properly developed, the theory of variety-based bornology could find an application in cancer-related research. More precisely, at the end of Section 7 in [16], its authors mention that a “systematic study of continuous spectrum of fractal dimension can put more light on several fractal organisms/objects observed in tissues of cancer patients”. Moreover, it appears that “in practical applications, we can meet a bornological space instead of a metric one” [16]. Our papers [26, 27, 28] together with the present one, could provide a starting point for building a convenient framework for dimension theory for variety-based bornological spaces (e.g., for the already mentioned concept of Hausdorff dimension for bornological spaces of [6]).

This manuscript is based in both category theory and universal algebra, relying more on the former. The necessary categorical background can be found in [2, 17]. For algebraic notions, we recommend [9, 14, 30]. Although we tried to make the paper as much self-contained as possible, it is expected from the reader to be acquainted with basic concepts of category theory, e.g., with that of reflective subcategory.

2. Algebraic and categorical preliminaries

For convenience of the reader, this section briefly recalls those algebraic and categorical concepts, which are necessary for the understanding of this paper.

Variety-based approach to bornology relies on the notion of universal algebra (shortened to algebra), which is thought to be a set, equipped with a family of operations, satisfying certain identities. The theory of universal algebra calls a class of finitary algebras (induced by a set of finitary operations), closed under the formation of homomorphic images, subalgebras and direct products, a variety. In view of the structures common in, e.g., lattice-valued topology (where set-theoretic unions are replaced with joins), we consider infinitary algebraic theories, extending the approach of varieties to cover our needs (cf. [25, 30]).

Definition 1. Let $\Omega = (n_\lambda)_{\lambda \in \Lambda}$ be a (possibly proper or empty) class of cardinal numbers. An Ω -algebra is a pair $(A, (\omega_\lambda^A)_{\lambda \in \Lambda})$, which comprises a set A and a family of maps $A^{n_\lambda} \xrightarrow{\omega_\lambda^A} A$ (n_λ -ary primitive operations on A). An Ω -homomorphism $(A, (\omega_\lambda^A)_{\lambda \in \Lambda}) \xrightarrow{\varphi} (B, (\omega_\lambda^B)_{\lambda \in \Lambda})$ is a map $A \xrightarrow{\varphi} B$, which makes the diagram

$$\begin{array}{ccc} A^{n_\lambda} & \xrightarrow{\varphi^{n_\lambda}} & B^{n_\lambda} \\ \omega_\lambda^A \downarrow & & \downarrow \omega_\lambda^B \\ A & \xrightarrow{\varphi} & B \end{array}$$

commute for every $\lambda \in \Lambda$. $\mathbf{Alg}(\Omega)$ is the construct of Ω -algebras and Ω -homomorphisms. ■

Every concrete category of this paper has the underlying functor $|-|$ to the respective ground category (e.g., the category **Set** of sets and maps in case of constructs), the latter mentioned explicitly in each case.

Definition 2. Let \mathcal{M} (resp. \mathcal{E}) be the class of Ω -homomorphisms with injective (resp. surjective) underlying maps. A *variety of Ω -algebras* is a full subcategory of $\mathbf{Alg}(\Omega)$, which is closed under the formation of products, \mathcal{M} -subobjects (subalgebras) and \mathcal{E} -quotients (homomorphic images). The objects (resp. morphisms) of a variety are called *algebras* (resp. *homomorphisms*). ■

Definition 3. Given a variety \mathbf{A} , a *reduct* of \mathbf{A} is a pair $(\|\!-\!\|, \mathbf{B})$, where \mathbf{B} is a variety such that $\Omega_{\mathbf{B}} \subseteq \Omega_{\mathbf{A}}$ (every primitive operation of \mathbf{B} is a primitive operation of \mathbf{A}), and $\mathbf{A} \xrightarrow{\|\!-\!\|} \mathbf{B}$ is a concrete functor. The pair $(\mathbf{A}, \|\!-\!\|)$ is called then an *extension* of \mathbf{B} . ■

The following example illustrates the concept of variety with several well-known constructs.

Example 4.

- (1) **Lat** $_{\perp}$ is the variety of *lattices with the smallest element*, i.e., partially ordered sets (posets), which have binary \wedge and finite (including the empty) \vee .
- (2) **CSLat** (Ξ) is the variety of Ξ -*semilattices*, i.e., posets, which have arbitrary $\Xi \in \{\wedge, \vee\}$.
- (3) **SQuant** is the variety of *semi-quantales*, i.e., \vee -semilattices, equipped with a binary operation \otimes [32].
- (4) **SFrm** is the variety of *semi-frames*, i.e., \vee -semilattices, with singled out finite meets [32].
- (5) **QSFrm** is the variety of *quantic semi-frames*, i.e., semi-frames, which are also semi-quantales, satisfying the conditions $a \otimes \top_A = \top_A \otimes a = a$ (*strict two-sidedness*) and $a \otimes \perp_A = \perp_A \otimes a = \perp_A$ for every $a \in A$.
- (6) **Frm** is the variety of *frames*, i.e., semi-frames A , which additionally satisfy the distributivity condition $a \wedge (\vee S) = \vee_{s \in S} (a \wedge s)$ for every $a \in A$ and every $S \subseteq A$ [22].
- (7) **CLat** is the variety of *complete lattices*, i.e., posets A , which are both \vee - and \wedge -semilattices.
- (8) **CBAlg** is the variety of *complete Boolean algebras*, i.e., complete lattices A such that $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ for every $a, b, c \in A$, equipped with a unary operation $A \xrightarrow{(-)^*} A$ such that $a \vee a^* = \top_A$ and $a \wedge a^* = \perp_A$ for every $a \in A$, where \top_A (resp. \perp_A) is the largest (resp. smallest) element of A . ■

We notice that, e.g., **CLat** is a reduct of **CBAlg**, and both **CSLat** (Ξ) and **Lat** $_{\perp}$ are reducts of **CLat**. In the next step, following [15], we recall the concept of ideal in universal algebras. We underline, however, immediately that while [15] works with classical universal algebras (which have a set of finitary operations), we adapt the respective definition of ideal to our case of generalized varieties. Moreover, for the sake of convenience (and following [15]), given a set X and a cardinal number n (not necessarily finite), the elements of the powerset X^n will be denoted \vec{x} , or sometimes (for better clarity) $\vec{x}_{i \in n}$. Additionally, given an element $x_0 \in X$, \vec{x}_0 will denote the element of X^n , every component of which is x_0 .

Definition 5. Let \mathbf{A} be a variety of algebras, the elements of which have a 0-ary operation \perp .

- (1) A term $p(\vec{x}, \vec{y})$ is an (\mathbf{A}) -*ideal term in \vec{y}* if $p(\vec{x}, \vec{\perp}) = \perp$ is an identity in \mathbf{A} .

- (2) A non-empty subset \mathcal{I} of an \mathbf{A} -algebra A is called an (\mathbf{A} -)ideal provided that for every ideal term $p(\vec{x}, \vec{y})$ in \vec{y} , $\vec{a} \in A^n$, $\vec{i} \in \mathcal{I}^m$, it follows that $p(\vec{a}, \vec{i}) \in \mathcal{I}$. ■

Every ideal contains the constant \perp , since the (constant) term \perp is an ideal term. Moreover, in the varieties **Rng** (rings) and **Lat** $_{\perp}$, ideals in the sense of Definition 5 coincide with their respective classical analogues. To use it later on in the paper, we recall the definition of lattice ideals.

Definition 6. Given a **Lat** $_{\perp}$ -object L with the smallest element \perp_L , a subset $\mathcal{I} \subseteq L$ is said to be an *ideal* of L provided that the following conditions are fulfilled:

- (1) for every finite subset $S \subseteq L$, $\bigvee S \in \mathcal{I}$;
- (2) for every $a \in L$ and every $i \in \mathcal{I}$, $a \wedge i \in \mathcal{I}$. ■

Definition 6(1) implies that every lattice ideal is non-empty, i.e., $\perp_L \in \mathcal{I}$. Moreover, Definition 6 suggests two possible (but not the only ones) ideal terms, i.e., $\bigvee \vec{y}$ (the term takes finite sequences and does not depend on \vec{x}) and $x \wedge y$. All the other lattice ideal terms are generated by the just mentioned two. For example, one can build ideal terms of the form $(x_1 \wedge y_1) \vee (x_2 \wedge y_2)$ or $(x \wedge y_1) \vee y_2$.

To use it later on in the paper, we recall a convenient property of ideals in universal algebras from [15].

Proposition 7 ([15]). *Let \mathbf{A} be a variety with a 0-ary operation, and let A be an \mathbf{A} -algebra.*

- (1) *The intersection of a family of ideals of A is an ideal.*
- (2) *Given a subset $S \subseteq A$, the ideal generated by S (denoted $\langle S \rangle_{\mathbf{A}}$) consists of all $p(\vec{a}, \vec{s})$, where $p(\vec{x}, \vec{y})$ is an ideal term in \vec{y} , $\vec{a} \in A^n$, and $\vec{s} \in S^m$.*

In the last step, we introduce a particular construction, which is related to ideals in universal algebras.

Definition 8. Given a variety of algebras \mathbf{A} with a 0-ary operation, \mathbf{A}_{it} is the construct, whose objects are \mathbf{A} -ideals, and whose morphisms are maps $\mathcal{I}_1 \xrightarrow{\varphi} \mathcal{I}_2$, which preserve the ideal terms as follows:

- (1) for every ideal term $p(\vec{y})$ in \vec{y} , $\varphi(p(\vec{i})) = p(\varphi(\vec{i}))$ for every $\vec{i} \in \mathcal{I}_1^n$;
- (2) for every ideal term $p(\vec{x}, \vec{y})$ in \vec{y} , there exist an ideal term $t(\vec{x} \cup \vec{z}, \vec{y})$ in \vec{y} (in which the notation “ $\vec{x} \cup \vec{z}$ ” means that one adds more variables to \vec{x} , e.g., taking x_1, x_2, z_1, z_2 instead of x_1, x_2 , but not to \vec{y}), and a family of ideal terms $\overrightarrow{s_u(\vec{v}_u, \vec{w}_u)}_{u \in \vec{z}}$ in \vec{w}_u such that $\varphi(p(\vec{a}, \vec{i})) = t(\varphi(\vec{a}) \cup \varphi(\overrightarrow{s_u(\vec{a}'_u, \vec{i}'_u)}_{u \in \vec{z}}), \varphi(\vec{i}))$ for every $\vec{a} \in \mathcal{I}_1^n$, $\vec{i} \in \mathcal{I}_1^m$, where $\vec{a}'_u \subseteq \vec{a}$, $\vec{i}'_u \subseteq \vec{i}$. ■

The main result, which could be easily obtained with the help of Definition 8 and which will be used throughout the paper, can be stated now as follows.

Proposition 9. *Given an \mathbf{A}_{it} -morphism $\mathcal{I}_1 \xrightarrow{\varphi} \mathcal{I}_2$, for every ideal $\mathcal{J}_2 \subseteq \mathcal{I}_2$, the set $\mathcal{J}_1 = \{a \in \mathcal{I}_1 \mid \varphi(a) \in \mathcal{J}_2\}$ is an ideal.*

PROOF. It follows directly from the two items of Definition 8 that \mathcal{J}_1 is closed under the ideal terms. □

For convenience of the reader, we notice that, eventually, we could have replaced the characterization of \mathbf{A}_{it} -morphisms in Definition 8 with the one in Proposition 9, namely, with the property that the preimage of an ideal is an ideal. The main intuition for Definition 8, however, will be provided while discussing the topic of powerset theories in the next section. At this place, we just mention the following easy result.

Proposition 10. *Given a variety \mathbf{A} with a 0-ary operation, there exists a non-full embedding $\mathbf{A} \xrightarrow{E} \mathbf{A}_{it}$, which is defined by $E(A_1 \xrightarrow{\varphi} A_2) = A_1 \xrightarrow{\varphi} A_2$.*

PROOF. Given an \mathbf{A} -homomorphism $A_1 \xrightarrow{\varphi} A_2$ and an ideal term $p(\vec{x}, \vec{y})$ in \vec{y} , it follows that $\varphi(p(\vec{a}, \vec{i})) = p(\varphi(\vec{a}), \varphi(\vec{i}))$ for every $\vec{a} \in A_1^n$, $\vec{i} \in A_1^m$. □

For the sake of convenience, later on, we will not distinguish between \mathbf{A} and its image in \mathbf{A}_{it} under E .

To conclude the preliminaries, we recall from [2] the definition of reflective subcategory (the dual case of coreflective subcategories is left to the reader).

Definition 11. Let \mathbf{A} be a subcategory of \mathbf{B} , and let B be a \mathbf{B} -object.

- (1) An \mathbf{A} -*reflection* (or \mathbf{A} -*reflection arrow*) for B is a \mathbf{B} -morphism $B \xrightarrow{r} A$ from B to an \mathbf{A} -object A with the following universal property: for every \mathbf{B} -morphism $B \xrightarrow{f} A'$ from B into some \mathbf{A} -object A' , there exists a unique \mathbf{A} -morphism $A \xrightarrow{f'} A'$ such that the triangle

$$\begin{array}{ccc} B & \xrightarrow{r} & A \\ & \searrow f & \downarrow f' \\ & & A' \end{array}$$

commutes.

- (2) \mathbf{A} is called a *reflective subcategory* of \mathbf{B} provided that each \mathbf{B} -object has an \mathbf{A} -reflection. ■

To give the reader more intuition, we provide one example of reflective subcategories.

Example 12. The category **Mon** of monoids is a reflective subcategory of the category **Sgr** of semigroups. Given a semigroup (X, \circ) , the extension $(X, \circ) \hookrightarrow (X \cup \{e\}, \hat{\circ}, e)$, which is obtained by adding a unit element $e \notin X$ of the operation $\hat{\circ}$, is a **Mon**-reflection for (X, \circ) . ■

3. Variety-based bornology

This section introduces our new approach to bornology, which is based in varieties of algebras. To provide more intuition for the next developments, we begin by recalling the concept of bornological space from [18].

We notice first that there exists the so-called covariant powerset functor $\mathbf{Set} \xrightarrow{\mathcal{P}} \mathbf{Set}$, which is defined by $\mathcal{P}(X \xrightarrow{f} Y) = \mathcal{P}X \xrightarrow{\mathcal{P}f} \mathcal{P}Y$, where $\mathcal{P}f(S) = \{f(s) \mid s \in S\}$.

Definition 13. A *bornological space* is a pair (X, \mathcal{B}) , where X is a set and \mathcal{B} (a *bornology* on X) is a subfamily of $\mathcal{P}X$ (the elements of which are called *bounded sets*), which satisfies the following axioms:

- (1) $X = \bigcup \mathcal{B}$ ($= \bigcup_{B \in \mathcal{B}} B$);
- (2) if $B \in \mathcal{B}$ and $D \subseteq B$, then $D \in \mathcal{B}$;
- (3) if $\mathcal{S} \subseteq \mathcal{B}$ is finite, then $\bigcup \mathcal{S} \in \mathcal{B}$.

Given bornological spaces (X_1, \mathcal{B}_1) and (X_2, \mathcal{B}_2) , a map $X_1 \xrightarrow{f} X_2$ is called *bounded* provided that $\mathcal{P}f(B_1) \in \mathcal{B}_2$ for every $B_1 \in \mathcal{B}_1$. **Born** is the construct of bornological spaces and bounded maps. ■

To start a short discussion, we notice that given a map $X \xrightarrow{f} Y$, while $\mathcal{P}X, \mathcal{P}Y$ are complete Boolean algebras, the map $\mathcal{P}f$ is just a $\mathbf{CSLat}(\vee)$ -homomorphism, i.e., $\mathcal{P}(\bigcup \mathcal{S}) = \bigcup_{S \in \mathcal{S}} \mathcal{P}(S)$ for every $\mathcal{S} \subseteq \mathcal{P}X$. We underline, moreover, that given $S, T \in \mathcal{P}X$, $\mathcal{P}f(S \cap T) = (\mathcal{P}f(S) \cap \mathcal{P}f(S \cap T)) \cap \mathcal{P}f(T)$, which takes us back to Definition 8 (recall the discussion after Definition 6 on ideal terms for lattices, and notice that $(u \wedge v) \wedge w$ is a lattice ideal term in w). As a consequence, we could have considered (verification is easy) the functor $\mathbf{Set} \xrightarrow{\mathcal{P}} \mathbf{CBAlg}_{it}$ instead of just $\mathbf{Set} \xrightarrow{\mathcal{P}} \mathbf{Set}$.

To continue, we notice that items (2), (3) of Definition 13 ensure that every bornology on a set X is a lattice ideal of $\mathcal{P}X$, but not a complete lattice ideal of $\mathcal{P}X$ in the sense of Definition 5 (no closure under arbitrary joins). Thus, bornologies on sets are the elements of the variety $\mathbf{Lat}_{\perp it}$ and not \mathbf{CBAlg}_{it} .

Lastly, we notice that the first item of Definition 13 just says that the \mathbf{CBAlg}_{it} -ideal of $\mathcal{P}X$, generated by \mathcal{B} , contains the largest element of $\mathcal{P}X$ (which implies then that $\langle \mathcal{B} \rangle_{\mathbf{CBAlg}_{it}}$ is the whole $\mathcal{P}X$).

The above discussion gives rise to the next variety-based analogue of the classical approach to bornology.

3.1. Variety-based powerset and bornological theories

From now on, we assume that every variety \mathbf{A} has 0-ary operations \perp and \top , and every our employed reduct \mathbf{B} of \mathbf{A} preserves the operation \perp (but not necessarily \top). One should also recall Proposition 10.

Definition 14. A *powerset theory* in a category \mathbf{X} (*ground category* of the theory) is a functor $\mathbf{X} \xrightarrow{P} \mathbf{A}_{it}$ to the category of ideals of a variety \mathbf{A} , with the additional property that PX lies in \mathbf{A} for every $X \in \mathbf{X}$. ■

The following example illustrates the just introduced notion.

Example 15.

- (1) The classical covariant powerset functor $\mathbf{Set} \xrightarrow{P} \mathbf{Set}$ provides a powerset theory $\mathbf{Set} \xrightarrow{P} \mathbf{CBAlg}_{it}$.
- (2) Given a \vee -semilattice L , there is a functor $\mathbf{Set} \xrightarrow{P_L} \mathbf{CSLat}(\vee)$, $P_L(X \xrightarrow{f} Y) = L^X \xrightarrow{P_L f} L^Y$, where $(P_L f(\alpha))(y) = \bigvee_{f(x)=y} \alpha(x)$ [32]. For every "suitable" extension \mathbf{A} of $\mathbf{CSLat}(\vee)$ (i.e., **SQuant**, **SFrm**, **QSFrm**, **Frm**, **CLat**, **CBAlg**), one gets then a powerset theory $\mathbf{Set} \xrightarrow{P_A} \mathbf{A}_{it}$ (which will be denoted $P_A^{\mathbf{A}}$). In particular, the two-element Boolean algebra $2 = \{\perp, \top\}$ provides the powerset theory from the previous item. For convenience of the reader, we also emphasize here that in case of the variety **QSFrm**, both $x \otimes y$ and $y \otimes x$ are ideal terms in y , for which one can easily show that $P_L f(\alpha \otimes \beta) = (P_L f(\alpha) \wedge P_L f(\alpha \otimes \beta)) \wedge P_L(\beta)$ and use the fact that $x \wedge y$ is an ideal term in y (notice that, by the definition of **QSFrm**, $(P_L f(\alpha \otimes \beta))(y) = \bigvee_{f(x)=y} (\alpha(x) \otimes \beta(x)) \leq \bigvee_{f(x)=y} (\alpha(x) \otimes \top_A) = \bigvee_{f(x)=y} \alpha(x) = (P_L f(\alpha))(y)$). In case of the variety **SQuant**, however, $x \otimes y$ (also $y \otimes x$) is not an ideal term in y .
- (3) Given a subcategory \mathbf{S} of $\mathbf{CSLat}(\vee)$, there is a functor $\mathbf{Set} \times \mathbf{S} \xrightarrow{P_S} \mathbf{CSLat}(\vee)$, which is defined by $P_S((X, L) \xrightarrow{(f, \varphi)} (Y, M)) = L^X \xrightarrow{P_S(f, \varphi)} M^Y$, where $(P_S f(\alpha))(y) = \bigvee_{f(x)=y} \varphi \circ \alpha(x)$ [32]. For every "suitable" extension \mathbf{A} of $\mathbf{CSLat}(\vee)$ (i.e., **SQuant**, **SFrm**, **QSFrm**, **Frm**, **CLat**, **CBAlg**), one obtains then a powerset theory $\mathbf{Set} \times \mathbf{S} \xrightarrow{P_S} \mathbf{A}_{it}$ (which will be denoted $P_S^{\mathbf{A}}$). If \mathbf{S} is the subcategory of the form $L \xrightarrow{1_L} L$, then one gets the powerset theory of the previous item. ■

For convenience of the reader, we notice the crucial difference between bornological and topological powerset theories [32, 36]: while the former extend the covariant powerset functor, the latter extend the so-called contravariant powerset functor of the form $\mathbf{Set} \xrightarrow{Q} \mathbf{Set}^{op}$, which is given by $Q(X \xrightarrow{f} Y) = P_X \xrightarrow{Qf} P_Y$, $(Qf)^{op}(S) = \{x \in X \mid f(x) \in S\}$. Since Q has, eventually, the form $\mathbf{Set} \xrightarrow{Q} \mathbf{CBAlg}^{op}$, the general form of a (topological) powerset theory is a functor $\mathbf{X} \xrightarrow{P} \mathbf{A}^{op}$, where \mathbf{A} is a variety of algebras.

We are ready to define bornological theories, which "forget" certain algebraic structure of "powersets".

Definition 16. Given a powerset theory $\mathbf{X} \xrightarrow{P} \mathbf{A}_{it}$ and a reduct \mathbf{B} of \mathbf{A} , a *bornological theory* in \mathbf{X} induced by the pair (P, \mathbf{B}) is the functor $\mathbf{X} \xrightarrow{T} \mathbf{B}_{it}$, which is defined as the composition $\mathbf{X} \xrightarrow{P} \mathbf{A}_{it} \xrightarrow{\|-\|_{it}} \mathbf{B}_{it}$. ■

As an example of bornological theories, we mention the classical one $\mathbf{Set} \xrightarrow{T} \mathbf{Lat}_{\perp, it}$, which is induced by the powerset theory $\mathbf{Set} \xrightarrow{P} \mathbf{CBAlg}_{it}$ and the reduct \mathbf{Lat}_{\perp} of **CBAlg**.

3.2. Variety-based bornological spaces

With bornological theories in hand, this subsection introduces variety-based bornological spaces. The reader should recall the existence of the 0-ary operation \top in every variety \mathbf{A} we use.

Definition 17. Let T be a bornological theory in a category \mathbf{X} . $\mathbf{Born}(T)$ is the concrete category over \mathbf{X} , whose objects (*T-bornological spaces* or *T-spaces*) are pairs (X, τ) , where X is an \mathbf{X} -object and τ (*T-bornology* on X) is an ideal of TX such that $\top \in \langle \tau \rangle_{\mathbf{A}}$, and whose morphisms (*T-bounded X-morphisms*) $(X_1, \tau_1) \xrightarrow{f} (X_2, \tau_2)$ are \mathbf{X} -morphisms $X_1 \xrightarrow{f} X_2$ such that $Tf(\alpha) \in \tau_2$ for every $\alpha \in \tau_1$ (*T-boundedness*). ■

The next example illustrates the just introduced concept of variety-based bornology.

Example 18.

- (1) $\mathbf{Born}((\mathcal{P}, \mathbf{Lat}_\perp))$ is isomorphic to the category \mathbf{Born} of bornological spaces and bounded maps of [18].
- (2) $\mathbf{Born}((\mathcal{P}_L^{\mathbf{CLat}}, \mathbf{Lat}_\perp))$ is isomorphic to the category $L\text{-Born}$ of [1].
- (3) $\mathbf{Born}((\mathcal{P}_S^{\mathbf{CLat}}, \mathbf{Lat}_\perp))$ is isomorphic to the category $\mathbf{L-Born}$ of [28]. ■

We have already mentioned that every category of variety-based topological spaces is topological over its ground category [35, 36]. In [28, 26], we showed that not every category of lattice-valued bornological spaces is topological, providing the necessary and sufficient conditions (on the respective lattices) for this property. Below, we show sufficient conditions for the category $\mathbf{Born}(T)$ to be topological over \mathbf{X} .

To provide more intuition for the subsequent general requirements, we start with a motivating discussion, which recalls some well-known (and simple) properties of complete lattices and their induced powersets.

First, given a complete lattice L and a lattice ideal $\mathcal{I} \subseteq L$ such that $\top_L \in \langle \mathcal{I} \rangle_{\mathbf{CLat}}$, there exists a subset $S \subseteq \mathcal{I}$ with $\bigvee S = \top_L$ (cf. Proposition 7). We can consider the operation \bigvee as a term $t(\vec{x})$ of no fixed arity.

Second, given complete lattices L, M , every \bigvee -semilattice morphism $L \xrightarrow{\varphi} M$ has a right adjoint (in the sense of posets) map $|L| \xrightarrow{\varphi^\top} |M|$, which is defined by $\varphi^\top(b) = \bigvee \{a \in L \mid \varphi(a) \leq b\}$, and which, additionally, is \bigwedge -preserving. In particular, $\varphi \circ \varphi^\top \leq 1_M$, which implies that given a lattice ideal $\mathcal{I} \subseteq M$, $\varphi \circ \varphi^\top(i) \in \mathcal{I}$ for every $i \in \mathcal{I}$. Further, given a map $X \xrightarrow{f} Y$, Example 15 (3) provides a \bigvee -preserving map $L^X \xrightarrow{\mathcal{P}_S(f, \varphi)} M^Y$, which (by the above formula) has a right adjoint $|M^Y| \xrightarrow{(\mathcal{P}_S(f, \varphi))^\top} |L^X|$ given by $(\mathcal{P}_S(f, \varphi))^\top(\beta) = \varphi^\top \circ \beta \circ f$, and which, moreover, is a complete lattice homomorphism provided that φ^\top is \bigvee -preserving.

Third, with the notations of the above item, for every subset $\mathcal{L} \subseteq L^X$, there exists the meet $\bigwedge \mathcal{L}$ such that $(\mathcal{P}_S(f, \varphi))(\bigwedge \mathcal{L}) \leq (\mathcal{P}_S(f, \varphi))(\alpha)$ for every $\alpha \in \mathcal{L}$, i.e., $(\mathcal{P}_S(f, \varphi))(\bigwedge \mathcal{L})$ belongs to the lattice ideal of M^Y generated by $(\mathcal{P}_S(f, \varphi))(\alpha)$ for every $\alpha \in \mathcal{L}$. Additionally, $\bigwedge \mathcal{L} = \underline{\top}_L$ provided that $\alpha = \underline{\top}_L$ for every $\alpha \in \mathcal{L}$. We can consider the operation \bigwedge as a term $t(\vec{x})$ with no fixed arity.

Fourth, in [28, 26], we showed that the category $L\text{-Born}$ (recall Example 18 (2)) is a topological construct iff the complete lattice L is ideally completely distributive at \top_L , which is defined as follows.

Definition 19. A complete lattice L is called *ideally completely distributive at \top_L* provided that for every non-empty family $\{S_i \mid i \in I\}$ of lattice ideals of L , $\bigwedge_{i \in I} (\bigvee S_i) = \top_L$ implies $\bigvee_{h \in H} (\bigwedge_{i \in I} h(i)) = \top_L$, where H is the set of all *choice maps* on $\bigcup_{i \in I} S_i$, i.e., maps $I \xrightarrow{h} \bigcup_{i \in I} S_i$ such that $h(i) \in S_i$ for every $i \in I$. ■

For convenience of the reader, we notice that ideal complete distributivity at \top does not even imply distributivity; conversely, every completely distributive lattice [13] is ideally completely distributive at \top , but there exists an infinitely distributive lattice L (namely, $a \wedge (\bigvee S) = \bigvee_{s \in S} (a \wedge s)$ and $a \vee (\bigwedge S) = \bigwedge_{s \in S} (a \vee s)$ for every $a \in L$ and every $S \subseteq L$), which is not ideally completely distributive at \top [28].

We introduce now four requirements on our variety-based bornological setting, which correspond to the above-mentioned four items.

Req. 1 Variety \mathbf{A} has an ideal term $p(\vec{x})$ of no fixed arity (for every \mathbf{A} -algebra A and every $\vec{a} \in A^n$, $p(\vec{a})$ is defined) such that given an \mathbf{A} -algebra A and an ideal $\mathcal{I} \subseteq A$ containing \top , $p(\vec{i}) = \top$ for some $\vec{i} \in \mathcal{I}^n$.

Req. 2 Every \mathbf{A}_{it} -morphism $A_1 \xrightarrow{\varphi} A_2$ has a map $|A_2| \xrightarrow{\varphi^\top} |A_1|$ such that for every \mathbf{B} -ideal $\mathcal{I} \subseteq A_2$ and every $i \in \mathcal{I}$, $\varphi \circ \varphi^\top(i) \in \mathcal{I}$. If $PX_1 \xrightarrow{\varphi} PX_2$ for some \mathbf{X} -objects X_1, X_2 , then φ^\top has two additional properties: first, $\varphi^\top(p(\vec{x})) = p(\varphi^\top(\vec{x}))$ for every $\vec{x} \in (PX_1)^n$ such that $p(\vec{x}) = \top$, where $p(\vec{y})$ is the ideal term from Req. 1; and, second, φ^\top preserves \top .

Req. 3 Variety \mathbf{A} has a term $t(\vec{x})$ of no fixed arity such that, first, for every \mathbf{A}_{it} -morphism $A_1 \xrightarrow{\varphi} A_2$, $\varphi(t(\vec{a})) \in \langle \varphi(b) \rangle_{\mathbf{B}}$ for every $b \in \vec{a}$ (in which the notation “ $b \in \vec{a}$ ” means that b is one of the components of the sequence \vec{a}), and, second, $t(\vec{a}) = \top$ provided that $b = \top$ for every $b \in \vec{a}$.

Req. 4 For every \mathbf{X} -object X , PX is *ideally completely distributive* at \top , i.e., given a family $\vec{y}_j \in (PX)^{n_j}$ with $j \in J$, $t(\overrightarrow{p(\vec{y}_j)_{j \in J}}) = \top$ implies $p(\overrightarrow{t(h(\vec{j})_{j \in J})_{h \in H}}) = \top$, with H the set of choice maps on $\bigcup_{j \in J} \vec{y}_j$.

The necessary requirements in hand, we can now prove the main results of this subsection.

Theorem 20. *If Req. 1 – 4 hold, then the category $\mathbf{Born}(T)$ is topological over \mathbf{X} .*

PROOF. Since the category $\mathbf{Born}(T)$ is clearly amnesic, by [2, Proposition 21.5], it will be enough to show that every structured $|-|-$ -source $(X \xrightarrow{f_j} |(X_j, \tau_j)|)_{j \in J}$ has an $|-|-$ -initial lift. Define $\tau = \{\alpha \in TX \mid Tf_j(\alpha) \in \tau_j \text{ for every } j \in J\}$. We have to show that, first, τ is an ideal of TX , and, second, $\top \in \langle \tau \rangle_{\mathbf{A}}$.

The first claim follows immediately from Propositions 7, 9. To show that $\top \in \langle \tau \rangle_{\mathbf{A}}$, we notice first that for every $j \in J$, by Req. 1, there exists $\vec{y}_j \in \tau_j^{n_j}$ such that $p(\overrightarrow{y_j}) = \top_j$. Let H be the set of choice functions on $\bigcup_{j \in J} \vec{y}_j$. For every $h \in H$, by Req. 2, 3, we define $\alpha_h = t(\overrightarrow{(Tf_j)^+ h(\vec{j})_{j \in J}})$. To show that $\alpha_h \in \tau$, we notice that given $j_0 \in J$, $Tf_{j_0}(\alpha_h) = Tf_{j_0}(t(\overrightarrow{(Tf_j)^+ h(\vec{j})_{j \in J}})) \stackrel{\text{Req. 3}}{\in} \langle Tf_{j_0} \circ (Tf_{j_0})^+(h(j_0)) \rangle_{\mathbf{B}} \stackrel{\text{Req. 2}}{\subseteq} \tau_{j_0}$. As a consequence, we get (by Req. 4) that $t(\overrightarrow{p((Tf_j)^+(\vec{y}_j)_{j \in J})}) \stackrel{\text{Req. 2}}{\cong} t(\overrightarrow{(Tf_j)^+(p(\vec{y}_j)_{j \in J}))} = t(\overrightarrow{(Tf_j)^+(\top_j)_{j \in J}}) \stackrel{\text{Req. 2}}{\cong} t(\overrightarrow{\top}) \stackrel{\text{Req. 3}}{\cong} \top$ implies $\top = p(\overrightarrow{t((Tf_j)^+(h(\vec{j})_{j \in J}))_{h \in H}}) = p(\overrightarrow{\alpha_{h \in H}})$, i.e., $\top \in \langle \tau \rangle_{\mathbf{A}}$. \square

We notice that Theorem 20 implies, in particular, one part (the sufficiency) of the result of [28] on the topological nature of the category $\mathbf{L-Born}$ (recall Example 18 (3)).

Definition 21. \mathbf{L}^+ is the subcategory of \mathbf{CLat} , the objects of which are complete lattices L , which are ideally completely distributive at \top_L , and whose morphisms $L_1 \xrightarrow{\psi} L_2$ are such that the map $L_2 \xrightarrow{\psi^+} L_1$ has the following property: $\psi^+(\bigvee S) = \bigvee_{\beta \in S} \psi^+(\beta)$ for every $S \subseteq L_2$ with $\bigvee S = \top_{L_2}$. \blacksquare

From Theorem 20, one then immediately gets the next result (notice that validity of Req. 1 – 4 in this particular case served as our main motivation for their introduction).

Theorem 22. *If \mathbf{L} is a subcategory of \mathbf{L}^+ , then the category $\mathbf{L-Born}$ is topological.*

3.3. Variety-based bornological systems

In this subsection, we introduce a variety-based bornological analogue of the concept of topological system of S. Vickers [37]. We notice, however, immediately that [28] has already introduced lattice-valued bornological systems, motivated by the notion of lattice-valued topological system of [10, 11]. It is the purpose of this subsection, to provide a bornological analogue of variety-based topological systems of [35, 36].

Definition 23. Given a bornological theory $\mathbf{X} \xrightarrow{T} \mathbf{B}_{it}$, $\mathbf{BornSys}(T)$ is the comma category $(1_{\mathbf{B}_{it}} \downarrow T)$, concrete over the product category $\mathbf{X} \times \mathbf{B}_{it}$, whose objects (*T-bornological systems* or *T-systems*) are triples (X, κ, B) , which are made by \mathbf{B}_{it} -morphisms $B \xrightarrow{\kappa} TX$ such that $\top \in \langle \mathcal{P}\kappa(B) \rangle_{\mathbf{A}}$, and whose morphisms (*T-bounded morphisms*) $(X_1, \kappa_1, B_1) \xrightarrow{(f, \varphi)} (X_2, \kappa_2, B_2)$ are $\mathbf{X} \times \mathbf{B}_{it}$ -morphisms $(X_1, B_1) \xrightarrow{(f, \varphi)} (X_2, B_2)$, making the diagram

$$\begin{array}{ccc} B_1 & \xrightarrow{\varphi} & B_2 \\ \kappa_1 \downarrow & & \downarrow \kappa_2 \\ TX_1 & \xrightarrow{Tf} & TX_2 \end{array}$$

commute. \blacksquare

In the following, we provide some examples of variety-based bornological systems.

Example 24.

- (1) $\mathbf{BornSys}((\mathcal{P}, \mathbf{Lat}_\perp))$ provides an analogue of the category $\mathbf{BornSys}$ of bornological systems and bounded morphisms of [28].
- (2) $\mathbf{BornSys}((\mathcal{P}_L^{\mathbf{CLat}}, \mathbf{Lat}_\perp))$ provides an analogue of the category $L\text{-BornSys}$ of [28].
- (3) $\mathbf{BornSys}((\mathcal{P}_S^{\mathbf{CLat}}, \mathbf{Lat}_\perp))$ provides an analogue of the category $\mathbf{L-BornSys}$ of [28]. ■

We notice that our variety-based approach does not fully incorporate the lattice-valued system setting of [28]. More precisely, in [28], the maps φ and κ_i , employed in the diagram of Definition 23, preserve different algebraic structure, i.e., do not come from the same category. In particular, the definition of κ_i is more restrictive than that of φ (even in view of Definition 23, which puts an additional condition on κ_i). The next subsection, however, shows that variety-based setting is more convenient for dealing with the so-called spatialization procedure for bornological systems.

3.4. Bornological systems versus bornological spaces

In [37], S. Vickers showed that the category of topological spaces is isomorphic to a full coreflective subcategory of the category of topological systems. The respective functor $\mathbf{TopSys} \xrightarrow{Spat} \mathbf{Top}$ (from topological systems to topological spaces) was called the system spatialization procedure. It is the main purpose of this subsection, to show a bornological analogue of the above functor $Spat$.

Proposition 25. *There exists a full embedding $\mathbf{Born}(T) \xhookrightarrow{E} \mathbf{BornSys}(T)$, $E((X_1, \tau_1) \xrightarrow{f} (X_2, \tau_2)) = (X_1, e_{\tau_1}, \tau_1) \xrightarrow{(f, \overline{Tf})} (X_2, e_{\tau_2}, \tau_2)$, where $\tau_i \xhookrightarrow{e_{\tau_i}} TX_i$ is the embedding, and \overline{Tf} is the restriction $(Tf)|_{\tau_1^{\tau_2}}$.*

PROOF. The functor is correct on both objects and morphism, which follows from the commutative diagram

$$\begin{array}{ccc}
 \tau_1 & \xrightarrow{\overline{Tf}=Tf|_{\tau_1^{\tau_2}}} & \tau_2 \\
 e_{\tau_1} \downarrow & & \downarrow e_{\tau_2} \\
 TX_1 & \xrightarrow{Tf} & TX_2.
 \end{array}$$

The same diagram takes care of fullness (every \mathbf{B}_{it} -morphism $\tau_1 \xrightarrow{\varphi} \tau_2$, which makes the above diagram commute, will be then necessarily the restriction of Tf). □

Proposition 26. *There exists a functor $\mathbf{BornSys}(T) \xrightarrow{Spat} \mathbf{Born}(T)$ defined by $Spat((X_1, \kappa_1, B_1) \xrightarrow{(f, \varphi)} (X_2, \kappa_2, B_2)) = (X_1, \tau_1 = \langle \mathcal{P}\kappa_1(B_1) \rangle_{\mathbf{B}}) \xrightarrow{f} (X_2, \tau_2 = \langle \mathcal{P}\kappa_2(B_2) \rangle_{\mathbf{B}})$.*

PROOF. To show that $Spat$ is correct on objects, we notice that $\langle \tau_i \rangle_{\mathbf{A}} = \langle \langle \mathcal{P}\kappa(B_i) \rangle_{\mathbf{B}} \rangle_{\mathbf{A}} = \langle \mathcal{P}\kappa(B_i) \rangle_{\mathbf{A}} \ni \top$.

To show that the functor is correct on morphisms, we consider the commutative diagram of Definition 23. Given an ideal term $p(\vec{x}, \vec{y})$, for every $\vec{x} \in (TX_1)^n$ and every $\vec{b} \in B_1^m$, $Tf(p(\vec{x}, \kappa_1(\vec{b}))) = p'(\vec{z}, Tf \circ \kappa_1(\vec{b})) = p'(\vec{z}, \kappa_2 \circ \varphi(\vec{b})) \in \langle \mathcal{P}\kappa_2(B_2) \rangle_{\mathbf{B}}$, which (in view Proposition 7) proves the claim. □

The main result of this subsection is then as follows.

Theorem 27. *$Spat$ is a left-adjoin-left-inverse to E .*

PROOF. Given a $\mathbf{BornSys}(T)$ -object (X, κ, B) , $ESpat(X, \kappa, B) = E(X, \tau = \langle \mathcal{P}\kappa(B) \rangle_{\mathbf{B}}) = (X, e_\tau, \tau)$, and, moreover, the diagram

$$\begin{array}{ccc}
 B & \xrightarrow{\overline{\kappa}=\kappa|^\tau} & \tau = \langle \mathcal{P}\kappa(B) \rangle_{\mathbf{B}} \\
 \kappa \downarrow & & \downarrow e_\tau \\
 TX & \xrightarrow{1_{TX}} & TX
 \end{array}$$

commutes. Thus, $(X, \kappa, B) \xrightarrow{(1_X, \bar{\kappa})} ESpat(X, \kappa, B)$ is a **BornSys**(T)-morphism.

To show that $((1_X, \bar{\kappa}), Spat(X, \kappa, B))$ is an E -universal arrow for (X, κ, B) , we take a **BornSys**(T)-morphism $(X, \kappa, B) \xrightarrow{(f, \varphi)} (E(X', \tau') = (X', e_{\tau'}, \tau'))$ and show that $(Spat(X, \kappa, B) = (X, \tau = \langle \mathcal{P}\kappa(B) \rangle_{\mathbf{B}})) \xrightarrow{f} (X', \tau')$ is a **Born**(T)-morphism. Commutativity of the diagram

$$\begin{array}{ccc} B & \xrightarrow{\varphi} & \tau' \\ \kappa \downarrow & & \downarrow e_{\tau'} \\ TX & \xrightarrow{Tf} & TX' \end{array}$$

gives then that for every ideal term $p(\vec{x}, \vec{y})$, every $\vec{x} \in (TX)^n$ and every $\vec{b} \in B^m$, $Tf(p(\vec{x}, \kappa(\vec{b}))) = p'(z, Tf \circ \kappa(\vec{b})) = p'(z, e_{\tau'} \circ \varphi(\vec{b})) = p'(z, \varphi(\vec{b})) \in \tau'$, which was to show.

It is easy to see that the triangle

$$\begin{array}{ccc} (X, \kappa, B) & \xrightarrow{(1_X, \bar{\kappa})} & ESpat(X, \kappa, B) \\ & \searrow (f, \varphi) & \downarrow Ef \\ & & E(X', \tau') \end{array}$$

commutes, and that f is the unique morphism with such property.

To show that $Spat$ is a left inverse to E , we notice that given a **Born**(T)-object (X, τ) , $SpatE(X, \tau) = Spat(E, e_{\tau}, \tau) = (X, \langle \mathcal{P}e_{\tau}(\tau) \rangle_{\mathbf{B}}) = (X, \tau)$. \square

As an immediate consequence of Propositions 25, 26 and Theorem 27, we get the following result.

Corollary 28. ***Born**(T) is isomorphic to a full reflective subcategory of **BornSys**(T).*

We are currently unable to verify, whether the reflective subcategory of Corollary 28 is actually epireflective or (regular epi)-reflective, which depends on whether the \mathbf{B}_{it} -morphism $B \xrightarrow{\bar{\kappa}} \langle \mathcal{P}\kappa(B) \rangle_{\mathbf{B}}$ of the proof of Theorem 27 is a (regular) epimorphism. More precisely, we presently lack a proper characterization of (regular) epimorphisms in the category \mathbf{B}_{it} , which will be the subject of our further study.

4. Conclusion and future work

In this paper, we introduced a new approach to (lattice-valued) bornology, which is based in varieties of algebras, and which is motivated by variety-based topology of [35, 36]. Our main idea here (similar to the case of topology) is to provide a common setting for different lattice-valued approaches to bornology. In particular, our variety-based approach incorporates the settings of M. Abel and A. Šostak [1] as well as our previous lattice-valued approach of [28]. Moreover, in this new setting, it is possible to get a bornological analogue of the notion of topological system of S. Vickers [37], and show that the category of bornological spaces is isomorphic to a full reflective subcategory of the category of bornological systems. We would like to end the paper with several open problems, which concern its topic of study.

4.1. Topological nature of the category of bornological spaces

In Theorem 20 of this paper, we showed sufficient conditions for the category **Born**(T) of variety-based bornological spaces to be topological over its ground category \mathbf{X} . Moreover, in [28], we showed the necessary and sufficient conditions for the category **L-Born** of lattice-valued variable-basis bornological spaces to be topological. The first open problem of this paper can be formulated then as follows.

Problem 29. What are the necessary and sufficient conditions for the category **Born**(T) to be topological over its ground category \mathbf{X} ? \blacksquare

4.2. The nature of the category of bornological systems

In [35], we showed that the category of variety-based topological systems is essentially algebraic (in the sense of [2, Definition 23.5]), which, taken together with the fact that the topological category of variety-based topological spaces is isomorphic to a full coreflective subcategory of the category of variety-based topological systems, provides then an embedding of topology into algebra [33]. The next open problem of this paper can be formulated therefore as follows.

Problem 30. What is the nature of the category $\mathbf{BornSys}(T)$ of variety-based bornological systems? ■

4.3. Localification procedure for bornological systems

In [37], S. Vickers provided two procedures for topological systems, namely, spatialization and localification. The former has already been considered in this paper, whereas the latter not. Briefly speaking, the category of locales is isomorphic to a full reflective subcategory of the category of topological systems. The functor $\mathbf{TopSys} \xrightarrow{Loc} \mathbf{Loc}$ (from topological systems to locales) is then called localification procedure for topological systems. In [34], we provided a variety-based generalization of the localification procedure. The last open problem of this paper is then as follows.

Problem 31. Provide a variety-based localification procedure for bornological systems. ■

We notice that one part of the above procedure is almost trivial.

Proposition 32. *There exists a functor $\mathbf{BornSys}(T) \xrightarrow{Loc} \mathbf{Bit}$, which is defined by $Loc((X_1, \kappa_1, B_1) \xrightarrow{(f, \varphi)} (X_2, \kappa_2, B_2)) = B_1 \xrightarrow{\varphi} B_2$.*

It is the functor in the opposite direction, which causes the main problem. We also notice that the term “localification” itself, which stems from “locale” in the topological setting, is not quite correct in case of bornology. In [28], we have proposed the term *bornale* for the underlying algebraic structures of lattice-valued bornology. Thus, the respective procedure could be called then “bornalification”.

The above open problems will be addressed in our next papers on the topic of variety-based bornology.

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