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Abstract

We consider an integrated planning problem that combines production, inventory and inbound transportation decisions in a context where several suppliers each provide a subset of the components necessary for the production of a final product at a central plant. We provide a mixed integer programming formulation of the problem and propose several families of valid inequalities to strengthen the linear programming relaxation. We propose two new algorithms to separate the subtour elimination constraints for fractional solutions. The inequalities and separation procedures are used in a branch-and-cut algorithm. Computational experiments on a large set of generated test instances show that both the valid inequalities and the new separation procedures significantly improve the performance of the branch-and-cut algorithm.

Keywords: logistics, assembly routing problem, valid inequalities, subtour elimination constraints separation, branch-and-cut, integrated production and routing

1. Introduction

The literature on integrated planning in manufacturing industries highlights a significant potential for cost savings in the supply chain by combining production and transportation decisions (Viswanathan and Mathur 1997, Fumero and Vercellis 1999, Chen and Vairaktarakis 2005, Archetti and Speranza 2016). The problem of simultaneously planning the production at a plant and the outbound delivery routing is known in the literature as the production routing problem (PRP) (Archetti et al. 2011, Adulyasak et al. 2015). When the production plan at the plant is given and the decisions concern only the inventory and route planning, the problem is referred to as the inventory routing problem (IRP) (Andersson et al. 2010, Coelho et al. 2013). There exist many

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models and solution algorithms for these two problems. In contrast, few studies have considered the integration of production planning with inbound transportation for the collection of components from suppliers to assemble a final product.

When the assembly plant is responsible for organizing the inbound transportation of the various components, significant gains can be achieved by integrating production planning with inbound transportation (Carter and Ferrin 1996). Automotive industry examples are studied in Blumenfeld et al. (1987) and Florian et al. (2011) for US and German manufacturers. Fernie and Sparks (2004) indicate that in the retail industry the logistics system should be effectively integrated with the suppliers. More specifically, they highlight the need for the optimization and management of the entire supply chain of retailers to be a single entity to obtain cost reduction advantages and service enhancements. Closing the supply chain loop is another example where the collection of the end-of-life products should be coordinated with the disassembly planning (Guide and Van Wassenhove 2009).

We study the assembly routing problem (ARP) which considers a joint planning problem with a central plant that produces a final product to satisfy a dynamic but deterministic demand. The plant collects the necessary components from several suppliers, each providing a subset of the components. The plant coordinates the scheduling of the production as well as the routing decisions and shipment quantities from the suppliers. The aim is to minimize the total costs of production, inventory and routing subject to several types of capacity constraints. The planning is done over a finite and discrete time horizon. The quantities available at the suppliers are assumed to be known in advance. The plant has a limited capacity for the production and no backlogging or stockouts are allowed. Both the plant and the suppliers can carry inventory. The plant has separate and capacitated inbound and outbound storage areas for the incoming components from suppliers and for the final product, respectively. Each supplier has a global storage capacity for its own components. The plant manages a limited fleet of capacitated vehicles to handle the shipment of components from the suppliers to the plant. Similar to the basic variants of the IRP and PRP, we do not allow a supplier to be visited by more than one vehicle in a specific period (i.e., no split pickups).

Some studies in the literature consider the optimization of the inbound transportation and inventory decisions without taking the production planning at the central plant into account. Popken (1994) and Berman and Wang (2006) study a single-period inbound logistics problem. They consider a multicommodity network with the origin (suppliers), destination (plant), and transshipment terminal nodes. The origin-destination commodity flows are supposed to be optimally routed through this network using at most one terminal node. The cost function includes the transportation and pipeline inventory costs for all supplier-plant pairs. The optimization of the inventory decisions together with the explicit inbound vehicle routes through multiple planning periods is studied in Moin et al. (2011) and Mjirda et al. (2014). Considering the automotive parts supply chain, these studies investigate the case of a single assembly plant for which multiple suppliers each provide a distinct part type.

A number of studies investigate the coordination of the inbound vehicle routes with the production rate in a just-in-time (JIT) environment where no end-period inventory exists in the planning horizon. Vaidyanathan et al. (1999) and Satoglu and Sahin (2013) study the parts delivery to an assembly line with the objective of minimizing the material handling equipment requirements in a central warehouse. Qu et al. (1999) and Sindhuchao et al. (2005) study the joint replenishment of multiple items in an inbound material-collection system for a central warehouse under the assumption of an infinite planning horizon. Chuah and Yingling (2005), Ohlmann et al. (2007), Stacey et al. (2007) and Natarajarathinam et al. (2012) consider a JIT supply pickup problem for an automotive assembly plant to minimize the inventory and transportation costs. Jiang et al. (2010) study a similar problem taking the storage space limit into account. Yücel et al. (2013) consider the problem of transporting specimens from different sites to the central processing facility of a clinical testing company. Lamsal et al. (2016) study a sugarcane harvest logistics problem in Brazil that requires the continuous operation of the production mill. Therefore, the inbound flow of raw material should never terminate.

One observes that the ARP includes a lot-sizing substructure with additional inventory constraints together with the distribution routing decisions in each period. Similar to the ARP, an inventory substructure exists in the uncapacitated lot-sizing problem (LSP) with inventory bounds which is well-studied in the literature. This problem was first introduced by Love (1973). Atamtürk and Küçükyavuz (2008) propose an $O(n^2)$ dynamic programming algorithm. Van Den Heuvel and Wagelmans (2008) show that the problem is equivalent to the LSP with a remanufacturing option, the LSP with production time windows, and the LSP with cumulative capacities. Di Summa and

Wolsey (2010) consider a variable upper bound on the initial inventory and give valid inequalities and extended formulations to describe the convex hull. More recently, Hwang and van den Heuvel (2012) and Phouratsamay et al. (2018) study this problem and propose polynomial and pseudo-polynomial algorithms for different cost structures. Akbalik et al. (2015) study the multi-item LSP with stationary production capacity, time-dependent inventory bounds and concave costs as well as a global capacitated storage space for all the items. They show that the problem is NP-hard even when each item has stationary and identical production cost and capacity over periods. Also, other integrated problems such as the IRP (Archetti et al. 2007, Solyalı and Süral 2011, Avella et al. 2015), maritime IRP (Agra et al. 2013), and PRP (Archetti et al. 2011, Adulyasak et al. 2014) consider bounded inventory in the problem structure. Even though these integrated problems all show some similarities with respect to the inventory structure, they possess a very different lot sizing structure. More specifically, the IRP and PRP have a distribution structure, whereas the ARP is based on an assembly structure. Furthermore, another difference is that the ARP considered in this paper takes into account a given rate of supply at the suppliers.

To the best of our knowledge, there are two papers that studied a problem close to the one being addressed in this paper. A general case with multiple components and products is introduced by Hein and Almeder (2016). The authors consider two scenarios. In the first scenario, the plant is allowed to keep the components in stock while in the second scenario, which represents a JIT environment, the components that arrive at the plant must be used immediately in production. They examine both scenarios under the traditional sequential planning approach and under the integrated approach. In the sequential planning process, an LSP is solved first to obtain the production plan for the final product. Then, in the second step, they solve an IRP for the first scenario and one vehicle routing problem (VRP) for each period in the second scenario. The computational experiments are performed on randomly generated instances with either 4 suppliers, 8 components, 3 final products, and 5 periods or 6 suppliers, 12 components, 4 final products, and 10 periods. They report cost savings of up to 12% with the integrated planning approach compared to the classical sequential approach. According to this study, one may expect a higher potential for cost savings in the JIT scenario when applying the integrated approach. Because the authors did not consider the holding cost at the suppliers in their study, the integrated decision making is entirely focused on the costs associated with the plant. This is appropriate when the suppliers and the assembly plant are separate organizations and the assembly plant is not concerned with the inventory costs at the suppliers.

In the case where both the suppliers and the assembly plant belong to the same firm, one should ideally take into account the suppliers' inventory costs and capacities in the integrated decision making process. Chitsaz et al. (2019) study the case with multiple components and one final product but consider the inventory costs and storage capacity of the suppliers as well as a component storage area at the plant. They assume that every supplier provides a unique component. Consequently, a one-to-one relationship exists between the suppliers and components. The authors develop a three-phase decomposition-based matheuristic that iteratively solves different subproblems. They apply their algorithm not only to the ARP, but also to the IRP and the PRP with the same parameter setting. The computational experiments show that this algorithm returns high quality solutions for the ARP instances and outperforms existing heuristics on large-scale multi-vehicle instances of the IRP and PRP. The algorithm finds new best-known solutions to many standard test instances of these two problems.

We extend the model of Chitsaz et al. (2019) to consider the case where each supplier may provide a subset of the components necessary for the final product and some components can be obtained from more than one supplier. This is the first contribution of this paper. Second, we develop several new valid inequalities to strengthen the linear programming (LP) relaxation of the mixed integer programming formulation of the problem. Although several of the proposed inequalities are inspired from existing lot-sizing inequalities, a novelty is that some of the inequalities use the known supply instead of the known demand. Third, we present novel algorithms to efficiently separate the subtour elimination constraints for the LP solutions that contain fractional routes, which can be adapted for other vehicle routing problems with the same feature. The inequalities and separation procedures are used in a branch-and-cut algorithm (BC). We generate a large test bed consisting of small to large instances with diverse ranges for the number of suppliers, products and planning periods. Finally, we analyze the impact of each class of valid inequalities on the value of the LP relaxation and on the final solution. Our extensive computational experiments show that both the valid inequalities and the new separation procedures notably enhance the performance of the branch-and-cut algorithm.

The remainder of the paper is organized as follows. We formally define the ARP and express it mathematically in Section 2. Section 3 is devoted to the presentation of the inequalities and to the proof of their validity. In Section 4, we present the upper bound generation procedure. To separate the subtour elimination constraints for our multi-period VRP, we present two heuristic algorithms in Section 5. The generation of the test instances and computational experiments are presented in Section 6. Finally, Section 7 concludes the paper.

2. Problem Definition and Mathematical Formulation

We consider a many-to-one assembly system with n suppliers represented by the set $N = \{1, ..., n\}$. The planning horizon includes l discrete time periods forming the set $T = \{1, .., l\}$. To produce the final product, k distinct components, represented by the set $K = \{1, ..., k\}$, are required. We extend the basic ARP introduced in Chitsaz et al. (2019) by assuming that each supplier i may provide a subset of the components $K_i \subseteq K$, where $K = \bigcup_i K_i$. Moreover, each component k can be provided by a subset of suppliers $N_k \subseteq N$, where $N = \bigcup_k N_k$. We define the problem on a complete undirected graph with the node set $N^+ = N \cup \{0\}$, where 0 represents the plant, and the edge set $E = \{(i, j) : i, j \in N^+, i < j\}$. We let $K^+ = K \cup \{0\}$ represent the set of all items, where 0 represents the final product. The suppliers as well as the central plant each have a global storage area for the components and may have some component inventory at hand at the beginning of the planning horizon. Moreover, the central plant has a separate storage space for the final product. A fleet of m homogeneous vehicles, each with a capacity of Q, is available to transport the components from the suppliers to the plant.

The decisions to make include whether or not to produce the final product and the quantity to be produced at the plant in each period, the supplier visit schedule and order in each vehicle route, and the shipment quantities from the suppliers to the plant. The manufacturing plant needs to minimize the production, inventory and transportation costs simultaneously for the entire planning horizon. The complete list of notations is presented in Table 1.

A compact formulation for the ARP can be written as the following $\mathcal{M}_{ARP}$ model:

$$(\mathcal{M}_{ARP}) \ \min \sum_{t \in T} \Big(up_t + fy_t + \sum_{k \in K^+} h_{0k} I_{0kt} + \sum_{i \in N} \sum_{k \in K_i} h_{ik} I_{ikt} + \sum_{(i,j) \in E} c_{ij} x_{ijt} \Big) \tag{1}$$

s.t.

$$I_{00,t-1} + p_t = d_t + I_{00t} \quad \forall t \in T \tag{2}$$

Table 1: ARP notation list

Sets:	
N^+	Set of nodes, $N^+ = \{0, ..., n\}$, where 0 represents the plant, and $N = N^+ \setminus \{0\}$ represents the set of suppliers.
E	Set of edges, $E = \{(i,j) : i,j \in N^+, i < j\}$.
K	Set of components indexed by $k \in \{1, ..., \|K\|\}$. We let $K^+ = K \cup \{0\}$.
K_i	Set of available components at supplier $i \in N$, $K_i \subseteq K$.
N_k	Set of suppliers that provide component $k \in K$, $N_k \subseteq N$.
T	Set of time periods, indexed by $t \in T = \{1, .., l\}$.
$E(S)$	Set of edges $(i,j) \in E$ such that $i,j \in S$, where $S \subseteq N^+$ is a given set of nodes.
$\delta(S)$	Set of edges incident to a node set S, $\delta(S) = \{(i,j) \in E : i \in S, j \notin S$ or $i \notin S, j \in S\}$.
Decision variables:	
p_t	Production quantity in period t at the plant.
y_t	Equal to 1 if there is production at the plant in period t, 0 otherwise.
I_{ikt}	Inventory of component $k \in K_i$ at supplier $i \in N$ at the end of period t.
I_{0kt}	Inventory of component or final product $k \in K^+$ at the plant at the end of period t.
x_{ijt}	Number of times a vehicle traverses the edge $(i,j) \in E$ in period t.
z_{it}	Equal to 1 if node $i \in N$ is visited in period t, 0 otherwise.
z_{0t}	Number of vehicles dispatched from the plant in period t.
q_{ikt}	Shipment quantity of component $k \in K$ from node $i \in N_k$ to the plant in period t.
Parameters:	
f, u	Fixed setup and unit production costs, respectively.
h_{ik}	Unit holding cost of item k at the plant or at supplier $i \in N+$.
c_{ij}	Transportation cost between nodes i and j, $(i,j) \in E$.
m	Fleet size.
C, Q	Production and vehicle capacity, respectively.
s_{ikt}	Supply of component $k \in K$ at node $i \in N_k$ in period t.
$s_{ikt_1t_2}$	Cumulative supply of component $k \in K$ at node $i \in N_k$ from period t_1 to period t_2 (inclusive), $t_1, t_2 \in T, t_1 \leq t_2$.
b_k	Unit size of component $k \in K$.
d_t	Demand for the final product at the plant in period t.
$d_{t_1t_2}$	Cumulative demand for the final product at the plant from period t_1 to period t_2 (inclusive), $t_1, t_2 \in T, t_1 \leq t_2$.
L_i	Global inventory capacity at supplier $i \in N$ for the components $k \in K_i$.
L	Global inventory capacity at the plant for the components $k \in K$.
L_0	Inventory capacity at the plant for the final product.
I_{ik0}	Initial inventory of component $k \in K$ available at supplier $i \in N_k$.
I_{0k0}	Initial inventory of component or final product $k \in K^+$ available at the plant.

$$I_{0k,t-1} + \sum_{i \in N_k} q_{ikt} = p_t + I_{0kt} \quad \forall k \in K, \forall t \in T \tag{3}$$

$$I_{ik,t-1} + s_{ikt} = q_{ikt} + I_{ikt} \quad \forall i \in N, \forall k \in K_i, \forall t \in T \tag{4}$$

$$p_t \leq C y_t \quad \forall t \in T \tag{5}$$

$$I_{00t} \leq L_0 \quad \forall t \in T \tag{6}$$

$$\sum_{k \in K} b_k I_{0kt} \leq L \quad \forall t \in T \tag{7}$$

$$\sum_{k \in K_i} b_k I_{ikt} \leq L_i \quad \forall i \in N, \forall t \in T \tag{8}$$

$$z_{0t} \leq m \quad \forall t \in T \tag{9}$$

$$\sum_{k \in K_i} b_k q_{ikt} \leq Q z_{it} \quad \forall i \in N, \forall t \in T \tag{10}$$

$$\sum_{(j,j')\in\delta(i)} x_{jj't} = 2z_{it} \quad \forall i \in N^+, \forall t \in T \tag{11}$$

$$Q \sum_{(i,j)\in E(S)} x_{ijt} \leq \sum_{i\in S} \Big(Qz_{it} - \sum_{k\in K_i} b_k q_{ikt}\Big) \quad \forall S \subseteq N, |S| \geq 2, \forall t \in T \tag{12}$$

$$p_t \geq 0, y_t \in \{0,1\}, z_{0t} \in \mathbb{Z} \quad \forall t \in T \tag{13}$$

$$I_{0kt} \geq 0 \quad \forall k \in K^+, \forall t \in T \tag{14}$$

$$I_{ikt}, q_{ikt} \geq 0 \quad \forall i \in N, \forall k \in K_i, \forall t \in T \tag{15}$$

$$x_{ijt} \in \{0,1\} \quad \forall (i,j) \in E : i \neq 0, \forall t \in T \tag{16}$$

$$x_{0it} \in \{0,1,2\}, z_{it} \in \{0,1\} \quad \forall i \in N, \forall t \in T. \tag{17}$$

The objective function (1) minimizes the total production, setup, inventory, and transportation costs. The inventory costs include both component inventories at the suppliers and at the plant, as well as the final product at the plant. The set of constraints (2) ensures the final product inventory flow while constraints (3) do the same for each component at the plant. Constraints (4) guarantee the inventory flow balance for each component at each supplier. Constraints (5) force a setup at the plant in each period where production takes place. They also impose a maximum limit on the production quantity. Constraints (6) consider the storage capacity of the final product at the plant. Constraints (7) impose the shared storage capacity of the components at the plant. The shared storage capacity of components at each supplier is enforced by constraints (8). Constraints (9) impose the limit on the fleet size. Constraints (10) force a vehicle visit whenever components are shipped from a certain node to the plant. The total component shipment quantity from each supplier in each period will also be limited by the vehicle capacity. Constraints (11) are the degree constraints. Constraints (12) are the subtour elimination constraints (SEC). These constraints are the modified version of the VRP capacity-cuts (Toth and Vigo 2001, Iori et al. 2007). They require each route to be connected to the plant and the total shipments on each route to not exceed the vehicle capacity. There exists an exponential number of these constraints. They are referred to in the literature as generalized fractional subtour elimination constraints (GFSEC) (Adulyasak et al. 2014). Constraints (13)-(17) are domain constraints.

3. Strengthening the LP Relaxation Bound

We present valid inequalities to improve the LP relaxation of $\mathcal{M}_{ARP}$. Moreover, we present the links between these inequalities and related polyhedral studies in the literature. The polyhedral structure of the LSP and VRP has been researched extensively. Barany et al. (1984) give a complete linear description of the convex hull of the solutions for the uncapacitated LSP. Pochet (1988), Miller et al. (2000), and Atamtürk and Muñoz (2004) present inequalities for the capacitated LSP with unlimited storage capacity. Atamtürk and Küçükyavuz (2005) investigate the polyhedral structure of the lot-sizing problem with inventory bounds and fixed costs. The polyhedral study of multiechelon LSP with intermediate demands is given in Zhang et al. (2012). The uncapacitated LSP is a special case of fixed charge network design (Van Roy and Wolsey 1985). Gendron et al. (1999) and Küçükyavuz (2005) study polyhedral approaches for capacitated multicommodity network design and fixed-charge network flow problems, respectively. Chouman et al. (2016) present cut-set-based inequalities for multicommodity capacitated fixed-charge network design problems. Similarly, many polyhedral studies are presented in the literature for different variants of the VRP. Cornuejols and Harche (1993) and Ralphs et al. (2003) study the capacitated variant and Belenguer et al. (2000) investigate the split delivery VRP.

Three classes of valid inequalities are presented to improve the LP relaxation bound for the $\mathcal{M}_{ARP}$ model. The first class contains *(l,S,WW)*-type inequalities. The second one concerns the bounds on the variables. We present the proof of the propositions in Section 1 of the online supplementary material. The last class includes general inequalities for the ARP. Propositions 1, 2 and 7 present inequalities derived from the particular structure of the underlying LSP for each component k (Pochet and Wolsey 2006). These inequalities take advantage of the aggregated available inventory of each component k at the suppliers (that provide component k) and the production plant for each period $t \in T$.

3.1. *(l,S,WW)-type inequalities*

The *(l,S)* inequalities were introduced in Barany et al. (1984) and provide the convex hull of the single-item uncapacitated LSP. In the (l, S) inequalities, l refers to a period ($l \leq |T|$) where T is the number of periods, and S is a subset of periods $\{1, ..., l\}$ not necessarily connected ($S \subseteq \{1, ..., l\}$) such as periods $\{1, 3, 7\}$ when $l = 10$. For a numerical example, we refer to Pochet and Wolsey (2006), pp. 122-123. Although there is an exponential number of these constraints for a general cost structure, Pochet and Wolsey (1994) showed that under the Wagner-Whitin (WW) cost condition it is sufficient to consider only $O(l^2)$ inequalities to describe the convex hull of the single item uncapacitated lot-sizing problem which are referred to as (l, S, WW) inequalities. The WW non-speculative cost structure requires the sum of unit production and inventory costs in every period to be larger than or equal to the unit production cost in the next period. Therefore, when the unit production costs are the same for all periods, the WW cost condition holds because the inventory costs are nonnegative. We first present the known (l, S, WW) inequalities applied to the lot-sizing structure (2) and (5):

$$\sum_{e=t_1}^{t_2} p_e \leq I_{00t_2} + \sum_{e=t_1}^{t_2} d_{et_2} y_e \quad \forall t_1, t_2 \in T, t_1 \leq t_2. \tag{18}$$

These inequalities link the production and setup variables at the plant with the predetermined downstream demand in order to improve the LP relaxation lower bound. Next, we derive three new families of valid inequalities for the ARP. The new inequalities are inspired from the standard (l, S, WW) inequalities, but present some novelties. In Proposition 1, we develop new inequalities that link the production and setup variables at the plant with the known upstream supply. The structure of the proof (given in Section 1 of the online supplementary material) follows a similar structure as for the (l, S) inequalities (Pochet and Wolsey 2006), but with an inverted logic as it takes into account the known supply at the suppliers. Moreover, in Propositions 2 and 3 we propose new inequalities linking the shipment quantities and node visit variables with the given supply and demand, respectively. The novelty in the structure of these constraints is that, for a given period, the shipment variables are defined for each supplier-component combination, whereas the supplier visit variables are only related to the supplier. There is no setup-type constraint in the model that directly links each component shipment variable to its supplier visit variable. This is different from a traditional lot-sizing structure.

Proposition 1. *Inequalities*

$$\sum_{e=t_1}^{t_2} p_e \leq I_{0k,t_1-1} + \sum_{i \in N_k} I_{ik,t_1-1} + \sum_{e=t_1}^{t_2} \sum_{i \in N_k} s_{ikt_1e} y_e \quad \forall k \in K, \forall t_1, t_2 \in T, t_1 \leq t_2 \tag{19}$$

are valid for the $\mathcal{M}_{ARP}$.

Notice that although both inequalities (18) and (19) provide bounds on the total production quantities, the first set of inequalities considers the cumulative demand and the remaining product inventory at the last period (t_2) while the second set of inequalities takes the cumulative component supply and the available inventory at the beginning of the first period (t_1) into account.

Proposition 2. *Inequalities*

$$\sum_{e=t_1}^{t_2} q_{ike} \leq I_{ik,t_1-1} + \sum_{e=t_1}^{t_2} s_{ikt_1e} z_{ie} \quad \forall i \in N, \forall k \in K_i, \forall t_1, t_2 \in T, t_1 \leq t_2 \tag{20}$$

are valid for the $\mathcal{M}_{ARP}$.

Proposition 3. *Inequalities*

$$\sum_{e=t_1}^{t_2} \sum_{i \in N_k} q_{ike} \leq I_{00t_2} + I_{0kt_2} + \sum_{e=t_1}^{t_2} d_{et_2} \sum_{i \in N_k} z_{ie} \quad \forall k \in K, \forall t_1, t_2 \in T, t_1 \leq t_2 \tag{21}$$

are valid for the $\mathcal{M}_{ARP}$.

Both inequalities (20) and (21) provide bounds on the total shipment quantities. The first set of inequalities considers the cumulative component supply and the available inventory at the beginning of the first period (t_1) at each supplier while the second set of inequalities takes the cumulative demand and the remaining product and component inventory at the plant in the last period (t_2) into account.

3.2. Bounds on variables

The bounds we propose in this subsection are linked to the cut-set type inequalities. Atamtürk and Küçükyavuz (2005) observe that *(l,S)* inequalities may not cut off fractional LP extreme solutions for lot-sizing with inventory bounds and fixed costs if for the subset of periods S incoming or outgoing inventory is at capacity. They introduce cut-set type inequalities to enforce one production setup for a certain number of periods. We introduce inequalities that are both a generalization and an extension of the cut-set type inequalities. We generalize the cut-set type inequalities to provide integer lower bounds on the number of required production setups from period $e = 1$ to $t \in T$ (Proposition 4). We further extend these cut-set type inequalities to enforce integer lower bounds on the number of vehicles dispatched (Proposition 5), and supplier visits from period $e = 1$ to $t \in T$ (Propositions 6-7).

Let $\mathcal{Q}_{it}$ (measured in required space) be a parameter equal to the sum of cumulative supply of components and the initial inventory of the components at supplier i minus its available storage capacity, i.e.,

$$\mathcal{Q}_{it} = \sum_{k \in K_i} b_k(s_{ik1t} + I_{ik0}) - L_i.$$

Proposition 4. *Inequalities*

$$\left\lceil \frac{\max\left\{0, d_{1t} - I_{000}, (\sum_{k \in K} b_k I_{0k0} + \sum_{i \in N} \max\{0, \mathcal{Q}_{it}\} - L)/\sum_{k \in K} b_k\right\}}{\min\{C, \max_{e \in \{1,\ldots,t\}}\{d_e\} + L_0\}} \right\rceil \leq \sum_{e=1}^{t} y_e \quad \forall t \in T \quad (22)$$

are valid for $\mathcal{M}_{ARP}$.

Notice that $\sum_{k \in K} b_k$ in the last expression of the LHS of the inequalities (22) represents the total required space by the components which are required to produce one unit of the final product. Next, we present valid inequalities for the lower bound on the total number of necessary vehicles dispatched from period $e = 1$ to t.

Proposition 5. *Inequalities*

$$\left\lceil \frac{1}{Q} \max\left\{ \sum_{k \in K} b_k \max\{0, d_{1t} - I_{000} - I_{0k0}\}, \sum_{i \in N} \max\{0, \mathcal{Q}_{it}\} \right\} \right\rceil \leq \sum_{e=1}^{t} z_{0e} \quad \forall t \in T \qquad (23)$$

are valid for $\mathcal{M}_{ARP}$.

Next, we present valid inequalities for a lower bound on the total number of necessary node visits from period $e = 1$ to t in the following proposition.

Proposition 6. *Inequalities*

$$\left\lceil \frac{\max\{0, \mathcal{Q}_{it}\}}{\min\left\{Q, L_i + \max_{e \in \{1,\ldots,t\}}\{\sum_{k \in K_i} b_k s_{ike}\}, \sum_{k \in K_i} b_k(I_{ik0} + s_{ik1t})\right\}} \right\rceil \leq \sum_{e=1}^{t} z_{ie} \quad \forall i \in N, \forall t \in T \qquad (24)$$

are valid for $\mathcal{M}_{ARP}$.

At any supplier, when the initial inventories plus the cumulative supply of components in the first t periods exceed the storage capacity, inequalities (24) provide a lower bound on the number of required visits to that supplier during these periods. The cumulative shipments from the supplier in the first t periods is limited first by the vehicle capacity, second by the available storage plus the maximum total component supply in any of those periods, and third by the sum of the initial inventories and the total supply of all components during these periods.

Proposition 7. *Inequalities*

$$\left\lceil \frac{\max\{0, d_{1t} - I_{000} - I_{0k0}\}}{\min\left\{\frac{Q}{b_k}, \max_{i \in N_k}\{I_{ik0} + s_{ik1t}\}\right\}} \right\rceil \leq \sum_{e=1}^{t} \sum_{i \in N_k} z_{ie} \quad \forall k \in K, \forall t \in T \tag{25}$$

are valid for $\mathcal{M}_{ARP}$.

For the periods whose cumulative demand cannot be satisfied from the initial product inventory and in the case where the initial inventory of a given component is not sufficient for the production, inequalities (25) force visits to the nodes which supply that specific component. The cumulative shipments of a component from any of the associated suppliers in the first t periods is limited not only by the vehicle capacity but also by the maximum of the initial inventory of that component plus the total supply of the component from those suppliers in the same periods. It is possible to state inequalities (24)-(25) for the edge variables (x_{ijt}) instead of node visits (z_{it}). This leads to identical constraints due to the degree constraints (11).

3.3. General inequalities

Without the SECs (12) added a priori to the model (e.g., as in the case of a BC algorithm), it may happen that the plant would not be connected to the other visited nodes in certain periods. In these cases, the following inequalities impose a positive value on the number of dispatched vehicles and hence on the degree of the plant if any node is visited in the same period:

$$z_{it} \leq z_{0t} \quad \forall i \in N, \forall t \in T. \tag{26}$$

Another type of SEC is Dantzig-Fulkerson-Johnson (DFJ), which can be represented for the $\mathcal{M}_{ARP}$ as follows:

$$\sum_{(i,j) \in E(S)} x_{ijt} \leq \sum_{i \in S} z_{it} - z_{et} \quad \forall S \subseteq N, |S| \geq 2, \forall e \in S, \forall t \in T. \tag{27}$$

DFJ inequalities are referred to in the literature as connectivity constraints (Laporte 1986), infeasible-path constraints (Ascheuer et al. 2000, Iori et al. 2007), or clique constraints (Bektaş and Gouveia 2014). They were first proposed by Dantzig et al. (1954) for the travelling salesman problem (TSP). These inequalities imply that the number of edges that can be chosen from the set of all edges with both endpoints in a subset of nodes S cannot be more than $|S| - 1$. The cardinality of these inequalities is exponential and thus they cannot be added a priori to the model in practical applications. Both GFSECs and DFJs can be added to the model at the same time. Observe that DFJs do not impose the vehicle capacity. Archetti et al. (2007) and Archetti et al. (2018) employ DFJ constraints for the IRP, and Archetti et al. (2011) and Adulyasak et al. (2014) use them for the PRP. The following inequalities enforce node visits for each edge traversal:

$$x_{ijt} \leq z_{it} \quad and \quad x_{ijt} \leq z_{jt} \quad \forall (i,j) \in E(N), \forall t \in T. \tag{28}$$

Inequalities (26) and (28) are used by Archetti et al. (2007) for the IRP, and by Archetti et al. (2011) and Adulyasak et al. (2014) for the PRP. Inequalities (28) are special cases of DFJs for node pairs (Gendreau et al. 1998), which can be added to the model a priori due to their polynomial cardinality.

4. Generating Upper Bounds

We adapted the unified matheuristic proposed in Chitsaz et al. (2019) and applied it to the generalized ARP, where each supplier provides a subset of the components, to obtain high quality feasible solutions as well as cutoff values that can be used to prune branches in our BC algorithm. This matheuristic (CCJ-DH) works by decomposing the problem into three separate subproblems and solving them iteratively. The first subproblem is a special LSP which determines a setup schedule with an approximation of the total transportation cost using the number of dispatched vehicles. The second subproblem returns node visits and shipment quantities. The latter employs another approximation of the total transportation cost using the node visit transportation cost. Finally, the third subproblem considers a separate VRP for each period t.

The solutions of the routing subproblems are used to update the node visit cost approximation in the second subproblem for the next iteration. This procedure is repeated to reach a local optimum. Then, a change in the setup schedule is imposed to explore other parts of the feasible solution space and diversify the search. The algorithm uses diversification constraints (Fischetti et al. 2004) to generate both new setup schedules using the first subproblem, and new node visit patterns using the second subproblem. The method terminates when a stopping condition is met. We present the detailed adaptation of CCJ-DH in Section 2 of the online supplementary material.

5. Separating Fractional Multi-Period Subtour Elimination Constraints

Subtour elimination constraints (12) belong to the family of capacity-cut constraints (CCC) which were developed for the capacitated VRP (Toth and Vigo 2001, Iori et al. 2007). The RHS of these constraints represents the number of vehicles required to serve the subset of nodes for which the inequality is applied. Depending on how the RHS is computed, different classes of this set of constraints can be obtained. The direct use of the fractional RHS results in the *fractional capacity inequalities*. This class of capacity constraints can be separated by solving a series of max-flow or min-cut problems in polynomial time (Semet et al. 2014). The next three classes of CCCs need specific algorithms and their separation is known to be NP-complete (Augerat 1995). When the RHS is rounded up, one obtains the *rounded capacity inequalities*. Using the optimal value of the bin-packing problem (where the weights of the items are equal to the shipment sizes and the bin capacity is equivalent to the vehicle capacity) in the RHS results in the *weak capacity inequalities*. Finally, computing the minimum number of required vehicles results in *global capacity constraints* and gives the tightest form.

Unlike the other types of CCCs, the quantities in the RHS of GFSECs are not given parameters but node visit (z_{it}) and shipment quantity (q_{ikt}) variables. For the non-vehicle index formulations of the IRP and the PRP, GFSECs are necessary to maintain the vehicle capacity of each route. To the best of our knowledge, there is no exact algorithm to separate GFSECs in polynomial time and it is not known whether separating GFSECs is NP-hard or not. Instead, a weak form of them (with $z_{it} = 1$) is usually separated using separation procedures designed for the TSP and VRP CCCs. Most of the BC algorithms in the IRP and the PRP literature use the separation procedure of Padberg and Rinaldi (1991) or heuristics that are included in the CVRPSEP package of Lysgaard et al. (2004). The procedures of Padberg and Rinaldi (1991) and Lysgaard et al. (2004) were originally developed for the TSP and the VRP, respectively. The algorithm of Padberg and Rinaldi (1991) is used by Archetti et al. (2007, 2011), Solyalı and Süral (2011), Avella et al. (2015) and Archetti et al. (2018). The CVRPSEP package is used by Adulyasak et al. (2014). If a violated inequality is found by one of these procedures, one has to check whether the corresponding GFSEC is violated or not (Solyalı and Süral 2011). In Section 3 of the online supplementary material, we present two examples for the LP solutions to the routing problem containing fractional values for the node visit (z_{it}) and edge traversal (x_{ijt}) variables. One example shows the case where a non-violated subtour elimination constraint is returned. The other example demonstrates the case where a violated subtour elimination constraint cannot be identified when the weak GFSEC is separated.

The separation problem for GFSECs in the ARP is to find a subset of nodes $S \subseteq N$ with cardinality greater than or equal to 2 ($|S| \geq 2$) for which the corresponding constraint is violated by the fractional solution. In each period t, the non-zero z^* and x^* values of the optimal LP solution form a subgraph $G^t(N^t, E^t)$. Each node in G^t has a shipment volume of $\sum_{k \in K_i} b_k q^*_{ikt}$. In order to define the separation problem, let the binary variable v_i be equal to 1 if and only if node $i \in N^t$ is selected and binary variable w_{ij} be equal to 1 if and only if edge $(i,j) \in E^t$ is chosen. We formulate the GFSECs separation problem for each period t as follows:

$$(\mathcal{S}^t_{GFSEC}) \quad \min \sum_{i \in N^t} (Qz^*_{it} - \sum_{k \in K_i} b_k q^*_{ikt}) v_i - Q \sum_{(i,j) \in E(N^t)} x^*_{ijt} w_{ij} \tag{29}$$

s.t.

$$\sum_{i \in N^t} v_i \geq 2 \tag{30}$$

$$w_{ij} \leq v_i \quad \forall (i,j) \in E^t \tag{31}$$

$$w_{ij} \leq v_j \quad \forall (i,j) \in E^t \tag{32}$$

$$v_i, w_{ij} \in \{0,1\} \quad \forall i \in N^t, \forall (i,j) \in E^t. \tag{33}$$

Since G^t is defined for $(i,j) \in E^t$, it may not be a complete subgraph nor a connected one. Observe that any feasible solution to this problem which has a strictly negative value returns one or more violated GFSECs. Notice that unlike the separation problem for the VRP CCCs, this problem is independent of the plant's (depot's) adjacent edges (x_{0it}). Moreover, the problem $\mathcal{S}^t_{GFSEC}$ is separable over the disconnected elements of the subgraph of period t, as was first implemented by Laporte et al. (1985) for the VRP under capacity and distance constraints.

To separate violated GFSECs with fractional node degrees, we propose two heuristics which can also be adapted for other vehicle routing problems. We define $e = (i_e, j_e) \in E^t$, the index of edges in the subgraph edge set of period t. We initialize sets $\Omega_1, ..., \Omega_{|E^t|}$ indexed by ϵ, and populate each Ω_ϵ with edge $\epsilon \in E^t$. We define $\Phi(\Omega_\epsilon)$ as the set of nodes corresponding to all the edges in Ω_ϵ. Let $\mathcal{C}_i = Qz^*_{it} - \sum_{k \in K_i} b_k q^*_{ikt}$ represent the node cost and $\mathcal{C}^e = Q \sum_{(i,j) \in E(N^t)} x^*_{ijt}$ the edge gain. The first algorithm (Algorithm $\mathcal{A}1$) finds violated GFSECs (for each period t) by adding to set Ω_ϵ the edge e which has the least marginal cost $(\mathcal{C}_{i_e} + \mathcal{C}_{j_e} - \mathcal{C}^e)$, not necessarily a negative cost, at each iteration. We only check for $e > \epsilon$ to force every initial set Ω_ϵ to deal with a different subset of edges. Otherwise, different sets eventually may end up with the same result. Notice that the last set, $\Omega_{|E^t|}$, will not examine other edges.

Algorithm 1: GFSEC Separation Procedure: $\mathcal{A}1$

Initialize $|E^t|$ sets Ω_ϵ, for all $\epsilon \in E^t$
for all $\epsilon \in \{1, ..., |E^t|\}$ **do**
    for all $e \in E^t \setminus \Omega_\epsilon, e > \epsilon$ **do**
        $e^* = \arg\min_e \{\mathcal{C}_{i_e} + \mathcal{C}_{j_e} - \mathcal{C}^e\}$
        $\Omega_\epsilon \leftarrow \Omega_\epsilon \cup \{e^*\}$
        if $\Phi(\Omega_\epsilon)$ introduces a violated GFSEC **and** $\Phi(\Omega_\epsilon)$ is not found yet **then**
            Add $\Phi(\Omega_\epsilon)$ to the list of violated GFSECs
        end if
    end for
end for
return the list of violated GFSECs

The second algorithm (Algorithm $\mathcal{A}2$) has a similar structure as $\mathcal{A}1$ with the difference that it terminates the search procedure for each set Ω_ϵ when the set returns the first violated GFSEC and then proceeds to the next set. Moreover, Algorithm $\mathcal{A}2$ does not accept the node sets which have (node) overlap with the violated GFSECs found earlier in the current call of the algorithm. Because every violated GFSEC needs to have at least two nodes, there is an explicit upper bound of $|N^t|/2$ on the number of violated GFSECs that $\mathcal{A}2$ returns for each period t.

6. Computational Experiments

The experiments were performed on the Calcul Québec computing infrastructure with Intel Xeon X5650 @ 2.67 GHz processors and a memory limit of 25 GB. The BC procedure is implemented in C++ using the CPLEX 12.6 callable library. All experiments are performed in sequential form using one thread. The algorithm applies the valid inequalities at the root node and adds GFSECs and DFJs at each node of the search tree as cutting planes whenever they are violated by more than 0.1 unit. To separate GFSECs, we either use CVRPSEP, $\mathcal{A}1$ or $\mathcal{A}2$. When a violated GFSEC is found, the BC method also adds the corresponding DFJ. In our experiments we set a time limit of one hour both for the BC and for CCJ-DH. We run the BC experiments with and without the CCJ-DH cutoff values to measure the performance of both methods in providing upper bounds.

We introduce a diverse set of instances to better study and evaluate the performance of the BC. We present the test bed generation procedure for the ARP in Section 6.1. We analyze the performance of CCJ-DH on the new instances in Section 6.2. We report the sensitivity analysis of the effect of valid inequalities on the LP relaxation of the $\mathcal{M}_{ARP}$ model, and the performance of the BC in Section 6.3. The performance analysis of the BC with different separation procedures is presented in Section 6.4. In Section 4 of the online supplementary material, we report the performance of the BC on the existing large instances of Chitsaz et al. (2019) and compare our results with the two lower bounding methods presented in that paper.

6.1. ARP Tests Instances

Two out of three ARP data sets introduced in Chitsaz et al. (2019) include instances with 50 and 100 suppliers, all with 6 periods. Therefore, they are too large to be solved by our exact algorithm. Moreover, those instances only consider the case where every supplier provides a unique component. To cover the general case of the ARP presented in this paper, and to test the BC on different sizes of instances, we generated three new classes of instances. The first class includes instances where each supplier provides a unique component type. The second class represents the case where each supplier provides a subset of components. The third class corresponds to the situation in which one single component is offered by all suppliers. Each class includes data sets with five different planning horizons ranging from 4 to 12 periods with a step of two. For each planning horizon we consider eight different numbers of suppliers, increasing by steps of 3. For each combination of the number of planning periods and suppliers we randomly generated five instances. Overall, 600 instances are generated for three classes, five planning horizons, eight numbers of suppliers, and five instances per category. As a result, the test bed includes small to large size instances. The rest of the specifications for the ARP instances are developed similar to the practices of Archetti et al. (2011) for the PRP. Table 2 presents an overview of the ARP instance parameters.

6.2. Performance of the Heuristic

Table 3 shows the performance of the adapted CCJ-DH on different classes of the new ARP instances compared to the BC when using the best-bound node selection strategy and algorithm $\mathcal{A}1$ for separating fractional subtours, and with the imposed time limit of one hour. The second

Table 2: ARP test instances*

Class	1	2	3
Number of instances	200	200	200
Number of periods: l		4 to 12	
Number of suppliers: n (for $l = 4$)		18 to 39	
Number of suppliers: n (for $l = 6$)		15 to 36	
Number of suppliers: n (for $l = 8$)		12 to 33	
Number of suppliers: n (for $l = 10$)		9 to 30	
Number of suppliers: n (for $l = 12$)		6 to 27	
Number of components: k	n	$0.4n$	1
Number of vehicles: m		UL‡	
Vehicle capacity: Q		$2\max_i L_i$	
Demand (final product): $d_t = d$		Constant and UDRI$^{\dagger\dagger}[50, 100]$	
Production capacity: C		UDRI$^{\dagger\dagger}[d, 3d]$	
Component supply: $s_{ikt} = s_{ik}$		Constant and UDRI$^{\dagger\dagger}[5, 0.5d]$	
Component size: b_k		UDRI$^{\dagger\dagger}[1, 2]$	
Plant inventory capacity for final product: L_0		UDRI$^{\dagger\dagger}[2d, 3d]$	
Plant inventory capacity for components: L		$\sum_{i\in N} L_i$	
Supplier inventory capacity: L_i		$\sum_{k\in K_i} b_k(I_{ik0} + 2s_{ik})$	
Plant initial inventory of final product: I_{000}		UDRI$^{\dagger\dagger}[0, 1.5d]$	
Plant initial inventory of components: I_{0k0}		UDRI$^{\dagger\dagger}[I_k^{*\dagger}, I_k^{*\dagger} + 0.5d]$	
Supplier initial inventory: I_{ik0}		UDRI$^{\ddagger\dagger}[0, d]$	
Unit production cost: u		$h_{00}/5$	
Production setup cost: f		$150u$	
Plant unit final product holding cost: h_{00}		UDRI$^{\dagger\dagger}[\sum_{k\in K} h_{0k}, 1.5\sum_{k\in K} h_{0k}]$	
Plant unit component holding cost: h_{0k}		$\max_i h_{ik}$	
Supplier unit holding cost for each component: h_{ik}		UDRI$^{\dagger\dagger}[1, 5]$	
Supplier and plant x,y coordinates		UDRI$^{\dagger\dagger}[0, 1000]$	
Travel distance		SA‡†	
Unit transportation cost		1	

* Adapted from Chitsaz et al. (2019)

† $I_k^* = \max\{0, l(d - \sum_{i\in N_k} s_{ik}) - I_{000}\}$, ‡ Unlimited, †† Uniformly Distributed Random Integer,
‡† Similar to Archetti et al. (2011)

column in this table presents the number of instances (#). The rest of the columns show the number of best upper bounds (#BUB) found by CCJ-DH, the average solution time (CPU), and the gaps of the heuristic solution with respect to the upper bound (Gap UB) and lower bound (Gap LB) obtained by the BC, respectively. The results highlight the fact that the instances of the second class need significantly more computing time. In these instances, each supplier provides multiple components. There are consequently more shipment variables (q_{ikt}), which results in a larger lot-sizing part compared to the instances in the two other classes. For the instances that are not solved to optimality by BC (larger instances), the matheuristic finds 122 best upper bounds (BUB) out of 161 instances (all classes). For these instances, CCJ-DH is able to improve the UBs found by the BC by 59%, 62.2% and 15.5% on average for the instances in the first, second and third class, respectively. For the instances solved to optimality, the heuristic provides high quality solutions within 1.2%, 1.2% and 1.6% of the optimal solution for the first, second and third class, respectively.

Table 3: Summary of the CCJ-DH results

Data Set	#	#BUB	CPU	Gap UB† (%)	Gap LB‡ (%)
Class 1					
Not Optimal	51	43	248.9	-59.04	2.74
Optimal	149	1	119.6	1.19	1.19
Total	200	44	152.6	-14.17	1.59
Class 2					
Not Optimal	81	66	2963.1	-62.24	3.62
Optimal	119	4	1786.3	1.22	1.22
Total	200	70	2262.9	-24.48	2.2
Class 3					
Not Optimal	29	13	90.8	-15.54	2.86
Optimal	171	5	44.1	1.55	1.55
Total	200	18	50.9	-0.93	1.74

† Gap UB = (UB_{CCJ-DH} - UB_{BC}) / UB_{BC}
‡ Gap LB = (UB_{CCJ-DH} - LB_{BC}) / LB_{BC}

6.3. Analysis of Valid Inequalities

To evaluate the effect of applying valid inequalities, we solve the LP relaxation of the $\mathcal{M}_{ARP}$ model where the SECs (12) are relaxed. We present in Table 4 the average LP solution times and values when no valid inequality is added to the model (None), and compare it with the cases where known valid inequalities (Known) from the literature (i.e., (18), (26)-(27)), or all valid inequalities (All) (i.e., (18)-(27)) are added to the model. Each row in this table shows the results for a period-supplier size combination. For the ease of comparison, the LP solution values are presented as a percentage of the BUB (LP%) for each instance. The average LP solution values without the valid inequalities vary in the range 63% to 65.9% for different classes and this range increases to 70.8% to 76.9% when the known inequalities are added and further to 88.7% to 90.2% with all valid inequalities added to the model. This is a significant improvement which is obtained at the expense of longer LP solution times. The average CPU times grow by a factor of 34, 22 and 10 for the instances in the first, second and third class, respectively when comparing the formulation without the valid inequalities to the formulation with all inequalities. We present details on the average LP solution values with and without considering each valid inequality type in the model in Section 5 of the online supplementary material.

We also compare the effect of the valid inequalities on the BC performance. In Table 5, we report a summary of the results on the performance of the BC when the default or the best-bound node selection strategies are employed, and either no inequality (None), only known inequalities (Known) or all inequalities (All) are applied. In all of these experiments we used algorithm $\mathcal{A}1$ to

Table 4: Effect of valid inequalities on LP solution

	Class 1							Class 2							Class 3						
	Set	None		Known		All		Set	None		Known		All		Set	None		Known		All	
l/n	Size	CPU	LP%	CPU	LP%	CPU	LP%	Size	CPU	LP%	CPU	LP%	CPU	LP%	Size	CPU	LP%	CPU	LP%	CPU	LP%
4/18	5	0.004	60.4	0.016	69.9	0.022	86.6	5	0.01	71.9	0.02	82	0.042	92.8	5	0	68.1	0.01	70.9	0.012	92.5
4/21	5	0.01	57.2	0.028	70.3	0.03	86.3	5	0.012	69	0.026	77.2	0.066	89.7	5	0	66.5	0.01	68.9	0.012	90.6
4/24	5	0.004	56.5	0.032	68.9	0.038	86.3	5	0.016	64.6	0.044	78.9	0.074	91.3	5	0.002	64.7	0.01	68.5	0.018	92.9
4/27	5	0	59.1	0.034	70.4	0.05	86.6	5	0.02	66.7	0.066	81.5	0.122	92.9	5	0.006	65.3	0.018	68	0.028	94.3
4/30	5	0.01	62.1	0.066	76.6	0.058	91	5	0.034	68.7	0.12	80.9	0.196	92.6	5	0.008	67	0.022	71	0.018	93.9
4/33	5	0.004	61	0.084	73.7	0.076	89.7	5	0.04	69.4	0.12	80.7	0.246	92.3	5	0.008	64.6	0.022	68.9	0.03	92.9
4/36	5	0.01	61.2	0.094	72.5	0.1	87.9	5	0.052	65.6	0.194	77.8	0.294	91.7	5	0.002	61.5	0.03	67.8	0.032	92.3
4/39	5	0.008	53.9	0.112	64.2	0.13	83.3	5	0.074	55.2	0.362	70.6	0.478	88.4	5	0.006	46.1	0.034	53.9	0.078	88.7
6/15	5	0.01	67.5	0.014	79.5	0.044	92.4	5	0.012	72.9	0.032	82.4	0.092	92.7	5	0.002	70.4	0.016	74	0.016	92
6/18	5	0.002	65.8	0.032	74.2	0.056	89	5	0.012	63.1	0.046	77.9	0.148	90.6	5	0.006	69.3	0.024	73.2	0.03	89.9
6/21	5	0.008	56.4	0.068	72.4	0.106	87.4	5	0.026	73.1	0.104	79.5	0.258	90.9	5	0.01	63.6	0.03	69.6	0.034	88.2
6/24	5	0.006	60.3	0.05	74.3	0.114	90	5	0.034	72.8	0.152	84.2	0.434	93.2	5	0.004	65.9	0.028	68.8	0.032	88.4
6/27	5	0.006	63.5	0.078	76.4	0.154	91.3	5	0.056	56.7	0.262	76.1	0.428	89.7	5	0.006	67.3	0.032	71.9	0.042	91
6/30	5	0.01	60.5	0.15	74.7	0.194	89.8	5	0.09	59.8	0.29	73.7	0.75	90.3	5	0.01	60.9	0.05	67.3	0.046	90.5
6/33	5	0.016	55.9	0.176	69.7	0.264	88	5	0.116	59.4	0.566	76.7	1.1	90.7	5	0.01	65.5	0.072	69	0.056	87.1
6/36	5	0.014	54	0.154	74	0.31	89.7	5	0.206	53.8	0.72	75.6	0.952	91.8	5	0.01	60.3	0.074	70.2	0.088	89.3
8/12	5	0.01	69.7	0.01	79.3	0.062	91.7	5	0.008	73.7	0.034	84	0.082	92.1	5	0.002	73.4	0.016	74.9	0.018	91
8/15	5	0.002	68.9	0.016	79.5	0.092	91.5	5	0.018	71.1	0.064	83.5	0.256	92.6	5	0.008	65.8	0.024	72.7	0.032	89.3
8/18	5	0.01	64.6	0.032	79.3	0.118	92.2	5	0.02	76.4	0.09	82.9	0.386	92.2	5	0.002	71.5	0.038	76.3	0.044	89.8
8/21	5	0.01	62.7	0.078	75.5	0.228	88.4	5	0.042	63	0.154	78.2	0.692	90.2	5	0.006	67.7	0.038	71.1	0.044	87.9
8/24	5	0.012	65.4	0.18	77.7	0.33	90.4	5	0.048	58	0.262	73.4	0.98	88.7	5	0.012	63.5	0.058	68.1	0.076	85.3
8/27	5	0.012	66.6	0.206	80	0.408	91.2	5	0.082	52.3	0.37	71.1	0.88	90.1	5	0.01	71.5	0.044	74.7	0.064	89.3
8/30	5	0.018	61.3	0.166	74.5	0.34	89.7	5	0.178	60.6	0.604	79.4	2.258	91.9	5	0.016	70.6	0.096	74.8	0.108	88
8/33	5	0.022	63	0.36	74.4	0.614	86.9	5	0.242	63.8	1.17	79.6	3.008	91.9	5	0.01	65.4	0.122	73.3	0.17	87.4
10/9	5	0.008	67	0.012	83.1	0.074	93.5	5	0.006	69.6	0.014	80.1	0.074	90.6	5	0.004	66	0.016	73	0.032	88.8
10/12	5	0	67.3	0.02	78.8	0.126	92	5	0.01	62	0.038	74.1	0.204	88.3	5	0.004	64.2	0.022	70.6	0.044	85.8
10/15	5	0.002	64.5	0.066	79.4	0.21	90.7	5	0.02	60.8	0.096	77.2	0.454	90.1	5	0.006	67.3	0.036	73.8	0.068	87.4
10/18	5	0.012	68.2	0.124	80.8	0.37	90.8	5	0.034	51.8	0.158	70.4	1.136	90.6	5	0.008	63	0.048	67.9	0.092	84
10/21	5	0.014	67.3	0.132	80.7	0.52	91.7	5	0.056	65	0.252	79.2	2.104	90.8	5	0.008	65.7	0.086	67.7	0.126	85.6
10/24	5	0.016	64.2	0.238	77	0.682	89.9	5	0.136	59	0.502	74.5	3.27	90.8	5	0.01	65.8	0.078	70.3	0.158	86.1
10/27	5	0.02	64.6	0.298	74.9	0.886	87.8	5	0.174	62.2	0.906	77.6	5.01	89.8	5	0.018	67.7	0.108	72.1	0.178	87.1
10/30	5	0.026	62.8	0.382	74.4	1.094	88.2	5	0.278	52.6	1.296	66.2	8.244	82.5	5	0.018	66.3	0.142	72.5	0.244	86.9
12/6	5	0.004	71.2	0.01	83.3	0.058	93.1	5	0	70.5	0.01	79.7	0.042	89.3	5	0.002	70.4	0.01	74.3	0.032	88.2
12/9	5	0	63.8	0.014	76	0.126	88.5	5	0.008	68.7	0.022	77.7	0.108	89.8	5	0.004	69.5	0.024	74.1	0.098	87.6
12/12	5	0.006	61	0.028	78.4	0.244	91.1	5	0.01	65.2	0.046	76.1	0.474	89.3	5	0.008	67.6	0.042	72.2	0.106	85.8
12/15	5	0.008	66.2	0.106	82.4	0.452	93	5	0.03	55.3	0.144	74	1.352	90.6	5	0.008	68.7	0.052	71.8	0.09	84.5
12/18	5	0.012	68.6	0.216	80.7	0.726	91.6	5	0.048	52.4	0.284	72.4	2.012	85.4	5	0.01	65.7	0.066	70.8	0.134	86.2
12/21	5	0.016	63.9	0.274	74.5	1.096	87.9	5	0.084	52.5	0.42	62.5	4.228	82.9	5	0.012	65.2	0.1	70.4	0.196	85.8
12/24	5	0.022	66.2	0.262	79.5	1.474	90.6	5	0.124	56.5	0.688	73.6	5.074	88.2	5	0.012	66.3	0.138	72.4	0.212	87.4
12/27	5	0.03	56.8	0.334	77.7	1.474	91.1	5	0.188	54.6	0.702	73.1	9.02	89.6	5	0.02	60.1	0.17	69.7	0.286	86.9
Total	200	0.010	63.0	0.119	76.1	0.339	89.7	200	0.066	63.0	0.286	76.9	1.426	90.2	200	0.008	65.9	0.051	70.8	0.081	88.7

Note. l/n: Number of periods/Number of suppliers separate SECs (12) and (27). This table presents the number of optimal solutions (#Opt), CPU time, the average lower bound values as a percentage of the upper bound obtained by the BC without applying the CCJ-DH cutoffs (%UB) and as a percentage of the BUB (%BUB) for each BC scenario and each class. To calculate the BUB for each BC scenario, we considered the upper bounds obtained by either that BC scenario or CCJ-DH.

The results indicate that the BC returns better results, in terms of the number of optimal solutions, average solution time, and optimality gap, when all inequalities are applied and the best-bound node selection strategy is selected. The BC returns better %UB with the default node selection strategy on all classes of instances. This highlights the fact that without applying CCJ-DH cutoffs, the default node selection strategy performs better than the best-bound. By comparing %UB and %BUB for each node selection strategy and each class, one observes the effect of applying CCJ-DH cutoffs within the BC. The best-bound node selection strategy results in better average lower bounds and consequently better results for %BUB.

On the instances of the first class, applying all inequalities and the best-bound node selection strategy enables the BC to obtain 149 (out of 200) optimal solutions in an average of 1422 seconds compared to 52 optimal solutions when known inequalities are employed, and only 8 optimal solutions when no valid inequality is considered. On the harder instances of the second class, the BC finds 119 optimal solutions within the time limit when all inequalities are added to the model while it is able to find 64 optimal solutions with known inequalities and only 5 optimal solutions without the valid inequalities. The same difference in the performance of the BC exists on the instances of the third class where 171 optimal solutions are found with all valid inequalities compared to 107 optimal solutions with known inequalities, and 14 optimal solutions without the valid inequalities. Overall, compared to the cases with no or only known inequalities, using all inequalities in BC with both node selection strategies notably increases the number of optimal solutions and significantly improves the %UB and %BUB for all classes. These results show that our new valid inequalities make a substantial difference in the success of the BC.

The detailed results for the same scenarios of the BC are presented in Tables 6 and 7. Similarly, in all of these experiments we used algorithm $\mathcal{A}1$ to separate SECs (12) and (27). These tables present CPU, %UB, and %BUB for every period-supplier combination group of each instance class. The number of instances (out of five) that are not solved to optimality is specified in parentheses within the %BUB figures.

Table 5: Summary of the results of the BC with the default and the best-bound node selection strategies, and with and without the valid inequalities on different instance classes*

Node Selection	Valid Ineq.	Class 1 Size	#Opt	CPU	%UB	%BUB	Class 2 Size	#Opt	CPU	%UB	%BUB	Class 3 Size	#Opt	CPU	%UB	%BUB
Default	None	200	11	3157	69.6	96.7	200	5	3234	65.4	95.2	200	22	3045	79.6	95.9
	Known	200	51	2576	86.3	96.8	200	44	2729	83.9	95.2	200	107	1912	96.1	97.5
	All	200	103	1980	91.2	99	200	69	2420	85	97.9	200	155	1205	98.3	99.5
Best-Bound	None	200	8	3207	56.5	97.3	200	5	3260	36.9	96.3	200	14	3098	64.5	96.6
	Known	200	52	2578	57.3	97.3	200	64	2418	61.8	96.3	200	107	1872	89.8	98.1
	All	200	149	1422	84.7	99.4	200	119	1976	74.4	98.7	200	171	938	97.4	99.8

* Separation procedure used for all BC scenarios: algorithm $\mathcal{A}1$

Size: Number of instances, None: With no inequality, Known: With known inequalities (18), (26) and (27),

All: With all inequalities (18)-(27)

6.4. Analysis of Different Separation Procedures

In Table 8, we present the performance of the BC with all valid inequalities added when the CVRPSEP package, $\mathcal{A}1$ and $\mathcal{A}2$ are applied to separate SECs (12) and (27). We used the best-bound node selection strategy for all these experiments. In this table we report CPU, %BUB and the number of instances that are not solved to optimality (inside the parentheses) for each combination of the period-supplier setting. One observes that both of our separation procedures outperform the CVRPSEP package by enabling the BC to find more optimal solutions within the time limit. The results in this table suggest that the BC is capable of closing the optimality gap for many more period-supplier combinations in each class with a better solution time when it uses $\mathcal{A}1$ and $\mathcal{A}2$ compared to when it employs the CVRPSEP package. Furthermore, the BC with $\mathcal{A}2$ is performing better on larger instances compared to the case with $\mathcal{A}1$. This is why we use $\mathcal{A}2$ in our BC when we apply it to solve the large ARP instances of Chitsaz et al. (2019) presented in Section 4 of the online supplementary material. The BC is capable of solving instances with up to 4 periods and 33 nodes, 6 periods and 30 nodes, 8 periods and 27 nodes, 10 periods and 24 nodes, and 12 periods and 21 nodes within the time limit.

Moreover, in Table 9 we present more details on the BC performance. For each SEC separation procedure and for each class, this table shows #Opt, the average number of explored nodes in the search tree (#Node), the average number of added GFSECs (GFS), the average amount of violation for the added GFSECs (AV^{GFS}), the average number of added DFJs (DFJ), the average amount of violation for the added DFJs (AV^{DFJ}), and information about the number of cuts that are added automatically by CPLEX: cover cuts (Cover), flow cover cuts (Flow), clique cuts

Table 6: Detailed results of the BC with the best-bound node selection strategy, and with and without the valid inequalities*

Set	Class 1									Class 2									Class 3								
	CPU			%UB			%BUB			CPU			%UB			%BUB			CPU			%UB			%BUB		
l/n	None	Known	All	None	Known	All	None	Known	All	None	Known	All	None	Known	All	None	Known	All	None	Known	All	None	Known	All	None	Known	All
4/18	2778	1782	265	98.4	99.7	100	$99.1^{(4)}$	$99.7^{(1)}$	100	3296.8	2214.9	622.9	96.3	99.1	100	$98.1^{(5)}$	$99.1^{(2)}$	100	1483.2	54.9	27.8	99	100	100	$99^{(2)}$	100	100
4/21	3296	1558	317	77.7	98.4	100	$98.2^{(5)}$	$99.3^{(2)}$	100	3295.9	1582.5	892.5	95.8	99.8	100	$97.6^{(5)}$	$99.8^{(2)}$	100	2237.9	699	151.7	97.8	100	100	$98.2^{(3)}$	100	100
4/24	3294	3299	750	91.3	70.8	100	$97.8^{(5)}$	$97.6^{(5)}$	100	3292.2	3026.1	1156.4	77.5	96.9	100	$97.4^{(5)}$	$97.9^{(4)}$	100	3297.4	126.1	30.2	91.3	100	100	$98.7^{(5)}$	100	100
4/27	3295	2943	741	88.7	93.7	100	$97.7^{(5)}$	$98.5^{(3)}$	100	3297	2687.4	587.2	73.7	96.6	100	$97.4^{(5)}$	$97.2^{(4)}$	100	3292.8	346.4	72.7	77.4	100	100	$98.2^{(5)}$	100	100
4/30	3295	3295	812	76	57.4	100	$97.8^{(5)}$	$97.6^{(5)}$	100	3295.3	3298.1	1494.1	77.9	79.8	100	$97.1^{(5)}$	$95.7^{(5)}$	100	3291.5	2153.4	294.1	97.9	98.4	100	$98.2^{(5)}$	$99.4^{(2)}$	100
4/33	3295	3295	1374	80.6	46.8	99.8	$97.7^{(5)}$	$96.8^{(5)}$	$99.8^{(1)}$	3294.2	3297.9	2178.9	43.6	60.1	99.9	$97^{(5)}$	$94.5^{(5)}$	$99.9^{(1)}$	3294.8	1829	509.7	83	96.4	100	$97.2^{(5)}$	$99.2^{(2)}$	100
4/36	3292	3296	2663	39.5	32.3	75.6	$96.5^{(5)}$	$97^{(5)}$	$98.8^{(4)}$	3293.9	3296.2	2343.1	25.4	41.3	79.7	$95.9^{(5)}$	$94.4^{(5)}$	$99.1^{(3)}$	3293.2	3183.6	1229.3	59.8	79.4	99.1	$95.4^{(5)}$	$96.7^{(4)}$	$99.5^{(1)}$
4/39	3295	3292	2716	41	19	48.1	$95.3^{(5)}$	$95.4^{(5)}$	$97.5^{(4)}$	3294.8	3294.4	3293.7	13	33.2	65	$94.2^{(5)}$	$91.5^{(5)}$	$97.5^{(5)}$	3298.4	2793.9	1407.4	9.1	88.6	99.3	$93.8^{(5)}$	$95^{(4)}$	$99.3^{(1)}$
6/15	3296	996	450	97.9	99.9	100	$99.4^{(5)}$	$99.9^{(1)}$	100	3296.6	963.6	424.1	89.9	99.9	100	$97.5^{(5)}$	$99.9^{(1)}$	100	2955.4	1002.5	285.9	99.2	99.8	100	$99.2^{(4)}$	$99.8^{(1)}$	100
6/18	3295	2675	562	91.8	98.2	100	$98^{(5)}$	$98.9^{(3)}$	100	3296.4	2134.9	818.3	58.3	97.7	100	$97.2^{(5)}$	$98.1^{(3)}$	100	3293.6	467.3	100.9	96.5	100	100	$98.5^{(5)}$	100	100
6/21	3295	3290	830	65.6	44.2	100	$96.7^{(5)}$	$97^{(5)}$	100	3295.6	3297.7	1514.7	60	95.2	100	$96.8^{(5)}$	$96.4^{(5)}$	100	3295	1351.7	221.8	91.7	99.8	100	$97^{(5)}$	$99.8^{(2)}$	100
6/24	3297	3297	1050	41.7	57	100	$97.6^{(5)}$	$96.7^{(5)}$	100	3297.1	3296.3	2320.3	13.6	35.8	99.9	$96.7^{(5)}$	$94.4^{(5)}$	$99.9^{(1)}$	3294.4	1724.6	311.6	81.2	97.5	100	$96^{(5)}$	$97.5^{(2)}$	100
6/27	3296	3295	1092	62.9	36.9	100	$97.8^{(5)}$	$95.7^{(5)}$	100	3295.4	3295.3	2292.9	22.4	57.1	99.8	$96.6^{(5)}$	$95.2^{(5)}$	$99.8^{(1)}$	3296.2	1530.5	420.2	74.1	98.9	100	$95.8^{(5)}$	$99.7^{(1)}$	100
6/30	3293	3296	1639	40.9	39.7	99.5	$97.2^{(5)}$	$96.4^{(5)}$	$99.5^{(2)}$	3296.5	3296.4	2862	26	5.3	71.1	$95.5^{(5)}$	$92.4^{(5)}$	$97.7^{(4)}$	3297	2098.8	606.4	37.6	98.4	100	$95.4^{(5)}$	$98.8^{(2)}$	100
6/33	3296	3295	3297	45.4	32.4	46.3	$96.1^{(5)}$	$95.2^{(5)}$	$98.1^{(5)}$	3295.9	3295.9	3293.9	11.9	5.7	39	$94.3^{(5)}$	$90.8^{(5)}$	$97.1^{(5)}$	3294.7	3297.3	1948.7	34.4	91.4	99.8	$95.4^{(5)}$	$95.7^{(5)}$	$99.8^{(2)}$
6/36	3297	3294	3297	40.1	24.8	34.5	$94.2^{(5)}$	$93.2^{(5)}$	$96.7^{(5)}$	3292.6	3295.5	3296.8	29.7	29.2	24.2	$93.7^{(5)}$	$92.4^{(5)}$	$96.8^{(5)}$	3297.7	3295.1	2665.7	30.2	86.1	97.6	$94.6^{(5)}$	$93.5^{(5)}$	$98.6^{(4)}$
8/12	3295	117	78	79.5	100	100	$99.3^{(5)}$	100	100	3297.2	440.1	570	78.6	100	100	$98.3^{(5)}$	100	100	2683.7	692.8	663.7	99.6	100	100	$99.6^{(4)}$	100	100
8/15	3296	978	252	76.5	99.9	100	$98.8^{(5)}$	$99.9^{(1)}$	100	3293.8	893	1074.1	50.8	100	100	$97.5^{(5)}$	100	100	3294.7	187.5	272.1	97.8	100	100	$98.1^{(5)}$	100	100
8/18	3297	2712	962	66.6	88	100	$98.1^{(5)}$	$98.4^{(4)}$	100	3294.1	2886	1446.2	0	77.8	100	$97.2^{(5)}$	$98.3^{(4)}$	100	3292.4	1735.9	292.1	77.8	99.4	100	$97.5^{(5)}$	$99.7^{(2)}$	100
8/21	3295	3293	1037	51.1	33.3	100	$97.4^{(5)}$	$97.2^{(5)}$	100	3294.2	2805.9	1591.1	14.9	93.8	100	$97.5^{(5)}$	$98^{(4)}$	100	3297.7	2705.7	652	80.6	97.1	100	$97.2^{(5)}$	$98.6^{(4)}$	100
8/24	3295	3295	1141	56.8	27.6	100	$97.4^{(5)}$	$96.8^{(5)}$	100	3292.4	3293.7	3138	11.8	53.3	78.6	$96.5^{(5)}$	$95.4^{(5)}$	$98.3^{(4)}$	3293.8	2593.5	1053.4	47.1	75.7	100	$95.7^{(5)}$	$97.3^{(3)}$	100
8/27	3293	3296	1807	30	25.7	100	$96.8^{(5)}$	$95.7^{(5)}$	100	3299.2	3297	3297.1	34.7	42.2	46.2	$95.7^{(5)}$	$94.8^{(5)}$	$97.4^{(5)}$	3293.8	2457.5	1196.3	50.8	77.8	99.9	$96.3^{(5)}$	$98.2^{(3)}$	$99.9^{(1)}$
8/30	3297	3295	2850	32.4	38.1	57.7	$95.8^{(5)}$	$94.4^{(5)}$	$98.7^{(4)}$	3297.3	3298.5	3296.5	17.2	12.8	20.6	$95.1^{(5)}$	$93.2^{(5)}$	$96.4^{(5)}$	3296.3	3221.5	1828.6	11.1	40	63.9	$94.6^{(5)}$	$96.4^{(3)}$	$98.6^{(2)}$
8/33	3297	3293	3296	14.7	12.8	13.1	$93.4^{(5)}$	$93.8^{(5)}$	$96.4^{(5)}$	3293.5	3296	3297.4	12.1	12	19.6	$93.1^{(5)}$	$92.2^{(5)}$	$95.9^{(5)}$	3295.4	3292.6	2804.9	27.9	74.4	97.5	$94.1^{(5)}$	$96.1^{(5)}$	$99.3^{(4)}$
10/9	2738	208	237	99.7	100	100	$99.7^{(3)}$	100	100	3296.6	93.5	120.7	95.6	100	100	$97.5^{(5)}$	100	100	961.8	671.9	86	99.8	99.8	100	$99.8^{(1)}$	$99.8^{(1)}$	100
10/12	3293	724	437	59.5	98.1	100	$99.1^{(5)}$	$99.6^{(1)}$	100	3296.6	162.5	274.5	72.9	100	100	$97.4^{(5)}$	100	100	3294.4	679.7	275.2	78.9	99.9	100	$98.1^{(5)}$	$99.9^{(1)}$	100
10/15	3293	2831	511	65.2	95	100	$97.7^{(5)}$	$99.3^{(3)}$	100	3296.7	513.4	1522.4	17.3	100	100	$97.7^{(5)}$	100	100	3294	777.9	467.7	97.8	100	100	$97.9^{(5)}$	$100^{(1)}$	100
10/18	3297	3296	745	59.8	38.9	100	$97.3^{(5)}$	$97.5^{(5)}$	100	3292.5	3172.4	1914.6	8.4	97.7	100	$96.6^{(5)}$	$99^{(4)}$	100	3294.9	2548.3	615.5	58.4	99.8	100	$96.2^{(5)}$	$99.8^{(2)}$	100
10/21	3297	3296	1104	59	52.8	100	$97^{(5)}$	$97^{(5)}$	100	3295.2	2805.3	2944.4	0	78.5	79.5	$97^{(5)}$	$97.7^{(4)}$	$98.6^{(4)}$	3296.6	3294.5	817.1	50.2	94.7	100	$94.4^{(5)}$	$96.1^{(5)}$	100
10/24	3296	3297	2477	30.5	9.8	70.2	$96.6^{(5)}$	$95.6^{(5)}$	$98.9^{(2)}$	3295.5	3297.9	3293.5	0	8.2	27.8	$95.4^{(5)}$	$92.9^{(5)}$	$97.4^{(5)}$	3297.9	2692.3	1164.6	15.3	93.7	100	$93.9^{(5)}$	$94.7^{(4)}$	100
10/27	3291	3294	3073	28.7	28.1	37.5	$94.7^{(5)}$	$95.1^{(5)}$	$97.2^{(4)}$	3296	3292.6	3292.3	0	0	0	$93.3^{(5)}$	$91.8^{(5)}$	$95.1^{(5)}$	3293.8	3292.5	2403.7	22.8	94.3	99.5	$95^{(5)}$	$96.3^{(5)}$	$99.5^{(2)}$
10/30	3298	3296	3294	7.4	6.8	13.5	$94.1^{(5)}$	$94.3^{(5)}$	$97.2^{(5)}$	3296.2	3297.8	3293.7	5.4	5.4	23.9	$92.3^{(5)}$	$91.3^{(5)}$	$95.3^{(5)}$	3294	3294.3	2755.4	0	35.7	82.9	$93.7^{(5)}$	$94^{(5)}$	$98.8^{(3)}$
12/6	1241	6	10	99.9	100	100	$99.9^{(1)}$	100	100	1889.7	8.8	17.5	100	100	100	100	100	100	1554.4	4.8	14.8	99.6	100	100	$99.7^{(2)}$	100	100
12/9	2906	354	281	59.8	100	100	$99^{(4)}$	100	100	3292.1	155.2	317.5	57.6	100	100	$97.4^{(5)}$	100	100	3297.6	128.9	144.6	99.1	100	100	$99.2^{(5)}$	100	100
12/12	3293	140	606	36.4	100	100	$98.9^{(5)}$	100	100	3292.4	214.2	562.2	51.2	100	100	$97.6^{(5)}$	100	100	3293.4	303.4	313.2	91	100	100	$97.9^{(5)}$	100	100
12/15	3295	2727	686	60	77	100	$98.2^{(5)}$	$99.4^{(4)}$	100	3292.2	1381.9	1902.7	0	80	80	$97.2^{(5)}$	$99.7^{(1)}$	$99.7^{(1)}$	3292.9	2663.8	860.8	76.9	98	100	$96.3^{(5)}$	$98.7^{(3)}$	100
12/18	3295	3298	969	32.2	41	100	$97.9^{(5)}$	$97.6^{(5)}$	100	3293.6	2642.7	2581.4	0	40	60	$96.6^{(5)}$	$98^{(3)}$	$98.9^{(2)}$	3296.6	2067.8	1061.7	77.5	98.6	100	$96.1^{(5)}$	$99.6^{(2)}$	100
12/21	3297	3296	2686	41.2	37.6	100	$97.9^{(5)}$	$97^{(5)}$	$100^{(1)}$	3296.5	3298.1	3296.1	9.2	29	26.5	$96.2^{(5)}$	$96.3^{(5)}$	$97.2^{(5)}$	3294.8	3024.3	1837.1	25	95	100	$94.4^{(5)}$	$96.6^{(4)}$	$100^{(1)}$
12/24	3295	3296	3206	34	20.8	66.8	$96.5^{(5)}$	$95^{(5)}$	$98.8^{(4)}$	3293.2	3296.9	3296.5	15.2	0	25.5	$94.4^{(5)}$	$92.7^{(5)}$	$95.5^{(5)}$	3296.3	3293.7	2375.3	21.5	25.2	79.7	$93.6^{(5)}$	$93.6^{(5)}$	$98.9^{(3)}$
12/27	3297	3294	3295	0	8.8	25.6	$94.1^{(5)}$	$94.4^{(5)}$	$97^{(5)}$	3296.3	3290.9	3293.6	7.6	7.6	7.7	$93^{(5)}$	$91.9^{(5)}$	$94.8^{(5)}$	3295	3294.3	3297.2	13.8	56.8	77.5	$93.1^{(5)}$	$93.8^{(5)}$	$98.3^{(5)}$
Total	3207	2578	1422	56.5	57.3	84.7	$97.3^{(192)}$	$97.3^{(148)}$	$99.4^{(51)}$	3260	2417.7	1975.6	36.9	61.8	74.4	$96.3^{(195)}$	$96.3^{(136)}$	$98.7^{(81)}$	3097.7	1871.8	938.4	64.5	89.8	97.4	$96.6^{(186)}$	$98.1^{(93)}$	$99.8^{(29)}$

* Separation procedure used for all BC scenarios: algorithm $\mathcal{A}1$

l/n: Number of periods/number of suppliers, None: With no inequality, Known: With known inequalities (18), (26) and (27), All: With all inequalities (18)-(27)

The numbers in parentheses present the number of instances out of five that are not solved to optimality within the time limit

Table 7: Detailed results of the BC with the default node selection strategy, and with and without the valid inequalities*

Set	Class 1								
	CPU			%UB			%BUB		
l/n	None	Known	All	None	Known	All	None	Known	All
4/18	2855	1175	438	98.9	99.8	100	$98.9^{(4)}$	$99.8^{(1)}$	100
4/21	3298	1509	524	95.9	99.7	100	$96.6^{(5)}$	$99.7^{(1)}$	100
4/24	3293	3294	834	96.6	98.3	100	$97.2^{(5)}$	$98.3^{(5)}$	100
4/27	3298	2329	1218	96.1	98.4	100	$96.9^{(5)}$	$98.7^{(3)}$	100
4/30	3295	3294	1599	96.5	96.1	99.7	$97.5^{(5)}$	$96.7^{(5)}$	$99.7^{(2)}$
4/33	3296	3291	1198	74.3	95.5	99.5	$96.6^{(5)}$	$96^{(5)}$	$99.5^{(1)}$
4/36	3294	3299	2950	56.8	48.9	97.7	$95.7^{(5)}$	$95.1^{(5)}$	$98.2^{(4)}$
4/39	3292	3296	3294	47.9	72.2	83.5	$94.6^{(5)}$	$94.5^{(5)}$	$97.9^{(5)}$
6/15	2172	851	718	99.4	99.7	99.9	$99.4^{(3)}$	$99.7^{(1)}$	$99.9^{(1)}$
6/18	3293	2248	633	98.1	99.4	100	$98.1^{(5)}$	$99.4^{(3)}$	100
6/21	3292	3290	3028	94.1	96.5	99.7	$95.9^{(5)}$	$96.7^{(5)}$	$99.7^{(4)}$
6/24	3295	3291	1867	73.8	95.3	99.7	$96.6^{(5)}$	$96.1^{(5)}$	$99.7^{(2)}$
6/27	3296	3296	2563	65.9	93.7	99.5	$97.2^{(5)}$	$95.1^{(5)}$	$99.7^{(3)}$
6/30	3298	3291	2458	70.5	87.6	95.5	$96.2^{(5)}$	$95.1^{(5)}$	$98.7^{(3)}$
6/33	3295	3298	3296	41.5	70.4	78.4	$95.6^{(5)}$	$93.7^{(5)}$	$97.4^{(5)}$
6/36	3295	3298	3298	55.1	64.1	71.8	$93.4^{(5)}$	$91.2^{(5)}$	$95.8^{(5)}$
8/12	3003	159	71	99.2	100	100	$99.2^{(4)}$	100	100
8/15	3298	1246	707	98	100	100	$98.3^{(5)}$	$100^{(1)}$	100
8/18	3298	3298	1927	91.7	98.5	99.9	$97.4^{(5)}$	$98.5^{(5)}$	$99.9^{(2)}$
8/21	3298	3298	2378	61	96	99.9	$96.3^{(5)}$	$96.7^{(5)}$	$99.9^{(2)}$
8/24	3298	3298	1826	61.4	96.5	99.6	$96.3^{(5)}$	$96.5^{(5)}$	$99.6^{(2)}$
8/27	3298	3298	3298	29.6	93.6	97.8	$95.9^{(5)}$	$95^{(5)}$	$98.5^{(5)}$
8/30	3296	3298	3295	32.1	71.4	86.3	$95.2^{(5)}$	$93^{(5)}$	$98^{(5)}$
8/33	3293	3298	3296	15	57.8	54.1	$93.1^{(5)}$	$93^{(5)}$	$95.3^{(5)}$
10/9	2678	179	191	99.5	100	100	$99.5^{(4)}$	100	100
10/12	3297	1098	976	98.9	99.9	100	$98.9^{(5)}$	$99.9^{(1)}$	$100^{(1)}$
10/15	3297	2609	892	95.1	99.4	100	$96.8^{(5)}$	$99.4^{(3)}$	100
10/18	3295	3297	2416	87.9	97.6	99.9	$96.5^{(5)}$	$97.6^{(5)}$	$99.9^{(3)}$
10/21	3294	3296	2282	72.9	95.7	99.8	$96.3^{(5)}$	$96.5^{(5)}$	$99.8^{(2)}$
10/24	3295	3294	3276	44.5	55.5	95	$95.7^{(5)}$	$95^{(5)}$	$97.7^{(4)}$
10/27	3296	3296	3294	18.9	68.7	49.2	$93.6^{(5)}$	$93.2^{(5)}$	$96.4^{(5)}$
10/30	3291	3291	3298	24	33.7	36.8	$93.5^{(5)}$	$92.5^{(5)}$	$96.2^{(5)}$
12/6	1439	7	11	99.5	100	100	$99.5^{(2)}$	100	100
12/9	2080	377	469	98.6	100	100	$98.7^{(2)}$	100	100
12/12	3298	844	810	98.6	100	100	$98.9^{(5)}$	$100^{(1)}$	100
12/15	3299	2745	1930	76.5	99.4	99.9	$97.3^{(5)}$	$99.4^{(4)}$	$99.9^{(2)}$
12/18	3298	3296	2728	23.5	91	98.6	$97^{(5)}$	$96.6^{(5)}$	$99^{(4)}$
12/21	3298	3292	3298	60	91.4	97.8	$97.1^{(5)}$	$96.1^{(5)}$	$98.3^{(5)}$
12/24	3298	3296	3298	20.5	44.8	75.2	$95.9^{(5)}$	$94.2^{(5)}$	$97.5^{(5)}$
12/27	3296	3291	3300	17.5	43.9	34.8	$93.4^{(5)}$	$93.1^{(5)}$	$95.9^{(5)}$
Total	3157	2576	1980	69.6	86.3	91.2	$96.7^{(189)}$	$96.8^{(149)}$	$99^{(97)}$

Set	Class 2								
	CPU			%UB			%BUB		
l/n	None	Known	All	None	Known	All	None	Known	All
4/18	3295	2199.1	862.6	96.3	99.1	99.9	$96.8^{(5)}$	$99.1^{(2)}$	$99.9^{(1)}$
4/21	3296.5	1646.9	288.3	97.5	99.6	100	$97.5^{(5)}$	$99.6^{(2)}$	100
4/24	3292	2893.2	1118.5	94.5	97.5	99.9	$96^{(5)}$	$97.5^{(4)}$	$99.9^{(1)}$
4/27	3297.5	3244.6	1142.2	94.2	96.2	99.8	$96.3^{(5)}$	$96.4^{(4)}$	$99.8^{(1)}$
4/30	3297.8	3297.1	1677.6	74.4	92.6	99.1	$95.6^{(5)}$	$93.9^{(5)}$	$99.3^{(2)}$
4/33	3297.9	3297.1	3026.9	76.1	91.9	99.1	$95.2^{(5)}$	$92.9^{(5)}$	$99.1^{(4)}$
4/36	3296.5	3297.4	3296.5	63.3	90.1	78.9	$94.3^{(5)}$	$92.9^{(5)}$	$98.1^{(5)}$
4/39	3297.9	3296	3299.4	56.4	83.9	83.5	$92.5^{(5)}$	$89.9^{(5)}$	$96.6^{(5)}$
6/15	3297.7	1508	906.8	94.4	99.1	100	$95.8^{(5)}$	$99.1^{(2)}$	100
6/18	3297.6	2906	2102.7	94.7	97.2	99.8	$95.9^{(5)}$	$97.2^{(4)}$	$99.8^{(1)}$
6/21	3292.1	3297.7	2840	88.5	95.2	98.7	$95.5^{(5)}$	$95.6^{(5)}$	$98.9^{(4)}$
6/24	3296.2	3297.7	3254.4	52.3	90.5	98.4	$95.1^{(5)}$	$92.4^{(5)}$	$98.4^{(4)}$
6/27	3297.5	3297.9	3298.1	55.4	91.1	98.1	$95.7^{(5)}$	$93.5^{(5)}$	$98.1^{(5)}$
6/30	3297.9	3296.8	2969.5	50.9	85.3	92.3	$93.7^{(5)}$	$90.6^{(5)}$	$96.4^{(4)}$
6/33	3294.1	3297.8	3295	33.1	48.6	61.9	$93^{(5)}$	$90.2^{(5)}$	$95.5^{(5)}$
6/36	3292.5	3297.9	3297.4	49	69.8	72.3	$93.6^{(5)}$	$92^{(5)}$	$96.1^{(5)}$
8/12	3296.6	1320.5	929.8	97.6	99.6	99.9	$97.8^{(5)}$	$99.6^{(1)}$	$99.9^{(1)}$
8/15	3296.8	2499.9	1317.4	93.1	99	99.9	$95.4^{(5)}$	$99^{(3)}$	$99.9^{(1)}$
8/18	3294.9	3297.8	2467.6	90.9	94.2	99	$94.9^{(5)}$	$95.5^{(5)}$	$99^{(3)}$
8/21	3293.9	3297.7	3247.3	35.5	93.3	98.4	$95.7^{(5)}$	$95.6^{(5)}$	$99^{(4)}$
8/24	3292.2	3297.7	3296.2	45	86.6	95.9	$95.1^{(5)}$	$92.4^{(5)}$	$96.8^{(5)}$
8/27	3290.8	3295.2	3293.6	74.4	79.1	66.3	$94.9^{(5)}$	$93.9^{(5)}$	$96.6^{(5)}$
8/30	3290.9	3297.7	3296.5	24.7	68.8	74.4	$94.4^{(5)}$	$92.8^{(5)}$	$95.8^{(5)}$
8/33	3292.8	3293.7	3290.9	58.1	38.6	54.8	$92.9^{(5)}$	$91.1^{(5)}$	$95.5^{(5)}$
10/9	3297.7	77.5	104.2	97.5	100	100	$97.5^{(5)}$	100	100
10/12	3297.9	950.9	848.3	95.6	100	100	$96.3^{(5)}$	$100^{(1)}$	$100^{(1)}$
10/15	3298.1	2874.3	3067.1	95.4	99.5	99.3	$96.4^{(5)}$	$99.5^{(3)}$	$99.3^{(4)}$
10/18	3296.4	3295.9	3294.1	51	96	98.4	$94.5^{(5)}$	$96.3^{(5)}$	$98.7^{(5)}$
10/21	3297.8	3292.5	3292.6	17.4	76.9	78.2	$95.2^{(5)}$	$95.6^{(5)}$	$96.8^{(5)}$
10/24	3298.5	3294.2	3294.2	16.9	80.8	82.6	$93.6^{(5)}$	$90.3^{(5)}$	$95.2^{(5)}$
10/27	3291	3297.9	3292.8	48.4	33.4	34.5	$92.6^{(5)}$	$90.8^{(5)}$	$94.3^{(5)}$
10/30	3294.4	3298.3	3295.1	33.3	57.2	23.9	$92.5^{(5)}$	$90.9^{(5)}$	$94.9^{(5)}$
12/6	817.2	10	15.8	100	100	100	100	100	100
12/9	3297.9	158	379.6	97.5	100	100	$97.6^{(5)}$	100	100
12/12	3294.8	1383.9	2255.1	93.8	99.5	99.1	$95.9^{(5)}$	$99.5^{(1)}$	$99.3^{(2)}$
12/15	3295.2	3060.6	2652.8	52.9	98.1	98.7	$95.5^{(5)}$	$98.4^{(4)}$	$98.7^{(3)}$
12/18	3292.2	3297.6	3297.2	0	91.6	95.6	$94.9^{(5)}$	$95.1^{(5)}$	$96.7^{(5)}$
12/21	3297.7	3297.9	3297.6	17.9	44.6	69.2	$94.1^{(5)}$	$94^{(5)}$	$94^{(5)}$
12/24	3297.7	3297.7	3294.5	48.5	49.3	43.6	$93.6^{(5)}$	$91.2^{(5)}$	$94.7^{(5)}$
12/27	3291.2	3298.1	3291.1	59.9	41.9	8.7	$92.9^{(5)}$	$91.5^{(5)}$	$94.4^{(5)}$
Total	3233.5	2728.9	2419.7	65.4	83.9	85	$95.2^{(195)}$	$95.2^{(156)}$	$97.9^{(131)}$

Set	Class 3								
	CPU			%UB			%BUB		
l/n	None	Known	All	None	Known	All	None	Known	All
4/18	1331.6	56.6	13.5	99.7	100	100	$99.7^{(1)}$	100	100
4/21	2294	680.1	215.2	97.4	100	100	$97.9^{(3)}$	100	100
4/24	3297.7	65.1	32.1	96.4	100	100	$97.7^{(5)}$	100	100
4/27	2952.6	470.6	48.6	95.6	100	100	$97.4^{(4)}$	100	100
4/30	2994.9	2804.2	248.6	94.1	98.5	100	$97.3^{(4)}$	$99.1^{(3)}$	100
4/33	3296.2	2047.8	208.5	95	99.3	100	$97^{(5)}$	$99.3^{(2)}$	100
4/36	3294.7	3294.9	936.6	80.1	95.5	99.5	$94.8^{(5)}$	$96.5^{(5)}$	$99.5^{(1)}$
4/39	3295.8	2530.1	1546.1	50.3	92.4	98.5	$92.7^{(5)}$	$94.5^{(3)}$	$98.5^{(1)}$
6/15	2878.3	617.8	243.2	99.1	100	100	$99.1^{(3)}$	100	100
6/18	3296.5	383.3	59.6	97.8	100	100	$97.9^{(5)}$	100	100
6/21	3297.4	1250.3	156	94.7	100	100	$96.5^{(5)}$	$100^{(1)}$	100
6/24	3293.6	1877.4	617.5	93.1	97.4	100	$95^{(5)}$	$97.4^{(2)}$	100
6/27	3296.3	1854.6	375.7	90.2	99.6	100	$95.4^{(5)}$	$99.6^{(1)}$	100
6/30	3297.7	2086.8	570.5	70.2	99	100	$94.3^{(5)}$	$99.1^{(2)}$	100
6/33	3295.3	3295.2	1532.2	83.1	93.7	99.7	$95.5^{(5)}$	$94.3^{(5)}$	$99.7^{(2)}$
6/36	3290.9	3297.5	2488.2	51.9	87.6	98.9	$94^{(5)}$	$91.1^{(5)}$	$98.9^{(3)}$
8/12	2239	485.7	776.6	99.9	100	100	$99.9^{(2)}$	100	$100^{(1)}$
8/15	3293.4	146.2	375.5	97.1	100	100	$97.5^{(5)}$	100	100
8/18	3295.4	2079.7	372.7	96	99.9	100	$96.9^{(5)}$	$99.9^{(1)}$	100
8/21	3295.2	2739.4	951.7	94.6	97.7	100	$96.5^{(5)}$	$97.8^{(4)}$	100
8/24	3293.7	2246.6	1442.3	55.2	95.1	99.7	$94.9^{(5)}$	$96.2^{(3)}$	$99.7^{(1)}$
8/27	3292.3	2387.4	1641.2	90.9	97	99.9	$95.4^{(5)}$	$97.6^{(3)}$	$99.9^{(1)}$
8/30	3296.8	3297.3	2481.2	68.1	89.7	88.4	$93.6^{(5)}$	$93.5^{(5)}$	$98.2^{(3)}$
8/33	3294.2	3297.8	3032.6	53.6	75.9	96.1	$93.3^{(5)}$	$94.2^{(5)}$	$98.8^{(3)}$
10/9	1029.2	691.6	155.3	99.7	99.7	100	$99.7^{(1)}$	$99.7^{(1)}$	100
10/12	2767.6	704.9	705.8	98.1	99.6	99.7	$98.1^{(4)}$	$99.6^{(1)}$	$99.7^{(1)}$
10/15	3296.2	1235.7	887.9	97.3	99.9	100	$97.6^{(5)}$	$99.9^{(1)}$	$100^{(1)}$
10/18	3294.8	2192.1	1170.9	92.8	98.9	99.9	$94.9^{(5)}$	$98.9^{(2)}$	$99.9^{(1)}$
10/21	3294.9	3296	1784.8	83.2	95.4	100	$93.1^{(5)}$	$95.6^{(5)}$	100
10/24	3296.2	2695.9	2477.4	52.5	92.8	99.8	$92.5^{(5)}$	$93.2^{(4)}$	$99.8^{(2)}$
10/27	3290.8	3297.6	3050.5	20.9	93.9	98.8	$94.3^{(5)}$	$95.4^{(5)}$	$98.8^{(4)}$
10/30	3294.1	3298	3272	17.5	85.8	82.9	$92.6^{(5)}$	$91.9^{(5)}$	$98.2^{(4)}$
12/6	1471.6	6.5	12.6	99.5	100	100	$99.5^{(2)}$	100	100
12/9	2977.1	169.2	341.2	99.1	100	100	$99.1^{(4)}$	100	100
12/12	3295.5	767.8	957.2	97.5	100	100	$97.7^{(5)}$	100	100
12/15	3295.1	2274.8	1053.1	94.2	99.6	100	$95.2^{(5)}$	$99.6^{(2)}$	100
12/18	3296.2	2852.8	2242.9	92.6	97.8	99.6	$95.5^{(5)}$	$98.1^{(3)}$	$99.6^{(3)}$
12/21	3297.8	3098.8	3115.9	25.7	94.6	98.9	$92.2^{(5)}$	$95.8^{(4)}$	$99.1^{(3)}$
12/24	3296.8	3297.8	3296.7	49.4	87.7	95	$92.5^{(5)}$	$92.5^{(5)}$	$96.1^{(5)}$
12/27	3294.2	3298.1	3295.9	20.5	82.3	77.2	$91.6^{(5)}$	$91.6^{(5)}$	$97.5^{(5)}$
Total	3044.8	1911.8	1204.6	79.6	96.1	98.3	$95.9^{(178)}$	$97.5^{(93)}$	$99.5^{(45)}$

* Separation procedure used for all BC scenarios: algorithm $\mathcal{A}1$

l/n: Number of periods/number of suppliers, None: With no inequality, Known: With known inequalities (18), (26) and (27), All: With all inequalities (18)-(27)

The numbers in parentheses present the number of instances out of five that are not solved to optimality within the time limit

Table 8: Performance of the BC with different separation procedures*

Set	Class 1						Class 2						Class 3					
	CVRPSEP		$\mathcal{A}1$		$\mathcal{A}2$		CVRPSEP		$\mathcal{A}1$		$\mathcal{A}2$		CVRPSEP		$\mathcal{A}1$		$\mathcal{A}2$	
l/n	CPU	%BUB	CPU	%BUB	CPU	%BUB	CPU	%BUB	CPU	%BUB	CPU	%BUB	CPU	%BUB	CPU	%BUB	CPU	%BUB
4/18	1446	$99.9^{(1)}$	265	100	444	100	1304	$99.8^{(1)}$	623	100	830	$99.9^{(1)}$	80	100	28	100	29	100
4/21	959	$99.6^{(2)}$	317	100	123	100	832	$99.8^{(1)}$	893	100	990	$100^{(1)}$	236	100	152	100	84	100
4/24	1981	$99.7^{(2)}$	750	100	942	100	2089	$99.7^{(3)}$	1156	100	1277	$100^{(1)}$	48	100	30	100	29	100
4/27	1984	$99.9^{(2)}$	741	100	190	100	1472	$100^{(1)}$	587	100	617	100	137	100	73	100	42	100
4/30	2500	$99.5^{(4)}$	812	100	311	100	1838	$99.4^{(2)}$	1494	100	1187	$100^{(1)}$	530	$100^{(1)}$	294	100	247	100
4/33	2876	$99.4^{(3)}$	1374	$99.8^{(1)}$	772	$99.5^{(1)}$	2726	$98.8^{(3)}$	2179	$99.9^{(1)}$	2054	$99.7^{(2)}$	399	100	510	100	94	100
4/36	3298	$97.5^{(5)}$	2663	$98.8^{(4)}$	2715	$99.1^{(4)}$	2901	$98.2^{(4)}$	2343	$99.1^{(3)}$	1821	$99.3^{(2)}$	1059	$99.7^{(1)}$	1229	$99.5^{(1)}$	743	$99.6^{(1)}$
4/39	3298	$96.2^{(5)}$	2716	$97.5^{(4)}$	2230	$98.7^{(3)}$	3294	$96.9^{(5)}$	3294	$97.5^{(5)}$	3298	$98.8^{(5)}$	1669	$99^{(2)}$	1407	$99.3^{(1)}$	983	$99.5^{(1)}$
6/15	755	$99.9^{(1)}$	450	100	724	$100^{(1)}$	1557	$99.8^{(2)}$	424	100	252	100	697	$100^{(1)}$	286	100	487	100
6/18	1976	$99.6^{(2)}$	562	100	483	100	1363	$99.9^{(1)}$	818	100	946	$99.9^{(1)}$	296	100	101	100	105	100
6/21	3295	$98.1^{(5)}$	830	100	974	100	2673	$99.2^{(3)}$	1515	100	1539	100	2034	$99.8^{(2)}$	222	100	257	100
6/24	3106	$99.7^{(4)}$	1050	100	1445	$99.9^{(1)}$	3078	$99.2^{(4)}$	2320	$99.9^{(1)}$	2519	$99.8^{(3)}$	2855	$99.4^{(4)}$	312	100	273	100
6/27	2848	$99.2^{(4)}$	1092	100	805	100	2765	$98.6^{(3)}$	2293	$99.8^{(1)}$	1530	$99.2^{(1)}$	1847	$99.9^{(2)}$	420	100	124	100
6/30	2510	$98.7^{(4)}$	1639	$99.5^{(2)}$	1517	$99.2^{(2)}$	2854	$96.3^{(4)}$	2862	$97.7^{(4)}$	2740	$97.7^{(4)}$	2120	$99.2^{(3)}$	606	100	241	100
6/33	3297	$97.9^{(5)}$	3297	$98.1^{(5)}$	3298	$98.4^{(5)}$	3298	$95.8^{(5)}$	3294	$97.1^{(5)}$	3296	$97.2^{(5)}$	3297	$99^{(5)}$	1949	$99.8^{(2)}$	1148	$99.9^{(1)}$
6/36	3295	$95.8^{(5)}$	3297	$96.7^{(5)}$	3293	$97.3^{(5)}$	3297	$96.4^{(5)}$	3297	$96.8^{(5)}$	3297	$97.2^{(5)}$	2639	$98.2^{(4)}$	2666	$98.6^{(4)}$	2032	$99.6^{(2)}$
8/12	176	100	78	100	80	100	882	$99.9^{(1)}$	570	100	327	100	777	$100^{(1)}$	664	100	973	$100^{(1)}$
8/15	520	100	252	100	175	100	1640	$99.7^{(2)}$	1074	100	1100	$99.9^{(1)}$	1073	$100^{(1)}$	272	100	229	100
8/18	2029	$99.5^{(3)}$	962	100	1076	$99.9^{(1)}$	2188	$99.7^{(2)}$	1446	100	1358	$99.7^{(1)}$	1135	100	292	100	218	100
8/21	2977	$99.1^{(4)}$	1037	100	845	100	2366	$99.4^{(3)}$	1591	100	1785	$100^{(1)}$	2475	$99^{(3)}$	652	100	709	100
8/24	2305	$98.2^{(3)}$	1141	100	793	100	3295	$97.4^{(5)}$	3138	$98.3^{(4)}$	2994	$98.4^{(4)}$	2856	$99^{(4)}$	1053	100	1145	$99.9^{(1)}$
8/27	3296	$98.4^{(5)}$	1807	100	1767	$99.7^{(1)}$	3297	$95.7^{(5)}$	3297	$97.4^{(5)}$	3295	$97.4^{(5)}$	1542	$99.3^{(1)}$	1196	$99.9^{(1)}$	1045	$99.9^{(1)}$
8/30	3297	$98^{(5)}$	2850	$98.7^{(4)}$	2843	$99^{(4)}$	3296	$96^{(5)}$	3296	$96.4^{(5)}$	3297	$96.5^{(5)}$	2725	$96.9^{(4)}$	1829	$98.6^{(2)}$	1863	$99.3^{(2)}$
8/33	3297	$95.4^{(5)}$	3296	$96.4^{(5)}$	3298	$97.2^{(5)}$	3296	$95^{(5)}$	3297	$95.9^{(5)}$	3298	$96.2^{(5)}$	3291	$97.3^{(5)}$	2805	$99.3^{(4)}$	2288	$99.4^{(3)}$
10/9	415	100	237	100	471	100	489	100	121	100	516	100	209	100	86	100	120	100
10/12	697	100	437	100	716	100	795	$99.9^{(1)}$	275	100	273	100	322	100	275	100	222	100
10/15	1503	$99.8^{(1)}$	511	100	290	100	2641	$99.3^{(2)}$	1522	100	1374	$100^{(1)}$	746	$99.9^{(1)}$	468	100	726	$100^{(1)}$
10/18	2803	$98.7^{(4)}$	745	100	602	100	2520	$99.7^{(3)}$	1915	100	2468	$99.9^{(2)}$	2858	$100^{(2)}$	615	100	652	100
10/21	2728	$97.8^{(4)}$	1104	100	978	100	2914	$98.1^{(4)}$	2944	$98.6^{(4)}$	2895	$98.6^{(4)}$	2221	$98.5^{(3)}$	817	100	568	100
10/24	3296	$97.5^{(5)}$	2477	$98.9^{(2)}$	2130	$99.2^{(2)}$	3292	$97.1^{(5)}$	3294	$97.4^{(5)}$	3294	$98^{(5)}$	1802	$99.7^{(1)}$	1165	100	594	100
10/27	3297	$96^{(5)}$	3073	$97.2^{(4)}$	2775	$97.7^{(3)}$	3294	$93.6^{(5)}$	3292	$95.1^{(5)}$	3298	$95.3^{(5)}$	3250	$99.1^{(4)}$	2404	$99.5^{(2)}$	2391	$99.6^{(3)}$
10/30	3297	$96.1^{(5)}$	3294	$97.2^{(5)}$	3298	$97.4^{(5)}$	3298	$94.3^{(5)}$	3294	$95.3^{(5)}$	3298	$95.1^{(5)}$	3294	$96.9^{(5)}$	2755	$98.8^{(3)}$	1866	$99.3^{(2)}$
12/6	24	100	10	100	13	100	18	100	17	100	22	100	14	100	15	100	12	100
12/9	862	$100^{(1)}$	281	100	399	100	777	$99.7^{(1)}$	318	100	246	100	804	100	145	100	196	100
12/12	925	$99.9^{(1)}$	606	100	312	100	891	100	562	100	538	100	492	100	313	100	378	100
12/15	1510	$99.7^{(1)}$	686	100	420	100	2607	$98.6^{(3)}$	1903	$99.7^{(1)}$	2542	$99.5^{(1)}$	2992	$99.3^{(4)}$	861	100	822	100
12/18	2610	$99.7^{(2)}$	969	100	824	100	2841	$97.6^{(4)}$	2581	$98.9^{(2)}$	2613	$98.4^{(3)}$	2754	$98.2^{(4)}$	1062	100	844	100
12/21	3069	$99^{(4)}$	2686	$100^{(1)}$	2142	$99.9^{(1)}$	3292	$96.4^{(5)}$	3296	$97.2^{(5)}$	3298	$97^{(5)}$	3297	$97.2^{(5)}$	1837	$100^{(1)}$	1910	100
12/24	3297	$97.4^{(5)}$	3206	$98.8^{(4)}$	3063	$98.9^{(3)}$	3294	$93.6^{(5)}$	3296	$95.5^{(5)}$	3295	$95.2^{(5)}$	3296	$95.3^{(5)}$	2375	$98.9^{(3)}$	2440	$98.4^{(3)}$
12/27	3295	$95.7^{(5)}$	3295	$97^{(5)}$	3298	$97^{(5)}$	3300	$92.3^{(5)}$	3294	$94.8^{(5)}$	3298	$94.7^{(5)}$	3291	$96.1^{(5)}$	3297	$98.3^{(5)}$	3119	$98.4^{(4)}$
Total	2264	$98.6^{(122)}$	1422	$99.4^{(51)}$	1322	$99.5^{(52)}$	2347	$98^{(123)}$	1976	$98.7^{(81)}$	1973	$98.7^{(95)}$	1688	$99.1^{(83)}$	938	$99.8^{(29)}$	806	$99.8^{(26)}$

* Best-bound node selection strategy is used for all these experiments

l/n: Number of periods/number of suppliers,

The numbers in parentheses present the number of instances that are not solved to optimality within the time limit

Table 9: Summary of added SECs and CPLEX cuts for different classes of instances when different separation procedures are applied*

Sep	Class	Size	#Opt	#Node	GFS	AV^{GFS}	DFJ	AV^{DFJ}	Cover	Flow	Clique	MIR	Path	ImplBd	ZeroHalf	LiftProj
CVRPSEP	1	200	78	7016	561.3	0.4	3432.3	0.62	172.2	254.2	19.2	745.9	26.1	69.9	295.9	17.8
	2	200	77	2898	209.1	0.4	1607.3	0.75	156.1	628.5	1.4	2010.5	89	377.4	151.7	24.4
	3	200	117	4452	562.3	0.42	4753.7	0.76	120.4	232.2	3.3	661.1	2.2	68.4	137.7	22.2
	Total	600	272	4768	442.2	0.41	3252.6	0.71	149.5	373.8	7.9	1146.4	39.5	173.6	194.4	21.5
A1	1	200	149	3940	981.2	0.29	4528	0.4	96.6	133.1	16.1	349.8	8	44.1	93.2	16.2
	2	200	119	2295	1024.9	0.24	3958.7	0.37	99.6	359.9	1.3	1034.8	39.3	253.7	68.3	17.5
	3	200	171	1887	748.9	0.22	3839.1	0.42	56.5	114.1	3.3	359	0.8	39.7	45.4	13.4
	Total	600	439	2707	918.3	0.25	4108.6	0.4	84.3	202.4	6.9	581.2	16	112.5	69	15.7
A2	1	200	148	5013	432.1	0.21	1473	0.44	127.8	187.6	18.1	510.3	13.2	58.2	168	14.7
	2	200	105	1962	349.3	0.18	1148.5	0.43	110	419.1	1.4	1320.2	45.2	304.4	79.6	17.6
	3	200	174	2047	305.9	0.19	1481.8	0.48	78.2	173.5	3.3	535.9	1	50.1	70.5	13.5
	Total	600	427	3007	362.4	0.2	1367.7	0.45	105.3	260.1	7.6	788.8	19.8	137.5	106	15.3

* Best-bound node selection strategy is used for all these experiments
Sep: Separation procedure

(Clique), mixed integer rounding cuts (MIR), flow path cuts (Path), implied bound cuts (ImplBd), zero-half cuts (ZeroHalf), and lift-and-project cuts (LiftProj). The results indicate that for each class the BC has to explore many more nodes and finds fewer optimal solutions when it employs the CVRPSEP package compared to when it uses one of the proposed separation procedures. Another observation is that the average violation amount of the SECs (both GFSECs and DFJs) found by the CVRPSEP package is higher than the ones found by the other separation procedures. The reason is that CVRPSEP is not able to find violated SECs in the initial stages of the search tree because the node visit values are small in a fractional solution. In other words, because the CVRPSEP package is not effective on the initial fractional solutions, the BC explores more different node visit patterns within the search tree. The same is also true for other types of cuts that are generated by CPLEX. Overall, the performance of the BC when it uses one of the proposed separation algorithms, $\mathcal{A}1$ or $\mathcal{A}2$, is better than when it employs CVRPSEP.

The results in Tables 5-9 indicate that instances in the second class are generally harder and it takes longer for the BC method to solve them (higher average CPUs and lower %UB and %BUB). Within the specified time limit, the BC obtains fewer optimal solutions for the instances in this class compared to when it is applied to the instances in the first and the third class. Instances in the third class are relatively easier to solve compared to the other ones. The BC method obtains the largest number of optimal solutions and lowest average gaps for the instances in this class within the smallest average solution time.

7. Summary

We generalized the assumptions of the assembly routing problem (ARP) to the case where each supplier may provide a subset of the components necessary for production. We presented a mixed integer linear programming model for this problem. We also developed many randomly generated test instances for this problem, for which we obtained good quality upper bounds by adapting the matheuristic of Chitsaz et al. (2019). To solve the problem to optimality, we proposed several types of valid inequalities and analyzed their performance with respect to the LP solution value of the model. Based on the valid inequalities, we proposed a branch-and-cut algorithm and performed extensive experiments to analyze different aspects of the algorithm. In addition, we have developed two algorithms to separate multi-period fractional capacity cut constraints and compared their efficiency with the state-of-the-art separation procedures of Lysgaard et al. (2004) for the single-period VRPs.

Our extensive computational experiments indicate that applying our newly developed valid inequalities significantly improves the performance of the branch-and-cut algorithm. Furthermore, the performance of the branch-and-cut algorithm is substantially enhanced when it employs any of our new separation procedures compared to the case when it uses the separation procedures offered in Lysgaard et al. (2004).

An interesting avenue for future research on the ARP is to compare different reformulations. The ARP is an integrated problem that considers lot-sizing (with an assembly structure) and capacitated vehicle routing problems at the same time. Beside the standard formulation for the LSP, it is possible to consider echelon stock, facility location, and shortest path, among others (Pochet and Wolsey 2006). Available formulations for the VRP (Toth and Vigo 2014) are standard, single-/two-/multi-commodity formulations as well as path-based formulations. These result in a large number of promising possibilities to present reformulations for the ARP.

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Online Supplementary Material

A Branch-and-Cut Algorithm for an Assembly Routing Problem

1. Proofs

Proposition 1. *Inequalities*

$$\sum_{e=t_1}^{t_2} p_e \leq I_{0k,t_1-1} + \sum_{i\in N_k} I_{ik,t_1-1} + \sum_{e=t_1}^{t_2}\sum_{i\in N_k} s_{ikt_1e} y_e \quad \forall k \in K, \forall t_1, t_2 \in T, t_1 \leq t_2 \tag{19}$$

are valid for the $\mathcal{M}_{ARP}$*.*

Proof. The inequalities for $\sum_{e=t_1}^{t_2} y_e = 0$ are trivial because $\sum_{e=t_1}^{t_2} p_e = 0$. Otherwise, let θ be the last period in which the production setup is performed, i.e., $\theta = \max_e\{t_1 \leq e \leq t_2 | y_e = 1\}$. Then,

$$\begin{aligned}
\sum_{e=t_1}^{t_2} p_e &= \sum_{e=t_1}^{\theta} p_e \\
&= \sum_{e=t_1}^{\theta} (I_{0k,e-1} - I_{0ke} + \sum_{i\in N_k} q_{ike}) \\
&= \sum_{e=t_1}^{\theta} \Big(I_{0k,e-1} - I_{0ke} + \sum_{i\in N_k} (I_{ik,e-1} - I_{ike} + s_{ike})\Big) \\
&= I_{0k,t_1-1} - I_{0k\theta} + \sum_{i\in N_k} (I_{ik,t_1-1} - I_{ik\theta} + s_{ikt_1\theta}) \\
&\leq I_{0k,t_1-1} + \sum_{i\in N_k} I_{ik,t_1-1} + \sum_{i\in N_k} s_{ikt_1\theta} \\
&= I_{0k,t_1-1} + \sum_{i\in N_k} I_{ik,t_1-1} + \sum_{i\in N_k} s_{ikt_1\theta} y_\theta \\
&\leq I_{0k,t_1-1} + \sum_{i\in N_k} I_{ik,t_1-1} + \sum_{e=t_1}^{t_2}\sum_{i\in N_k} s_{ikt_1e} y_e.
\end{aligned}$$

The first four equations follow from the definition of θ, constraints (3), constraints (4), and the definition of $s_{ikt_1t_2}$, respectively. The first inequality holds due to the non-negativity of inventory variables. The next equation is valid because $y_\theta = 1$. The last inequality holds because the y_e variables are nonnegative. □

Proposition 2. *Inequalities*

$$\sum_{e=t_1}^{t_2} q_{ike} \leq I_{ik,t_1-1} + \sum_{e=t_1}^{t_2} s_{ikt_1e} z_{ie} \quad \forall i \in N, \forall k \in K_i, \forall t_1, t_2 \in T, t_1 \leq t_2 \tag{20}$$

are valid for the $\mathcal{M}_{ARP}$.

Proof. If $\sum_{e=t_1}^{t_2} z_{ie} = 0$, then the supplier i will not be visited during periods t_1 to t_2. Therefore, for these periods no shipment is possible ($\sum_{e=t_1}^{t_2} q_{ike} = 0$) and inequalities (20) are satisfied. Otherwise, let θ be the last period in which the supplier i will be visited, i.e., $\theta = \max_e\{t_1 \leq e \leq t_2 | z_{ie} = 1\}$. Then,

$$\begin{aligned}
\sum_{e=t_1}^{t_2} q_{ike} &= \sum_{e=t_1}^{\theta} q_{ike} \\
&= \sum_{e=t_1}^{\theta} (I_{ik,e-1} - I_{ike} + s_{ike}) \\
&= I_{ik,t_1-1} - I_{ik\theta} + s_{ikt_1\theta} \\
&\leq I_{ik,t_1-1} + s_{ikt_1\theta} \\
&= I_{ik,t_1-1} + s_{ikt_1\theta} z_{i\theta} \\
&\leq I_{ik,t_1-1} + \sum_{e=t_1}^{t_2} s_{ikt_1e} z_{ie}.
\end{aligned}$$

The first three equations hold due to the definition of θ, constraints (4), and the definition of $s_{ikt_1t_2}$, respectively. The first inequality is valid because of the non-negativity of inventory variables. The next equality is valid for the reason that $z_{i\theta} = 1$. The last inequality holds because the z_{ie} variables are nonnegative. □

Proposition 3. *Inequalities*

$$\sum_{e=t_1}^{t_2} \sum_{i \in N_k} q_{ike} \leq I_{00t_2} + I_{0kt_2} + \sum_{e=t_1}^{t_2} d_{et_2} \sum_{i \in N_k} z_{ie} \quad \forall k \in K, \forall t_1, t_2 \in T, t_1 \leq t_2 \tag{21}$$

are valid for the $\mathcal{M}_{ARP}$.

Proof. If $\sum_{e=t_1}^{t_2} \sum_{i \in N_k} z_{ie} = 0$, then no visit to the suppliers $i \in N_k$ will be made during periods t_1 to t_2 and hence no shipment of component k is possible during this period ($\sum_{e=t_1}^{t_2} \sum_{i \in N_k} q_{ike} = 0$). Then, inequalities (21) are satisfied because the left-hand-side (LHS) will be equal to zero and the inventory variables in the right-hand-side (RHS) are nonnegative. Otherwise, let θ be the first period in which at least one node $i \in N_k$ is visited, i.e., $\theta = \min_e\{t_1 \leq e \leq t_2 | \sum_{i \in N_k} z_{ie} \geq 1\}$.

Then,

$$\begin{aligned}
\sum_{e=t_1}^{t_2}\sum_{i\in N_k} q_{ike} &= \sum_{e=\theta}^{t_2}\sum_{i\in N_k} q_{ike} \\
&= \sum_{e=\theta}^{t_2}(I_{0ke} - I_{0k,e-1} + p_e) \\
&= \sum_{e=\theta}^{t_2}\Big(I_{0ke} - I_{0k,e-1} + (I_{00e} - I_{00,e-1} + d_e)\Big) \\
&= I_{00t_2} - I_{00,\theta-1} + I_{0kt_2} - I_{0k,\theta-1} + d_{\theta t_2} \\
&\le I_{00t_2} + I_{0kt_2} + d_{\theta t_2} \\
&\le I_{00t_2} + I_{0kt_2} + d_{\theta t_2}\sum_{i\in N_k} z_{i\theta} \\
&\le I_{00t_2} + I_{0kt_2} + \sum_{e=\theta}^{t_2} d_{et_2}\sum_{i\in N_k} z_{ie} \\
&= I_{00t_2} + I_{0kt_2} + \sum_{e=t_1}^{t_2} d_{et_2}\sum_{i\in N_k} z_{ie}.
\end{aligned}$$

The first four equations follow from the definition of θ, constraints (3), constraints (2), and the definition of $d_{t_1t_2}$, respectively. The first inequality holds due to the non-negativity of inventory variables. The next inequality is valid because at least one node is visited in period θ, i.e., $\sum_{i\in N_k} z_{i\theta} \ge 1$. The last inequality is valid since the z_{ie} variables are nonnegative. The last equation holds due to the assumption that θ is the first period in which at least one node $i \in N_k$ is visited. □

Lemma 1. *Inequalities*

$$\max\{0, \mathcal{Q}_{it}\} \le \sum_{e=1}^{t}\sum_{k\in K_i} b_k q_{ike} \quad \forall i \in N, t \in T$$

are valid for $\mathcal{M}_{ARP}$.

Proof. We have

$$\begin{aligned}
\mathcal{Q}_{it} &\le \sum_{k\in K_i} b_k(s_{ik1t} + I_{ik0}) - \sum_{k\in K_i} b_k I_{ikt} \\
&= \sum_{k\in K_i} b_k \sum_{e=1}^{t}(s_{ike} + I_{ik,e-1} - I_{ike})
\end{aligned}$$

$$= \sum_{e=1}^{t} \sum_{k \in K_i} b_k q_{ike},$$

where the inequality follows from the storage capacity constraints (8), and the equations hold due to the definition of $s_{ikt_1t_2}$ and constraints (4), respectively. Because only a strictly positive $\mathcal{Q}_{it}$ triggers the shipment to the plant, we obtain:

$$\max\{0, \mathcal{Q}_{it}\} \leq \textstyle\sum_{e=1}^{t} \sum_{k \in K_i} b_k q_{ike}.$$

$\square$

Proposition 4. *Inequalities*

$$\left\lceil \frac{\max\left\{0, d_{1t} - I_{000}, (\sum_{k \in K} b_k I_{0k0} + \sum_{i \in N} \max\{0, \mathcal{Q}_{it}\} - L)/\sum_{k \in K} b_k\right\}}{\min\{C, \max_{e \in \{1,\ldots,t\}}\{d_e\} + L_0\}} \right\rceil \leq \sum_{e=1}^{t} y_e \quad \forall t \in T \tag{22}$$

are valid for $\mathcal{M}_{ARP}$.

Proof. We first obtain two lower bounds on the cumulative production from period 1 to t.

$$\begin{aligned} \sum_{e=1}^{t} p_e &= \sum_{e=1}^{t} (d_e + I_{00e} - I_{00,e-1}) \\ &= d_{1t} + I_{00t} - I_{000} \\ &\geq d_{1t} - I_{000}. \end{aligned}$$

The first and the second equations hold because of constraints (2), and the definition of $d_{t_1t_2}$, respectively. The inequality is valid due to the non-negativity of the inventory variables. Moreover,

$$\begin{aligned} \sum_{k \in K} b_k \sum_{e=1}^{t} p_e &= \sum_{k \in K} b_k \sum_{e=1}^{t} (I_{0k,e-1} - I_{0ke} + \sum_{i \in N_k} q_{ike}) \\ &= \sum_{k \in K} b_k I_{0k0} - \sum_{k \in K} b_k I_{0kt} + \sum_{i \in N} \sum_{e=1}^{t} \sum_{k \in K_i} b_k q_{ike} \\ &\geq \sum_{k \in K} b_k I_{0k0} - L + \sum_{i \in N} \max\{0, \mathcal{Q}_{it}\}. \end{aligned}$$

The first equation follows from constraints (3). The second equation is obtained by rearranging the terms. The inequality holds based on the component storage capacity at the suppliers and Lemma 1. Next, we determine two upper bounds on the cumulative production from period 1 to t. The cumulative production amount forces a minimum number of production setups due to production capacity constraints (5): $\sum_{e=1}^{t} p_e \leq C \sum_{e=1}^{t} y_e$. Then, we present another expression for the minimum number of required production setups:

$$\begin{aligned}
\sum_{e=1}^{t} p_e &\leq \sum_{e=1}^{t} (d_e + I_{00e}) y_e \\
&\leq \sum_{e=1}^{t} \max_{e' \in \{1,\ldots,t\}} \{d_{e'} + I_{00e'}\} y_e \\
&= \max_{e' \in \{1,\ldots,t\}} \{d_{e'} + I_{00e'}\} \sum_{e=1}^{t} y_e \\
&\leq \big(\max_{e' \in \{1,\ldots,t\}} \{d_{e'}\} + L_0 \big) \sum_{e=1}^{t} y_e.
\end{aligned}$$

The first inequality is valid since $p_t = d_t + I_{00t} - I_{00t-1} \leq d_t + I_{00t}$, and the fact that $y_t = 0$ forces $p_t = 0$. The second inequality and the equation hold trivially. The last inequality is valid because of the product storage capacity (L_0). Combining the two parts of the proof, we obtain:

$$\max \Big\{ 0, d_{1t} - I_{000}, \big(\sum_{k \in K} b_k I_{0k0} + \sum_{i \in N} \max\{0, \mathcal{Q}_{it}\} - L \big) / \sum_{k \in K} b_k \Big\} \leq \\ \sum_{e=1}^{t} p_e \leq \min \big\{ C, \max_{e \in \{1,\ldots,t\}} \{d_e\} + L_0 \big\} \sum_{e=1}^{t} y_e.$$

□

Proposition 5. *Inequalities*

$$\left\lceil \frac{1}{Q} \max \Big\{ \sum_{k \in K} b_k \max\{0, d_{1t} - I_{000} - I_{0k0}\}, \sum_{i \in N} \max\{0, \mathcal{Q}_{it}\} \Big\} \right\rceil \leq \sum_{e=1}^{t} z_{0e} \quad \forall t \in T \tag{23}$$

are valid for $\mathcal{M}_{ARP}$.

Proof. We obtain the first expression as follows:

$$\begin{aligned}
\sum_{e=1}^{t} Q z_{0e} &\geq \sum_{e=1}^{t} \sum_{k \in K} \sum_{i \in N_k} b_k q_{ike} \\
&= \sum_{e=1}^{t} \sum_{k \in K} b_k (d_e + I_{00e} - I_{00,e-1} + I_{0ke} - I_{0k,e-1}) \\
&= \sum_{k \in K} b_k (d_{1t} + I_{00t} - I_{000} + I_{0kt} - I_{0k0}) \\
&\geq \sum_{k \in K} b_k (d_{1t} - I_{000} - I_{0k0}).
\end{aligned}$$

The first inequality is valid since the LHS is the total capacity of the dispatched vehicles from period $e = 1$ to t, and the RHS is the total shipped amount over the same periods, all components and all suppliers. The first equation follows from constraints (3), and by replacing the p_t variables using constraints (2). The second equation is valid due to the definition of $d_{t_1 t_2}$. The second inequality holds due to the non-negativity of inventory variables. Next, we have

$$\begin{aligned}\sum_{e=1}^{t} Q z_{0e} &\geq \sum_{e=1}^{t} \sum_{i \in N} \sum_{k \in K_i} b_k q_{ike} \\ &\geq \sum_{i \in N} \max\{0, \mathcal{Q}_{it}\},\end{aligned}$$

where the first inequality is valid because of the total fleet capacity, and the second inequality follows from Lemma 1. □

Proposition 6. *Inequalities*

$$\left\lceil \frac{\max\{0, \mathcal{Q}_{it}\}}{\min\left\{Q, L_i + \max_{e \in \{1,\ldots,t\}}\{\sum_{k \in K_i} b_k s_{ike}\}, \sum_{k \in K_i} b_k (I_{ik0} + s_{ik1t})\right\}} \right\rceil \leq \sum_{e=1}^{t} z_{ie} \quad \forall i \in N, \forall t \in T \tag{24}$$

are valid for $\mathcal{M}_{ARP}$*.*

Proof. Based on Lemma 1 we know that

$$\max\{0, \mathcal{Q}_{it}\} \leq \textstyle\sum_{e=1}^{t} \sum_{k \in K_i} b_k q_{ike}.$$

Now, we present upper bounds on the cumulative shipments from node i during period 1 to t. The vehicle capacity constraints (10) provide the first upper bound: $\sum_{e=1}^{t} \sum_{k \in K_i} b_k q_{ike} \leq Q \sum_{e=1}^{t} z_{ie}$. Next, we have

$$\begin{aligned}\sum_{e=1}^{t} \sum_{k \in K_i} b_k q_{ike} &\leq \sum_{e=1}^{t} (L_i + \sum_{k \in K_i} b_k s_{ike}) z_{ie} \\ &\leq \sum_{e=1}^{t} \big(L_i + \max_{e' \in \{1,\ldots,t\}} \{\sum_{k \in K_i} b_k s_{ike'}\}\big) z_{ie} \\ &= \big(L_i + \max_{e' \in \{1,\ldots,t\}} \{\sum_{k \in K_i} b_k s_{ike'}\}\big) \sum_{e=1}^{t} z_{ie}.\end{aligned}$$

Where the first inequality follows from $\sum_{k \in K_i} b_k q_{ikt} \leq L_i + \sum_{k \in K_i} b_k s_{ikt}$ which is valid due to constraints (4) and (8), and the fact that $z_{it} = 0$ forces $\sum_{k \in K_i} b_k q_{ikt} = 0$. The second inequality and the equation hold trivially. Moreover, we have

$$\begin{aligned}\sum_{e=1}^{t}\sum_{k\in K_i} b_k q_{ike} &\le \sum_{e=1}^{t}\sum_{k\in K_i} b_k(I_{ik0}+s_{ik1e})z_{ie}\\ &\le \sum_{e=1}^{t}\sum_{k\in K_i} b_k\big(I_{ik0}+\max_{e'\in\{1,\dots,t\}}\{s_{ik1e'}\}\big)z_{ie}\\ &= \sum_{e=1}^{t}\sum_{k\in K_i} b_k(I_{ik0}+s_{ik1t})z_{ie}\\ &= \sum_{k\in K_i} b_k(I_{ik0}+s_{ik1t})\sum_{e=1}^{t} z_{ie}.\end{aligned}$$

Where the first inequality is valid for the reason that $q_{ike} \le I_{ik0} + s_{ik1e}$ which is valid due to constraints (4), the definition of $s_{ikt_1t_2}$, and the fact that $z_{it} = 0$ forces $\sum_{k\in K_i} b_k q_{ikt} = 0$. The second inequality holds trivially. The first equation follows from $\max_{e'\in\{1,\dots,t\}}\{s_{ik1e'}\} = s_{ik1t}$. The second equation holds trivially. Consequently, we obtain

$$\begin{aligned}\max\{0,\mathcal{Q}_{it}\} &\le \sum_{e=1}^{t}\sum_{k\in K_i} b_k q_{ike}\\ &\le \min\big\{Q, L_i+\max_{e\in\{1,\dots,t\}}\{\sum_{k\in K_i} b_k s_{ike}\}, \sum_{k\in K_i} b_k(I_{ik0}+s_{ik1t})\big\}\sum_{e=1}^{t} z_{ie}.\end{aligned}$$

□

Proposition 7. *Inequalities*

$$\left\lceil \frac{\max\{0, d_{1t}-I_{000}-I_{0k0}\}}{\min\big\{\frac{Q}{b_k}, \max_{i\in N_k}\{I_{ik0}+s_{ik1t}\}\big\}}\right\rceil \le \sum_{e=1}^{t}\sum_{i\in N_k} z_{ie} \quad \forall k\in K, \forall t\in T \tag{25}$$

are valid for $\mathcal{M}_{ARP}$.

Proof. We have

$$d_{1t}-I_{000}-I_{0k0} \le \textstyle\sum_{e=1}^{t}\sum_{i\in N_k} q_{ike},$$

which can be obtained by replacing p_t using constraints (2) in constraints (3), and the non-negativity of the inventory variables. Next, we have

$$\textstyle\sum_{e=1}^{t}\sum_{i\in N_k} q_{ike} \le \frac{Q}{b_k}\sum_{e=1}^{t}\sum_{i\in N_k} z_{ie},$$

which is valid due to $b_k q_{ikt} \leq Q z_{it}$. Furthermore, we have

$$\begin{aligned}\sum_{i \in N_k} \sum_{e=1}^{t} q_{ike} &\leq \sum_{i \in N_k} (I_{ik0} + s_{ik1t}) \sum_{e=1}^{t} z_{ie} \\ &\leq \sum_{i \in N_k} \max_{i' \in N_k} \{I_{i'k0} + s_{i'k1t}\} \sum_{e=1}^{t} z_{ie} \\ &= \max_{i' \in N_k} \{I_{i'k0} + s_{i'k1t}\} \sum_{i \in N_k} \sum_{e=1}^{t} z_{ie}.\end{aligned}$$

Where the first inequality comes from constraints (4), and by checking for $\sum_{e=1}^{t} z_{ie} = 0$ and $\sum_{e=1}^{t} z_{ie} \geq 1$. The second inequality and the equation are valid trivially. Finally, we obtain

$$\begin{aligned}\max\{0, d_{1t} - I_{000} - I_{0k0}\} &\leq \sum_{e=1}^{t} \sum_{i \in N_k} q_{ike} \\ &\leq \min\Big\{\frac{Q}{b_k}, \max_{i \in N_k} \{I_{ik0} + s_{ik1t}\}\Big\} \sum_{e=1}^{t} \sum_{i \in N_k} z_{ie}.\end{aligned}$$

□

2. Adaptation of CCJ-DH

In this section, we present the adaptation of CCJ-DH (Chitsaz et al. 2019) to the generalized version of the ARP. The algorithm decomposes the problem into three distinct subproblems. The framework of the heuristic is presented in Figure 1.

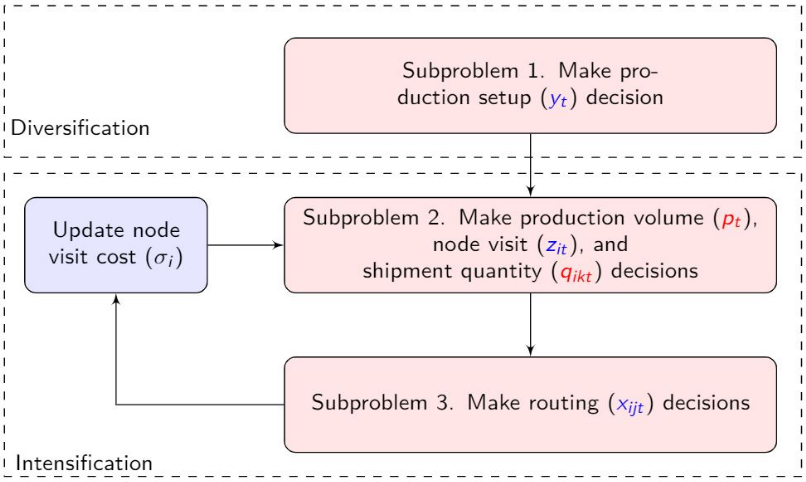


Figure 1: CCJ-DH framework

The first subproblem returns a setup schedule. It uses an approximate transportation cost based on the number of vehicles dispatched from the plant. This results in the following objective function:

$$\min \sum_{t \in T} \left(up_t + f y_t + \sum_{k \in K^+} h_{0k} I_{0kt} + \sum_{i \in N} \sum_{k \in K_i} h_{ik} I_{ikt} + \sigma_{0t} z_{0t} \right) \tag{26}$$

where σ_{0t} is the cost of each vehicle dispatch. This objective function does not include any routing decision and hence constraints (11)-(12) become redundant. To impose the aggregate fleet capacity in the first subproblem, the algorithm adds the following constraints to constraints (3)-(10), and (13)-(15):

$$\sum_{i \in N} \sum_{k \in K_i} b_k q_{ikt} \leq Q z_{0t} \quad \forall t \in T. \tag{27}$$

After solving this subproblem using CPLEX, the algorithm fixes the setup schedule and uses it as a given parameter in the second subproblem.

The second subproblem returns node visit and shipment quantity decisions. The algorithm employs another approximation of the transportation cost in the objective function based on the cost associated with visiting each supplier (node). This results in the following objective function:

$$\min \sum_{t \in T} \Big(up_t + \sum_{k \in K^+} h_{0k} I_{0kt} + \sum_{i \in N} \sum_{k \in K_i} h_{ik} I_{ikt} + \sum_{i \in N} \sigma_{it} z_{it} \Big) \tag{28}$$

where σ_{it} represents the node visit cost estimation. Similarly as in the first subproblem, this subproblem ignores the routing decisions. To enforce the vehicle capacity and to make sure that the shipments can be packed into the available vehicles, the algorithm considers the following constraints as well as constraints (3)-(8), (10), and (14)-(15) in the second subproblem:

$$\sum_{i \in N} \sum_{k \in K_i} b_k q_{ikt} \leq \lambda_t m Q \quad \forall t \in T. \tag{29}$$

Here, $\lambda_t = 1 - \frac{2}{n}$ is a parameter. CCJ-DH solves this subproblem using CPLEX. Having the node visit and the shipment quantity decisions fixed for each time period, the algorithm solves one capacitated VRP for each period as the third subproblem. CCJ-DH uses the tabu search heuristic of Cordeau et al. (1997) to solve the VRPs.

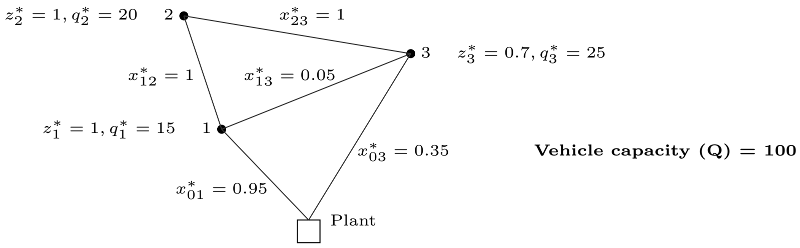
To intensify the search, CCJ-DH updates the node visit cost estimates (σ_{it}) for the next iteration. The algorithm uses two estimation mechanisms. The first mechanism is the cheapest insertion cost among all existing routes. The second mechanism splits the cost of each route (in each period) over its nodes proportional to their direct shipment cost. In this mechanism, if a node is not visited in a certain period, the algorithm considers the direct shipment cost as the estimated cost for that node. CCJ-DH switches between these two mechanisms after using each for 7 consecutive iterations.

To diversify the search, the algorithm adds a local branching type cut (Fischetti et al. 2004) to the set of constraints in the first subproblem in order to consider a new setup schedule. The stopping condition for the overall algorithm is a maximum of 200 intensification iterations. To perform a diversification, CCJ-DH considers two stopping conditions: a maximum of 80 intensification iterations, or 60 intensification iterations without incumbent solution improvement.

3. Examples for Fractionally Violated and Non-Violated Subtours

Figure 2 shows an example where CVRPSEP returns a violated VRP CCC which is a non-violated ARP GFSEC in the ARP (or the IRP and the PRP). Figure 3 shows an example for the case that a fractionally violated GFSEC or DFJ in the ARP (or the IRP and the PRP) cannot be found if the node visit variables (z_{it}) are not considered.

Figure 2: A violated VRP CCC which is a non-violated GFSEC.



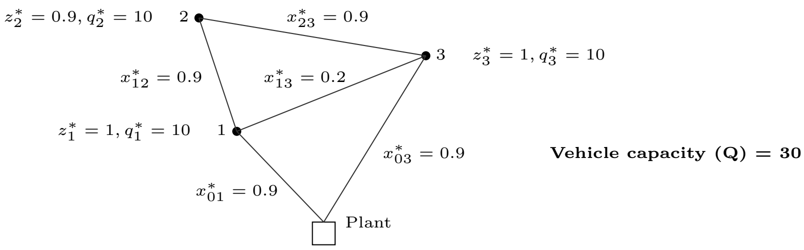
Violated VRP subtour, $S = \{1, 2, 3\}$: $1 + 1 + 0.05 = 2.05 > |S| - 1 = |3| - 1 = 2$

$$\textbf{LHS} = Q \sum_{(i,j)\in E(S)} x_{ij}^* = 100 * (1 + 1 + 0.05) = \mathbf{205}$$

$$\textbf{RHS} = \sum_{i\in S} (Qz_i^* - \sum_{k\in K_i} b_k q_{ik}^*) = 100 * (1 + 1 + 0.7) - (15 + 20 + 25) = \mathbf{210}$$

LHS $<$ **RHS** Satisfied (non-violated) fractional ARP GFSEC

Figure 3: Violated ARP GFSEC and DFJ which is a non-violated VRP CCC and DFJ.



Non-violated VRP DFJ, $S = \{1, 2, 3\}$: $0.9 + 0.9 + 0.2 = 2 = |S| - 1 = |3| - 1 = 2$

Non-violated VRP CCC:

$$\textbf{LHS} = Q \sum_{(i,j)\in E(S)} x_{ij}^* = 30 * (0.9 + 0.9 + 0.2) = \mathbf{60}$$

$$\textbf{RHS} = \sum_{i\in S} (Q - \sum_{k\in K_i} b_k q_{ik}^*) = 3 * (30 - 10) = \mathbf{60}$$

LHS $=$ **RHS** Satisfied (non-violated) fractional VRP CCC

Violated ARP DFJ, $S = \{1, 2, 3\}$: $0.9 + 0.9 + 0.2 = 2 > (z_1^* + z_2^* + z_3^*) - z_1^* = (1 + 0.9 + 1) - 1 = 1.9$

LHS $>$ **RHS** Violated fractional ARP DFJ

Violated ARP GFSEC: $S = \{1, 2, 3\}$: $\textbf{LHS} = Q \sum_{(i,j)\in E(S)} x_{ij}^* = 30 * (0.9 + 0.9 + 0.2) = \mathbf{60}$

$$\textbf{RHS} = \sum_{i\in S} (Qz_i^* - \sum_{k\in K_i} b_k q_{ik}^*) = 30 * (1 + 0.9 + 1) - (10 + 10 + 10) = \mathbf{57}$$

LHS $>$ **RHS** Violated fractional ARP GFSEC

4. Results on the Large ARP Instances of Chitsaz et al. (2019)

Chitsaz et al. (2019) presented two lower bounding methods for the ARP. The first method (BC-T) is a truncated BC with a time limit of 12 hours. BC-T uses the best-bound node selection strategy. It adds inequalities (26) and (28) a priori to the model, and SECs (12) and (27) dynamically through the search using the CVRPSEP package for separation. The second method (MIP-CP) relaxes SECs (12) from the model and solves the resulting MIP. Then, it iteratively adds the violated SECs (12) as cutting planes for the resulting integral subtours and re-solves the new MIP. A time limit of five hours is set for this method.

In Table 10, we present the performance of CCJ-DH, BC-T, and MIP-CP, and compare them with our BC. In these experiments, the BC uses all inequalities and implements algorithm $\mathcal{A}2$ to separate SECs. Two branching node selection strategies are examined: balanced between optimality and feasibility (default) or the best-bound node selection. Because BC-T is able to solve the small instances with 14 suppliers in the first set (MV-C1) to optimality in a very short time, we did not apply our BC to these instances. Columns four to six present the results for CCJ-DH: CPU, #BUB, and the average solution value as a percentage of the best lower bound obtained by the BC method (%BLB). Columns 7 to 11 show the results for BC-T: CPU, #BUB, the number of best lower bounds (#BLB), %UB, and %BUB. Columns 12 to 14 show the results for MIP-CP which only generates lower bounds: CPU, #BLB, and %BUB. Columns 15 to 19, and 20 to 24 include similar results as columns 7 to 11 for the BC of this paper with the default and with the best-bound node selection strategies, respectively.

Columns under #BUB and %UB for the BC-T and our BC methods reflect the results without considering the CCJ-DH cutoffs. The comparison of columns under %UB and %BUB for each of the BC-T and our BC methods shows the effectiveness of CCJ-DH in finding upper bounds for these large instances. Most of the BUBs for the instances with $n = 50$ and all of the BUBs for the instances with $n = 100$ are obtained by CCJ-DH. BC-T is unable to find upper bounds for the instances with $n = 100$. Therefore, it returns zero under column %UB in all four classes of these instances. Our BC with the best-bound node selection strategy is performing better than with the default node selection strategy. Moreover, it outperforms the two other methods presented in Chitsaz et al. (2019), both in terms of number of BLBs, and %BUBs.

Finally, we present more details on the performance of our BC with the default and with the

Table 10: Comparison of the BC performance with the lower bounding methods presented in Chitsaz et al. (2019)

			Chitsaz et al. (2019)												BC (This paper)										
			CCJ-DH			BC-T					MIP-CP				Default					Best-Bound					
n	Class	Size	CPU	#BUB	%BLB	CPU†	#BUB	#BLB	%UB	%BUB	CPU‡	#BLB	%BUB		CPU††	#BUB	#BLB	%UB	%BUB	CPU††	#BUB	#BLB	%UB	%BUB	
50	1	120	602.8	116	99	43200	2	0	52	98.3	18000	0	97.9		3600	1	0	47.6	98.6	3600	1	120	23	99	
	2	120	592.4	112	99	43200	7	1	52.1	98.5	18000	0	97.9		3600	0	1	40.6	98.6	3600	1	118	23.7	99	
	3	120	467.8	119	96.2	43200	1	0	35.4	93.9	18000	0	91.7		3600	0	2	29.5	94.6	3600	0	118	10.1	96.2	
	4	120	914.4	109	99.3	43200	10	1	72.6	99	18000	24	99		3600	1	0	51.3	98.9	3600	0	95	24	99.3	
	Total	480	644.4	456	98.3	43200	20	2	53	97.4	18000	24	96.6		3600	2	3	42.3	97.7	3600	2	451	20.2	98.4	
100	1	120	2966.6	120	97.9	43200	0	9	0	97.1	18000	4	97.3		3600	0	25	1.4	97.1	3600	0	82	3.4	97.9	
	2	120	2931.6	120	97.9	43200	0	8	0	97.1	18000	2	97.3		3600	0	15	2.6	97.4	3600	0	95	2.6	97.9	
	3	120	1971.3	120	91.4	43200	0	8	0	89.7	18000	1	89.2		3600	0	26	0.3	90.5	3600	0	85	0	91.3	
	4	120	4212.6	120	98.6	43200	0	14	0	97.4	18000	46	98.3		3600	0	9	2.5	97.7	3600	0	51	2.6	98.5	
	Total	480	3020.5	480	96.5	43200	0	39	0	95.3	18000	53	95.5		3600	0	75	1.7	95.7	3600	0	313	2.2	96.4	
Total		960	1832.4	936	97.4	43200	20	41	26.5	96.4	18000	77	96.1		3600	2	78	22	96.7	3600	2	764	11.2	97.4	

Size: Number of instances, † Time limit = 12 hours and maximum 30 GB memory, ‡ Time limit = 5 hours and maximum 30 GB memory, †† Time limit = one hour and maximum 25 GB memory best-bound node selection strategies in Table 11. In this table we present #Node, GFS, AV^{GFS}, DFJ, and AV^{DFJ}. Although within the default node selection strategy the BC explores more nodes, the best-bound strategy returns better lower bounds. Another interesting observation is that the method with the default node selection strategy applies more GFSECs and DFJs with almost the same average violation on the instances with $n = 50$. This reflects the fact that the method with the default node selection strategy explores some nodes that do not contribute much to improve the lower bound.

Table 11: Summary of the results of the BC on the large ARP instances of Chitsaz et al. (2019) with different node selection strategies

Node Selection	n	Class	Size	%UB	%BUB	#Node	GFS	AV^{GFS}	DFJ	AV^{DFJ}
Default	50	1	120	47.6	98.6	2014.3	1625	0.21	6039	0.4
	50	2	120	40.6	98.6	1778.9	1533	0.21	5666	0.4
	50	3	120	29.5	94.6	1547	1814	0.21	5882	0.39
	50	4	120	51.3	98.9	2434.6	1069	0.22	5640	0.48
	Total		480	42.3	97.7	1944.2	1510	0.21	5806	0.42
	100	1	120	1.4	97.1	4.6	1939	0.28	3549	0.37
	100	2	120	2.6	97.4	5.3	2032	0.28	3728	0.36
	100	3	120	0.3	90.5	0.6	2263	0.25	3859	0.32
	100	4	120	2.5	97.7	35.8	1346	0.32	3429	0.48
	Total		480	1.7	95.7	11.5	1896	0.28	3641	0.38
Best-Bound	50	1	120	23	99	987.1	1160	0.22	3907	0.39
	50	2	120	23.7	99	1070.1	1146	0.22	4047	0.39
	50	3	120	10.1	96.2	653	1336	0.22	3760	0.37
	50	4	120	24	99.3	2255.2	700	0.24	3969	0.5
	Total		480	20.2	98.4	1242.1	1085	0.23	3921	0.41
	100	1	120	3.4	97.9	1.7	1921	0.28	3668	0.38
	100	2	120	2.6	97.9	1.3	2098	0.28	3730	0.37
	100	3	120	0	91.3	0.1	2140	0.26	3970	0.33
	100	4	120	2.6	98.5	22.6	1442	0.32	3664	0.48
	Total		480	2.2	96.4	6.4	1899	0.28	3757	0.39

Size: Number of instances, Time limit = 1 hour

5. Detailed Results on Effect of Valid Inequalities

Each type of valid inequality introduced in Section 3 of the main paper has a different effect on the LP relaxation value and solution time of the $\mathcal{M}_{ARP}$ model. To evaluate the effect of applying different inequality types, we performed a sensitivity analysis considering different scenarios. We consider the effect on the LP solution value when only one inequality type is added to the model. Also, we evaluate the effect when all types of valid inequalities but one are added. Furthermore, we consider the cases where no valid inequality (None), known valid inequalities (Known) from the literature (i.e., (18), (26), and (28)), or all valid inequalities (All) (i.e., (18)-(26), and (28)) are added to the model. Similar to the results presented in Table 4, we present the obtained lower bound as a percentage of the best upper bound found by the BC method or CCJ-DH. Tables 12, 13 and 14 present the results for each class of instances. Each column number in these tables refers to the associated valid inequality type number presented in Section 3 of the paper. For the first class of instances, inequalities (18), (21) and (24) have the greatest impact. For the second and third classes of instances, inequalities (18), (22) and (24) show the largest LP solution value improvements.

Table 12: Effect of individual valid inequality types on average LP solution value as a percentage of BUB (class 1)

			Including only one type											Excluding only one type										
	Set		(l,S,WW)-type				Var Bnd				Gen Ineq			(l,S,WW)-type				Var Bnd				Gen Ineq		
$\mathcal{C}/l/n$	Size	None	(18)	(19)	(20)	(21)	(22)	(23)	(24)	(25)	(26)	(28)	Known	(18)	(19)	(20)	(21)	(22)	(23)	(24)	(25)	(26)	(28)	All
1/4/18	5	60.4	69.6	66.3	66.1	66.1	65.5	62	67.4	60.4	60.4	60.7	69.9	82.8	86.2	84.4	84.9	84.4	86	85.2	86.6	86.6	84.2	86.6
1/4/21	5	57.2	69.9	60.8	60.9	61.5	64.6	59.8	63.1	57.3	57.3	57.6	70.3	77.6	86.1	84.4	84.9	82.5	85.3	85.3	86.3	86.3	84.1	86.3
1/4/24	5	56.5	68.5	61	61	61.9	62	58.7	63	56.5	56.5	56.8	68.9	78.6	85.6	84.2	84.8	83.2	85.6	85.3	86.3	86.3	83.6	86.3
1/4/27	5	59.1	70.1	62.4	63.4	64	65.1	60.9	65.1	59.1	59.1	59.3	70.4	78.5	85.9	84.6	85.4	83.3	85.8	85.6	86.6	86.6	84.7	86.6
1/4/30	5	62.1	76.3	65.2	65.6	66.1	68.9	63.1	68.4	62.1	62.1	62.3	76.6	80.6	90.8	89.9	90.1	87.1	90.7	89.4	91	91	88.9	91
1/4/33	5	61	73.4	64.3	65.4	65.8	67.4	62.7	67.9	61	61	61.2	73.7	80.8	89.2	88.2	88.6	86	89	88	89.7	89.7	88.2	89.7
1/4/36	5	61.2	72.3	66.7	66.2	66.9	66	62.2	67.6	61.2	61.2	61.4	72.5	82.1	87.5	85.7	86.2	85	87.4	86.9	87.9	87.9	85.9	87.9
1/4/39	5	53.9	63.7	58.2	58.4	59.2	61.9	57	59.4	53.9	54	54.4	64.2	78.4	82.4	81.3	82	79.6	82.3	82.2	83.3	83.3	80.4	83.3
1/6/15	5	67.5	79.1	71.3	70.8	72.2	71.1	70.1	72.2	67.5	67.6	67.8	79.5	85.9	92.3	91.2	90.4	91.1	91	91.3	92.4	92.4	89.8	92.4
1/6/18	5	65.8	74	67.8	70.2	72.7	68.3	68	72.4	65.8	65.8	66.1	74.2	83.8	89	87.7	86.2	87.8	87.7	87.7	89	89	87	89
1/6/21	5	56.4	72	63.4	60.7	61.8	61.7	58	62.7	56.4	56.4	56.7	72.4	79.3	86.6	85.7	85.8	85.3	86.9	86.1	87.4	87.4	85.4	87.4
1/6/24	5	60.3	74	63.9	64.8	67.3	65.5	62.4	66.1	60.3	60.4	60.6	74.3	81.4	89.9	88.4	87	87.7	89.4	89.4	90	90	88	90
1/6/27	5	63.5	76.2	67.3	67.9	69.2	67.9	64.6	69.8	63.5	63.5	63.7	76.4	82.7	90.7	89.9	89.3	89.2	91.1	89.9	91.3	91.3	89	91.3
1/6/30	5	60.5	74.3	65.6	65.6	67.4	64.4	62.5	66.5	60.5	60.5	60.9	74.7	82.7	89.6	87.9	87	89	89.1	89.2	89.8	89.8	87.1	89.8
1/6/33	5	55.9	69.2	61.3	60.8	65.8	61.1	58.8	61.9	55.9	56	56.2	69.7	82.1	86.9	86.7	85.1	85.8	87.2	86.8	88	87.8	86.2	88
1/6/36	5	54	73.6	59.8	58.8	60.1	60.7	56.8	60.9	54	54.2	54.3	74	77.7	89.7	88.1	87.3	87.6	89	88.5	89.7	89.7	87.5	89.7
1/8/12	5	69.7	79	72.1	72.9	75.6	72.4	72	74.3	69.7	69.8	70	79.3	85.8	91.6	90.9	89.1	90.4	90.6	90.8	91.7	91.7	89.9	91.7
1/8/15	5	68.9	79.1	70.6	72	74.4	72.6	70.2	74.2	69.1	69	69.3	79.5	84.4	91.2	91	89.8	89.6	91.4	89.6	91.5	91.5	89.6	91.5
1/8/18	5	64.6	78.9	68.1	67.5	71.3	68	66.4	68.7	64.7	64.7	64.9	79.3	82.4	92.2	91.4	88.7	90.3	91.8	91.4	92.2	92.1	90.2	92.2
1/8/21	5	62.7	75.3	68.2	66.7	67.4	65.7	63.7	67.4	62.7	62.7	62.8	75.5	80.6	86.9	86.6	86.7	87.4	88.2	87.7	88.4	88.3	86.9	88.4
1/8/24	5	65.4	77.5	73.1	70	70.2	68.5	67.3	70.4	65.4	65.5	65.6	77.7	86	89.8	88.3	88.7	89.9	90.3	89.9	90.4	90.3	88.2	90.4
1/8/27	5	66.6	79.7	71.3	70.5	70.9	69.5	68.2	71.9	66.6	66.7	66.9	80	84.1	90.8	89.7	89.7	90.7	91	90.1	91.2	91.2	89.4	91.2
1/8/30	5	61.3	73.8	62.8	64.6	69.4	65.2	63.7	66.9	61.4	61.4	61.8	74.5	80.8	89.5	89.1	86.9	87.7	89.2	88.4	89.7	89.6	86.8	89.7
1/8/33	5	63	74.1	69.1	66.9	68.1	66.2	64.7	67.8	63	63	63.3	74.4	82.3	86	85.1	85.1	86.6	86	86.2	86.9	86.9	84.8	86.9
1/10/9	5	67	82.7	68.2	69.2	72.5	71.2	68.3	71.2	67.3	67.1	67.3	83.1	81.8	93.3	93.2	91	92.2	93.3	92.1	93.5	93.4	92.1	93.5
1/10/12	5	67.3	78.3	68.7	70.4	74.1	71.1	68.9	71.8	67.4	67.4	67.8	78.8	84.1	91.8	91.4	89	90.1	91.9	90.9	92	91.9	89.5	92
1/10/15	5	64.5	79	67.9	67.5	68.6	67.7	66.1	69	64.6	64.6	64.8	79.4	79.6	89.9	89.7	88.8	89.6	90.5	89.8	90.7	90.5	89.1	90.7
1/10/18	5	68.2	80.6	71.8	71.9	71.8	71	69.1	73.2	68.2	68.2	68.3	80.8	82.2	90.3	89.4	89.9	90.1	90.6	90	90.8	90.7	89.4	90.8
1/10/21	5	67.3	80.5	71.2	71.1	72.5	70	68.3	72.2	67.3	67.3	67.4	80.7	83.1	91.7	90.4	89.1	90.7	91.6	91	91.7	91.6	90.3	91.7
1/10/24	5	64.2	76.7	69.1	68.2	69.4	69.3	66.2	69.6	64.2	64.3	64.4	77	83.4	89.4	88.7	88.1	89	89.4	89	89.9	89.9	88.1	89.9
1/10/27	5	64.6	74.5	67.8	68.7	70.5	66.8	67.4	69.2	64.6	64.7	64.9	74.9	81.4	87.5	86.1	85.3	87.1	86.4	87.5	87.8	87.8	86.2	87.8
1/10/30	5	62.8	74	65.9	67.7	69.6	65.5	65.4	68.3	62.8	62.8	63.1	74.4	81.6	87.8	86.7	85.7	87.6	86.8	87.6	88.2	88.2	86.1	88.2
1/12/6	5	71.2	83	73.1	74.2	74.6	74.4	73.1	75.8	71.2	71.3	71.4	83.3	84.6	93	92.2	91.8	92.6	92.8	92.1	93.1	93	91.4	93.1
1/12/9	5	63.8	75.6	67.4	68.1	70.9	66.1	66	68.7	63.8	63.8	64.1	76	82.2	88.2	87.1	86	88.1	87.5	87.7	88.5	88.5	86.8	88.5
1/12/12	5	61	78.1	63.5	64	68.2	65	62.3	66.1	61.2	61	61.3	78.4	78.2	90.7	90.4	88.8	89.4	91	90	91.1	91	89.4	91.1
1/12/15	5	66.2	82.2	69.7	69.2	69.7	70.3	67.1	70.9	66.3	66.3	66.5	82.4	81.6	92.7	92	91.8	91.5	92.9	91.7	93	92.9	91.2	93
1/12/18	5	68.6	80.4	71.8	71.8	72	72	69.7	73.6	68.7	68.6	68.8	80.7	83.5	90.9	90.5	90.6	90.8	91.5	90.3	91.6	91.6	89.5	91.6
1/12/21	5	63.9	74	65.9	67.8	70.6	68.4	64.6	68.9	64.4	64	64.4	74.5	81.8	87.2	86.7	86.7	86.1	87.8	86.5	87.9	87.7	86.1	87.9
1/12/24	5	66.2	79.3	70.4	69.5	72.3	68.6	66.8	70.2	66.2	66.2	66.4	79.5	82.1	89.5	89.5	87.6	90.2	90.5	90.1	90.6	90.5	88.8	90.6
1/12/27	5	56.8	77.1	61.7	60.6	65.5	62.8	58.5	61.9	57.1	57	57.4	77.7	79.6	90.4	90.2	88	90.2	90.9	90.3	91.1	90.8	88.9	91.1
Total	200	63	75.7	66.9	66.9	68.7	67.3	64.8	68.4	63.1	63.1	63.3	76.1	81.8	89.3	88.4	87.7	88.1	89.2	88.7	89.7	89.7	87.7	89.7

Note. $\mathcal{C}/l/n$: Class/Number of periods/Number of suppliers, Var Bnd: Bounds on the variables, Gen Ineq: General inequalities

Table 13: Effect of individual valid inequality types on average LP solution value as a percentage of BUB (class 2)

			Including only one type											Excluding only one type										
	Set		(l,S,WW)-type				Var Bnd				Gen Ineq			(l,S,WW)-type				Var Bnd				Gen Ineq		
$\mathcal{C}/l/n$	Size	None	(18)	(19)	(20)	(21)	(22)	(23)	(24)	(25)	(26)	(28)	Known	(18)	(19)	(20)	(21)	(22)	(23)	(24)	(25)	(26)	(28)	All
2/4/18	5	71.9	81.9	71.9	74.9	72	76.6	72.7	78.5	71.9	71.9	72	82	85.9	92.8	91.8	92.7	91.2	92.4	87.8	92.8	92.7	91.7	92.8
2/4/21	5	69	76.9	69	71.5	69.2	75.6	70.7	75.1	69	69.1	69.2	77.2	85.1	89.7	88.6	89.6	86.3	89.1	85.6	89.7	89.6	88.6	89.7
2/4/24	5	64.6	78.7	64.6	67.9	65	71.7	65.7	71.5	64.6	64.7	64.8	78.9	82	91.3	90.2	91.1	89	90.8	86.4	91.3	91.2	89.7	91.3
2/4/27	5	66.7	81.3	66.7	70.1	66.9	73.7	67.7	73	66.7	66.7	66.8	81.5	83.1	92.9	91.7	92.8	90.7	92.5	88.7	92.9	92.8	91.7	92.9
2/4/30	5	68.7	80.8	68.7	72.5	68.9	74.6	69.7	75.9	68.7	68.7	68.9	80.9	85	92.6	91.3	92.6	91.2	92.4	87.9	92.6	92.6	91.4	92.6
2/4/33	5	69.4	80.6	69.4	73.4	69.6	75.1	70.3	76.2	69.4	69.4	69.5	80.7	85.3	92.3	90.7	92.2	91	92	87.8	92.3	92.2	91.1	92.3
2/4/36	5	65.6	77.5	65.6	70.4	65.7	71.1	67.6	73.3	65.6	65.7	65.8	77.8	83.7	91.7	89.9	91.7	90.1	90.7	87.1	91.7	91.7	90.2	91.7
2/4/39	5	55.2	70.3	58.2	60.9	55.9	64.9	56.7	62.6	55.2	55.2	55.4	70.6	79.2	88.3	85.2	88.4	85	87.9	84.9	88.4	88.4	86.7	88.4
2/6/15	5	72.9	82.2	72.9	77.1	73.1	76.8	74.2	77	72.9	72.9	73	82.4	85.6	92.7	90.3	92.7	91.1	92.2	90.4	92.7	92.7	91.5	92.7
2/6/18	5	63.1	77.6	63.1	68.1	63.3	68.9	64.6	68.4	63.1	63.1	63.3	77.9	79.8	90.6	87.9	90.6	88.6	90.1	87.1	90.6	90.6	88.6	90.6
2/6/21	5	73.1	79.3	73.1	77.3	73.2	76.4	74.6	78.4	73.1	73.1	73.2	79.5	86.2	90.9	88.4	90.8	89.3	90.2	87.8	90.9	90.8	89.5	90.9
2/6/24	5	72.8	84	72.8	75.8	72.8	76.5	74.5	76.2	72.8	72.8	72.9	84.2	84.1	93.2	91.5	93.2	91.6	92.1	90.8	93.2	93.2	92.2	93.2
2/6/27	5	56.7	75.8	57	64.1	57.8	62.7	57.8	63.1	56.8	56.8	56.9	76.1	75.1	89.7	86	89.7	88.4	89.3	86.9	89.7	89.5	87.9	89.7
2/6/30	5	59.8	73.3	61.5	66.2	60.6	67	61.8	66.1	59.8	59.8	60	73.7	81.8	90.2	86.4	90.2	87.8	89.8	87.6	90.3	90.2	88	90.3
2/6/33	5	59.4	76.4	59.4	66.3	60	65.4	61.1	65	59.4	59.5	59.6	76.7	77.7	90.7	87	90.6	88.8	90.1	88.5	90.7	90.4	88.6	90.7
2/6/36	5	53.8	75.4	54.1	61.5	54.2	61.3	55.4	61.8	53.8	53.8	53.9	75.6	75.5	91.8	88.1	91.7	90.1	91.3	88.1	91.8	91.6	89.9	91.8
2/8/12	5	73.7	83.8	73.7	76.8	73.8	76.6	75.2	76.8	73.7	73.7	73.9	84	83.6	92.1	90.2	92	90.8	91.2	90.1	92.1	92	91.1	92.1
2/8/15	5	71.1	83.4	71.1	74.9	71.2	75.7	72.5	74.5	71.1	71.2	71.2	83.5	83.3	92.6	90.2	92.6	91.1	91.9	90.5	92.6	92.6	91.7	92.6
2/8/18	5	76.4	82.7	76.4	80.3	76.5	79.9	77.1	80.3	76.4	76.4	76.6	82.9	87.8	92.2	89.5	92.2	90.9	91.9	89.6	92.2	92.2	90.9	92.2
2/8/21	5	63	77.7	63.1	67	63.5	72.3	65.6	68	63.1	63.2	63.3	78.2	82.6	90.2	88.6	90.2	88.2	89.1	87.9	90.2	90.1	88.6	90.2
2/8/24	5	58	73.1	58.5	64.8	59	66.9	59.7	64.3	58.1	58	58.1	73.4	78.5	88.7	85.4	88.7	85.3	88.5	86.7	88.7	88.6	87.5	88.7
2/8/27	5	52.3	70.8	53.3	60.6	53.4	62.3	54.6	60.9	52.4	52.4	52.6	71.1	78.4	90.1	86.1	90.1	87.7	89.5	87.2	90.1	89.9	87.5	90.1
2/8/30	5	60.6	79.2	60.6	66.7	61	67.1	61.8	66.4	60.6	60.6	60.7	79.4	77.6	91.9	89	91.9	90.1	91.8	89.7	91.9	91.8	90.5	91.9
2/8/33	5	63.8	79.3	63.9	69.1	64.1	71.3	65.3	68.4	63.8	63.8	63.9	79.6	81.8	91.9	88.7	91.9	90.2	91.1	89.6	91.9	91.9	90.1	91.9
2/10/9	5	69.6	79.9	69.6	75	69.7	72.3	70.8	73.6	69.6	69.6	69.8	80.1	81.7	90.6	87.2	90.6	89.5	89.9	88.9	90.6	90.6	89.2	90.6
2/10/12	5	62	73.7	62	67.6	62.6	70.9	63.7	68.8	62	62.1	62.3	74.1	82.5	88.3	86.2	88.3	85.8	88	85.5	88.3	88.2	86.1	88.3
2/10/15	5	60.8	77	62.2	67.7	61.5	69.5	61.6	67	60.8	60.8	61	77.2	81.1	89.9	86.6	90.1	88.9	90	88	90.1	89.9	88.2	90.1
2/10/18	5	51.8	70	52.6	61.7	53.1	59.8	52.8	60.7	51.9	51.9	52.1	70.4	76.4	90.6	85.6	90.6	87.6	90.5	87.8	90.6	90.5	87.3	90.6
2/10/21	5	65	79	65	71.3	65.3	71.6	66.2	70.1	65	65.1	65.1	79.2	81.7	90.8	87	90.8	89.6	90.4	89.4	90.8	90.7	89.6	90.8
2/10/24	5	59	74.1	60.2	68.1	59.4	65.1	60.7	65.9	59	59.1	59.3	74.5	79.8	90.8	84.9	90.8	90	90.4	89.1	90.8	90.7	88.4	90.8
2/10/27	5	62.2	77.3	62.2	68	62.4	66.5	64	67.1	62.2	62.2	62.4	77.6	77.2	89.8	86.6	89.8	88.2	88.7	87.8	89.8	89.8	88.3	89.8
2/10/30	5	52.6	65.6	56.4	56.7	54.3	60.5	55.6	58.3	52.7	52.9	53	66.2	73	81	80.4	82.5	79.6	81.9	80.6	82.5	82.4	81.1	82.5
2/12/6	5	70.5	79.5	70.5	75.7	70.6	73.3	72.4	74.3	70.5	70.6	70.8	79.7	82.4	89.3	86.2	89.3	88.7	88.1	87.8	89.3	89.3	88.2	89.3
2/12/9	5	68.7	77.5	69.5	74.8	69	73.5	70.5	73.1	68.7	68.7	68.9	77.7	84.1	89.8	85.9	89.8	88.8	88.8	88.2	89.8	89.8	88.4	89.8
2/12/12	5	65.2	76	65.7	73	65.8	69.9	65.9	71.6	65.2	65.2	65.3	76.1	81.8	89.1	85.2	89.2	88.6	89.1	87.4	89.3	89.2	87.7	89.3
2/12/15	5	55.3	73.8	58.1	64.4	56.5	61.2	56.6	62.4	55.4	55.4	55.6	74	77.3	90.6	85.3	90.6	89.5	90.5	88.5	90.6	90.3	87.4	90.6
2/12/18	5	52.4	71.6	52.8	54.9	53.2	68.5	56.2	57.6	52.8	53	52.8	72.4	77.7	85.4	85	85.4	81.3	84.9	83.2	85.4	85.1	84.7	85.4
2/12/21	5	52.5	62	53.7	58	54.5	64.4	55.5	60.5	52.9	52.8	52.9	62.5	78.8	82.9	80.6	82.8	78.4	82.4	79.1	82.9	82.6	80.4	82.9
2/12/24	5	56.5	73.5	56.6	65.1	57.8	63.7	58.2	64	56.6	56.6	56.7	73.6	76.2	88.2	84.6	88.2	86.5	88.1	86.6	88.2	88	86.9	88.2
2/12/27	5	54.6	72.9	56.9	64.5	55.9	63.2	55.6	62.4	54.7	54.6	54.7	73.1	77.5	89.5	84.2	89.6	88.8	89.5	88.4	89.6	89.4	87.6	89.6
Total	200	63	76.6	63.6	68.6	63.6	69.6	64.6	69	63.1	63.1	63.2	76.9	81	90.2	87.4	90.2	88.3	89.7	87.5	90.2	90.1	88.6	90.2

Note. $\mathcal{C}/l/n$: Class/Number of periods/Number of suppliers, Var Bnd: Bounds on the variables, Gen Ineq: General inequalities

Table 14: Effect of individual valid inequality types on average LP solution value as a percentage of BUB (class 3)

			Including only one type											Excluding only one type										
	Set		(l,S,WW)-type				Var Bnd				Gen Ineq			(l,S,WW)-type				Var Bnd				Gen Ineq		
$\mathcal{C}/l/n$	Size	None	(18)	(19)	(20)	(21)	(22)	(23)	(24)	(25)	(26)	(28)	Known	(18)	(19)	(20)	(21)	(22)	(23)	(24)	(25)	(26)	(28)	All
3/4/18	5	68.1	70.4	68.1	68.8	68.1	74.3	69.9	82.1	68.1	68.3	68.3	70.9	91.9	92.5	92.4	92.5	88	92	77.9	92.5	92.5	90.2	92.5
3/4/21	5	66.5	68.4	66.5	66.8	66.6	74.6	68.4	78.2	66.6	66.7	66.9	68.9	89.8	90.6	90.5	90.6	83.5	89.9	78	90.6	90.5	88.1	90.6
3/4/24	5	64.7	68.1	64.7	65.8	64.7	76.4	66.1	77.3	64.7	64.9	65	68.5	91.5	92.9	92.7	92.9	83	92.5	81.1	92.9	92.8	91.6	92.9
3/4/27	5	65.3	67.6	65.3	66.1	65.3	78.2	66.2	78.4	65.3	65.4	65.5	68	94.3	94.3	94.2	94.3	83.6	94.2	80.6	94.3	94.1	92.2	94.3
3/4/30	5	67	70.5	67	67.2	67	77.4	68.6	79.4	67	67.1	67.3	71	92.6	93.9	93.9	93.9	85.5	93.1	81.2	93.9	93.9	92.1	93.9
3/4/33	5	64.6	68.3	64.6	65.2	64.6	73.9	66.3	78.5	64.6	64.8	64.9	68.9	91.6	92.9	92.8	92.9	85.9	92.5	78.4	92.9	92.9	89.9	92.9
3/4/36	5	61.5	66.8	61.5	62.1	61.7	71.6	65.6	75.9	61.7	62	62.2	67.8	91.5	92.3	92.1	92.3	86.6	90.3	78.1	92.3	92.3	90.2	92.3
3/4/39	5	46.1	53.5	46.1	48.2	46.2	62.2	48	66.7	46.2	46.2	46.4	53.9	87.1	88.7	88.4	88.7	78.1	88	69.6	88.7	88.5	85.7	88.7
3/6/15	5	70.4	73.5	70.4	71.2	70.5	76.8	72.2	81.3	70.4	70.6	70.8	74	91	92	91.9	92	87.7	91.4	81.2	92	91.9	90.2	92
3/6/18	5	69.3	72.9	69.3	70.4	69.3	75.4	70.3	79.5	69.3	69.4	69.5	73.2	89	89.9	89.6	89.9	86.4	89.5	79.6	89.9	89.7	87.2	89.9
3/6/21	5	63.6	69	63.6	65.5	63.7	70.8	65.6	74.2	63.7	63.8	63.9	69.6	86.9	88.2	87.7	88.2	85.1	87.5	77.8	88.2	87.7	84.9	88.2
3/6/24	5	65.9	68.3	65.9	67.5	66	72.9	67.1	75.9	66	66	66.3	68.8	88.1	88.4	88.2	88.4	83.4	88.1	77.8	88.4	88.2	84.3	88.4
3/6/27	5	67.3	71.6	67.3	68	67.4	76.4	68.9	78.1	67.4	67.4	67.5	71.9	90.4	91	90.7	91	85.5	90.5	80.1	91	91	88.7	91
3/6/30	5	60.9	67	60.9	62.3	61	74.6	62	71.8	61	61	61.1	67.3	89.3	90.5	90.2	90.5	81.7	90.5	79.2	90.5	90	87.9	90.5
3/6/33	5	65.5	68	65.5	66.7	65.5	72.6	69.2	73.2	65.5	65.9	66.1	69	86.4	87.1	86.6	87.1	81.6	85.4	79.9	87.1	86.5	84.6	87.1
3/6/36	5	60.3	69.5	60.3	61.9	60.4	73.2	63.1	69.2	60.4	60.5	60.9	70.2	86.9	89.3	88.7	89.3	83.1	88.4	81.3	89.3	89.1	86.5	89.3
3/8/12	5	73.4	74.2	73.4	74.9	73.5	78.5	77.4	81.1	73.5	73.7	73.8	74.9	90.7	91	91	91	86.4	89.4	84.6	91	91	89.6	91
3/8/15	5	65.8	72.3	65.8	67.1	65.8	75.6	67.1	74.7	65.8	65.8	65.9	72.7	87	89.3	89.2	89.3	83.7	89	81.1	89.3	89.2	87.7	89.3
3/8/18	5	71.5	75.9	71.5	73.2	71.6	76.6	73.4	79.1	71.6	71.6	71.8	76.3	87.6	89.8	89.3	89.8	86.9	89.5	83.2	89.8	89.7	87.6	89.8
3/8/21	5	67.7	70.7	67.7	68.8	67.8	75.1	69.9	74.8	67.8	67.8	68	71.1	86.5	87.9	87.6	87.9	82	87.3	80.8	87.9	87.7	85.5	87.9
3/8/24	5	63.5	67.6	63.5	65.3	63.5	70.2	64.9	73.2	63.5	63.6	63.9	68.1	84.1	85.3	85.1	85.3	81.5	85.1	76	85.3	85.3	82.1	85.3
3/8/27	5	71.5	74.3	71.5	72	71.5	77	73.9	79.1	71.5	71.6	71.7	74.7	88.3	89.3	89	89.3	85.6	88.7	81.7	89.3	89.2	87.2	89.3
3/8/30	5	70.6	74.4	70.6	71.4	70.6	75.8	71.6	78.2	70.6	70.6	70.8	74.8	86.3	88	87.8	88	84.9	87.8	80.2	88	87.9	86.1	88
3/8/33	5	65.4	73	65.4	66.5	65.5	73.4	66.5	73.2	65.4	65.5	65.6	73.3	84.2	87.4	87.2	87.4	83.8	87.3	79.7	87.4	87	85.6	87.4
3/10/9	5	66	71.9	66	67.8	66.2	74.2	71.5	72.2	66.1	66.5	66.5	73	85.7	88.8	88.3	88.8	83.5	86.5	85.4	88.8	88.8	87.7	88.8
3/10/12	5	64.2	69.9	64.2	66.9	64.3	70.9	66.8	72.4	64.3	64.4	64.7	70.6	83.6	85.8	85	85.8	82.8	84.7	80.3	85.8	85.3	83.8	85.8
3/10/15	5	67.3	73.4	67.3	69.2	67.4	73.3	69.4	75	67.4	67.4	67.6	73.8	84.5	87.4	87	87.4	84.7	86.9	81.2	87.4	87.3	85.4	87.4
3/10/18	5	63	67.5	63	64.7	63.1	68.9	65.4	71.4	63	63.1	63.2	67.9	82	84	83.5	84	80.6	83.7	76.2	84	83.6	81.7	84
3/10/21	5	65.7	67.2	65.7	67.6	65.8	70.8	68.5	73.9	65.7	65.9	66	67.7	84.9	85.6	85.2	85.6	81.2	85.2	77.9	85.6	85.6	82.3	85.6
3/10/24	5	65.8	69.9	65.8	67.5	65.9	72.1	67.6	73.8	65.8	65.9	66.1	70.3	84.2	86.1	85.7	86.1	82	86.1	79	86.1	86	83.8	86.1
3/10/27	5	67.7	71.8	67.7	69.7	67.8	73.7	68.3	76.4	67.8	67.8	67.9	72.1	85.3	87.1	86.8	87.1	83.4	87.1	79.2	87.1	86.9	84.9	87.1
3/10/30	5	66.3	72.1	66.3	67.7	66.4	74.2	68.1	73.4	66.3	66.4	66.5	72.5	85	86.9	86.4	86.9	82.9	86.7	80.1	86.9	86.8	85	86.9
3/12/6	5	70.4	74	70.4	72.8	70.5	74.4	72.3	78.6	70.4	70.5	70.6	74.3	86.1	88.2	88	88.2	85.7	87.7	81.8	88.2	88	86.6	88.2
3/12/9	5	69.5	73.8	69.5	71	69.5	75	70.7	76	69.5	69.5	69.7	74.1	85.3	87.6	87.4	87.6	84	87.2	81.5	87.6	87.4	85.1	87.6
3/12/12	5	67.6	71.7	67.6	70.1	67.9	72.5	70.4	74.4	67.7	67.9	67.9	72.2	83.8	85.8	85.3	85.8	83	85.3	81	85.8	85.7	84.2	85.8
3/12/15	5	68.7	71.3	68.7	70.3	68.8	73.3	70.8	74.8	68.7	68.8	69	71.8	83.2	84.5	84.2	84.5	81.3	83.9	79.4	84.5	84.4	82.3	84.5
3/12/18	5	65.7	70.1	65.7	67.6	65.8	73.7	68.2	71.7	65.8	65.8	66.2	70.8	84.5	86.2	86	86.2	80.9	85.9	81.4	86.2	85.7	83.2	86.2
3/12/21	5	65.2	70.1	65.2	67.6	65.3	70.5	66.9	73.5	65.2	65.3	65.4	70.4	83.5	85.8	85.4	85.8	83.1	85.7	79	85.8	85.4	82.9	85.8
3/12/24	5	66.3	71.9	66.3	68.4	66.3	72	69.4	74.4	66.3	66.5	66.5	72.4	84.3	87.4	87.3	87.4	84.2	86.1	80.9	87.4	87.3	85.9	87.4
3/12/27	5	60.1	68.8	60.1	62.9	60.7	70.2	62.5	68.2	60.2	60.3	60.8	69.7	83.2	86.9	86.5	86.9	81.8	86.3	80.9	86.9	86.7	84.5	86.9
Total	200	65.9	70.3	65.9	67.4	66	73.6	68	75.3	65.9	66.1	66.2	70.8	87.1	88.7	88.4	88.7	83.7	88.1	79.9	88.7	88.5	86.4	88.7

Note. $\mathcal{C}/l/n$: Class/Number of periods/Number of suppliers, Var Bnd: Bounds on the variables, Gen Ineq: General inequalities

Highlights

- We study integrated production, inventory and inbound transport planning problem
- The suppliers each provide a subset of the components necessary for the production
- We provide a mixed integer programming formulation of the problem
- We propose several families of valid inequalities to strengthen the formulation
- We generate a large test bed consisting of small to large instances
- We analyze the impact of each family of valid inequalities