

# On Efficient Domination for Some Classes of $H$ -Free Chordal Graphs

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## Abstract

A vertex set  $D$  in a finite undirected graph  $G$  is an *efficient dominating set* (*e.d.s.* for short) of  $G$  if every vertex of  $G$  is dominated by exactly one vertex of  $D$ . The *Efficient Domination* (ED) problem, which asks for the existence of an e.d.s. in  $G$ , is known to be NP-complete even for very restricted graph classes such as for  $2P_3$ -free chordal graphs while it is solvable in polynomial time for  $P_6$ -free chordal graphs (and even for  $P_6$ -free graphs). A standard reduction from the NP-complete Exact Cover problem shows that ED is NP-complete for a very special subclass of chordal graphs generalizing split graphs. The reduction implies that ED is NP-complete e.g. for double-gem-free chordal graphs while it is solvable in linear time for gem-free chordal graphs (by various reasons such as bounded clique-width, distance-hereditary graphs, chordal square etc.), and ED is NP-complete for butterfly-free chordal graphs while it is solvable in linear time for  $2P_2$ -free graphs.

We show that (weighted) ED can be solved in polynomial time for  $H$ -free chordal graphs when  $H$  is net, extended gem, or  $S_{1,2,3}$ .

**Keywords:** Weighted efficient domination;  $H$ -free chordal graphs; NP-completeness; net-free chordal graphs; extended-gem-free chordal graphs;  $S_{1,2,3}$ -free chordal graphs; polynomial time algorithm; clique-width.

## 1 Introduction

Let  $G = (V, E)$  be a finite undirected graph. A vertex  $v$  *dominates* itself and its neighbors. A vertex subset  $D \subseteq V$  is an *efficient dominating set* (*e.d.s.* for short) of  $G$  if every vertex of  $G$  is dominated by exactly one vertex in  $D$ ; for any e.d.s.  $D$  of  $G$ ,  $|D \cap N[v]| = 1$  for every  $v \in V$  (where  $N[v]$  denotes the closed neighborhood of  $v$ ). Note that not every graph has an e.d.s.; the EFFICIENT DOMINATING SET (ED) problem asks for the existence of an e.d.s. in a given graph  $G$ .

The EXACT COVER problem asks for a subset  $\mathcal{F}'$  of a set family  $\mathcal{F}$  over a ground set, say  $V$ , containing every vertex in  $V$  exactly once. In particular, this means that the elements of  $\mathcal{F}'$  form a partition of  $V$ , i.e., for every two distinct elements  $U, W \in \mathcal{F}'$ ,  $U \cap W = \emptyset$  and  $\bigcup_{X \in \mathcal{F}'} X = V$ . Thus, EXACT COVER is a partition problem since it asks for a subset  $\mathcal{F}'$  of  $\mathcal{F}$  which forms a partition of  $V$  (however, in [21], the problem PARTITION is a distinct problem [SP12]). As shown by Karp [23], EXACT COVER is NP-complete even for set families containing only 3-element subsets of  $V$  (see problem X3C [SP2] in [21]).

Clearly, ED is EXACT COVER for the closed neighborhood hypergraph of  $G$ . The notion of efficient domination was introduced by Biggs [3] under the name *perfect code*. The ED problem is motivated by various applications, including coding theory and resource allocation in parallel computer networks; see e.g. [1–3, 16, 24–26, 29, 30, 32, 33].

In [1, 2], it was shown that the ED problem is NP-complete. Moreover, ED is NP-complete for  $2P_3$ -free chordal unipolar graphs [18, 31, 33].

In this paper, we will also consider the following weighted version of the ED problem:

WEIGHTED EFFICIENT DOMINATION (WED)

*Instance:* A graph  $G = (V, E)$ , vertex weights  $\omega : V \rightarrow \mathbb{N} \cup \{\infty\}$ .

*Task:* Find an e.d.s. of minimum finite total weight,  
or determine that  $G$  contains no such e.d.s.

The relationship between WED and ED is analyzed in [7].

For a set  $\mathcal{F}$  of graphs, a graph  $G$  is called  $\mathcal{F}$ -free if  $G$  contains no induced subgraph isomorphic to a member of  $\mathcal{F}$ . In particular, we say that  $G$  is  $H$ -free if  $G$  is  $\{H\}$ -free. Let  $H_1 + H_2$  denote the disjoint union of graphs  $H_1$  and  $H_2$ , and for  $k \geq 2$ , let  $kH$  denote the disjoint union of  $k$  copies of  $H$ . For  $i \geq 1$ , let  $P_i$  denote the chordless path with  $i$  vertices, and let  $K_i$  denote the complete graph with  $i$  vertices (clearly,  $P_i = K_i$  for  $i = 1, 2$ ). For  $i \geq 4$ , let  $C_i$  denote the chordless cycle with  $i$  vertices.

For indices  $i, j, k \geq 0$ , let  $S_{i,j,k}$  denote the graph with vertices  $u, x_1, \dots, x_i, y_1, \dots, y_j, z_1, \dots, z_k$  such that the subgraph induced by  $u, x_1, \dots, x_i$  forms a  $P_{i+1}$  ( $u, x_1, \dots, x_i$ ), the subgraph induced by  $u, y_1, \dots, y_j$  forms a  $P_{j+1}$  ( $u, y_1, \dots, y_j$ ), and the subgraph induced by  $u, z_1, \dots, z_k$  forms a  $P_{k+1}$  ( $u, z_1, \dots, z_k$ ), and there are no other edges in  $S_{i,j,k}$ . Thus, claw is  $S_{1,1,1}$ , chair is  $S_{1,1,2}$ , and  $P_k$  is isomorphic to  $S_{0,0,k-1}$ . Claw will also be denoted by  $K_{1,3}$ , and its *midpoint* is the vertex with degree 3 in the claw.

$H$  is a *linear forest* if every component of  $H$  is a chordless path, i.e.,  $H$  is claw-free and cycle-free.

$H$  is a *co-chair* if it is the complement graph of a chair.  $H$  is a *P* if  $H$  has five vertices such that four of them induce a  $C_4$  and the fifth is adjacent to exactly one of the  $C_4$ -vertices.  $H$  is a *co-P* if  $H$  is the complement graph of a  $P$ .  $H$  is a *bull* if  $H$  has five vertices such that four of them induce a  $P_4$  and the fifth is adjacent to exactly the two mid-points of the  $P_4$ .  $H$  is a *net* if  $H$  has six vertices such that five of them induce a bull and the sixth is adjacent to exactly the vertex of the bull with degree 2.  $H$  is a *diamond* if  $H$  has four vertices such that only two of them are nonadjacent. The diamond will also be denoted by  $K_4 - e$ .  $H$  is a *gem* if  $H$  has five vertices such that four of them induce a  $P_4$  and the fifth is adjacent to all of the  $P_4$  vertices.  $H$  is a *co-gem* if  $H$  is the complement graph of a gem.

For a vertex  $v \in V$ ,  $N(v) = \{u \in V : uv \in E\}$  denotes its (*open*) *neighborhood*, and  $N[v] = \{v\} \cup N(v)$  denotes its *closed neighborhood*. A vertex  $v$  *sees* the vertices in  $N(v)$  and *misses* all the others. The *non-neighborhood* of a vertex  $v$  is  $\overline{N}(v) := V \setminus N[v]$ . For  $U \subseteq V$ ,  $N(U) := \bigcup_{u \in U} N(u) \setminus U$  and  $\overline{N}(U) := V \setminus (U \cup N(U))$ .

We say that for a vertex set  $X \subseteq V$ , a vertex  $v \notin X$  has a *join* (resp., *co-join*) to  $X$  if  $X \subseteq N(v)$  (resp.,  $X \subseteq \overline{N}(v)$ ). Join (resp., *co-join*) of  $v$  to  $X$  is denoted by  $v \textcircled{1} X$  (resp.,  $v \textcircled{0} X$ ). Correspondingly, for vertex sets  $X, Y \subseteq V$  with  $X \cap Y = \emptyset$ ,  $X \textcircled{1} Y$  denotes  $x \textcircled{1} Y$  for all  $x \in X$  and  $X \textcircled{0} Y$  denotes  $x \textcircled{0} Y$  for all  $x \in X$ . A vertex  $x \notin U$  *contacts*  $U$  if  $x$  has a neighbor in  $U$ . For vertex sets  $U, U'$  with  $U \cap U' = \emptyset$ ,  $U$  *contacts*  $U'$  if there is a vertex in  $U$  contacting  $U'$ .

If  $v \notin X$  but  $v$  has neither a join nor a co-join to  $X$ , then we say that  $v$  *distinguishes*  $X$ . A set  $H$  of at least two vertices of a graph  $G$  is called *homogeneous* if  $H \neq V(G)$  and every

vertex outside  $H$  is either adjacent to all vertices in  $H$ , or to no vertex in  $H$ . Obviously,  $H$  is homogeneous in  $G$  if and only if  $H$  is homogeneous in the complement graph  $\overline{G}$ . A graph is *prime* if it contains no homogeneous set. In [8, 12], it is shown that the WED problem can be reduced to prime graphs.

A graph  $G$  is *chordal* if it is  $C_i$ -free for any  $i \geq 4$ .  $G = (V, E)$  is *unipolar* if  $V$  can be partitioned into a clique and the disjoint union of cliques, i.e., there is a partition  $V = A \cup B$  such that  $G[A]$  is a complete subgraph and  $G[B]$  is  $P_3$ -free.  $G$  is a *split graph* if  $G$  and its complement graph are chordal. Equivalently,  $G$  can be partitioned into a clique and an independent set. It is well known that  $G$  is a split graph if and only if it is  $(2P_2, C_4, C_5)$ -free [19].

It is well known that ED is NP-complete for claw-free graphs (even for  $(K_{1,3}, K_4 - e)$ -free perfect graphs [28]) as well as for bipartite graphs (and thus for triangle-free graphs) [29] and for chordal graphs [18, 31, 33]. Thus, for the complexity of ED on  $H$ -free graphs, the most interesting cases are when  $H$  is a linear forest. Since (W)ED is NP-complete for  $2P_3$ -free graphs and polynomial for  $(P_5 + kP_2)$ -free graphs [8, 9], the class of  $P_6$ -free graphs was the only open case. It was finally solved in [13, 14] by a direct polynomial time approach (and in [27] by an indirect one).

It is well known that for a graph class with bounded clique-width, ED can be solved in polynomial time [17]. Thus we only consider ED on  $H$ -free chordal graphs for which the clique-width is unbounded. For example, the clique-width of  $H$ -free chordal graphs is unbounded for claw-free chordal graphs while it is bounded if  $H \in \{\text{bull, gem, co-gem, co-chair}\}$ . In [4], the clique-width of  $H$ -free chordal graphs is classified for all but two stubborn cases.

For graph  $G = (V, E)$ , let  $d_G(x, y)$  denote the distance between  $x$  and  $y$  (i.e., the shortest length of a path between  $x$  and  $y$ ) in  $G$ . The square  $G^2$  has the same vertex set  $V$  as  $G$ , and two vertices  $x, y \in V$ ,  $x \neq y$ , are adjacent in  $G^2$  if and only if  $d_G(x, y) \leq 2$ . The WED problem on  $G$  can be reduced to Maximum Weight Independent Set (MWIS) on  $G^2$  (see [7, 10, 12, 30]).

While the complexity of ED for  $2P_3$ -free chordal graphs is NP-complete (as mentioned above), it was shown in [5] that WED is solvable in polynomial time for  $P_6$ -free chordal graphs, since the square of every  $P_6$ -free chordal graph  $G$  with e.d.s. is also chordal.

It is well known [20] that MWIS is solvable in linear time for chordal graphs.

However, there are still many cases of graphs  $H$  for which the complexity of WED in  $H$ -free chordal graphs is open.

## 2 WED is NP-Complete for Chordal Hereditary Satgraphs

It is well known [15] that WED is solvable in linear time for split graphs. In this section, we show that ED is NP-complete for a slight generalization of split graphs, namely a subclass of chordal hereditary satgraphs: A graph  $G$  is called a *satgraph* (described by Zverovich in [34]) if there exists a partition  $A \cup B = V(G)$  such that

- (i)  $A$  induces a complete subgraph (possibly,  $A = \emptyset$ ),
- (ii)  $G[B]$  is an induced matching (possibly,  $B = \emptyset$ ), and
- (iii) there are no triangles  $(a, b, b')$ , where  $a \in A$  and  $b, b' \in B$ .

In [34], Zverovich characterized the class of hereditary satgraphs as the class of  $\mathcal{Z}_{SAT}$ -free graphs where the set  $\mathcal{Z}_{SAT}$  consists of the graphs  $F_1, F_2, \dots, F_{21}$  shown in Figure 3 of [34]. Hereby,  $F_i$  for  $i \in \{1, 2, 4, 7, 8, 13, 14, 15, 16, 18, 19, 20, 21\}$  contain  $C_4, C_5, C_6$  or  $C_7$ .

The eight remaining  $F_i$ , namely  $F_3, F_5, F_6, F_9, F_{10}, F_{11}, F_{12}, F_{17}$  are presented in Figure 1.

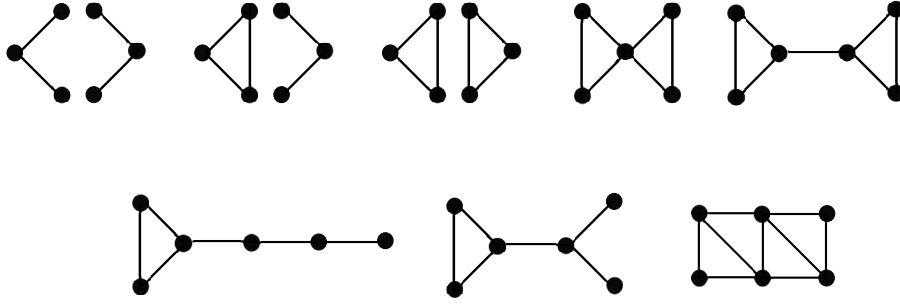


Figure 1:  $2P_3$ ,  $K_3 + P_3$ ,  $2K_3$ , butterfly, extended butterfly, extended co- $P$ , extended chair, and double-gem

**Lemma 1.** *ED is NP-complete for  $(2P_3, K_3 + P_3, 2K_3, \text{butterfly}, \text{extended butterfly}, \text{extended co-}P, \text{extended chair}, \text{double-gem})$ -free chordal and unipolar graphs.*

**Proof.** The reduction from X3C to Efficient Domination will show that ED is NP-complete for this special subclass of chordal graphs.

Let  $H = (V, \mathcal{E})$  with  $V = \{v_1, \dots, v_n\}$  and  $\mathcal{E} = \{e_1, \dots, e_m\}$  be a hypergraph with  $|e_i| = 3$  for all  $i \in \{1, \dots, m\}$ . Let  $G_H$  be the following reduction graph:

$V(G_H) = V \cup X \cup Y$  such that  $X = \{x_1, \dots, x_m\}$ ,  $Y = \{y_1, \dots, y_m\}$  and  $V, X, Y$  are pairwise disjoint. The edge set of  $G_H$  consists of all edges  $v_i x_j$  whenever  $v_i \in e_j$ . Moreover  $V$  is a clique in  $G_H$ ,  $X$  is an independent subset in  $G_H$ , and every  $y_i$ ,  $i = 1, \dots, m$ , is only adjacent to  $x_i$ .

Clearly,  $H = (V, \mathcal{E})$  has an exact cover if and only if  $G_H$  has an e.d.s.  $D$ : For an exact cover  $\mathcal{E}'$  of  $H$ , every  $e_i \in \mathcal{E}'$  corresponds to vertex  $x_i \in D$ , and every  $e_i \notin \mathcal{E}'$  corresponds to vertex  $y_i \in D$ . Conversely, let  $D$  be an e.d.s. in  $G_H$ . If  $D \cap V \neq \emptyset$ , say without loss of generality,  $v_1 \in V \cap D$  and  $v_1 \in e_1$  then  $v_1$  dominates  $x_1$  and  $y_1$  cannot be dominated which is a contradiction. Thus, we have  $D \cap V = \emptyset$ , and now,  $D \cap X$  corresponds to an exact cover of  $H$ .

Clearly,  $G_H$  is chordal and unipolar. Since any induced  $P_3$  or  $K_3$  in  $G_H$  has a vertex in  $V$ , the reduction shows that  $G_H$  is not only  $2P_3$ -free but also  $F$ -free for various other graphs  $F$  such as  $K_3 + P_3$ ,  $2K_3$ , butterfly, extended butterfly, extended co- $P$ , extended chair, and double-gem as shown in Figure 1.  $\square$

The reduction implies that WED is NP-complete e.g. for double-gem-free chordal graphs while it is solvable in linear time for gem-free chordal graphs (since gem-free chordal graphs are distance-hereditary and thus, their clique-width is at most 3 as shown in [22]), and WED is NP-complete for butterfly-free chordal graphs while it is solvable in linear time for  $2P_2$ -free graphs [12].

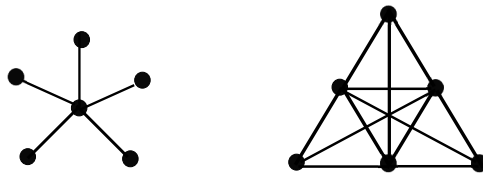


Figure 2:  $K_{1,5}$  and  $K_3 \circledast 3K_1$

**Lemma 2.** *ED is NP-complete for  $K_{1,5}$ -free chordal graphs and for  $K_3 \oplus 3K_1$ -free chordal graphs.*

**Proof.** The Exact Cover problem remains NP-complete if no element occurs in more than three subsets (see X3C [SP2] in [21]). With respect to the standard reduction, recall that  $V(G_H) = V \cup X \cup Y$ ,  $V$  is a clique in  $G_H$ , for each hyperedge  $e_i \in \mathcal{E}$ , there is exactly one vertex  $x_i \in X$  that corresponds to  $e_i$ ,  $X$  is independent in  $G_H$ , and for every  $y_i \in Y$ ,  $x_i$  is the only neighbor of  $y_i$  in  $G_H$ .

We first claim that every midpoint of a claw in  $G_H$  is in  $V$ : Let  $a, b, c, d$  induce a claw in  $G_H$  with midpoint  $a$ . Then obviously,  $a \notin Y$ , at most one of  $b, c, d$  is in  $V$ , and if  $a \notin V$ , i.e.,  $a \in X$  then two of  $b, c, d$  are in  $V$  which is a contradiction.

Now  $G_H$  is  $K_{1,5}$ -free since for  $K_{1,5}$ , say with vertices  $a, b, c, d, e, f$  and midpoint  $a$ , we have  $a \in V$  and at most one of  $b, c, d, e, f$  is in  $V$ , say  $b \in V$  but then  $c, d, e, f \in X$  which is a contradiction to the Exact Cover condition that no element occurs in more than three subsets.

Finally, we claim that  $G_H$  is  $K_3 \oplus 3K_1$ -free: Let  $a, b, c, d, e, f$  induce a  $K_3 \oplus 3K_1$  such that  $a, b, c$  induce a  $K_3$  and  $d, e, f$  induce a  $3K_1$ . Then each of  $a, b, c$  are midpoint of a claw, and thus,  $a, b, c \in V$ . Moreover, at most one of  $d, e, f$  is in  $V$ , say  $e, f \in X$  but now,  $e$  and  $f$  have a join to the same hyperedge  $\{a, b, c\}$  which is a contradiction to the standard reduction.  $\square$

### 3 $G^2$ -Approach For Net-Free and Extended-Gem-Free Chordal Graphs

Motivated by the  $G^2$  approach in [5,6], and the result of Milanič [30] showing that for  $(S_{1,2,2}, \text{net})$ -free graphs  $G$ , its square  $G^2$  is claw-free, we show in this section that  $G^2$  is chordal for  $H$ -free chordal graphs with e.d.s. when  $H$  is a net or an extended gem (see Figure 3 - extended gem generalizes  $S_{1,2,2}$  and some other subgraphs), and thus, WED is solvable in polynomial time for these two graph classes.

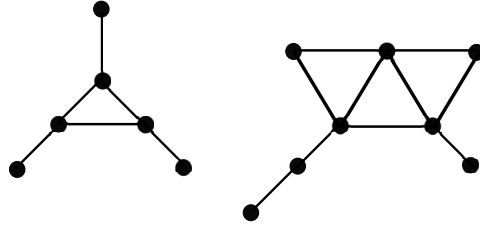


Figure 3: net and extended gem

**Claim 3.1.** *Let  $G$  be a chordal graph, and let  $v_1, \dots, v_k$ ,  $k \geq 4$ , induce a  $C_k$  in  $G^2$  with  $d_G(v_i, v_{i+1}) \leq 2$  and  $d_G(v_i, v_j) \geq 3$ ,  $i, j \in \{1, \dots, k\}$ ,  $|i - j| > 1$  (index arithmetic modulo  $k$ ). Then we have:*

- (i) *For each  $i \in \{1, \dots, k\}$ ,  $d_G(v_i, v_{i+1}) = 2$ .*
- (ii) *Let  $x_i$  be a common neighbor of  $v_i$  and  $v_{i+1}$  in  $G$  (an auxiliary vertex). Then for each  $i, j \in \{1, \dots, k\}$ ,  $i \neq j$ , we have  $x_i \neq x_j$ , and  $x_i x_{i+1} \in E(G)$ .*

*Proof.* (i): Suppose without loss of generality that  $d_G(v_1, v_2) = 1$ . Then, since  $d_G(v_1, v_3) \geq 3$  and  $d_G(v_k, v_2) \geq 3$ , we have  $d_G(v_2, v_3) = 2$  and  $d_G(v_k, v_1) = 2$ ; let  $x_2$  be a common neighbor of  $v_2, v_3$  and  $x_k$  be a common neighbor of  $v_k, v_1$ . Clearly,  $x_2 \neq x_k$  since  $d_G(v_k, v_2) \geq 3$ . Moreover,

$x_2v_1 \notin E$  since  $d_G(v_1, v_3) \geq 3$  and  $x_kv_2 \notin E$  since  $d_G(v_k, v_2) \geq 3$ . Now,  $x_kx_2 \notin E$  since otherwise  $x_k, v_1, v_2, x_2$  would induce a  $C_4$  in  $G$  but now in any case, the  $P_4$  induced by  $x_k, v_1, v_2, x_2$  leads to a chordless cycle in  $G$  which is a contradiction.

(ii): Clearly, as above, we have  $x_i \neq x_j$  for any  $i \neq j$ . Without loss of generality, suppose to the contrary that there is a non-edge  $x_kx_1 \notin E$ . Then, if  $x_k$  and  $x_1$  have a common neighbor  $x_i, i \neq k, 1$ , then  $x_k, v_1, x_1, x_i$  would induce a  $C_4$  in  $G$  which is a contradiction, and if  $x_k$  and  $x_1$  do not have any common neighbor  $x_i, i \neq k, 1$ , then a shortest path between  $x_1$  and  $x_k$  in  $G[\{x_1, v_2, x_2, v_3, \dots, x_{k-1}, v_k, x_k\}]$  together with  $v_1$  would again lead to a chordless cycle in  $G$  which is a contradiction.  $\square$

**Theorem 1.** *If  $G$  is a net-free chordal graph with e.d.s. then  $G^2$  is chordal.*

**Proof.** Let  $G = (V, E)$  be a net-free chordal graph and assume that  $G$  has an e.d.s.  $D$ . We first show that  $G^2$  is  $C_4$ -free:

Suppose to the contrary that  $G^2$  contains a  $C_4$ , say with vertices  $v_1, v_2, v_3, v_4$  such that  $d_G(v_i, v_{i+1}) \leq 2$  and  $d_G(v_i, v_{i+2}) \geq 3, i \in \{1, 2, 3, 4\}$  (index arithmetic modulo 4). By Claim 3.1, we have  $d_G(v_i, v_{i+1}) = 2$  for each  $i \in \{1, 2, 3, 4\}$ ; let  $x_i$  be a common neighbor of  $v_i, v_{i+1}$ . By Claim 3.1,  $x_i \neq x_j$  for  $i \neq j$ . Since  $G$  is chordal,  $x_1, x_2, x_3, x_4$  either induce a diamond or  $K_4$  in  $G$ .

Assume first that  $x_1, x_2, x_3, x_4$  induce a diamond in  $G$ , say with  $x_1x_3 \in E$  and  $x_2x_4 \notin E$ . We claim:

$$D \cap \{x_1, x_2, x_3, x_4\} = \emptyset. \quad (1)$$

*Proof.* First suppose to the contrary that  $x_1 \in D$ . Then by the e.d.s. property, we have  $v_3, v_4, x_2, x_3, x_4 \notin D$ . Since  $v_3$  and  $v_4$  have to be dominated by  $D$ , let  $d_3 \in D$  with  $d_3v_3 \in E$  and  $d_4 \in D$  with  $d_4v_4 \in E$ . Clearly,  $d_3 \neq x_2, x_3$  and  $d_4 \neq x_3, x_4$ . By the e.d.s. property,  $d_3$  and  $d_4$  are nonadjacent to the neighbors  $v_1, v_2, x_2, x_3, x_4$  of  $x_1$ . Thus,  $d_3 \neq d_4$  since otherwise  $x_1, x_2, v_3, d_3, v_4, x_4$  would induce a  $C_6$  in the chordal graph  $G$ . This implies  $d_3v_4 \notin E$  but now,  $v_2, x_2, v_3, d_3, x_3, v_4$  induce a net in  $G$  which is a contradiction. Thus,  $x_1 \notin D$  and correspondingly,  $x_3 \notin D$ .

Now suppose to the contrary that  $x_2 \in D$ . Then by the e.d.s. property,  $v_1, v_4, x_1, x_3, x_4 \notin D$ . Since  $v_1$  and  $v_4$  have to be dominated by  $D$ , let  $d_1 \in D$  with  $d_1v_1 \in E$  and  $d_4 \in D$  with  $d_4v_4 \in E$ . Clearly,  $d_1 \neq x_1, x_4$  and  $d_4 \neq x_3, x_4$ . By the e.d.s. property,  $d_1$  and  $d_4$  are nonadjacent to the neighbors  $v_2, v_3, x_1, x_3$  of  $x_2$ . Thus,  $d_1v_4 \notin E$  since otherwise  $d_1, v_1, x_1, x_3, v_4$  would induce a  $C_5$  in the chordal graph  $G$ , and analogously,  $d_4v_1 \notin E$ . Now, if  $d_1x_4 \notin E$  then  $d_1, v_1, x_1, v_2, x_4, v_4$  induce a net in  $G$ , and if  $d_1x_4 \in E$  then by the e.d.s. property,  $d_4x_4 \notin E$  and thus,  $d_4, v_4, x_3, v_3, x_4, v_1$  induce a net in  $G$ , which is a contradiction. Thus,  $x_2 \notin D$  and correspondingly,  $x_4 \notin D$ , and claim (1) is shown.  $\diamond$

Next we claim:

$$D \cap \{v_1, v_2, v_3, v_4\} = \emptyset. \quad (2)$$

*Proof.* Without loss of generality, suppose to the contrary that  $v_1 \in D$ . Then by the e.d.s. property, we have  $v_2, v_4, x_1, x_2, x_3, x_4 \notin D$ . Since  $v_2$  and  $v_4$  have to be dominated by  $D$ , let  $d_2 \in D$  with  $d_2v_2 \in E$  and  $d_4 \in D$  with  $d_4v_4 \in E$ . Since  $d_G(v_2, v_4) > 2$ , we have  $d_2 \neq d_4$ .

Moreover,  $d_2x_3 \notin E$  since otherwise,  $d_2, v_2, x_1, x_3$  induce a  $C_4$  in  $G$ . This implies  $d_2v_3 \notin E$  since otherwise,  $d_2, v_3, x_3, x_1, v_2$  induce a  $C_5$  in  $G$ .

Now, if  $d_2x_2 \notin E$  then  $d_2, v_2, x_2, v_3, x_1, v_1$  induce a net, and if  $d_2x_2 \in E$  then  $d_2, x_2, x_1, x_3, v_1, v_4$  induce a net, which is a contradiction.

Thus,  $v_1 \notin D$ , and correspondingly,  $v_2, v_3, v_4 \notin D$ , and claim (2) is shown.  $\diamond$



Let  $d_i \in D$  be the  $D$ -neighbor of  $v_i$ . By (1) and (2) and the distance properties, we have  $d_i \neq v_j, x_j, i, j \in \{1, 2, 3, 4\}$ . Next we claim that  $d_1, d_2, d_3, d_4$  are pairwise distinct:

$$|\{d_1, d_2, d_3, d_4\}| = 4. \quad (3)$$

*Proof.* Since  $d_G(v_1, v_3) > 2$  and  $d_G(v_2, v_4) > 2$ , we have  $d_1 \neq d_3$  and  $d_2 \neq d_4$ . Thus,  $|\{d_1, d_2, d_3, d_4\}| \geq 2$ .

If without loss of generality,  $d_1 = d_4$ , i.e.,  $d_1v_1 \in E$  and  $d_1v_4 \in E$  then, since  $d_1, v_1, x_1, x_3, v_4$  do not induce a  $C_5$  in  $G$ , we have  $d_1x_1 \in E$  or  $d_1x_3 \in E$ , and if without loss of generality,  $d_1x_1 \in E$  and  $d_1x_3 \notin E$  then  $d_1, x_1, x_3, v_4$  induce a  $C_4$  in  $G$ . Thus,  $d_1x_1 \in E$  and  $d_1x_3 \in E$ .

This shows that if  $d_1v_1 \in E$  and  $d_1v_4 \in E$  then  $d_2 \neq d_3$ , and thus  $|\{d_1, d_2, d_3, d_4\}| \geq 3$ .

Now assume that  $|\{d_1, d_2, d_3, d_4\}| = 3$ , i.e.,  $d_1v_1 \in E$  and  $d_1v_4 \in E$ ,  $d_2v_3 \notin E$  and  $d_3v_2 \notin E$ . Recall  $d_1x_1 \in E$  and  $d_1x_3 \in E$ . Thus,  $d_2x_1 \notin E$ ,  $d_2x_3 \notin E$ ,  $d_3x_1 \notin E$ ,  $d_3x_3 \notin E$ .

If  $d_2x_2 \notin E$  then  $d_2, v_2, x_1, v_1, x_2, v_3$  induce a net in  $G$ , and if  $d_2x_2 \in E$  then  $d_3x_2 \notin E$  and thus,  $d_3, v_3, x_2, v_2, x_3, v_4$  induce a net in  $G$  which is a contradiction. Thus,  $d_1, d_2, d_3, d_4$  are pairwise distinct, and claim (3) is shown.  $\diamond$

If  $d_1x_1 \notin E$  and  $d_1x_4 \notin E$  then  $d_1, v_1, x_1, x_4, v_2, v_4$  induce a net in  $G$ , and correspondingly by symmetry, a similar statement can be made about  $d_i, x_{i-1}, x_i, i \neq 1$ . Thus, we can assume that for each  $i \in \{1, \dots, 4\}$ ,  $d_i$  sees at least one of  $x_{i-1}, x_i$  (index arithmetic modulo 4).

If  $d_1x_1 \in E$  and  $d_1x_4 \in E$  then clearly,  $d_2x_1 \notin E$  and  $d_4x_4 \notin E$  and thus, by the above, we can assume that  $d_2x_2 \in E$  and  $d_4x_3 \in E$  but now,  $d_2, x_2, v_3, x_3, d_3, d_4$  induce a net in  $G$ .

Thus, assume that  $d_1$  is adjacent to exactly one of  $x_1, x_4$ , say  $d_1x_1 \in E$  (which implies  $d_2x_1 \notin E$ ) and  $d_1x_4 \notin E$ . By symmetry, this holds for  $d_2, d_3, d_4$  as well, i.e.,  $d_2x_2 \in E$ ,  $d_3x_3 \in E$ , and  $d_4x_4 \in E$ . Then  $d_1, x_1, d_2, x_2, d_3, x_3$  induce a net in  $G$ .

Thus, when  $x_1, x_2, x_3, x_4$  induce a diamond in  $G$ , then  $G^2$  does not contain a  $C_4$  with vertices  $v_1, v_2, v_3, v_4$ .

Now assume that  $x_1, x_2, x_3, x_4$  induce a  $K_4$  in  $G$ . The proof is very similar as above. Again we claim:

$$D \cap \{x_1, x_2, x_3, x_4\} = \emptyset. \quad (4)$$

*Proof.* By symmetry, suppose to the contrary that  $x_1 \in D$ . Then by the e.d.s. property, we have  $v_3, v_4, x_2, x_3, x_4 \notin D$ . Since  $v_3$  and  $v_4$  have to be dominated by  $D$ , let  $d_3 \in D$  with  $d_3v_3 \in E$  and  $d_4 \in D$  with  $d_4v_4 \in E$ . By the e.d.s. property,  $d_3$  and  $d_4$  are nonadjacent to the neighbors  $v_1, v_2, x_2, x_3, x_4$  of  $x_1$ . Thus,  $d_3 \neq d_4$  since otherwise  $x_2, v_3, d_3, v_4, x_4$  would induce a  $C_5$  in the chordal graph  $G$ . This implies  $d_3v_4 \notin E$  but now,  $v_2, x_2, v_3, d_3, x_3, v_4$  induce a net in  $G$  which is a contradiction. Thus,  $x_1 \notin D$  and correspondingly,  $x_2, x_3, x_4 \notin D$ , and claim (4) is shown.  $\diamond$

Next we claim:

$$D \cap \{v_1, v_2, v_3, v_4\} = \emptyset. \quad (5)$$

*Proof.* Without loss of generality, suppose to the contrary that  $v_1 \in D$ . Then by the e.d.s. property, we have  $v_2, v_4, x_1, x_2, x_3, x_4 \notin D$ . Since  $v_2$  and  $v_4$  have to be dominated by  $D$ , let  $d_2 \in D$  with  $d_2v_2 \in E$  and  $d_4 \in D$  with  $d_4v_4 \in E$ . Since  $d_G(v_2, v_4) > 2$ , we have  $d_2 \neq d_4$ .

Moreover,  $d_2x_3 \notin E$  since otherwise,  $d_2, v_2, x_1, x_3$  induce a  $C_4$  in  $G$ . This implies  $d_2v_3 \notin E$  since otherwise,  $d_2, v_3, x_3, x_1, v_2$  induce a  $C_5$  in  $G$ .

Now, if  $d_2x_2 \notin E$  then  $d_2, v_2, x_2, v_3, x_1, v_1$  induce a net, and if  $d_2x_2 \in E$  then  $d_2, x_2, x_1, x_3, v_1, v_4$  induce a net, which is a contradiction.

Thus,  $v_1 \notin D$ , and correspondingly,  $v_2, v_3, v_4 \notin D$ , and claim (5) is shown.  $\diamond$

Again, let  $d_i \in D$  be the  $D$ -neighbor of  $v_i$ . By (4) and (5) and the distance properties, we have  $d_i \neq v_j, x_j, i, j \in \{1, 2, 3, 4\}$ . Next we claim that  $d_1, d_2, d_3, d_4$  are pairwise distinct:

$$|\{d_1, d_2, d_3, d_4\}| = 4. \quad (6)$$

*Proof.* Since  $d_G(v_1, v_3) > 2$  and  $d_G(v_2, v_4) > 2$ , we have  $d_1 \neq d_3$  and  $d_2 \neq d_4$ . Thus,  $|\{d_1, d_2, d_3, d_4\}| \geq 2$ .

If without loss of generality,  $d_1 = d_4$ , i.e.,  $d_1v_1 \in E$  and  $d_1v_4 \in E$  then, since  $d_1, v_1, x_1, x_3, v_4$  do not induce a  $C_5$  in  $G$ , we have  $d_1x_1 \in E$  or  $d_1x_3 \in E$ , and if without loss of generality,  $d_1x_1 \in E$  and  $d_1x_3 \notin E$  then  $d_1, x_1, x_3, v_4$  induce a  $C_4$  in  $G$ . Thus,  $d_1x_1 \in E$  and  $d_1x_3 \in E$ .

This shows that if  $d_1v_1 \in E$  and  $d_1v_4 \in E$  then  $d_2 \neq d_3$ , and thus  $|\{d_1, d_2, d_3, d_4\}| \geq 3$ .

Now assume that  $|\{d_1, d_2, d_3, d_4\}| = 3$ , i.e.,  $d_1v_1 \in E$  and  $d_1v_4 \in E$ ,  $d_2v_3 \notin E$  and  $d_3v_2 \notin E$ . Recall  $d_1x_1 \in E$  and  $d_1x_3 \in E$ . Thus,  $d_2x_1 \notin E$ ,  $d_2x_3 \notin E$ ,  $d_3x_1 \notin E$ ,  $d_3x_3 \notin E$ .

If  $d_2x_2 \notin E$  then  $d_2, v_2, x_1, v_1, x_2, v_3$  induce a net in  $G$ , and if  $d_2x_2 \in E$  then  $d_3x_2 \notin E$  and thus,  $d_3, v_3, x_2, v_2, x_3, v_4$  induce a net in  $G$  which is a contradiction. Thus,  $d_1, d_2, d_3, d_4$  are pairwise distinct, and claim (6) is shown.  $\diamond$

If  $d_1x_1 \notin E$  and  $d_1x_4 \notin E$  then  $d_1, v_1, x_1, x_4, v_2, v_4$  induce a net in  $G$ , and correspondingly by symmetry, a similar statement can be made about  $d_i, x_{i-1}, x_i, i \neq 1$ . Thus, we can assume that for each  $i \in \{1, \dots, 4\}$ ,  $d_i$  sees at least one of  $x_{i-1}, x_i$ .

If  $d_1x_1 \in E$  and  $d_1x_4 \in E$  then clearly,  $d_2x_1 \notin E$  and  $d_4x_4 \notin E$  and thus, by the above, we can assume that  $d_2x_2 \in E$  and  $d_4x_3 \in E$  but now,  $d_2, x_2, v_3, x_3, d_3, d_4$  induce a net in  $G$ .

Thus, assume that  $d_1$  is adjacent to exactly one of  $x_1, x_4$ , say  $d_1x_1 \in E$  (which implies  $d_2x_1 \notin E$ ) and  $d_1x_4 \notin E$ . By symmetry, this holds for  $d_2, d_3, d_4$  as well, i.e.,  $d_2x_2 \in E$ ,  $d_3x_3 \in E$ , and  $d_4x_4 \in E$ . Then  $d_1, x_1, d_2, x_2, d_3, x_3$  induce a net in  $G$ .

Thus, when  $x_1, x_2, x_3, x_4$  induce a  $K_4$  in  $G$ , then  $G^2$  does not contain a  $C_4$  with vertices  $v_1, v_2, v_3, v_4$ .

Now suppose to the contrary that  $G^2$  contains  $C_k, k \geq 5$ , say with vertices  $v_1, \dots, v_k$  such that  $d_G(v_i, v_{i+1}) \leq 2$  and  $d_G(v_i, v_j) \geq 3, i, j \in \{1, \dots, k\}, |i - j| > 1$  (index arithmetic modulo  $k$ ). By Claim 3.1, we have  $d_G(v_i, v_{i+1}) = 2$  for each  $i \in \{1, \dots, k\}$ ; let  $x_i$  be a common neighbor of  $v_i, v_{i+1}$ . Again, by Claim 3.1, the auxiliary vertices  $x_1, \dots, x_k$  are pairwise distinct and  $x_i x_{i+1} \in E$  for each  $i \in \{1, \dots, k\}$ .

Let  $x_i, x_j, x_l$  induce a triangle in  $G$ . We first claim:

- (i) If  $j = i + 1$  but  $|i - l| \geq 2$  and  $|j - l| \geq 2$  then  $x_i, x_j, x_l, v_i, v_{j+1}, v_l$  induce a net in  $G$ .
- (ii) If  $|i - j| \geq 2, |i - l| \geq 2$ , and  $|j - l| \geq 2$  then  $x_i, x_j, x_l, v_i, v_j, v_l$  induce a net in  $G$ .

Since  $G$  is chordal, there is a p.e.o.  $\sigma$  of  $G$ , and without loss of generality, assume that  $x_1$  is the leftmost vertex of  $x_1, \dots, x_k$  in  $\sigma$ . Then  $x_2x_k \in E$  since the neighborhood of  $x_1$  in  $x_2, \dots, x_k$  is a clique.

First assume that  $k = 5$ , and in this case,  $x_2x_5 \in E$ . Since  $x_2, x_3, x_4, x_5$  do not induce a  $C_4$  in  $G$ , we have  $x_2x_4 \in E$  or  $x_3x_5 \in E$ ; without loss of generality, assume that  $x_2x_4 \in E$ . But then,  $x_2, x_4, x_5$  induce a triangle as in case (i) of the previous claim, which would lead to a net, which is a contradiction. Next assume that  $k = 6$ , and in this case,  $x_2x_6 \in E$ . Then for the cycle  $x_2, x_3, x_4, x_5, x_6$  (which is no  $C_5$  in  $G$ ), the same argument works as for  $k = 5$ . Analogously, for every  $k \geq 7$ , it can be reduced to the case  $k - 1$  as for  $k = 6$ .

Note that for  $k \geq 5$ , we do not need the existence of an e.d.s. in  $G$ .

Thus, Theorem 1 is shown.  $\square$

In a very similar way, we can show:



**Theorem 2.** *If  $G$  is an extended-gem-free chordal graph with e.d.s. then  $G^2$  is chordal.*

**Proof.** Let  $G = (V, E)$  be an extended-gem-free chordal graph and assume that  $G$  has an e.d.s.  $D$ . We first show that  $G^2$  is  $C_4$ -free:

Suppose to the contrary that  $G^2$  contains a  $C_4$ , say with vertices  $v_1, v_2, v_3, v_4$  such that  $d_G(v_i, v_{i+1}) \leq 2$  and  $d_G(v_i, v_{i+2}) \geq 3$ ,  $i \in \{1, 2, 3, 4\}$  (index arithmetic modulo 4). By Claim 3.1, we have  $d_G(v_i, v_{i+1}) = 2$  for each  $i \in \{1, 2, 3, 4\}$ ; let  $x_i$  be a common neighbor of  $v_i, v_{i+1}$ . By Claim 3.1,  $x_i \neq x_j$  for  $i \neq j$ . Since  $G$  is chordal,  $x_1, x_2, x_3, x_4$  either induce a diamond or  $K_4$  in  $G$ .

Assume first that  $x_1, x_2, x_3, x_4$  induce a diamond in  $G$ , say with  $x_1x_3 \in E$  and  $x_2x_4 \notin E$ . We claim:

$$D \cap \{x_1, x_2, x_3, x_4\} = \emptyset. \quad (7)$$

*Proof.* First suppose to the contrary that  $x_1 \in D$ . Then by the e.d.s. property, we have  $v_3, v_4, x_2, x_3, x_4 \notin D$ . Since  $v_3$  and  $v_4$  have to be dominated by  $D$ , let  $d_3 \in D$  with  $d_3v_3 \in E$  and  $d_4 \in D$  with  $d_4v_4 \in E$ . Clearly,  $d_3 \neq x_2, x_3$  and  $d_4 \neq x_3, x_4$ . By the e.d.s. property,  $d_3$  and  $d_4$  are nonadjacent to the neighbors  $v_1, v_2, x_2, x_3, x_4$  of  $x_1$ . Thus,  $d_3 \neq d_4$  since otherwise  $x_1, x_2, v_3, d_3, v_4, x_4$  would induce a  $C_6$  in the chordal graph  $G$ . This implies  $d_3v_4 \notin E$  but now,  $v_1, x_1, x_3, v_4, x_4, v_2, v_3, d_3$  induce an extended gem which is a contradiction. Thus,  $x_1 \notin D$  and correspondingly,  $x_3 \notin D$ . Now suppose to the contrary that  $x_2 \in D$ . Then by the e.d.s. property, we have  $v_1, v_4, x_1, x_3, x_4 \notin D$ . Since  $v_1$  and  $v_4$  have to be dominated by  $D$ , let  $d_1 \in D$  with  $d_1v_1 \in E$  and  $d_4 \in D$  with  $d_4v_4 \in E$ . Clearly,  $d_1 \neq x_1, x_4$  and  $d_4 \neq x_3, x_4$ . By the e.d.s. property,  $d_1$  and  $d_4$  are nonadjacent to the neighbors  $v_2, v_3, x_1, x_3$  of  $x_2$ . Thus,  $d_1v_4 \notin E$  since otherwise  $d_1, v_1, x_1, x_3, v_4$  would induce a  $C_5$  in the chordal graph  $G$ , and analogously,  $d_4v_1 \notin E$ . Now,  $d_1, v_1, x_1, v_2, x_2, v_3, x_3, v_4$  induce an extended gem which is a contradiction. Thus,  $x_2 \notin D$  and correspondingly,  $x_4 \notin D$ , and claim (7) is shown.  $\diamond$

Next we claim:

$$D \cap \{v_1, v_2, v_3, v_4\} = \emptyset. \quad (8)$$

*Proof.* Without loss of generality, suppose to the contrary that  $v_1 \in D$ . Then by the e.d.s. property, we have  $v_2, v_4, x_1, x_2, x_3, x_4 \notin D$ . Since  $v_2$  and  $v_4$  have to be dominated by  $D$ , let  $d_2 \in D$  with  $d_2v_2 \in E$  and  $d_4 \in D$  with  $d_4v_4 \in E$ . Since  $d_G(v_2, v_4) > 2$ , we have  $d_2 \neq d_4$ .

Moreover,  $d_2x_3 \notin E$  since otherwise,  $d_2, v_2, x_1, x_3$  induce a  $C_4$  in  $G$ . This implies  $d_2v_3 \notin E$  since otherwise,  $d_2, v_3, x_3, x_1, v_2$  induce a  $C_5$  in  $G$ . But now,  $v_1, x_1, x_3, v_4, x_4, v_3, v_2, d_2$  induce an extended gem which is a contradiction. Thus, claim (8) is shown, i.e.,  $D \cap \{v_1, v_2, v_3, v_4\} = \emptyset$ .  $\diamond$

Let  $d_i \in D$  be the  $D$ -neighbor of  $v_i$ ,  $i = 1, \dots, 4$ . By (7) and (8), we have  $d_i \neq v_j, x_j$ ,  $i, j \in \{1, 2, 3, 4\}$ . Next we claim that  $d_1, d_2, d_3, d_4$  are pairwise distinct:

$$|\{d_1, d_2, d_3, d_4\}| = 4. \quad (9)$$

*Proof.* Since  $d_G(v_1, v_3) > 2$  and  $d_G(v_2, v_4) > 2$ , we have  $d_1 \neq d_3$  and  $d_2 \neq d_4$ . Thus,  $|\{d_1, d_2, d_3, d_4\}| \geq 2$ .

If without loss of generality,  $d_1 = d_4$ , i.e.,  $d_1v_1 \in E$  and  $d_1v_4 \in E$  then, since  $d_1, v_1, x_1, x_3, v_4$  do not induce a  $C_5$  in  $G$ , we have  $d_1x_1 \in E$  or  $d_1x_3 \in E$ , and if without loss of generality,  $d_1x_1 \in E$  and  $d_1x_3 \notin E$  then  $d_1, x_1, x_3, v_4$  induce a  $C_4$  in  $G$ . Thus,  $d_1x_1 \in E$  and  $d_1x_3 \in E$ .

This shows that if  $d_1v_1 \in E$  and  $d_1v_4 \in E$  then  $d_2 \neq d_3$ , and thus  $|\{d_1, d_2, d_3, d_4\}| \geq 3$ .

Now assume that  $|\{d_1, d_2, d_3, d_4\}| = 3$ , i.e.,  $d_1v_1 \in E$  and  $d_1v_4 \in E$ ,  $d_2v_3 \notin E$  and  $d_3v_2 \notin E$ . Recall  $d_1x_1 \in E$  and  $d_1x_3 \in E$ . Thus,  $d_2x_1 \notin E$ ,  $d_2x_3 \notin E$ ,  $d_3x_1 \notin E$ ,  $d_3x_3 \notin E$ . Then  $v_1, x_1, x_3, v_4, d_1, v_2, d_2, v_3$  induce an extended gem which is a contradiction.

Thus,  $d_1, d_2, d_3, d_4$  are pairwise distinct, and claim (9) is shown.  $\diamond$

If  $d_1x_1 \notin E$  and  $d_1x_4 \notin E$  then, since  $d_1, v_1, x_1, x_2$  do not induce a  $C_4$  in  $G$ , we have  $d_1x_2 \notin E$ , and accordingly, since  $d_1, v_1, x_4, x_3$  do not induce a  $C_4$  in  $G$ , we have  $d_1x_3 \notin E$ , but now  $d_1, v_1, x_1, v_2, x_2, v_3, x_3, v_4$  induce an extended gem in  $G$  which is a contradiction.

Thus, we can assume that for each  $i \in \{1, \dots, 4\}$ ,  $d_i$  sees at least one of  $x_{i-1}, x_i$  (index arithmetic modulo 4).

If  $d_1x_1 \in E$  and  $d_1x_4 \in E$  then clearly,  $d_2x_1 \notin E$  and  $d_4x_4 \notin E$  and thus, by the above, we can assume that  $d_2x_2 \in E$  and  $d_4x_3 \in E$  but now,  $v_2, x_1, v_1, x_4, v_4, x_3, v_3, d_3$  induce an extended gem in  $G$ .

Thus, assume that  $d_1$  is adjacent to exactly one of  $x_1, x_4$ , say  $d_1x_1 \in E$  (which implies  $d_2x_1 \notin E$ ) and  $d_1x_4 \notin E$ . By symmetry, this holds for  $d_2, d_3, d_4$  as well, i.e.,  $d_2x_2 \in E$ ,  $d_3x_3 \in E$ , and  $d_4x_4 \in E$ . Then  $v_1, x_1, v_2, x_2, v_3, x_3, v_4, d_4$  induce an extended gem in  $G$ .

Now assume that  $x_1, x_2, x_3, x_4$  induce a  $K_4$  in  $G$ . The proof is very similar as above. Again we claim:

$$D \cap \{x_1, x_2, x_3, x_4\} = \emptyset. \quad (10)$$

*Proof.* By symmetry, suppose to the contrary that  $x_1 \in D$ . Then by the e.d.s. property, we have  $v_3, v_4, x_2, x_3, x_4 \notin D$ . Since  $v_3$  and  $v_4$  have to be dominated by  $D$ , let  $d_3 \in D$  with  $d_3v_3 \in E$  and  $d_4 \in D$  with  $d_4v_4 \in E$ . By the e.d.s. property,  $d_3$  and  $d_4$  are nonadjacent to the neighbors  $v_1, v_2, x_2, x_3, x_4$  of  $x_1$ . Thus,  $d_3 \neq d_4$  since otherwise  $x_2, v_3, d_3, v_4, x_4$  would induce a  $C_5$  in the chordal graph  $G$ . This implies  $d_3v_4 \notin E$  but now,  $v_2, x_2, x_4, v_1, x_1, v_3, d_3, v_4$  induce an extended gem in  $G$  which is a contradiction. Thus,  $x_1 \notin D$  and correspondingly,  $x_2, x_3, x_4 \notin D$ , and claim (10) is shown.  $\diamond$

Next we claim:

$$D \cap \{v_1, v_2, v_3, v_4\} = \emptyset. \quad (11)$$

*Proof.* Without loss of generality, suppose to the contrary that  $v_1 \in D$ . Then by the e.d.s. property, we have  $v_2, v_4, x_1, x_2, x_3, x_4 \notin D$ . Since  $v_2$  and  $v_4$  have to be dominated by  $D$ , let  $d_2 \in D$  with  $d_2v_2 \in E$  and  $d_4 \in D$  with  $d_4v_4 \in E$ . Since  $d_G(v_2, v_4) > 2$ , we have  $d_2 \neq d_4$ .

Moreover,  $d_2x_3 \notin E$  since otherwise,  $d_2, v_2, x_1, x_3$  induce a  $C_4$  in  $G$ . This implies  $d_2v_3 \notin E$  since otherwise,  $d_2, v_3, x_3, x_1, v_2$  induce a  $C_5$  in  $G$ .

Now,  $v_1, x_1, x_3, v_4, x_4, v_2, d_2, v_3$  induce an extended gem which is a contradiction. Thus,  $v_1 \notin D$ , and correspondingly  $v_2, v_3, v_4 \notin D$ , and claim (11) is shown.  $\diamond$

Again, let  $d_i \in D$  be the  $D$ -neighbor of  $v_i$ ,  $i = 1, \dots, 4$ . By (10) and (11), we have  $d_i \neq v_j, x_j$ ,  $i, j \in \{1, 2, 3, 4\}$ . Next we claim that  $d_1, d_2, d_3, d_4$  are pairwise distinct:

$$|\{d_1, d_2, d_3, d_4\}| = 4. \quad (12)$$

*Proof.* Since  $d_G(v_1, v_3) > 2$  and  $d_G(v_2, v_4) > 2$ , we have  $d_1 \neq d_3$  and  $d_2 \neq d_4$ . Thus,  $|\{d_1, d_2, d_3, d_4\}| \geq 2$ .

If without loss of generality,  $d_1v_1 \in E$  and  $d_1v_4 \in E$  then, since  $d_1, v_1, x_1, x_3, v_4$  do not induce a  $C_5$  in  $G$ , we have  $d_1x_1 \in E$  or  $d_1x_3 \in E$ , and if  $d_1x_1 \in E$  and  $d_1x_3 \notin E$  then  $d_1, x_1, x_3, v_4$  induce a  $C_4$  in  $G$ . Thus,  $d_1x_1 \in E$  and  $d_1x_3 \in E$ .

This shows that if  $d_1v_1 \in E$  and  $d_1v_4 \in E$  then  $d_2 \neq d_3$ , and thus  $|\{d_1, d_2, d_3, d_4\}| \geq 3$ .

Now assume that  $|\{d_1, d_2, d_3, d_4\}| = 3$ , i.e.,  $d_1v_1 \in E$  and  $d_1v_4 \in E$ ,  $d_2v_3 \notin E$  and  $d_3v_2 \notin E$ . Since  $d_1x_1 \in E$  and  $d_1x_3 \in E$ ,  $v_1, x_1, x_3, v_4, d_1, v_2, d_2, v_3$  induce an extended gem in  $G$  which is a contradiction. Thus,  $d_1, d_2, d_3, d_4$  are pairwise distinct, and claim (12) is shown.  $\diamond$

If  $d_1x_1 \notin E$  then, since  $d_1, v_1, x_1, x_2$  do not induce a  $C_4$  in  $G$ , we have  $d_1x_2 \notin E$ , and analogously,  $d_1x_3 \notin E$ . But now  $v_2, x_1, x_3, v_3, x_2, v_1, d_1, v_4$  induce an extended gem in  $G$  which is a contradiction. Thus,  $d_1x_1 \in E$  and by symmetry,  $d_1x_4 \in E$  but now, by the e.d.s. property,

$d_2x_1 \notin E$  and  $d_2x_4 \notin E$ , and since  $d_2, v_2, x_1, x_3$  do not induce a  $C_4$ , we have  $d_2x_3 \notin E$ . But now,  $v_1, x_1, x_3, v_4, x_4, v_2, d_2, v_3$  induce an extended gem in  $G$  which is a contradiction.

Thus, when  $x_1, x_2, x_3, x_4$  induce a diamond or  $K_4$  in  $G$ , then  $G^2$  does not contain a  $C_4$  with vertices  $v_1, v_2, v_3, v_4$ .

Now suppose to the contrary that  $G^2$  contains  $C_k$ ,  $k \geq 5$ , say with vertices  $v_1, \dots, v_k$  such that  $d_G(v_i, v_{i+1}) \leq 2$  and  $d_G(v_i, v_j) \geq 3$ ,  $i, j \in \{1, \dots, k\}$ ,  $|i - j| > 1$  (index arithmetic modulo  $k$ ). By Claim 3.1, we have  $d_G(v_i, v_{i+1}) = 2$  for each  $i \in \{1, \dots, k\}$ ; let  $x_i$  be a common neighbor of  $v_i, v_{i+1}$ . Again, by Claim 3.1, the auxiliary vertices  $x_1, \dots, x_k$  are pairwise distinct and  $x_i x_{i+1} \in E$  for each  $i \in \{1, \dots, k\}$ .

Clearly, since  $G$  is chordal, there is an edge  $x_i x_{i+2} \in E$ . We claim:

$$\text{If } x_i x_{i+2} \in E \text{ then } x_i, x_{i+1}, x_{i+2} \notin D \text{ and } v_{i+1}, v_{i+2} \notin D. \quad (13)$$

*Proof.* Without loss of generality, let  $x_1 x_3 \in E$ . If  $x_2 \in D$  then clearly,  $v_1 \notin D$  and  $x_k, x_1 \notin D$ ; let  $d_1 \in D$  be a new vertex with  $d_1 v_1 \in E$ . Clearly,  $d_1 \odot \{x_1, v_2, x_2, v_3, x_3, v_4\}$  but now,  $x_1, v_2, x_2, v_3, x_3, v_4, v_1, d_1$  induce an extended gem. Thus,  $x_2 \notin D$ .

If  $x_1 \in D$  then clearly,  $v_4 \notin D$  and  $x_3, x_4 \notin D$ ; let  $d_4 \in D$  be a new vertex with  $d_4 v_4 \in E$ . Clearly,  $d_4 \odot \{v_1, x_1, v_2, x_2, v_3, x_3\}$  but now,  $v_1, x_1, v_2, x_2, v_3, x_3, v_4, d_4$  induce an extended gem. Thus,  $x_1 \notin D$  and correspondingly,  $x_3 \notin D$  by symmetry.

If  $v_2 \in D$  then clearly,  $v_1 \notin D$  and  $x_k, x_1 \notin D$ ; let  $d_1 \in D$  be a new vertex with  $d_1 v_1 \in E$ . As before,  $d_1 \odot \{x_1, v_2, x_2, v_3, x_3, v_4\}$  but now,  $d_1, v_1, x_1, v_2, x_2, v_3, x_3, v_4$  induce an extended gem. Thus,  $v_2 \notin D$  and correspondingly,  $v_3 \notin D$  by symmetry which shows (13).  $\diamond$

Next we claim:

$$\text{If } x_i x_{i+2} \in E \text{ then } x_{i+2} x_{i+4} \notin E \text{ and } x_{i-2} x_i \notin E. \quad (14)$$

*Proof.* Without loss of generality, let  $x_1 x_3 \in E$  and suppose to the contrary that  $x_3 x_5 \in E$ . Then by (13), there are new vertices  $d_3, d_4, d_5 \in D$ ,  $d_3, d_4, d_5 \notin \{v_3, v_4, v_5, x_2, x_3, x_4, x_5\}$ , with  $d_3 v_3 \in E$ ,  $d_4 v_4 \in E$  and  $d_5 v_5 \in E$ . We first claim that  $d_3 \neq d_4$ :

Suppose to the contrary that  $d_3 = d_4$ . If  $x_2 x_4 \in E$  then, since  $d_3, v_3, x_2, x_4, v_4$  do not induce a chordless cycle, we have  $d_3 x_2 \in E$  and  $d_3 x_4 \in E$ , but now,  $v_3, x_2, x_4, v_4, d_3, v_2, v_5, d_5$  induce an extended gem. Thus, let  $x_2 x_4 \notin E$ .

Since  $v_2, x_1, x_3, v_3, x_2, v_1, x_4, v_5$  do not induce an extended gem, we have  $x_1 x_4 \in E$ . Since  $d_3, v_3, x_2, x_1, x_4, v_4$  do not induce a chordless cycle, we have  $d_3 x_2 \in E$ ,  $d_3 x_1 \in E$ , and  $d_3 x_4 \in E$ . Thus, by the e.d.s. property,  $d_5 x_1 \notin E$ ,  $d_5 x_4 \notin E$ , and thus,  $d_5 v_1 \notin E$  since  $d_5, v_1, x_1, x_4, v_5$  do not induce a  $C_5$ . But now,  $x_2, x_1, x_4, v_4, d_3, v_1, v_5, d_5$  induce an extended gem which is a contradiction. Thus,  $d_3 \neq d_4$  is shown.

By the e.d.s. property,  $d_3 x_3 \notin E$  or  $d_4 x_3 \notin E$ . Recall that  $x_3 x_5 \in E$  was supposed, and thus, say without loss of generality,  $d_4 x_3 \notin E$ . Then by the chordality of  $G$ ,  $d_4 x_2 \notin E$  and  $d_4 x_1 \notin E$ , and clearly,  $d_4 \odot \{v_1, v_2, v_3\}$  but now,  $v_1, x_1, v_2, x_2, v_3, x_3, v_4, d_4$  induce an extended gem. Thus, (14) is shown.  $\diamond$

For a  $C_5$  in  $G^2$ , fact (14) leads to a  $C_4$  in  $G$  induced by  $x_1, x_3, x_4, x_5$  if  $x_1 x_3 \in E$ . Thus, from now on, let  $k \geq 6$ . We claim:

$$\text{If } x_i x_{i+2} \in E \text{ then } x_{i+1} x_{i+3} \notin E \text{ and } x_{i-1} x_{i+1} \notin E. \quad (15)$$

*Proof.* Without loss of generality, let  $x_1 x_3 \in E$  and suppose to the contrary that  $x_2 x_4 \in E$ . Then by (14),  $x_3 x_5 \notin E$  and  $x_4 x_6 \notin E$  as well as  $x_1 x_{k-1} \notin E$  and  $x_2 x_k \notin E$ , and since  $G$  is chordal,  $x_3 x_6 \notin E$  and  $x_2 x_{k-1} \notin E$ .

Since  $v_2, x_2, v_3, x_3, v_4, x_4, x_5, v_6$  does not induce an extended gem, we have  $x_2x_5 \in E$ . For  $k = 6$  this contradicts the fact that  $x_2x_{k-1} \notin E$ , i.e.,  $x_2x_5 \notin E$ . Thus, from now on, let  $k \geq 7$ .

Since  $v_2, x_2, x_3, x_4, v_5, x_5, x_6, v_7$  do not induce an extended gem, we have  $x_2x_6 \in E$  (recall  $x_3x_5 \notin E$ ,  $x_3x_6 \notin E$  and  $x_4x_6 \notin E$ ). For  $k = 7$ , this implies that  $x_1, x_2, x_6, x_7$  induce a  $C_4$  which is a contradiction. Thus, let  $k \geq 8$  but now,  $x_2, v_3, x_3, v_4, x_4, v_5, x_6, v_6$  induce an extended gem. Thus, (15) is shown.  $\diamond$

Recall that  $k \geq 6$ ; without loss of generality, let  $x_1x_3 \in E$ . Then by (14) and (15), we have  $x_2x_4 \notin E$ ,  $x_kx_2 \notin E$ , and  $x_3x_5 \notin E$ ,  $x_{k-1}x_1 \notin E$ . Since  $G$  is chordal, we have  $x_2x_5 \notin E$ .

Since  $v_2, x_1, x_3, v_3, x_2, x_4, v_5, v_1$  do not induce an extended gem, we have  $x_1x_4 \in E$ .

Since  $x_2, x_1, x_4, v_4, x_3, x_5, v_6, v_1$  do not induce an extended gem, we have  $x_1x_5 \in E$  (which, for  $k = 6$  contradicts the fact that  $x_{k-1}x_1 \notin E$ ) but now,  $v_2, x_1, x_3, v_3, x_2, x_5, v_5, v_4$  induce an extended gem.

Thus, Theorem 2 is shown.  $\square$

In the case of net-free chordal graphs, Theorem 1 generalizes the corresponding result for AT-free chordal graphs (i.e., interval graphs—see e.g. [11]).

By [7], and since MWIS is solvable in linear time for chordal graphs [20], we obtain:

**Corollary 1.** *WED is solvable in time  $\mathcal{O}(n^3)$  for net-free chordal graphs and for extended-gem-free chordal graphs.*

Theorems 1 and 2 and the subsequent lemma imply further polynomial cases for WED:

**Lemma 3** ([8, 9]). *If WED is solvable in polynomial time for  $F$ -free graphs then WED is solvable in polynomial time for  $(P_2 + F)$ -free graphs.*

This clearly implies the corresponding fact for  $(P_1 + F)$ -free graphs.

Recall Lemma 1 for  $H \in \{2P_3, K_3 + P_3, 2K_3, \text{butterfly}, \text{extended butterfly}, \text{extended co-P}, \text{extended chair}, \text{double-gem}\}$ . Now we consider induced subgraphs  $H' = H - x$  of  $H$  which are the following:

- $H = 2P_3$ :  $H' \in \{P_2 + P_3, P_3 + 2P_1\}$
- $H = K_3 + P_3$ :  $H' \in \{P_2 + P_3, K_3 + P_2, K_3 + 2P_1\}$
- $H = 2K_3$ :  $H' = P_2 + K_3$
- $H = \text{butterfly}$ :  $H' \in \{2P_2, \text{paw}\}$
- $H = \text{extended butterfly}$ :  $H' \in \{K_3 + P_2, \text{co-P}\}$
- $H = \text{extended co-P}$ :  $H' \in \{K_3 + P_2, P_5, \text{paw} + P_1, \text{co-P}\}$
- $H = \text{extended chair}$ :  $H' \in \{K_3 + 2P_1, P_2 + P_3, \text{chair}, \text{co-P}\}$
- $H = \text{double-gem}$ :  $H' \in \{\text{co-P}, \text{gem}\}$

**Corollary 2.** *For every proper induced subgraph  $H'$  of any graph  $H \in \{2P_3, K_3 + P_3, 2K_3, \text{butterfly}, \text{extended butterfly}, \text{extended co-P}, \text{extended chair}, \text{double-gem}\}$ , WED is solvable in polynomial time for  $H'$ -free chordal graphs.*

**Proof.** By [4], the clique-width of co-chair-free chordal graphs is bounded, and by [22], the clique-width of gem-free chordal graphs is bounded. By Theorem 2, WED is solvable in polynomial time for chair-free chordal graphs since chair is an induced subgraph of extended gem, and similarly, for co- $P$ -free chordal graphs. By Lemma 3, WED is solvable in polynomial time for  $(K_3 + P_2)$ -free chordal graphs and since the clique-width of  $K_3$ -free chordal graphs is bounded. In all other cases, we can use Lemma 3 and the fact that WED is solvable in polynomial time (even in linear time) for  $P_5$ -free graphs (and thus also for  $2P_2$ -free graphs).  $\square$

## 4 WED for $S_{1,2,3}$ -Free Chordal Graphs - a Direct Approach

By Lemma 1, and since  $S_{1,1,4}$  as well as  $S_{1,3,3}$  contain  $2P_3$  as an induced subgraph, WED is NP-complete for  $S_{1,1,4}$ -free chordal as well as for  $S_{1,3,3}$ -free chordal graphs. In this section, we give a polynomial-time solution for WED on  $S_{1,2,3}$ -free chordal graphs by a direct approach.

This generalizes WED for  $S_{1,2,2}$ -free chordal graphs as well as for  $S_{1,1,3}$ -free chordal graphs ( $S_{1,2,2}$  and  $S_{1,1,3}$  are induced subgraphs of extended gem—see Figure 3 and recall Theorem 2) and for  $P_6$ -free chordal graphs (recall [5, 6]).

Throughout this section, let  $G = (V, E)$  be a prime  $S_{1,2,3}$ -free chordal graph; recall that WED for  $G$  can be reduced to prime graphs [8, 9, 12]. For any vertex  $v \in V$ , let

$$Z^+(v) := \{u \in V : N[v] \subset N[u]\}, \text{ and}$$

$$Z^-(v) := \{u \in V : N[u] \subset N[v]\}.$$

Let us say that a vertex  $v \in V$  is a *maximal vertex* of  $G$  if  $Z^+(v) = \emptyset$ . Clearly,  $G$  has at least one maximal vertex.

**Lemma 4.** *Let  $v \in V$  be a maximal vertex of  $G$ . Then a minimum (finite) weight e.d.s.  $D$  with  $v \in D$  (if  $D$  exists) can be computed in polynomial time.*

**Proof.** Assume that  $D$  is a (possible) e.d.s. of finite weight of  $G$  with  $v \in D$ . Recall that  $G$  is prime (and thus, connected); then, by excluding the trivial case in which  $V = \{v\}$ ,  $G$  is not a clique. As usual, let  $N_0 = \{v\}$  and let  $N_1, \dots, N_t$  (for some natural  $t$ ) denote the distance levels of  $v$  in  $G$ . Then  $N_0, N_1, \dots, N_t$  is a partition of  $V$ . Clearly, since  $v \in D$ ,  $(N_1 \cup N_2) \cap D = \emptyset$ . Since  $G$  is chordal, we have:

**Claim 4.1.** *For every  $i \in \{1, \dots, t\}$  and every vertex  $x \in N_i$ ,  $N(x) \cap N_{i-1}$  is a clique, and in particular,  $x$  contacts exactly one component of  $G[N_{i-1}]$ .*

**Claim 4.2.**

- (i) *For any vertex  $u_1 \in N_1$ , there is a vertex  $z_1 \in N_1$  with  $z_1 u_1 \notin E$ .*
- (ii) *For any vertex  $u_2 \in N_2$ , with neighbor  $u_1 \in N_1$ , there is a vertex  $z_1 \in N_1$  with  $z_1 u_1 \notin E$  and  $z_1 u_2 \notin E$ .*
- (iii) *For any vertex  $u_i \in N_i$ ,  $i \geq 2$ , there is a chordless path  $P_{u_i v}$  with at least four vertices including  $u_i$  and  $v$ .*

*Proof.* Statement (i) holds since  $v$  is a maximal vertex of  $G$  and since the prime graph  $G$  is not a clique. Statement (ii) holds by (i) and since  $G$  is chordal. If  $i \geq 3$  then statement (iii) trivially holds by construction. If  $i = 2$  then it easily follows by (i) and (ii).  $\diamond$

**Claim 4.3.** *For any fixed  $i$ ,  $i \in \{2, \dots, t-1\}$ , let*

$X := \{x \in N_i : x \text{ has a neighbor in } D \cap N_{i+1}\}$ , let

$\mathcal{C}_X := \{Y_1, \dots, Y_q\}$  (for some natural  $q$ ) be the family of connected components of  $G[N_{i+1}]$  contacting  $X$ , and let

$X_i := \{x \in X : x \text{ contacts } Y_i\}$ ,  $i = 1, \dots, q$ .

Then the following statements hold:

- (i) For every  $x \in X$ ,  $x$  contacts exactly one of  $Y_1, \dots, Y_q$ , and thus, for  $i \neq j$ ,  $X_i \cap X_j = \emptyset$ , i.e.,  $X$  admits a partition  $\{X_1, \dots, X_q\}$  such that for  $h, k \in \{1, \dots, q\}$ ,  $k \neq h$ ,  $Y_h$  contacts  $X_h$  and does not contact  $X_k$ .
- (ii) For every  $h \in \{1, \dots, q\}$ ,  $|D \cap Y_h| = 1$ , say  $D \cap Y_h = \{d_h\}$ , and  $d_h$  dominates  $X_h \cup Y_h$ , i.e.,  $X_h \cup Y_h \subseteq N[d_h]$ .

*Proof.* (i): First we prove that for any  $x \in X$ ,  $x$  contacts exactly one of  $Y_1, \dots, Y_q$ : Without loss of generality, suppose to the contrary that  $x$  contacts  $Y_1$  and  $Y_2$ , and assume that the neighbor of  $x$  in  $D \cap N_{i+1}$ , say  $d$ , belongs to  $Y_1$ . Then let  $y$  be a neighbor of  $x$  in  $Y_2$ : By the e.d.s. property,  $y$  has a neighbor in  $D$ , say  $d'$ , with  $d' \neq d$ . Clearly, by the e.d.s. property and by definition of  $X$ , we have  $d' \notin X$  and  $xd' \notin E$  and thus, by Claim 4.1,  $d' \notin N_i$ .

Thus,  $d' \in N_{i+1} \cup N_{i+2}$ . Then  $d', y, d, x$ , and three further vertices of the path  $P_{xv}$  found by Claim 4.2 (iii) induce an  $S_{1,2,3}$ , which is a contradiction.

Thus, for  $i \neq j$ ,  $X_i \cap X_j = \emptyset$ , and (i) follows directly by the above and by definition of  $X$ ,  $X_i$  and  $\mathcal{C}_X$ .

(ii): First we prove that  $|D \cap Y_h| = 1$  (note that  $D \cap Y_h \neq \emptyset$ , by the proof of statement (i) of this claim: Suppose to the contrary that there are  $d, d' \in D \cap Y_h$ ,  $d \neq d'$ . Since  $G$  is connected and by definition of  $X$ , there are  $x \in X$  with  $xd \in E$  and  $x' \in X$  with  $x'd' \in E$ . By the e.d.s. property, the shortest path, say  $P$ , in  $Y_h$  from  $d$  to  $d'$  has at least two internal vertices, i.e., there exist  $a, b \in P$  with  $da \in E$  and  $bd' \in E$ . Since  $G$  is  $S_{1,2,3}$ -free, by Claim 4.2 (iii) and by the e.d.s. property,  $x$  is nonadjacent to all vertices of  $P \setminus \{d, a\}$ , while  $x'$  is nonadjacent to all vertices of  $P \setminus \{b, d'\}$ , which contradicts the fact that  $G$  is chordal. Thus,  $|D \cap Y_h| = 1$ ; let  $D \cap Y_h = \{d_h\}$ .

Next we claim that  $d_h$  dominates  $X_h$ : This follows by definition of  $X$ , by statement (i) of this claim, and by the e.d.s. property. By the way, by Claim 4.1,  $X_h$  is a clique.

Finally we claim that  $d_h$  dominates  $Y_h$ : Suppose to the contrary that there is a vertex  $y \in Y_h$  with  $yd_h \notin E$ . Since  $D \cap Y_h = \{d_h\}$ , we have  $y \notin D$ . Then there is  $d \in D$ ,  $d \neq d_h$ , with  $yd \in E$ . Let  $P'$  be a shortest path in  $Y_h$  between  $d_h$  and  $y$ , and let  $x \in X$  be adjacent to  $d_h$  (by the above,  $d_h$  dominates  $X_h$ ). Clearly, by the e.d.s. property,  $xd \notin E$ .

If  $xy \in E$  then by Claim 4.1,  $d \notin N_i$ , i.e.,  $d \in N_{i+1} \cup N_{i+2}$ ; then  $d, y, d_h, x$ , and three further vertices of the path  $P_{xv}$  found by Claim 4.2 (iii) induce an  $S_{1,2,3}$  which is a contradiction. Thus  $xy \notin E$ .

If  $d \in N_i$  then, by considering the (not necessarily induced) path formed by vertices  $x, d_h, P', y, d$ , we get a contradiction to the fact that  $G$  is chordal. Thus,  $d \in N_{i+1} \cup N_{i+2}$ . Then let  $y'$  be a neighbor of  $y$  in  $N_i$ ; clearly, by the e.d.s. property,  $y' \notin D$ .

Note that  $y'd_h \notin E$  (else  $d_h, y', y, d$ , and three further vertices of the path  $P_{y'v}$  found by Claim 4.2 (iii) induce an  $S_{1,2,3}$ ) and  $y'x \in E$ , else by considering the (not necessarily induced) path formed by  $x, d_h, P', y, y'$ , we get a contradiction since  $G$  is chordal.



Then there is  $d' \in D$  adjacent to  $y'$ . Clearly,  $d' \neq d_h$  by the above. Furthermore  $d' \neq d$ : Otherwise, if  $y'd \in E$  then  $d \in N_{i+1}$ , and then by considering the path between  $d_h$  and  $d$  in  $N_{i+1}$  (consisting of path  $P'$  in  $Y_h$  between  $d_h$  and  $y$  and additionally  $d$ ) we get a contradiction to the fact that  $G$  is chordal by an argument similar to the one above for showing that  $|D \cap Y_h| = 1$ .

If  $d' \in N_{i-1}$  then, since  $D \cap (N_1 \cup N_2) = \emptyset$ ,  $i \geq 4$ , and  $d_h, x, y, y'$ , and three further vertices of the path  $P_{y'v}$  found by Claim 4.2 (iii) containing  $d'$  induce an  $S_{1,2,3}$  which is a contradiction.

If  $d' \in N_i$  then  $i \geq 3$  since  $D \cap (N_1 \cup N_2) = \emptyset$ . Since  $G$  is chordal,  $y'$  and  $d'$  have a common neighbor in  $N_{i-1}$ , say  $z$ , and then  $zx \in E$  since otherwise  $d_h, x, y, y'$ , and three further vertices of the path  $P_{y'v}$  found by Claim 4.2 (iii) containing  $z$  induce an  $S_{1,2,3}$ . Now, since  $xz \in E$ , the vertices  $d', d_h, x, z$ , and three further vertices of the path  $P_{zv}$  found by Claim 4.2 (iii) induce an  $S_{1,2,3}$ , which is a contradiction.

Finally if  $d' \in N_{i+1}$  then  $d, y, d', y'$ , and three further vertices of the path  $P_{y'v}$  found by Claim 4.2 (iii) induce an  $S_{1,2,3}$ , which is a contradiction.

Thus, Claim 4.3 is shown.  $\diamond$

**Claim 4.4.** *For every component  $K$  of  $G[N_i]$ ,  $i \in \{3, \dots, t\}$ , we have*

- (i)  $|D \cap K| \leq 1$ , and
- (ii) if  $|D \cap K| = 1$ , say  $D \cap K = \{d\}$  then  $d$  dominates  $K$ .

*Proof.* (i): It can be proved similarly to the first paragraph of the proof of Claim 4.3 (ii).

(ii): It follows by Claim 4.3 (ii) since  $d$  (and thus  $K$ ) contacts a set of vertices of  $N_{i-1}$  which consequently have a neighbor in  $D \cap N_i$ .  $\diamond$

Now let us consider the problem of checking whether such an e.d.s.  $D$  of  $G$  with  $v \in D$  does exist. According to Claim 4.1, graph  $G$  can be viewed as a tree  $T$  rooted at  $\{v\}$ , whose nodes are the connected components of  $G[N_i]$  for  $i \in \{0, 1, \dots, t\}$  (recall  $N_0 := \{v\}$ ), such that two nodes are adjacent if and only if the corresponding connected components contact each other.

Then for any connected component  $K$  of  $G[N_i]$ ,  $i \in \{0, 1, \dots, t\}$ , let  $T(K)$  denote the vertex set of the induced subgraph of  $G$  corresponding to the subtree of  $T$  rooted at  $K$ . In particular  $N_0$  has a unique connected component (recall  $N_0 := \{v\}$ ), say  $K_0$ , so that  $T(K_0) = V$ .

According to Claim 4.4, let us say that a vertex  $d$  of  $G$  of finite weight, belonging to a connected component say  $K$  of  $G[N_i]$ ,  $i \in \{0, 1, \dots, t\}$ , is a *D-candidate* (or equivalently let us say that  $K$  admits a *D-candidate*  $d$ ) if

- (i)  $d$  dominates  $K$ , and
- (ii) there is an e.d.s. in  $G[T(K)]$  containing  $d$ .

**Claim 4.5.** *An e.d.s.  $D$  of  $G$  with  $v \in D$  does exist if and only if  $v$  is a D-candidate.*

*Proof.* It directly follows by the above.  $\diamond$

**Claim 4.6.** *Let  $K$  be a connected component of  $G[N_i]$ , for any fixed  $i \in \{1, \dots, t\}$ , and let  $d \in V(K)$  be a vertex of finite weight. Then let  $H_j := T(K) \cap N_j$  for  $i+1 \leq j \leq t$ , and let*

$$A := \{x \in H_{i+1} : xd \notin E\};$$

$$C_A = \{A'_1, \dots, A'_q\} \text{ be the family of connected components of } G[H_{i+2}] \text{ contacting } A;$$

$$B \text{ be the vertex set of connected components of } G[H_{i+2}] \text{ not contacting } A;$$

$\mathcal{C}_B = \{B'_1, \dots, B'_q\}$  be the family of connected components of  $G[H_{i+3}]$  contacting  $B$ .

Then the following statements hold:

- (i) If  $A = B = \emptyset$  then  $d$  is a  $D$ -candidate if and only if  $d$  dominates  $K$ .
- (ii) If  $A \neq \emptyset$  and  $B = \emptyset$  then  $d$  is a  $D$ -candidate if and only if  $d$  dominates  $K$ , Claim 4.3 (i) holds for  $A$  and for  $\mathcal{C}_A$ , and according to the notation of Claim 4.3,  $A$  admits a partition  $\{A_1, \dots, A_q\}$ , and each member  $A'_h$  of  $\mathcal{C}_A$  admits a  $D$ -candidate which dominates  $A_h \cup A'_h$  and does not contact  $N(d) \cap H_{i+1}$ .
- (iii) If  $A = \emptyset$  and  $B \neq \emptyset$  then  $B$  admits a partition  $\{B_1, \dots, B_q\}$ ,  $d$  is a  $D$ -candidate if and only if  $d$  dominates  $K$ , Claim 4.3 (i) holds for  $B$  and for  $\mathcal{C}_B$ , and according to the notation of Claim 4.3, each member  $B'_h$  of  $\mathcal{C}_B$  admits a  $D$ -candidate which dominates  $B_h \cup B'_h$ .
- (iv) If  $A \neq \emptyset$  and  $B \neq \emptyset$  then  $d$  is a  $D$ -candidate if and only if  $d$  dominates  $K$ , Claim 4.3 (i) holds for  $A$  and for  $\mathcal{C}_A$ , and according to the notation of Claim 4.3, each member  $A'_h$  of  $\mathcal{C}_A$  admits a  $D$ -candidate which dominates  $A_h \cup A'_h$  and does not contact  $N(d) \cap H_{i+1}$ , Claim 4.3 (i) holds for  $B$  and for  $\mathcal{C}_B$ , and according to the notation of Claim 4.3, each member  $B'_h$  of  $\mathcal{C}_B$  admits a  $D$ -candidate which dominates  $B_h \cup B'_h$ .

*Proof.* It follows by definition of  $D$ -candidate, by the e.d.s. property, by Claim 4.3, and by Claim 4.4; in particular by construction, each vertex of  $A$  contacts  $V(K) \setminus \{d\}$ , each vertex of  $B$  contacts  $N(d) \cap H_{i+1}$  and no member of  $\mathcal{C}_A$ , and then each member of  $\mathcal{C}_A$  contacts no member of  $\mathcal{C}_B$  by Claim 4.1.  $\diamond$

Then by Claims 4.5 and 4.6, one can check if e.d.s.  $D$  with  $v \in D$  does exist by the following procedure which can be executed in polynomial time:

**Procedure 4.1** ( $v$ -Maximal-WED).

**Input:** A maximal vertex  $v$  of  $G$ .

**Task:** A minimum weight e.d.s.  $D$  of  $G$  containing  $v$  (if it exists).

**begin**

Let  $N_0, N_1, \dots, N_t$  (for some natural  $t$ ), with  $N_0 = \{v\}$ , be the distance levels of  $v$  in  $G$ .

**for**  $i = t, t-1, \dots, 1, 0$  **do**

**begin**

for each component  $K$  of  $G[N_i]$ , detect all  $D$ -candidates in  $K$ , and for each  $D$ -candidate in  $K$ , say  $u$ , store (iteratively by the possible  $D$ -candidates in  $\mathcal{C}_A$  and in  $\mathcal{C}_B$ ) any minimum weight e.d.s. of  $G[T(K)]$  containing  $u$ ;

**end**

**if**  $v$  is a  $D$ -candidate **then** return “ $D$  does exist”

**else** return “ $D$  does not exist”.

**end**

This completes the proof of Lemma 4. □

**Theorem 3.** For  $S_{1,2,3}$ -free chordal graphs, WED is solvable in polynomial time.

**Proof.** Let us observe that, if all vertices of  $G$  are maximal, then by Lemma 4, the WED problem can be solved for  $G$  by computing a minimum finite weight e.d.s.  $D$  with  $v \in D$  (if  $D$  exists), for all  $v \in V$ .

Then let us focus on those vertices  $x$  which are not maximal, i.e., there is a vertex  $y$  with  $N[x] \subset N[y]$  (which means  $x \in Z^-(y)$ ). Thus, there is a maximal vertex  $v$  such that  $x \in Z^-(v)$ . In particular removing such maximal vertices  $v$  leads to new maximal vertices in the reduced graph. Recall that for any graph  $G = (V, E)$  and any e.d.s.  $D$  of  $G$ ,  $|D \cap N[x]| = 1$  for every  $x \in V$ .

**Fact 1.** *Let  $v \in V$  be a maximal vertex of  $G$ , with  $Z^-(v) \neq \emptyset$ , and let  $x \in Z^-(v)$ . If  $G$  has an e.d.s., say  $D$ , then  $D \cap (N(v) \setminus N(x)) = \emptyset$ .*

*Define a reduced weighted graph  $G^*$  from  $G$  as follows:*

- (i) *For each vertex  $x \in Z^-(v)$ , assign weight  $\infty$  to all vertices in  $N(v) \setminus N(x)$ , and*
- (ii) *remove  $v$ , i.e.,  $V(G^*) = V \setminus \{v\}$  (and reduce  $G^*$  to its prime connected components; recall that WED can be reduced to prime graphs).*

*Then the problem of checking if  $G$  has a finite (minimum weight) e.d.s. not containing  $v$  can be reduced to that of checking if  $G^*$  has a finite (minimum weight) e.d.s.*

*Proof.* The reduction is correct by the e.d.s. property and by definition of  $Z^-(v)$ . Moreover, by the e.d.s. property, by definition of  $Z^-(v)$  and by construction of  $G^*$ , every (possible) e.d.s. of finite weight of  $G^*$  contains exactly one vertex which is a neighbor of  $v$  in  $G$  since  $|D \cap N[x]| = 1$  for a vertex  $x \in Z^-(v)$ .  $\diamond$

Since the above holds in a hereditary way for any subgraph of  $G$ , and since WED for any graph  $H$  can be reduced to the same problem for the connected components of  $H$ , let us introduce a possible algorithm to solve WED for  $G$  in polynomial time.

**Algorithm 4.1** (WED- $S_{1,2,3}$ -Free-Chordal-Graphs).

**Input:** Graph  $G = (V, E)$ .

**Task:** A minimum (finite) weight e.d.s. of  $G$  (if it exists).

**begin**

Set  $W := \emptyset$ ;

**while**  $V \neq W$  **do**

**begin**

*take any maximal vertex of  $G$ , say  $v \in V$ , and set  $W := W \cup \{v\}$ ;*

*compute a minimum (finite) weight e.d.s. containing  $v$  in the connected component of  $G[V]$  with  $v$  (if it exists) {by Lemma 4 and Procedure 4.1};*

**if**  $Z^-(v) \neq \emptyset$  **then** {by Fact 1}

**begin**

**for each** vertex  $x \in Z^-(v)$ , assign weight  $\infty$  to all vertices in  $N(v) \setminus N(x)$ ;

remove  $v$  from  $V$ , i.e., set  $V := V \setminus \{v\}$

**end**

**end**

*if there exist some e.d.s. of finite weight of  $G$  (in particular, for each resulting set of e.d.s. candidates, check whether this is an e.d.s. of  $G$ ) then choose one of minimum weight, and return it else return “ $G$  has no e.d.s.”*

**end**

The correctness and the polynomial time bound of the algorithm is a consequence of the arguments above and in particular of Lemma 4 and Fact 1. This completes the proof of Theorem 3.  $\square$

It is still an open question how to generalize this approach. For example, the complexity of WED remains an open problem for  $S_{2,2,3}$ -free chordal as well as for  $S_{2,2,2}$ -free chordal graphs. However, for trees and forests  $T$ , there are only finitely many cases for the complexity of WED on  $T$ -free chordal graphs since WED on  $T$ -free chordal graphs is  $\text{NP}$ -complete if  $T$  contains an induced  $K_{1,5}$  or  $2P_3$ . In Figure 4, the maximum tree without induced  $K_{1,5}$  and  $2P_3$  is shown.

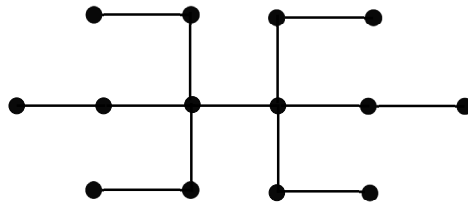


Figure 4: The maximum tree  $T$  for which the complexity of ED for  $T$ -free chordal graphs is open.

## 5 Conclusion

The results described in Theorems 1, 2, and 3 are still far away from a dichotomy for the complexity of ED on  $H$ -free chordal graphs. For chordal graphs  $H$  with four vertices, all cases are solvable in polynomial time as described in Lemma 5 below.

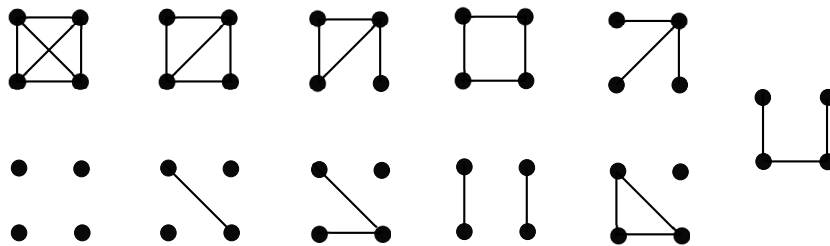


Figure 5: All graphs  $H$  with four vertices

For chordal graphs  $H$  with five vertices, the complexity of ED on  $H$ -free chordal graphs is still open for the following graphs as described in Lemma 5:

**Lemma 5.**

- (i) For every chordal graph  $H$  with exactly four vertices, WED is solvable in polynomial time for  $H$ -free chordal graphs.



Figure 6: Graphs  $H_1, \dots, H_4$  with five vertices for which ED is open for  $H$ -free chordal graphs

(ii) For every chordal graph  $H$  with exactly five vertices, the four cases described in Figure 6 are the only ones for which the complexity of WED is open for  $H$ -free chordal graphs.

**Proof.** (i): It is well known (see [4]) that for  $H \in \{K_4, K_4 - e, paw, P_4\}$ , the clique-width is bounded for  $H$ -free chordal graphs and thus, WED is solvable in polynomial time. By Theorem 2 as well as by Theorem 3, WED is solvable in polynomial time for claw-free chordal graphs.

By Lemma 3, WED is solvable in polynomial time for all other graphs  $H$  with four vertices (see Figure 5 for all such graphs; clearly,  $C_4$  is excluded).

(ii): For graphs  $H$  with five vertices, let  $v$  be one of its vertices. We consider the following cases for  $N(v)$  (and clearly exclude the cases when  $H$  is not chordal):

**Case 1.**  $|N(v)| = 4$  (i.e.,  $v$  is universal in  $H$ ):

Clearly, if  $H[N(v)]$  is a  $2P_2$  then  $H$  is a butterfly and thus, WED is  $\mathbb{NP}$ -complete. If  $H[N(v)]$  is a  $K_4$ , or paw, or  $P_4$ , or  $K_3 + P_1$ , then the clique-width is bounded [4]; in particular, if  $H[N(v)]$  is a paw or  $K_3 + P_1$  then  $H$  is an induced subgraph of  $\overline{K_{1,3} + 2P_1}$ , and according to Theorem 1 of [4], the clique-width is bounded. If  $H[N(v)]$  is  $P_3 + P_1$  then it is a special case of Theorem 2, where it is shown that this case can be solved in polynomial time. The other cases correspond to graphs  $H_1, \dots, H_4$  of Figure 6 (by Theorem 1 of [4], their clique-width is unbounded).

**Case 2.**  $|N(v)| = 0$  (i.e.,  $v$  is isolated in  $H$ ): By Lemma 3, and by Lemma 5 (i), WED is solvable in polynomial time.

In particular, for the same reason, WED is solvable in polynomial time whenever  $H$  is not connected (since in that case, at least one connected component of  $H$  has at most two vertices). Thus, from now on, we can assume that  $H$  is connected.

**Case 3.**  $|N(v)| = 3$  (and thus,  $|\overline{N(v)}| = 1$ ):

If  $v$  has exactly one non-neighbor in  $K_4$  then  $H = H_4$ . If  $v$  has exactly one non-neighbor in  $K_{1,3}$  with midpoint  $w$ , namely one of degree 1, then  $H[N(w)] = P_3 + P_1$  according to Case 1 (a special case of Theorem 2).

If  $v$  has exactly one non-neighbor in a diamond, namely one of degree 2, or exactly one non-neighbor in a paw, namely one of degree 1, then  $H$  is an induced subgraph of  $\overline{K_{1,3} + 2P_1}$ . Moreover, if  $v$  has exactly one non-neighbor in a paw, namely one of degree 2, then  $H$  is a gem, and if  $v$  has exactly one non-neighbor in  $P_4$ , namely one of degree 1, then  $H$  is a co-chair. If  $v$  has exactly one non-neighbor in  $P_1 + P_3$ , namely one of degree 1, then  $H$  is a bull. In all these cases, the clique-width is bounded according to Theorem 1 of [4].

In the remaining cases,  $H$  is a chair or co- $P$ , and thus, WED is solvable in polynomial time.

**Case 4.**  $|N(v)| = 2$  (and thus,  $|\overline{N(v)}| = 2$ ):

In one of the cases, namely if  $v$  is adjacent to the two vertices with degree 1 and with degree 3 in a paw,  $H$  is a butterfly and thus, WED is  $\mathbb{NP}$ -complete.

If  $v$  has exactly two neighbors in  $K_4$  or if  $v$  is adjacent to degree 2 and degree 3 vertices in diamond or if  $v$  is adjacent to the two degree 2 vertices in a paw or if  $v$  is adjacent to the two degree 2 vertices (midpoints) in a  $P_4$ , then by Theorem 1 of [4], the clique-width is bounded.

If  $v$  is adjacent to the two vertices of degree 3 of a diamond then  $H = H_3$ . If  $v$  is adjacent to degree 2 vertex  $u$  and degree 3 vertex  $w$  in a paw then for the degree 3 vertex  $w$ ,  $H[N(w)] = P_3 + P_1$  as above. If  $v$  is adjacent to degree 1 and degree 3 vertices in a claw then  $H = H_2$ .

In all other cases,  $H$  is a  $P_5$ , chair or co- $P$ , and thus, WED is solvable in polynomial time (by Theorem 2 for co- $P$ -free chordal graphs, and by Theorems 2 and 3, for  $P_5$ -free chordal graphs, and for chair-free chordal graphs).

**Case 5.**  $|N(v)| = 1$  (and thus,  $|\overline{N(v)}| = 3$ ):

Now  $v$  is adjacent to exactly one vertex of  $V \setminus \{v\}$ .

If  $v$  is adjacent to a degree 3 vertex  $w$  of a diamond then  $H[N(w)] = P_3 + P_1$  as above. If  $v$  is adjacent to a degree 3 vertex of a paw then  $H = H_2$ . If  $v$  is adjacent to a degree 3 vertex of a claw then  $H = H_1$ .

If  $v$  is adjacent to one vertex of  $K_4$  or one vertex of the diamond of degree 2 (co-chair) or one vertex of a paw of degree 2 (bull) then by Theorem 1 of [4], the clique-width is bounded.

In all other cases,  $H$  is a  $P_5$ , chair or co- $P$ , and thus, WED is solvable in polynomial time as above.  $\square$

Of course there are many larger examples of graphs  $H$  for which ED is open for  $H$ -free chordal graphs. In general, one can restrict  $H$  by various conditions such as diameter (if the diameter of  $H$  is at least 6 then  $H$  contains an induced  $2P_3$ ) and size of connected components (if  $H$  has at least two connected components of size at least 3 then  $H$  contains an induced  $2P_3$ ,  $K_3 + P_3$ , or  $2K_3$ ). It would be nice to classify the open cases in a more detailed way.

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