

A new sufficient condition of optimality for the two-machine flowshop problem

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Abstract

This paper focuses on the characterization of a subset of optimal sequences for the famous two-machine flowshop problem. Based on the relative order of the job processing times, two particular interval structures are defined so that each job is associated with an interval. Then, using the Allen's algebra, the interval relationships are analysed and a sufficient optimality condition is established providing a characterisation of a large subset of optimal sequences. This set necessarily includes any Johnson's sequences together with numerous other optimal job sequences.

Key words: Scheduling, robustness, permutation flowshop, interval structures, pyramids.

1 Introduction

Scheduling methods able to provide a set of solutions as opposed to methods providing only one solution are said robust (or flexible) since they tolerate unforeseen disturbances without any schedule re-calculation [13]. According to this definition, several robust scheduling methods have been proposed in the literature [17,18]. Here under, the famous two machine flowshop problem is studied with the aim to characterize without enumeration a large set of optimal sequences. Classically, we denote J the set of jobs to sequence on two machines M_1 and M_2 (p_{j1} and p_{j2} respectively indicate the processing times of each job $j \in J$ on M_1 and M_2). The selected criterion is the makespan C_{max} and only permutation sequences are considered (i.e. $F2|pmu|C_{max}$) since they are dominant for $Fm||C_{max}$, if $m \leq 3$ [12].

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Several related papers already dealt with this problem providing a single or several optimal solutions. A well-known result was given by Johnson [19] who demonstrated that any job sequence which respects the rule $\min(p_{i1}, p_{j2}) \leq \min(p_{j1}, p_{i2}) \Leftrightarrow i \prec j$ is optimal. Following this rule, a single optimal job sequence can be computed in $O(n \log n)$. The use of the algorithm proposed in [5] gives the list of all the sequences satisfying the Johnson's rule. Billaut and Lopez [6] also proposed an algorithm which enumerates, by job permutations in the optimal Johnson's sequences, the complete set of optimal sequences. A related approach based on the notion of *maximal sequence* was also proposed in [8]. In both cases, we notice that only small problems, with a dozen of jobs at most, can be solved due to a prohibitive time complexity of the algorithms. Indeed, the enumeration of the complete set of optimal sequences is a problem known to be NP-hard in the strong sense [16].

A $F2|prmu|C_{max}$ problem possibly have a very large set of optimal solutions. The complete enumeration of this set has to be avoided since it is obviously time consuming. As a consequence, the approaches aiming at characterizing, rather than enumerating, a subset of optimal solutions should be preferred. This classically implies to handle sequential flexibility by means of a partial order. A partial order P is defined by a pair $P = (X, \preceq_P)$ where the binary relation \preceq_P on $X \times X$ is reflexive, antisymmetric and transitive [11]. A partial order $P = (X, \preceq_P)$ is a *complete order* (or a *linear order*) if for every pair $(u, v) \in X \times X$ either $u \preceq_P v$ or $v \preceq_P u$ holds. Moreover, given a pair of partial orders $Q = (X, \preceq_Q)$ and $P = (X, \preceq_P)$, on the same set X , we call Q an *extension* of P if $u \preceq_P v$ implies $u \preceq_Q v$ for all $u, v \in X$.

The notion of *group sequence* [22] gives an interesting partial order. It allows the characterization of a set of solution specifying, for each resource, an ordered sequence of group, provided that the order of the elements of a group is not imposed. The main advantage of such a partial order lies on the capability to perform a worst case analysis and to determine the quality of the set of solutions with respect to a regular criterion. Of course, the larger the set of solutions is and the worse its quality is. The notion of group sequence has been widely used in the field of shop scheduling and intends to determine a family of solutions to be used during the schedule real time execution. Switching from one solution to the other allows to face the potential disruptions without any performance loss [21,1,10,3,23,7,20].

In both [4] and [16], the authors suggest to reuse the notion of group sequence for the $F2|prmu|C_{max}$ problem. The Baptiste's algorithm gives a group sequence characterizing a set of sub-optimal job sequences of which the worst execution time is bounded. On the basis of a given optimal job sequence, Esswein et al. also proposed a *Greedy Forward Grouping* algorithm, having the worst case complexity $O(n \log n)$, which determines an optimal group sequence of jobs while maximizing the flexibility (i.e. the number of groups is

minimized). In both previous works, the authors showed that the number of characterized optimal solutions can be very impressive.

The work presented in this paper is quite connected with the above approaches since we are also interested in characterizing a large set of optimal sequences for the $F2|prmu|C_{max}$ problem. Nevertheless, some new interesting features are provided. Firstly, a new partial order is defined which does not use the notion of group sequence. Moreover, our approach does not require any initial optimal sequence, since it can be directly stated that any sequence characterized with our partial order is optimal (the partial order is sufficient regarding the makespan criterion). Additionally, our partial order being defined on the basis of the relative order of the processing times p_{i1} and p_{i2} , the characterized set of sequences remains optimal even if the job processing times increase, provided that the relative order is unchanged. Such a feature is of interest in a robust context where scheduling approaches have to be quite insensitive to data variations. Also interesting is the fact that Johnson's sequences are always included in the set of characterized sequences.

The paper is organised as follows. First we recall some basic notions, used for the definition of our partial order and related to the analysis of interval structures. Then we focus on two particular interval structures for the $F2|prmu|C_{max}$ problem and we progressively expound several sufficient conditions of optimality, starting from the analysis of particular interval structures up to the general case. Lastly, an example is given.

2 Interval structure and basic concepts

An interval structure can be defined by a couple $\langle I, C \rangle$ with $I = \{i_1, \dots, i_n\}$ a set of intervals and C a set of constraints over $I \times I$. Each interval i_j is defined by its lower and upper bounds x_j and y_j . Any constraint between two intervals i_j and i_k can be expressed either by specifying a total order relation among the lower and upper bounds of the intervals or by directly using the relations of the algebra proposed by Allen [2] (see figure 1).

For instance, let us consider the interval structure with $I = \{A, B, C, D, E, F, G\}$ of the figure 2. We assume that C is defined by a total order such that $(x_A = x_B = x_G) < (x_C = x_D) < y_C < (y_B = x_F) < x_E < y_A < (y_D = y_E) < y_F = y_G$. The set of equivalent Allen's relations is represented on the table of the figure 3. Such a table can be computed with a time complexity $O(n(n-1)/2)$, n being the number of intervals.

Top and *base* [15] are two interesting notions related to the concept of interval structure.

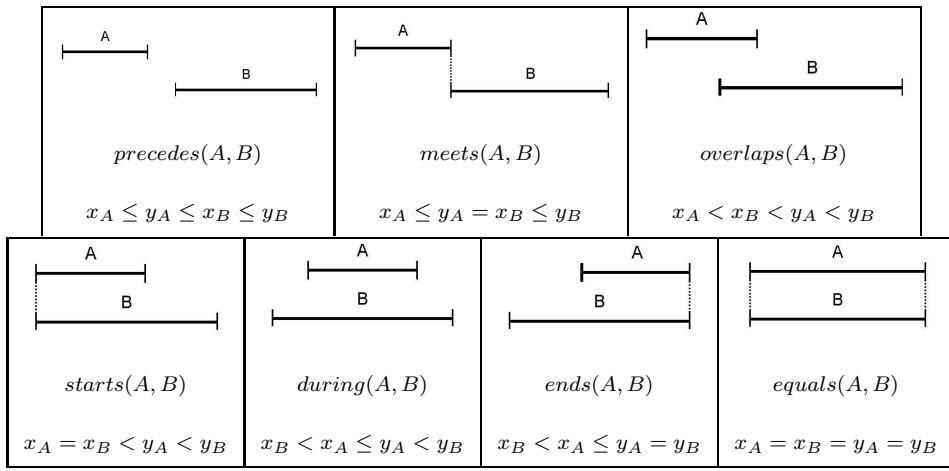


Fig. 1. Allen's relations

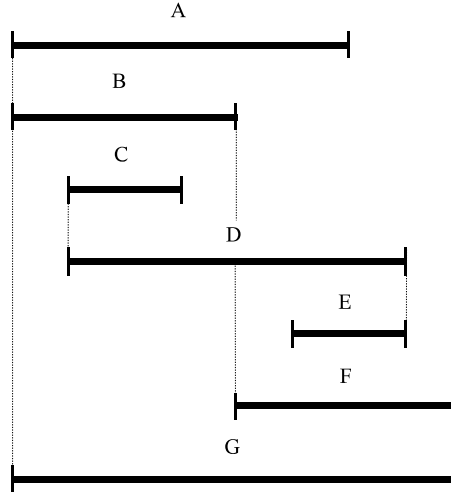


Fig. 2. An interval structure example

	A	B	C	D	E	F	G
A	$equals(A, A)$	$starts(B, A)$	$during(C, A)$	$overlaps(A, D)$	$overlaps(A, E)$	$overlaps(A, F)$	$starts(A, G)$
B	$starts(B, A)$	$equals(B, B)$	$during(C, B)$	$overlaps(B, D)$	$precedes(B, E)$	$meets(B, F)$	$starts(B, G)$
C	$during(C, A)$	$during(C, B)$	$equals(C, C)$	$starts(C, D)$	$precedes(C, E)$	$precedes(C, F)$	$during(C, G)$
D	$overlaps(A, D)$	$overlaps(B, D)$	$starts(C, D)$	$equals(D, D)$	$ends(E, D)$	$overlaps(D, F)$	$during(D, G)$
E	$overlaps(A, E)$	$precedes(B, E)$	$precedes(C, E)$	$ends(E, D)$	$equals(E, E)$	$during(E, F)$	$during(E, G)$
F	$overlaps(A, F)$	$meets(B, F)$	$precedes(C, F)$	$overlaps(D, F)$	$during(E, F)$	$equals(F, F)$	$ends(F, G)$
G	$starts(A, G)$	$starts(B, G)$	$during(C, G)$	$during(D, G)$	$during(E, G)$	$ends(F, G)$	$equals(G, G)$

Fig. 3. Allen's relation for the intervals $\{A, B, C, D, E, F, G\}$

Definition 1. A top of an interval structure $\langle I, C \rangle$ is an interval $t \in I$ such that $\forall i \in I$ the Allen's relation $during(i, t)$ never holds.

Definition 2. A base of an interval structure $\langle I, C \rangle$ is an interval $b \in I$ such that $\forall i \in I$ the Allen's relation $during(b, i)$ never holds.

These notions of top and base can be respectively used to define the notions of t -pyramid and b -pyramid.

Definition 3. Given a top t_α , a t-pyramid P_α related to t_α is the set of intervals $i \in I$ such that $\text{during}(t_\alpha, i)$.

Definition 4. Given a base b_α , a b-pyramid P_α related to b_α is the set of intervals such that $\text{during}(i, b_\alpha)$.

For illustration, let us consider the interval structure of the figure 2. It has three tops $\{C, D, E\}$ and four bases $\{A, B, F, G\}$. The involved t-pyramids are $P_C = \{B, A, G\}$, $P_D = \{G\}$ and $P_E = \{F, G\}$, and the b-pyramids are $P_A = \{C\}$, $P_B = \{C\}$, $P_F = \{E\}$ and $P_G = \{C, D, E\}$.

The concept of t-pyramid has been already used in the scheduling literature in order to determine a dominant partial order [14,9] for the single machine problem $(1|r_i, d_i|L_{max})$. In the following, the focus is put on the b-pyramid concept which gives a sufficient condition of optimality for the $F2|prmu|C_{max}$ problem.

3 Two interval structures for $F2|prmu|C_{max}$

Two particular interval structures I_1 and I_2 are defined:

- I_1 is the interval structure associated with the jobs $j \in J$ such that $p_{j1} \leq p_{j2}$;
- I_2 is the interval structure associated with the jobs $j \in J$ such that $p_{j2} \leq p_{j1}$.

An interval $i_j = [p_{j1}, p_{j2}]$ is associated to each job $j \in I_1$ and similarly, an interval $i_j = [p_{j2}, p_{j1}]$ is associated to each job $j \in I_2$. Therefore, a job j such that $p_{j1} = p_{j2}$ belongs to both interval structures I_1 and I_2 and its corresponding interval is a point. Moreover, one can see that the interval structures I_1 and I_2 do not change as long as the relative order of the processing times p_{j1} and p_{j2} remains unchanged.

In the following, we focus on the bases of I_1 and I_2 . We assume that the n_1 bases of I_1 are indexed according to the ascending order of their processing times on the first machine (in an arbitrary order in case of equality). Similarly, the n_2 bases of I_2 are indexed, starting from the index $n_1 + 1$, according to the descending order of their processing times on the second machine (in an arbitrary order in case of equality).

For the problem instance of the figure 4, the interval structures I_1 and I_2 are represented on the figure 5. First, we notice that I_1 contains three bases, $b_1 = 1$, $b_2 = 2$ and $b_3 = 3$, which involve three b-pyramids: $P_1 = \{5, 6\}$, $P_2 = \{6, 7\}$ and $P_3 = \{8\}$. The interval structure I_2 only contains a single base, $b_4 = 4$, which involves the b-pyramid $P_4 = \{7, 9\}$.

jobs j	1	2	3	4	5	6	7	8	9
p_{j1}	1	3	8	8	2	4	7	9	5
p_{j2}	6	8	12	2	4	5	7	11	3

Fig. 4. A $F2|prmu|C_{max}$ problem instance

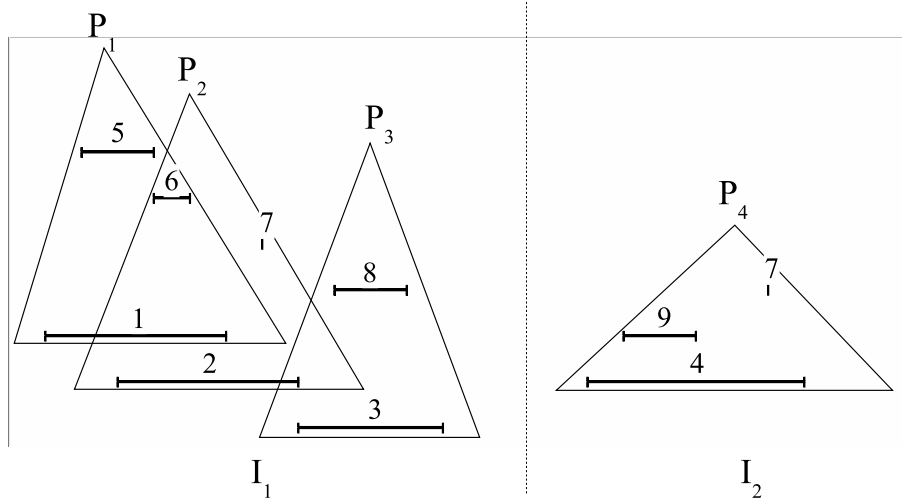


Fig. 5. Interval structures I_1 and I_2 for the problem of the figure 4

Before giving our general result, we need to consider two particular interval structures: we focus first on the case where a $F2|prmu|C_{max}$ problem is characterised by one single b-pyramid, then we consider the bi-pyramidal case.

4 The mono-pyramidal case

Let us consider a $F2|prmu|C_{max}$ problem with only one single base b and one single b-pyramid P_b . Two cases will be distinguished either P_b is a b-pyramid of I_1 or P_b is a b-pyramid of I_2 .

4.1 P_b is a b-pyramid of I_1

In this case, the following theorem can be stated.

Theorem 1. *Any job sequence which places the base b in the first position and the other jobs in any order is optimal with:*

$$C_{max} = p_{b1} + \sum_{j \in J} p_{j2}$$

Proof. We must prove that there is necessarily no idle time on M_2 when the base b is sequenced first. First, as b is the only base, any job $j \in J - \{b\}$ of I_1 obviously respects the inequality $p_{b1} < p_{j1} \leq p_{j2} < p_{b2}$. As illustrated on the figure 6, we note σ the sub-sequence of n jobs being already sequenced after the base b , and Δt_n the idle time on M_1 (i.e. $\Delta t_n = C_{\sigma+\{b\}} - \sum_{j \in \sigma+\{b\}} p_{j1}$). Proving that there is no idle time on M_2 is equivalent to prove that $\Delta t_n \geq p_{j1} \forall n$ and $\forall j \in J - \sigma - \{b\}$. The case where only the base b is sequenced (i.e. $\sigma = \emptyset$) is obvious since $\Delta t_1 = p_{b2}$ and therefore, $\Delta t_1 > p_{j1} \forall j \in J - \{b\}$. If we consider now the general case, then it is easy to see that $\Delta t_n = \Delta t_{n-1} + p_{j2} - p_{j1}$. Since $p_{j2} - p_{j1} \geq 0$ by hypothesis, we can deduce that $\Delta t_n \geq \Delta t_{n-1} \geq \dots \geq \Delta t_1 > p_{j1}$. Hence there is no idle time on M_2 and $C_{max} = p_{b1} + \sum_{j \in J} p_{j2}$. \square

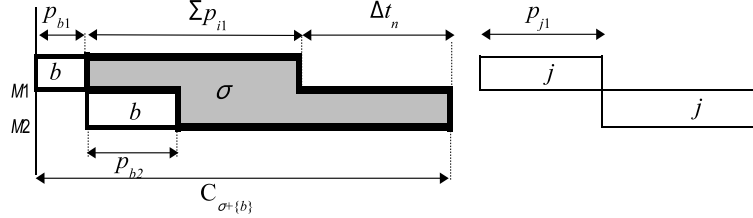


Fig. 6. Optimal job sequences for the mono-pyramidal case (P_b belongs to I_1)

4.2 P_b is a b -pyramid of I_2

In the case where b is a base of I_2 , a dual theorem can be stated.

Theorem 2. *Any job sequence which places the base b in the last position and the other jobs in any order is optimal with :*

$$C_{max} = p_{b2} + \sum_{j \in J} p_{j1}$$

Proof. The proof is similar to the previous one by permuting the processing times p_{j1} and p_{j2} (reverse problem formulation). Therefore, the optimal job sequences are the same but in the reverse order. \square

5 The bi-pyramidal case

We assume now that the problem is characterized by two bases b_1 and b_2 . We consider two cases either b_1 and b_2 belongs to the same interval structure (I_1 or I_2) or not.

5.1 The bases b_1 and b_2 belong to the same interval structure I_1

We assume that b_1 and b_2 are two bases belonging to I_1 such that $p_{b_11} \leq p_{b_21}$. As illustrated on the figure 7, we focus on job sequences in the form $b_1 \prec \sigma_1 \prec b_2 \prec \sigma_2$. We distinguish again two sub-cases either the relation $\text{precedes}(b_1, b_2)$ holds or not.

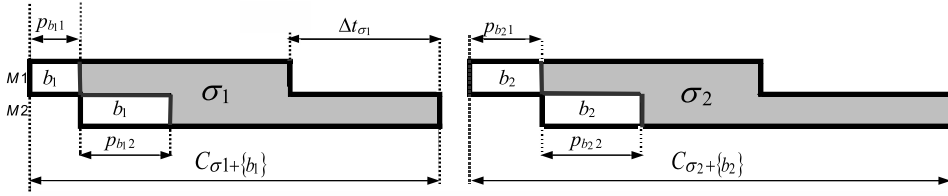


Fig. 7. Optimal job sequences for the by-pyramidal case (b_1 and b_2 belong to I_1)

5.1.1 The relation $\text{precedes}(b_1, b_2)$ holds

Since $\text{precedes}(b_1, b_2)$, obviously $P_{b_1} \cap P_{b_2} = \emptyset$ and every job $j \in J - \{b_1, b_2\}$ belongs to either P_{b_1} or P_{b_2} . We prove the following theorem:

Theorem 3. *If P_{b_1} and P_{b_2} are two b -pyramids of I_1 with $\text{precedes}(b_1, b_2)$, then any job sequence such that:*

- b_1 precedes b_2 ;
- all the jobs belonging to P_{b_1} are sequenced inside σ_1 in any order;
- all the jobs belonging to P_{b_2} are sequenced inside σ_2 in any order;

is optimal with:

$$C_{max} = \sum_{j \in \sigma_2 + \{b_2\}} p_{j2} + \max(p_{b_11} + \sum_{j \in \sigma_1 + \{b_1\}} p_{j2}, \sum_{j \in \sigma_1 + \{b_1\}} p_{j1} + p_{b_21}).$$

Proof. Let us focus on the Johnson's optimal sequence which arranges jobs in the ascending order of their p_{j1} . This optimal sequence is in the form $b_1 \prec \sigma_1 \prec b_2 \prec \sigma_2$ and obviously respects the three conditions of the theorem 3 since b_1 precedes b_2 , any job $j \in P_{b_1}$ is assigned to σ_1 and any job $j \in P_{b_2}$ is assigned to σ_2 . From the theorem 1, it can be stated that the values of $C_{\sigma_1 + \{b_1\}}$ and $C_{\sigma_2 + \{b_2\}}$ (hence the value of the optimal makespan), do not change whatever the job orders inside σ_1 and σ_2 are. Depending on the value of Δt_{σ_1} compared to p_{b_21} , the optimal makespan C_{max} has two possible expressions. If $\Delta t_{\sigma_1} \geq p_{b_21}$ then $C_{max} = p_{b_11} + \sum_{j \in \sigma_1 + \{b_1\}} p_{j2} + \sum_{j \in \sigma_2 + \{b_2\}} p_{j2}$. Otherwise, $C_{max} = \sum_{j \in \sigma_1 + \{b_1\}} p_{j1} + p_{b_21} + \sum_{j \in \sigma_2 + \{b_2\}} p_{j2}$. The value of the makespan is the greater between those, hence the C_{max} expression of the theorem 3. \square

5.1.2 The relation $\text{precedes}(b_1, b_2)$ does not hold

We assume that b_1 and b_2 are two bases of the interval structure I_1 such that $p_{b_11} \leq p_{b_21}$. Since the relation $\text{precedes}(b_1, b_2)$ does not hold, six Allen's relations remain possible: $\text{equals}(b_2, b_1)$, $\text{starts}(b_2, b_1)$, $\text{starts}(b_1, b_2)$, $\text{ends}(b_2, b_1)$, $\text{overlaps}(b_1, b_2)$ and $\text{meets}(b_1, b_2)$. For all these cases, we notice that some jobs can belong to both P_{b_1} and P_{b_2} .

Theorem 4. *If P_{b_1} and P_{b_2} are two b -pyramids of I_1 such that the relation $\text{precedes}(b_1, b_2)$ does not hold, then any job sequence such that:*

- b_1 precedes b_2 ;
- all the jobs only belonging to P_{b_1} are sequenced inside σ_1 in any order;
- all the jobs only belonging to P_{b_2} are sequenced inside σ_2 in any order;
- any job belonging both to P_{b_1} and P_{b_2} is sequenced anywhere either inside σ_1 or σ_2 in any order;

is optimal with:

$$C_{max} = p_{b_11} + \sum_{j \in J} p_{j2}.$$

Proof. The proof is quite similar to the previous one. Indeed, we also focus on the Johnson's optimal sequence which arranges the jobs in the ascending order of their p_{j1} . This optimal sequence is in the form $b_1 \prec \sigma_1 \prec b_2 \prec \sigma_2$ (see figure 7) and obviously respects the conditions of the theorem 4 since b_1 precedes b_2 , jobs belonging only to P_{b_1} are assigned to σ_1 and all the other jobs (belonging either to $P_{b_1} \cap P_{b_2}$ or only to P_{b_2}) are assigned to σ_2 . Since b_1 and b_2 are two bases such that $\text{precedes}(b_1, b_2)$ does not hold, the relation $p_{b_11} \leq p_{b_21} \leq p_{b_12}$ stands. Therefore, according to the proof of the theorem 1, one can deduce that $\Delta t_{\sigma_1} \geq p_{b_21}$. Hence the optimal makespan value is $C_{max} = C_{\sigma_1 + \{b_1\}} + C_{\sigma_2 + \{b_2\}} - p_{b_21} = p_{b_11} + \sum_{j \in \sigma_1 + \{b_1\}} p_{j2} + \sum_{j \in \sigma_2 + \{b_2\}} p_{j2}$. From the theorem 1, it can be stated that the values of $C_{\sigma_1 + \{b_1\}}$ and $C_{\sigma_2 + \{b_2\}}$ (hence the one of the optimal makespan), do not change whatever the job orders inside σ_1 and σ_2 are. Moreover, scheduling a job belonging both to P_{b_1} and P_{b_2} in σ_1 (instead of σ_2 as in the Johnson's sequence) does not decrease Δt_{σ_1} . So the relation $\Delta t_{\sigma_1} \geq p_{b_21}$ still holds and the C_{max} value does not change. Lastly, in the particular case where $p_{b_11} = p_{b_21}$, the base indexes can be swapped in the theorem. \square

5.2 The bases b_1 and b_2 belong to the same interval structure I_2

We assume here that b_1 and b_2 are two bases belonging to I_2 such that $p_{b_12} \geq p_{b_22}$. Then the two following theorems can be directly stated since

their proof is similar to both previous ones, by considering the reverse formulation of the problem. In this case, the structure of the optimal sequences is $\sigma_1 \prec b_1 \prec \sigma_2 \prec b_2$.

Theorem 5. *If P_{b_1} and P_{b_2} are two b -pyramids of I_2 with $\text{precedes}(b_2, b_1)$, then any job sequence such that:*

- b_1 precedes b_2 ;
- all the jobs belonging to P_{b_1} are sequenced inside σ_1 in any order;
- all the jobs belonging to P_{b_2} are sequenced inside σ_2 in any order;

is optimal with:

$$C_{max} = \sum_{j \in \sigma_1 + \{b_1\}} p_{j1} + \max(p_{b_22} + \sum_{j \in \sigma_2 + \{b_2\}} p_{j1}, \sum_{j \in \sigma_2 + \{b_2\}} p_{j2} + p_{b_12}).$$

Theorem 6. *If P_{b_1} and P_{b_2} are two b -pyramids of I_2 such that the relation $\text{precedes}(b_2, b_1)$ does not hold, then any job sequence such that:*

- b_1 precedes b_2 ;
- all the jobs only belonging to P_{b_1} are sequenced inside σ_1 in any order;
- all the jobs only belonging to P_{b_2} are sequenced inside σ_2 in any order;
- any job belonging both to P_{b_1} and P_{b_2} is sequenced anywhere either inside σ_1 or σ_2 in any order;

is optimal with:

$$C_{max} = p_{b_22} + \sum_{j \in J} p_{j1}.$$

5.3 The bases b_1 and b_2 do not belong to the same interval structure

In this section, we assume that b_1 is a base of I_1 and b_2 is a base of I_2 . We notice that only the jobs such that $p_{j1} = p_{j2}$ can belong both to P_{b_1} and P_{b_2} (their corresponding interval is a point). As illustrated on the figure 8, we focus on job sequences in the form $b_1 \prec \sigma_1 \prec \sigma_2 \prec b_2$. Then we state the following theorem:

Theorem 7. *If P_{b_1} is a b -pyramid of I_1 and P_{b_2} is a b -pyramid of I_2 , then any job sequence such that:*

- b_1 precedes b_2 ;
- all the jobs only belonging to P_{b_1} are sequenced inside σ_1 in any order;
- all the jobs only belonging to P_{b_2} are sequenced inside σ_2 in any order;
- any job belonging both to P_{b_1} and P_{b_2} is sequenced anywhere either inside σ_1 or σ_2 in any order;

is optimal with:

$$C_{max} = p_{b_{11}} + p_{b_{22}} + \max\left(\sum_{j \in \sigma_1 \cup \sigma_2} p_{j1} + p_{b_{21}}, \sum_{j \in \sigma_1 \cup \sigma_2} p_{j2} + p_{b_{12}}\right).$$

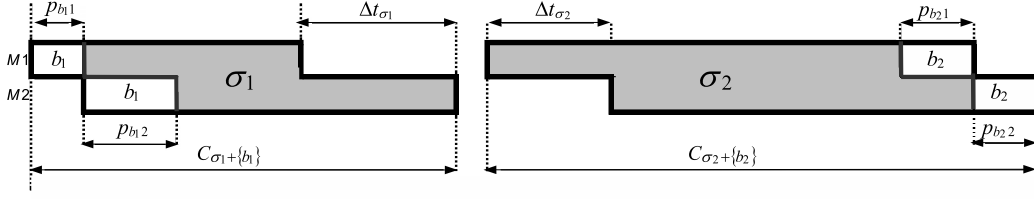


Fig. 8. Optimal job sequences for the by-pyramidal case (b_1 belong to I_1 and b_2 belong to I_2)

Proof. Once again, let us consider the Johnson's optimal sequence which arranges the jobs belonging to I_1 in the ascending order of their p_{j1} and the jobs belonging to I_2 in the descending order of their p_{j2} . This optimal sequence obviously respects the three conditions of the theorem 7 since b_1 precedes b_2 , any job $j \in P_{b_1}$ is assigned to σ_1 and any job $j \in P_{b_2}$ is assigned to σ_2 . From the theorem 1, one can see that the values of $C_{\sigma_1 + \{b_1\}}$ and $C_{\sigma_2 + \{b_2\}}$ (hence the one of the optimal makespan), do not change whatever the job orders inside σ_1 and σ_2 are. Additionally, if a job j belongs to both P_{b_1} and P_{b_2} ($p_{j1} = p_{j2}$), then it can be assigned either to σ_1 or σ_2 since whatever the assignment is, it neither modifies the values of Δt_{σ_1} and Δt_{σ_2} , nor the one of the makespan. Depending on the value of Δt_{σ_1} compared to Δt_{σ_2} , two cases may happen. If $\Delta t_{\sigma_1} \geq \Delta t_{\sigma_2}$ then $C_{max} = p_{b_{11}} + p_{b_{12}} + \sum_{j \in \sigma_1 \cup \sigma_2} p_{j2} + p_{b_{22}}$. Otherwise, $C_{max} = p_{b_{11}} + \sum_{j \in \sigma_1 \cup \sigma_2} p_{j1} + p_{b_{21}} + p_{b_{22}}$. Hence the general C_{max} expression of the theorem 7. \square

6 A sufficient condition of optimality for $F2|prmu|C_{max}$

Now we consider the general case where each interval structure I_1 and I_2 is characterized by any number of b-pyramids. As illustrated on the figure 9, we denote σ_j , with $j \in [1, n_1]$, the sub-sequence of jobs located between the bases b_j and b_{j+1} , provided that the bases of I_1 have been indexed according to the ascending order of their p_{j1} (in any order in case of equality). Similarly, we denote σ_k , with $k \in [n_1 + 1, n_1 + n_2]$, the sub-sequence of jobs located between the bases b_{k-1} and b_k , provided that the bases of I_2 have been indexed according to the descending order of their p_{j2} (in any order in case of equality). We focus on job sequences in the form: $b_1 \prec \sigma_1 \prec b_2 \prec \sigma_2 \prec \dots \prec b_{n_1} \prec \sigma_{n_1} \prec \sigma_{n_1+1} \prec b_{n_1+1} \prec \dots \prec \sigma_{n_1+n_2} \prec b_{n_1+n_2}$.

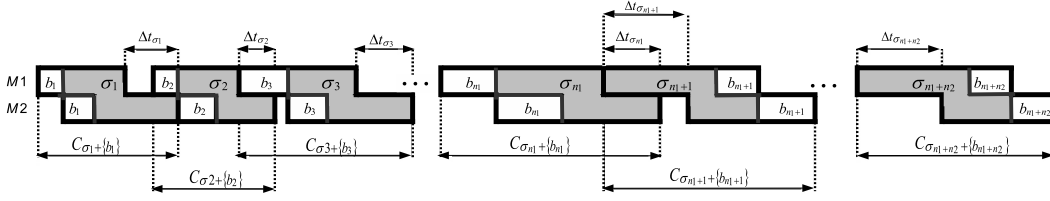


Fig. 9. Optimal job sequences for the general case

For each job j , we denote $u(j)$ (respectively $v(j)$) the index of the base of the first (respectively the last) b-pyramid to which the job j belongs. For a base b_i , we set $u(b_i) = v(b_i) = i$. Then we state the following general theorem.

Theorem 8. *Any job sequence such that:*

- *the bases of I_1 and I_2 are sequenced according to the ascending order of their indexes;*
- *any job j is sequenced inside any sub-sequence from $\sigma_{u(j)}$ to $\sigma_{v(j)}$ in any order;*

is optimal with:

$$\begin{aligned}
C_{max} = & C_{\sigma_1 + \{b_1\}} + \\
& \max(C_{\sigma_2 + \{b_2\}} - p_{b_2 1}, C_{\sigma_2 + \{b_2\}} - \Delta t_{\sigma_1}) + \\
& \max(C_{\sigma_3 + \{b_3\}} - p_{b_3 1}, C_{\sigma_3 + \{b_3\}} - \Delta t_{\sigma_2}) + \\
& \dots + \\
& \max(C_{\sigma_{n_1} + \{b_{n_1}\}} - p_{b_{n_1} 1}, C_{\sigma_{n_1} + \{b_{n_1}\}} - \Delta t_{\sigma_{n_1-1}}) + \\
& \max(C_{\sigma_{n_1+1} + \{b_{n_1+1}\}} - \Delta t_{\sigma_{n_1+1}}, C_{\sigma_{n_1+1} + \{b_{n_1+1}\}} - \Delta t_{\sigma_{n_1}}) + \\
& \max(C_{\sigma_{n_1+2} + \{b_{n_1+2}\}} - \Delta t_{\sigma_{n_1+2}}, C_{\sigma_{n_1+2} + \{b_{n_1+2}\}} - p_{b_{n_1+2} 2}) + \\
& \dots + \\
& \max(C_{\sigma_{n_1+n_2-1} + \{b_{n_1+n_2-1}\}} - \Delta t_{\sigma_{n_1+n_2-1}}, C_{\sigma_{n_1+n_2-1} + \{b_{n_1+n_2-1}\}} - p_{b_{n_1+n_2-2} 2}) + \\
& \max(C_{\sigma_{n_1+n_2} + \{b_{n_1+n_2}\}} - \Delta t_{\sigma_{n_1+n_2}}, C_{\sigma_{n_1+n_2} + \{b_{n_1+n_2}\}} - p_{b_{n_1+n_2-1} 2})
\end{aligned}$$

with:

$$C_{\sigma_i + \{b_i\}} = \begin{cases} p_{b_i 1} + \sum_{j \in \sigma_i + \{b_i\}} p_{j 2} & \text{if } i \in [1, n_1] \\ p_{b_i 2} + \sum_{j \in \sigma_i + \{b_i\}} p_{j 1} & \text{if } i \in [n_1 + 1, n_1 + n_2] \end{cases}$$

$$\Delta t_{\sigma_i} = \begin{cases} C_{\sigma_i + \{b_i\}} - \sum_{j \in \sigma_i + \{b_i\}} p_{j 1} & \text{if } i \in [1, n_1] \\ C_{\sigma_i + \{b_i\}} - \sum_{j \in \sigma_i + \{b_i\}} p_{j 2} & \text{if } i \in [n_1 + 1, n_1 + n_2] \end{cases}$$

Proof. We focus on the Johnson's sequence which arranges first the jobs of I_1 , according to the ascending order of their $p_{j 1}$, then the jobs of I_2 , according to

the descending order of their p_{j2} . This optimal sequence respects the theorem 8 since the base order matches the increasing order of the base index, any job $j \in I_1$ is assigned to $\sigma_{v(j)}$ and any job $j \in I_2$ is assigned to $\sigma_{u(j)}$. From the theorems 3, 4, 5, 6 and 7, one can easily deduce the optimal makespan expression given in the theorem 8. Moreover, since this expression gives the optimal makespan value for the particular previous Johnson's sequence, it also can be deduced that this value does not changed for any assignment of a non-base job j ranging from $\sigma_{u(j)}$ to $\sigma_{v(j)}$ (as seen in the proof of the theorem 4). \square

Now we intend to prove that the partial order defined by the theorem 8 is an extension of the partial order defined by the Johnson's rule.

Corollary 1. *Any Johnson's sequence is necessarily included in the set of job sequences characterized by the theorem 8.*

Proof. From the theorem 8, we state that a job i cannot precede a job j in any optimal sequence if and only if $u(i) > v(j)$. Indeed, if $u(i) \leq v(j)$ then either $u(i) \leq v(i) < u(j) \leq v(j)$ and i precedes j in every job sequence, or $u(i) \leq u(j) \leq v(i) \leq v(j)$ and job sequences exists such that i and j are assigned to the same sub-sequence σ_k with $u(j) \leq k \leq v(i)$ (hence i and j can be sequenced in any order). The relation $u(i) > v(j)$ (i.e. $i \not\prec j$) only stands in one of the following cases:

- j belongs to I_1 (but not to I_2) and i belongs to I_2 (but not to I_1), hence $p_{j2} > p_{j1}$ and $p_{i1} > p_{i2}$;
- i and j both belong to I_1 and either $p_{j1} < p_{i1} \leq p_{j2} < p_{i2}$ (i.e. *overlaps*(j, i) or *meets*(j, i)) or $p_{j1} < p_{j2} < p_{i1} < p_{i2}$ (i.e. *precedes*(j, i)), and there does not exist any base b such that both *during*(i, b) and *during*(j, b) (i and j do not belong to any common b-pyramid);
- i and j both belong to I_2 and either $p_{i2} < p_{j2} \leq p_{i1} < p_{j1}$ (i.e. *overlaps*(i, j) or *meets*(i, j)) or $p_{i2} < p_{i1} < p_{j2} < p_{j1}$ (i.e. *precedes*(i, j)), and there does not exist any base b such that both *during*(i, b) and *during*(j, b) (i and j do not belong to any common b-pyramid);

Thus one can see that in any above case the inequality $\min(p_{i1}, p_{j2}) > \min(p_{i2}, p_{j1})$ always holds. Therefore the theorem 8 only discards job precedences which do not satisfy the Johnson's rule. Hence the Johnson's sequences are conserved. \square

Also interesting is the impressive number of job sequences characterized by our theorem. Let S be the set of all the possible job assignments inside the sub-sequences σ . The cardinality of S is $\prod_{q=1}^m q^{n_q}$ [14], where m is the number of b-pyramids and n_q , the number of jobs belonging to exactly q b-pyramids.

For a particular job assignment $s \in S$, let $n_i(s)$ be the total number of jobs assigned to σ_i with respect of the theorem. Then the total number of job permutations is $\prod_{i=1}^m (n_i(s)!)$. Hence, the total number of characterized sequences is $\sum_{s \in S} (\prod_{i=1}^m n_i(s)!)$.

As illustration, for a problem with 20 jobs such that $m = 4$, $u(j) = v(j) \forall j \in J$ and $n_i = 4$, the theorem characterizes $(4!)^4 = 331776$ optimal sequences. Now for the same problem, if we assume that one (and only one) job belongs to two b-pyramids, then the number of characterized optimal sequences becomes $(4!)^4 + (3! * 5! * 4! * 4!) = 746496$.

7 Example

For theorem 8 illustration, let us take again the example of the figure 4. The value of the functions $u(j)$ and $v(j)$ are given for each job j on the table of the figure 10.

Jobs j	1	2	3	4	5	6	7	8	9
$u(j)$	1	2	3	4	1	1	2	3	4
$v(j)$	1	2	3	4	1	2	4	3	4

Fig. 10. $u(j)$ and $v(j)$ values

According to the theorem, we focus on job sequences in the form $b_1 \prec \sigma_1 \prec b_2 \prec \sigma_2 \prec b_3 \prec \sigma_3 \prec \sigma_4 \prec b_4$. Regarding the table of the figure 10, the possible assignments of the jobs are $5 \in \sigma_1$, $6 \in \sigma_1$ or $6 \in \sigma_2$, $7 \in \sigma_2$ or $7 \in \sigma_3$ or $7 \in \sigma_4$, $8 \in \sigma_3$ and $9 \in \sigma_4$. We recall that the order of jobs inside each sub-sequence σ_i does not impact the makespan value. Therefore, enumerating all the possibilities, we find thirteen optimal job sequences with the optimal makespan $C_{max} = 49$: $1 \prec 5 \prec 6 \prec 2 \prec 7 \prec 3 \prec 8 \prec 9 \prec 4$, $1 \prec 6 \prec 5 \prec 2 \prec 7 \prec 3 \prec 8 \prec 9 \prec 4$, $1 \prec 5 \prec 6 \prec 2 \prec 3 \prec 8 \prec 7 \prec 9 \prec 4$, $1 \prec 6 \prec 5 \prec 2 \prec 3 \prec 8 \prec 7 \prec 9 \prec 4$, $1 \prec 5 \prec 6 \prec 2 \prec 3 \prec 7 \prec 8 \prec 9 \prec 4$, $1 \prec 6 \prec 5 \prec 2 \prec 3 \prec 7 \prec 8 \prec 9 \prec 4$, $1 \prec 5 \prec 6 \prec 2 \prec 3 \prec 8 \prec 9 \prec 7 \prec 4$, $1 \prec 6 \prec 5 \prec 2 \prec 3 \prec 8 \prec 9 \prec 7 \prec 4$, $1 \prec 5 \prec 2 \prec 6 \prec 7 \prec 3 \prec 8 \prec 9 \prec 4^*$, $1 \prec 5 \prec 2 \prec 7 \prec 6 \prec 3 \prec 8 \prec 9 \prec 4$, $1 \prec 5 \prec 2 \prec 6 \prec 3 \prec 7 \prec 8 \prec 9 \prec 4^*$, $1 \prec 5 \prec 2 \prec 6 \prec 3 \prec 8 \prec 7 \prec 9 \prec 4^*$, $1 \prec 5 \prec 2 \prec 6 \prec 3 \prec 8 \prec 9 \prec 7 \prec 4$. For this problem instance, only the three asterisked sequences satisfy the Johnson's rule.

In order to illustrate the robustness of our approach, we slightly increase some job processing times in the above example. Then we analyse the consequences on the above set of optimal job sequences.

First let us suppose that, due to one disruption, p_{51} increases so that $p_{51} = 3$ (instead of 2). This modification does not affect the interval structures I_1 and I_2 since the Allen's relations between any pair of job intervals are maintained. Therefore the above set of optimal sequences is kept unchanged, as well as the C_{max} value (since C_{max} does not depend on p_{51} for this problem instance).

Now if we set $p_{52} = 5$ (instead of 4) then the interval structures I_1 and I_2 are again unchanged, as well as the set of optimal sequences. Nevertheless, the optimal makespan value is now $C_{max} = 60$ (instead of 59) for any job sequence.

Now we suppose that p_{62} increases so that $p_{62} = 7$ (instead of 5). In this case, the interval structure I_1 is modified because the job 6 does not belong any more to the b-pyramid P_{b_1} . Therefore any job sequence which assigns the job 6 to σ_1 has to be removed from the set of optimal sequences. Thus it only remains five optimal job sequences with the optimal makespan $C_{max} = 62$:

$1 \prec 5 \prec 2 \prec 6 \prec 7 \prec 3 \prec 8 \prec 9 \prec 4,$ $1 \prec 5 \prec 2 \prec 7 \prec 6 \prec 3 \prec 8 \prec 9 \prec 4,$
 $1 \prec 5 \prec 2 \prec 6 \prec 3 \prec 7 \prec 8 \prec 9 \prec 4,$ $1 \prec 5 \prec 2 \prec 6 \prec 3 \prec 8 \prec 7 \prec 9 \prec 4,$
 $1 \prec 5 \prec 2 \prec 6 \prec 3 \prec 8 \prec 9 \prec 7 \prec 4.$

We see through this simple scenario that our approach is relatively insensitive to data variations as long as these variations do not modify the bases of the interval structures. Of course if this condition is not respected then a new interval analysis has to be performed in order to re-compute dynamically the new b-pyramids.

8 Conclusion

In this paper, we established a new sufficient condition of optimality for the $F2|prmu|C_{max}$ problem. Based on two interval structures associated to the problem, the sufficient condition of optimality characterizes a set of optimal job sequences by determining the bases and the related b-pyramids of the interval structures. This sufficient condition is quite efficient since it ensures that any Johnson's sequence is always included in the set of characterized sequences, together with numerous other optimal job sequences. Furthermore, as the set of optimal sequences remains unchanged as long as the relative order of the processing times is conserved, this approach is relatively insensitive to processing time variations. For the future works, we want to reuse the results of this paper for solving more complex problems, starting with the $F2|r_i|C_{max}$ one, in order to define some new sufficient or dominant partial orders.

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