OPTIMIZATION IN GRAPHICAL SMALL CANCELLATION THEORY

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ABSTRACT. Gromov (2003) constructed finitely generated groups whose Cayley graphs contain all graphs from a given infinite sequence of expander graphs of unbounded girth and bounded diameter-to-girth ratio. These so-called *Gromov monster groups* provide examples of finitely generated groups that do not coarsely embed into Hilbert space, among other interesting properties. If graphs in Gromov's construction admit graphical small cancellation labellings, then one gets similar examples of Cayley graphs containing all the graphs of the family as isometric subgraphs. Osajda (2020) recently showed how to obtain such labellings using the probabilistic method. In this short note, we simplify Osajda's approach, decreasing the number of generators of the resulting group significantly.

1. Introduction

Given a finitely generated group Γ , with a finite set S of generators such that $S^{-1} = S$, the Cayley graph $\operatorname{Cay}(\Gamma, S)$ is the graph whose vertices are the elements of Γ , in which we add an edge between γ and $\gamma \cdot s$ for any $\gamma \in \Gamma$ and $s \in S$. Cayley graphs are a central object of study in geometric group theory. It turns out that a number of interesting properties of a group Γ do not depend of the choice of the generating set S. In particular, in order to show that Γ does not satisfy a given property of this type, it is sufficient to find one generating set S such that the corresponding Cayley graph $\operatorname{Cay}(\Gamma, S)$ has a pathological behaviour.

Consider a sequence $\mathcal{G} = (G_n)_{n \geq 1}$ of bounded degree graphs, whose girth (length of a shortest non-trivial cycle) tends to infinity. We say that the sequence is dgbounded if the ratio between the diameter and the girth of each G_n is bounded by a (uniform) constant, see [1]. Consider such a sequence \mathcal{G} . Gromov [5] proved that there is a finitely generated group Γ whose Cayley graph contains (in a certain metric sense) all the members of \mathcal{G} . By choosing \mathcal{G} as a family of suitable expander graphs, this implies that such a group Γ has a number of pathological properties, in particular related to coarse embeddings in Hilbert space, or to Guoliang Yu's property A. The construction has also been used very recently to disprove a conjecture on the twin-width of groups and hereditary graph classes [2]. Gromov [5] introduced the graphical small cancellation condition on the labellings. By the classical small cancellation theory, the existence of labellings of $\mathcal{G} = (G_n)_{n \geqslant 1}$ with the graphical small cancellation condition guarantees that in Gromov's construction each graph G_n embeds isometrically in the Cayley graph $Cay(\Gamma, S)$, which means that the embedding of each G_n in $Cay(\Gamma, S)$ is distance-preserving and thus in particular the graphs G_n appear as induced subgraphs in $Cay(\Gamma, S)$. Osajda [11] recently showed, using the probabilistic method, that the graphical small cancellation labellings do exist, under mild assumptions on $\mathcal{G} = (G_n)_{n \geq 1}$.

The authors are partially supported by the French ANR Project GrR (ANR-18-CE40-0032), TWIN-WIDTH (ANR-21-CE48-0014-01), and by LabEx PERSYVAL-lab (ANR-11-LABX-0025).

Given a sequence $\mathcal{G} = (G_n)_{n \geq 1}$ of graphs whose edges are labelled with elements from some set S, a word in \mathcal{G} is a sequence of labels that can be read along a path of some graph of \mathcal{G} . The main idea of graphical small cancellation theory is to assign labels from a finite set S to the edges of all the graphs from the sequence $\mathcal{G} = (G_n)_{n \geq 1}$, such that words in each G_n that are sufficiently long compared to the girth of G_n occur only once in all the sequence \mathcal{G} (this will be made more precise in the next section). The labels from S are then used as generators to define the group Γ whose relators are the words labelling the cycles of each G_n . The number of labels (the size of the set S) then gives an upper bound to the minimum number of generators of the group, and thus the degree of the associated Cayley graph (up to a multiplicative factor of two, if we do not require that S is closed under taking inverses). A natural problem is to minimize this number of generators.

The purpose of the present note is twofold: we present a simplified version of the proof of existence of the labelling of Osajda [11], and significantly decrease the number of generators (and thus the degree of the corresponding Cayley graph). Osajda's proof is based on an application of the Lovász Local Lemma. Instead, we use a self-contained counting argument popularized by Rosenfeld [12], and originally introduced in the field of combinatorics on words in the context of pattern avoidance. This allows us to cleanly handle all the different forbidden patterns at once, instead of sequentially, and greatly reduces the number of labels. We combine this with a significantly simpler (and stronger) analysis of intersecting patterns in order to a obtain a shorter argument that also produces much better bounds.

For the sake of concreteness, if we take $\mathcal{G} = (G_n)_{n \geq 1}$ to be the sequence of cubic Ramanujan graphs introduced by Chiu [3], which is likely to offer the best known parameters in terms of degree and diameter-to-girth ratio, our result leads to the existence of a group with 96 generators, whose Cayley graph (of maximum degree 96) contains all the graphs from \mathcal{G} as isometric subgraphs. For the same family, the construction of Osajda [11] uses about 10^{272} generators (although we note that some of the quick optimization steps we perform in Section 4 can also be carried directly in Osajda's proof, improving his bound to about 10^{70} generators).

2. Preliminaries

All the graphs we consider in the paper are initially undirected. Each graph G is then given an arbitrary orientation \vec{G} (i.e., the choice of a direction, for each edge of G). The results do not depend on the specific orientation, but the orientation is nevertheless crucial to define the relevant objects that we consider belows. Consider a set S which is closed under (formal) inverse (that is, there is an involution without fixed point between the elements of S, which we denote by $a \mapsto \bar{a}$). Consider also a labelling $\ell: E(G) \to S$ of the edges of G by the elements of S. We extend the labelling ℓ to the ordered pairs of adjacent vertices (x,y) in G as follows: if (x,y) is an arc of \bar{G} then $\ell(x,y)=\ell(xy)$ and otherwise $\ell(x,y)=\overline{\ell(xy)}$. The orientation \bar{G} is only used to define this extended labelling ℓ of the the ordered pairs of adjacent vertices, and will not be mentioned elsewhere. We say that the labelling ℓ is reduced if for any vertex $v \in V(G)$, and for any pair of distinct neighbors u, w of v in G, $\ell(v, u) \neq \ell(v, w)$. An ℓ -word (or simply a word, if ℓ is clear from the context) in G is obtained from a path P in G as follows: if $P = v_1, v_2, \ldots, v_k$, then $\ell(P) := \ell(v_1, v_2) \cdots \ell(v_{k-1}, v_k) \in L^*$ is the ℓ -word associated to P. The length of a path is its number of edges. We remark

that in this paper we consider paths as either a sequence of vertices, or a sequence of edges, depending on the context, and in particular any path $P = v_1, v_2, \ldots, v_k$ is distinct from the reverse path $\overline{P} := v_k, v_{k-1}, \ldots, v_1$.

The girth (length of a smallest cycle) of a graph G is denoted by $\operatorname{girth}(G)$, and its diameter (the maximum distance between two vertices of G) is denoted by $\operatorname{diam}(G)$. Let $\mathcal{G} = (G_n)_{n\geqslant 1}$ be a sequence of graphs. Let λ be a positive real number (for the main application in group theory we need $\lambda \in (0, \frac{1}{6}]$, but this will not be needed in the full generality of the results presented in this section and the next). Following the terminology of [11], a sequence of labellings $(\ell_n)_{n\geqslant 1}$ of the graphs from \mathcal{G} , with labels from some set S as above, is said to satisfy the $C'(\lambda)$ -small cancellation property if for all $n\geqslant 1$, ℓ_n is a reduced labelling of G_n and no word of length at least λ -girth (G_n) in G_n appears on a different path in \mathcal{G} . Small cancellation properties were initially introduced for groups, as a convenient tool to construct word-hyperbolic groups, see for instance Chapter V in [8]. The property $C'(\lambda)$ we use here is defined in the more general context of graphs, and is usually known as graphical cancellation property in the literature. In the remainder of the paper we will omit the "graphical" term, as there is no risk of confusion with the original small cancellation properties.

Osajda [11] recently proved that under mild assumptions, any sequence of bounded degree dg-bounded graphs of unbounded girth admits small cancellation labellings with a finite number of labels.

Theorem 2.1 ([11]). Let $\lambda \in (0, \frac{1}{6}]$ and A > 0 be real numbers, and let $\Delta \geqslant 3$ be an integer. Let $\mathcal{G} = (G_n)_{n\geqslant 1}$ be a sequence of graphs of maximum degree Δ such that $girth(G_n) \to \infty$ as $n \to \infty$, and $diam(G_n) \leqslant A \cdot girth(G_n)$ for any $n \geqslant 1$. Assume moreover that $1 < \lfloor \lambda \cdot girth(G_n) \rfloor < \lfloor \lambda \cdot girth(G_{n+1}) \rfloor$ for every $n \geqslant 1$. Let

$$L \geqslant 2e^4 \Delta^{2A/\lambda+2} \cdot (4e^4 \Delta)^{8A/\lambda+16}$$

be any even integer. Then G has a sequence of labellings satisfying the $C'(\lambda)$ -small cancellation property, with labels from a set S of size L.

The bound on L in Theorem 2.1 has two components: $2e^4\Delta^{2A/\lambda+2}$ comes from a first phase, where Osajda shows how to assign labels in each $G_n \in \mathcal{G}$, so that no word of G_n appears as a word of length at least $\lambda \cdot \operatorname{girth}(G_i)$ in some G_i , with i < n. The second component, $(4e^4\Delta)^{8A/\lambda+16}$, comes from a second phase where Osajda shows how to assign labels in each $G_n \in \mathcal{G}$, so that no word of G_n of length at least $\lambda \cdot \operatorname{girth}(G_n)$ appears twice in G_n . This second phase is significantly more involved, which explains the much larger label size. Our main contribution is the following.

- we use a counting argument instead of the Lovász Local Lemma. This allows us to assign labels in a single phase (resulting in an additive combination of the number of labels, instead of a multiplicative one), and optimize the multiplicative constants. Moreover, the resulting proof is completely self-contained.
- we provide a major simplification in the analysis of Osajda's second phase, showing that the long words appearing twice in G_n can be avoided with a number of labels of size comparable to Osajda's first phase.

Using results of Gromov (see [10, 6]), Theorem 2.1 leads to the following.

Corollary 2.2. Let $\lambda, A, \Delta, \mathcal{G}$ be as in Theorem 2.1. Then for any even integer $L \geqslant 2e^4\Delta^{2A/\lambda+2} \cdot (4e^4\Delta)^{8A/\lambda+16}$, there is a group Γ with a set S of L generators such

that the corresponding Cayley graph $Cay(\Gamma, S)$ contains isometric copies of all the graphs from \mathcal{G} .

As alluded to in the introduction, in applications we typically want \mathcal{G} to be a sequence of expander graphs. We omit the precise definition here, as it will not be necessary in this paper. We only mention that expansion can be defined in several essentially equivalent ways, using isoperimetric inequalities or spectral properties. Families of random regular graphs typically have these properties, but constructing explicit families of expander graphs has been an important problem in Mathematics, with major applications in Theoretical Computer Science. We refer the interested reader to the survey [7] for more on expander graphs.

A useful family \mathcal{G} for us is the sequence of cubic Ramanujan graphs introduced by Chiu [3]. These graphs are expander graphs (as Ramanujan graphs, they have the best possible spectral expansion), are Δ -regular with $\Delta = 3$, satisfy $girth(G_n) \to \infty$ as $n \to \infty$ (their girth is logarithmic in their number of vertices) and $diam(G) \leq \frac{3}{2} girth(G) + 5$ for any $G \in \mathcal{G}$. By discarding a bounded number of small graphs in the sequence, this implies that we have $diam(G) \leq (\frac{3}{2} + \epsilon) girth(G)$ for any $\epsilon > 0$ and any graph G in the sequence, and thus we can take $A \leq \frac{3}{2} + \epsilon$ for any $\epsilon > 0$.

3. SMALLER CANCELLATION LABELLINGS

Our main result is the following optimized version of Theorem 2.1.

Theorem 3.1. Let $\lambda, A, \Delta, \mathcal{G}$ be as in Theorem 2.1, that is $\lambda \in (0, \frac{1}{6}]$ and A > 0 are real numbers, $\Delta \geqslant 3$ is an integer, and $\mathcal{G} = (G_n)_{n\geqslant 1}$ is a sequence of graphs of maximum degree Δ such that $girth(G_n) \to \infty$ as $n \to \infty$, and $diam(G_n) \leqslant A \cdot girth(G_n)$ and $1 < |\lambda \cdot girth(G_n)| < |\lambda \cdot girth(G_{n+1})|$ for every $n \geqslant 1$. Let

$$L \geqslant 2(\Delta - 1) + 26(\Delta - 1)^{2A/\lambda + 2}$$

be any even integer. Then G has a sequence of labellings satisfying the $C'(\lambda)$ -small cancellation property, with labels from a set S of size L.

We note that the multiplicative constant of 26 in the bound on L can be optimized both for small values of Δ and asymptotically as $\Delta \to \infty$. We have chosen not to do so here for simplicity, and we remark that improving the factor 2 in the exponent of $(\Delta - 1)$ is a more rewarding challenge (see the next section). When $\Delta \to \infty$, the number L of labels in Theorem 3.1 grows as $O(\Delta^{2A/\lambda+2})$, and we will see in the next section that this can be easily improved to $O(\Delta^{A/\lambda+2})$. This is to be compared with the bound $O(\Delta^{10A/\lambda+18})$ of Theorem 2.1. In the next section we will also see several ways to improve the constants significantly when $\Delta = 3$, and the girth of the first graph in the sequence is already quite large.

Similarly as above, we obtain the following corollary.

Corollary 3.2. Let $\lambda, A, \Delta, \mathcal{G}, L$ be as in Theorem 3.1. Then there is a group Γ with a set S of L generators such that the corresponding Cayley graph $Cay(\Gamma, S)$ contains isometric copies of all the graphs from \mathcal{G} .

Using the family of cubic Ramanujan graphs of Chiu [3] mentioned at the end of the previous section, we can apply Corollary 3.2 with $\Delta = 3$, $A = \frac{3}{2}$ and $\lambda = \frac{1}{6}$. Then we obtain a group with a set of $L = 4 + 26 \cdot 2^{20} = 27262980$ generators such that the corresponding Cayley graph contains isometric copies of graphs from an infinite

family of expander graphs. We will see in Section 4 how to decrease this number of generators to 96.

If instead we apply Corollary 2.2 to the same family \mathcal{G} (and hence with the same parameters $\Delta = 3$, $A = \frac{3}{2}$ and $\lambda = \frac{1}{6}$), the resulting Cayley graph has degree more than 10^{272} .

We now prove our main result.

Proof of Theorem 3.1. Let $\alpha := 2(\Delta - 1)^{2A/\lambda + 2}$, and let

$$L \geqslant 2(\Delta - 1) + 13\alpha = 2(\Delta - 1) + 26(\Delta - 1)^{2A/\lambda + 2}$$

be an even integer. Let S be a set of L elements, closed under formal inverses (and such that each element $a \in S$ is different from its formal inverse \bar{a}). For any $n \geq 1$, let $\gamma_n := \lfloor \lambda \cdot \operatorname{girth}(G_n) \rfloor$. In particular $\gamma_n \leq \lambda \cdot \operatorname{girth}(G_n) \leq \gamma_n + 1$ for any $n \geq 1$, and thus

(1)
$$\frac{1}{\lambda} \leqslant \frac{\operatorname{girth}(G_n)}{\gamma_n} \leqslant \frac{1}{\lambda} + \frac{1}{\lambda \gamma_n} \leqslant \frac{2}{\lambda}.$$

We will sequentially assign labels from S to the edges of each of the graphs $(G_n)_{n\geqslant 1}$. Assume that for each i< n, we have already defined a labelling ℓ_i of the edges of G_i such that the sequence of labellings $(\ell_i)_{i< n}$ satisfies the $C'(\lambda)$ -small cancellation property. We now want to define a labelling ℓ_n of G_n so that the sequence $(\ell_i)_{i\leqslant n}$ of labellings of the graphs from $(G_i)_{i\leqslant n}$ still satisfies the $C'(\lambda)$ -small cancellation property.

For the proof it will be convenient to consider partial labellings of G_n , which are labellings of some subset F of edges of G_n . Equivalently, these are labellings of the edges of $G_n[F]$, the subgraph of G_n induced by the edges of F. We recall that each labelling $\ell(xy)$ of an edge xy yields two labellings $\ell(x,y)$ and $\ell(y,x)$ of the pairs (x,y) and (y,x) by elements of S that are formal inverse (and that whether $\ell(xy) = \ell(x,y)$ or $\ell(xy) = \ell(y,x)$ depends only on the orientation of the edge xy in some fixed but otherwise arbitrary orientation of the graph under consideration).

Let F be a non-empty subset of $E(G_n)$. We say that a labelling ℓ of $G_n[F]$ with labels from S is valid if it satisfies the following properties:

- (a) ℓ is a reduced labelling of $G_n[F]$,
- (b) for each $1 \leq i < n$, no ℓ_i -word of length at least γ_i in G_i appears as an ℓ -word in $G_n[F]$, and
- (c) no ℓ -word of length at least γ_n appears on two different paths of $G_n[F]$.

Let c(F) be the number of valid labellings ℓ of $G_n[F]$ with labels from S (when F is empty we conveniently define c(F) := 1). In the remainder of the proof we will show the following claim, which clearly implies that G_n has a labelling ℓ_n such that the sequence of labellings $(\ell_i)_{i \leq n}$ of $(G_i)_{i \leq n}$ still satisfies the $C'(\lambda)$ -small cancellation property, and thus we can find such labellings in all the graphs from \mathcal{G} .

Claim 3.3. For any non-empty
$$F \subseteq E(G_n)$$
 and any $e \in F$, $c(F) \geqslant \alpha \cdot c(F \setminus \{e\})$.

We prove the claim by induction on |F|. Recall that by assumption, $\gamma_i > 1$ for any $i \ge 1$, so the properties (a), (b), (c) above are trivially satisfied if F contains a single element e, which is assigned an arbitrary label from S. It follows that

 $c(\{e\}) = L \geqslant \alpha = \alpha \cdot c(\emptyset)$, as desired. So we can now assume that F contains at least two elements.

Assume that we have proved the claim for any $F' \subseteq E(G_n)$ with |F'| < |F|. Consider any edge $xy \in F$. Our goal in the remainder of the proof is to show that $c(F) \geqslant \alpha \cdot c(F \setminus \{xy\})$. Note that by the induction hypothesis, for any subset $F' \subseteq F$ containing xy,

(2)
$$c(F \setminus F') \leqslant \alpha^{1-|F'|} \cdot c(F \setminus \{xy\}).$$

Let \mathcal{L} denote the set of labellings ℓ of F with labels from S whose restriction to $F \setminus \{xy\}$ is valid, but such that ℓ itself is not. Then

(3)
$$c(F) = L \cdot c(F \setminus \{xy\}) - |\mathcal{L}|.$$

Consider first the subset $\mathcal{L}_a \subseteq \mathcal{L}$ of labellings of F that do not satisfy (a) above. Then by definition, for any $\ell \in \mathcal{L}_a$, x has a neighbor z different from y such that $\ell(x,y) = \ell(x,z)$, or y has a neighbor z different from x such that $\ell(y,x) = \ell(y,z)$. By assumption, the labelling ℓ^- of $F \setminus \{xy\}$ obtained from ℓ by discarding the label of xy is valid. Moreover, ℓ can be recovered in a unique way from ℓ^- and the edge xz or yz as above. As there are at most $2(\Delta - 1)$ choices for such an edge incident to xy, we obtain

$$(4) |\mathcal{L}_a| \leqslant 2(\Delta - 1) \cdot c(F \setminus \{xy\}).$$

For $1 \leq i \leq n-1$, let \mathcal{L}_i be the subset of labellings $\ell \in \mathcal{L}$ of F such that $G_n[F]$ contains a path P containing xy such that $\ell(P)$ coincides with some ℓ_i -word $\ell_i(Q)$ of length γ_i in G_i . Let \mathcal{L}_n be the subset of labellings $\ell \in \mathcal{L} \setminus \mathcal{L}_a$ of F such that $G_n[F]$ contains a path P containing xy such that $\ell(P)$ coincides with some ℓ -word $\ell(Q)$ of length γ_n in $G_n[F]$, for some path Q distinct from P.

For each $1 \leq i \leq n$ and each labelling $\ell \in \mathcal{L}_i$ as above, let ℓ^- denote the labelling of $F \setminus E(P)$ obtained from ℓ by discarding the labels of the edges of P. Then ℓ^- is a valid labelling of $F \setminus E(P)$. Moreover, if $1 \leq i \leq n-1$ or if i=n and P and Q are disjoint, then ℓ^- together with the paths P in G_n and Q in G_i (where each path is viewed as a sequence of edges) are sufficient to recover ℓ in a unique way.

Assume now that $\ell \in \mathcal{L}_n$ (so in particular ℓ is reduced), and the distinct paths P and Q of length γ_n in $G_n[F]$ such that $\ell(P) = \ell(Q)$ are not edge-disjoint. We first observe that $E(P) \cap E(Q)$ is a subpath of P and Q, since otherwise G_n would contain a cycle of length less than $2\gamma_n$, contradicting the assumption that $\text{girth}(G_n) \geqslant \frac{\gamma_n}{\lambda} \geqslant 6\gamma_n$. Let $P = x_0, x_1, \ldots, x_{\gamma_n}$ and $Q = y_0, y_1, \ldots, y_{\gamma_n}$. Then $\ell(x_i, x_{i+1}) = \ell(y_i, y_{i+1})$ for any $0 \leqslant i \leqslant \gamma_n - 1$. Our goal is to show that despite the fact that the edges of $E(P) \cap E(Q)$ have been unlabelled in ℓ^- , we can still recover ℓ from ℓ^- , P and Q.

Assume first that P and Q intersect in the same direction, that is there are integers $0 \le p, q \le \gamma_n - 1$ and $1 \le k \le \gamma_n - 1$ such that $x_{p+i} = y_{q+i}$ for any $0 \le i \le k$. Note that $p \ne q$ since otherwise we would have $x_p = y_p$ and $x_{p+k} = y_{p+k}$ and the fact that $\ell(x_{p-1}, x_p) = \ell(y_{p-1}, y_p)$ or $\ell(x_{p+k}, x_{p+k+1}) = \ell(y_{p+k}, y_{p+k+1})$ would contradict the fact that ℓ is reduced. Up to considering the reverse paths P and Q instead of P and Q, we can assume without loss of generality that q > p. Divide P into consecutive subpaths P_1 , $P \cap Q$, P_3 , and P_4 and divide Q into consecutive subpaths Q_1 , Q_2 , $P \cap Q$, and Q_4 , in such a way that $\ell(P_1) = \ell(Q_1)$ and $\ell(P_4) = \ell(Q_4)$ (see Figure 1, left). As P_1 and P_4 are edge-disjoint from $E(P) \cap E(Q)$, both $\ell(P_1)$ and $\ell(P_4)$ can be recovered from ℓ^- . Note that as we assumed that q > p, $|Q_2| > 0$, i.e. Q_2 has at least one edge. Let P' be the subpath of P obtained by concatenating $P \cap Q$ and P_3 .

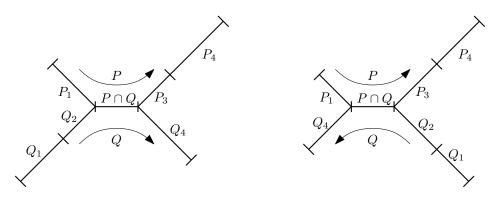


FIGURE 1. Two intersecting paths P and Q.

It remains to explain how to recover $\ell(P')$ from ℓ^- . For this, it suffices to observe that since $\ell(P) = \ell(Q)$, the prefix of $\ell(P')$ of size $|Q_2|$ must be equal to $\ell(Q_2)$. Then the prefix of $\ell(P')$ of size $2|Q_2|$ must be equal to $\ell(Q_2) \cdot \ell(Q_2)$. By iterating this observation, it follows that $\ell(P')$ is a prefix of the word $\ell(Q_2)^{\omega}$ (the concatenation of an infinite number of copies of $\ell(Q_2)$). Since Q_2 is edge-disjoint from $E(P) \cap E(Q)$, $\ell(P')$ (and thus $\ell(P)$) can be recovered from ℓ^- , P and Q, as desired.

We now assume that P and Q intersect in reverse directions, that is there are integers $0 \leqslant p, q \leqslant \gamma_n$ and $k \geqslant 1$ such that $x_{p+i} = y_{q-i}$ for any $0 \leqslant i \leqslant k$. We say that P and Q collide if there is an index i such that either $x_i = y_i$, or $x_i = y_{i+1}$ and $y_i = x_{i+1}$ (think of two particles following the trajectories of P and Q at the same speed). Assume for the sake of contradiction that P and Q collide. If $x_i = y_i$ for some index i, then ℓ is not reduced, which is a contradiction. Otherwise we have $\ell(x_i, x_{i+1}) = \ell(y_i, y_{i+1}) = \ell(x_{i+1}, x_i)$, which contradicts the fact that $\ell(x_i, x_{i+1}) = \ell(x_i, x_{i+1})$ $\overline{\ell(x_{i+1},x_i)}$ as for each $a\in S, \overline{a}\neq a$. So P and Q do not collide, and in particular $p \neq q$. We recall that \overleftarrow{P} and \overleftarrow{Q} denote the paths obtained by reversing P and Q, respectively. When we use this notation below we also write \overrightarrow{P} and \overrightarrow{Q} instead of Pand Q to avoid any confusion. Up to considering \overrightarrow{P} and \overrightarrow{Q} instead of \overrightarrow{P} and \overrightarrow{Q} , we can again assume without loss of generality that q > p. We divide P into consecutive subpaths P_1 , $\overrightarrow{P} \cap \overleftarrow{Q}$, P_3 and P_4 and we divide Q into consecutive subpaths Q_1 , Q_2 , $\overrightarrow{P} \cap \overrightarrow{Q}$, and Q_4 , in such a way that $\ell(P_1) = \ell(Q_1)$, $\ell(P_4) = \ell(Q_4)$ (see Figure 1, right). As before, P_1 and P_4 are edge-disjoint from $E(P) \cap E(Q)$, so both $\ell(P_1)$ and $\ell(P_4)$ can be recovered from ℓ^- . As P and Q do not collide, $|Q_2| > |\overrightarrow{P} \cap \overleftarrow{Q}|$, which implies that $\ell(\overrightarrow{P} \cap \overleftarrow{Q})$ is equal to a prefix of $\ell(Q_2)$, and can thus be recovered from ℓ^- . Finally, since $\ell(P) = \ell(Q)$, $\ell(P_3)$ is equal to $\ell(\overleftarrow{P} \cap \overrightarrow{Q})$, which is obtained by reading $\ell(\overrightarrow{P} \cap \overleftarrow{Q})$ backwards. Hence, $\ell(P)$ can be recovered from ℓ^- , P and Q, as desired.

For each $1 \leq i \leq n$ and each edge e in G_i there are at most $(\Delta - 1)^{\gamma_i - 1}$ paths of length γ_i containing e in which e is at a fixed position on the path. Hence, there are at at most $2\gamma_i(\Delta - 1)^{\gamma_i - 1}$ paths of length γ_i containing e (and in particular at most $2\gamma_i(\Delta - 1)^{\gamma_i - 1}$ choices for the path P in G_n containing xy when considering a labelling $\ell \in \mathcal{L}_i$). Moreover, each G_i has at most $1 + \Delta + \Delta(\Delta - 1) + \cdots + \Delta(\Delta - 1)^{\text{diam}(G_i) - 1}$ vertices, and thus at most

(5)
$$\frac{\Delta}{2} \cdot \left(1 + \Delta \frac{(\Delta - 1)^{\operatorname{diam}(G_i)} - 1}{\Delta - 2}\right) \leqslant \frac{3}{2} (\Delta - 1)^{\operatorname{diam}(G_i) + 2}$$

edges, using $\Delta \geqslant 3$ (the inequality is quite loose here, we have chosen the right-hand side mostly in order to simplify the computation later). It follows that each G_i has at most

(6)
$$\frac{3}{2}(\Delta - 1)^{\operatorname{diam}(G_i) + 2} \cdot 2(\Delta - 1)^{\gamma_i - 1} \leqslant 3(\Delta - 1)^{(2A/\lambda + 1)\gamma_i + 1}$$

paths of length γ_i (here the multiplicative factor γ_i disappears since we can count each path from its starting edge). It follows that there are at most $3(\Delta-1)^{(2A/\lambda+1)\gamma_i+1}$ choices for the path Q in G_i when considering a labelling $\ell \in \mathcal{L}_i$. Since $|E(P)| = \gamma_i$, it follows from (2) that for each labelling $\ell \in \mathcal{L}_i$, the number of valid labellings ℓ^- of $F \setminus E(P)$ is $c(F \setminus E(P)) \leq \alpha^{1-\gamma_i} \cdot c(F \setminus \{xy\})$. As each $\ell \in \mathcal{L}_i$ can be recovered from ℓ^- , P and Q in a unique way, we obtain

$$|\mathcal{L}_{i}| \leq 2\gamma_{i}(\Delta - 1)^{\gamma_{i} - 1} \cdot 3(\Delta - 1)^{(2A/\lambda + 1)\gamma_{i} + 1} \cdot \alpha^{1 - \gamma_{i}} \cdot c(F \setminus \{xy\})$$

$$\leq 6\gamma_{i}(\Delta - 1)^{(2A/\lambda + 2)\gamma_{i}} \cdot \alpha^{1 - \gamma_{i}} \cdot c(F \setminus \{xy\})$$

$$\leq 6\gamma_{i}(\alpha/2)^{\gamma_{i}} \cdot \alpha^{1 - \gamma_{i}} \cdot c(F \setminus \{xy\})$$

$$\leq 6\alpha \cdot \gamma_{i}(1/2)^{\gamma_{i}} \cdot c(F \setminus \{xy\}),$$

where we have used $\alpha = 2(\Delta - 1)^{2A/\lambda + 2}$ in the third inequality. As a consequence

(7)
$$\sum_{i=1}^{n} |\mathcal{L}_i| \leqslant 6\alpha \sum_{i=1}^{n} \gamma_i (1/2)^{\gamma_i} c(F \setminus \{xy\}) \leqslant 12\alpha \cdot c(F \setminus \{xy\}),$$

where we have used $\sum_{j=1}^{\infty} j(1/2)^j = 2$. As $\mathcal{L} = \mathcal{L}_a \cup \bigcup_{i=1}^n \mathcal{L}_i$, it follows from (4) and (7) that

$$|\mathcal{L}| \leq c(F \setminus \{xy\}) \cdot (2(\Delta - 1) + 12\alpha)$$

$$\leq c(F \setminus \{xy\})(L - \alpha),$$

by the definition of L. By (3), we have

$$c(F) = L \cdot c(F \setminus \{xy\}) - |\mathcal{L}|$$

$$\geqslant L \cdot c(F \setminus \{xy\}) - (L - \alpha)c(F \setminus \{xy\})$$

$$\geqslant \alpha \cdot c(F \setminus \{xy\}),$$

as desired. This completes the proof of Claim 3.3, which concludes the proof of Theorem 3.1.

4. Optimizing the number of generators

So far our goal was to optimize the construction of Osajda [11], while obtaining a result that is comparable to his (i.e., a result with the exact same set of initial assumptions). There are two quick ways to further optimize the number of labels in Theorem 3.1, if we have some control over the family \mathcal{G} .

The first way consists in removing all sufficiently small graphs from \mathcal{G} (we have done this already with the cubic Ramanujan graphs of Chiu [3], to argue that A was arbitrarily close to $\frac{3}{2}$ in this case). As the girth of the graphs in \mathcal{G} tends to infinity, the right-hand-side of (1) can be replaced by $\frac{1+\epsilon}{\lambda}$ for any $\epsilon > 0$. This allows to replace all instances of $2A/\lambda$ by $(1+\epsilon)A/\lambda$ in the proof, effectively dividing by 2 the exponent of the number of labels in the theorem. Using this observation in the case of the cubic Ramanujan graphs of Chiu [3], with $\lambda = 1/6$, we obtain $\alpha = 2 \cdot 2^{(1+\epsilon)\frac{3}{2}/\frac{1}{6}+2} \leq 4097$ for sufficiently small $\epsilon > 0$, and a number of labels $L \geqslant 2 \cdot 2 + 13 \cdot 4097 \approx 53266$ is sufficient.

A more efficient way to decrease the number of labels in the case of families of expander graphs with an explicit description consists in using a more precise bound on the number of edges in a graph $G_n \in \mathcal{G}$, as a function of $girth(G_n)$. In (5), we have used that $|E(G_n)| \leq \frac{3}{2}(\Delta-1)^{\operatorname{diam}(G_n)+2} \leq \frac{3}{2}(\Delta-1)^{A\operatorname{girth}(G_n)+2}$. However, better bounds are known for a number of families \mathcal{G} . This is the case for the cubic Ramanujan graphs of Chiu [3] mentioned in the previous section. The graphs G in this class satisfy $|E(G_n)| \leq \frac{3}{2} \cdot 2^{(3\operatorname{girth}(G_n)+6)/4}$, which is an improvement over the bound based on the diameter (recall that for these graphs $\Delta=3$ and A can be made arbitrarily close to $\frac{3}{2}$). Fix any real $\epsilon>0$, and recall that $\gamma_n=\lfloor\lambda\cdot\operatorname{girth}(G_n)\rfloor$. Using as in the previous paragraph the fact that the girth of the graphs from \mathcal{G} can be made arbitrarily large by discarding a constant number of graphs from the family, we can assume that $\gamma_n\epsilon>\gamma_1\epsilon$ is larger than any fixed constant, and thus $(3\operatorname{girth}(G_n)+6)/4\leq \frac{3+\epsilon}{4\lambda}\gamma_n$ and $|E(G_n)|\leq \frac{3}{2}\cdot 2^{\gamma_n\cdot(3+\epsilon)/4\lambda}$, for any $n\geqslant 1$. With $\lambda=1/6$, we obtain $|E(G_n)|\leq \frac{3}{2}\cdot 2^{(9+\epsilon)\gamma_n/2}$, for any $n\geqslant 1$. Substituting this bound in (6), we obtain that there are at most $3\cdot 2^{(11+\epsilon)\gamma_n/2-1}$, paths of length γ_n in G_n . Substituting this bound in the proof of Theorem 3.1, and defining $\alpha:=(1+\epsilon)2^{(13+\epsilon)/2}$, we obtain the following.

$$|\mathcal{L}_{i}| \leq 2\gamma_{i}2^{\gamma_{i}-1} \cdot 3 \cdot 2^{(11+\epsilon)\gamma_{i}/2-1} \cdot \alpha^{1-\gamma_{i}} \cdot c(F \setminus \{xy\})$$

$$\leq \frac{3\alpha}{2}\gamma_{i} \cdot 2^{(13+\epsilon)\gamma_{i}/2} \cdot \alpha^{-\gamma_{i}} \cdot c(F \setminus \{xy\})$$

$$\leq \frac{3\alpha}{2}\gamma_{i} \cdot (\frac{1}{1+\epsilon})^{\gamma_{i}} \cdot c(F \setminus \{xy\}),$$

As $\sum_{j=1}^{\infty} j \cdot (\frac{1}{1+\epsilon})^j$ converges, we can choose again γ_1 sufficiently large so that the truncated sum $\sum_{j=\gamma_1}^{\infty} j \cdot (\frac{1}{1+\epsilon})^j$ is arbitrarily small (say smaller than $\epsilon/(\frac{3\alpha}{2})$). We obtain $\sum_{i=1}^{n} |\mathcal{L}_i| \leqslant \epsilon \cdot c(F \setminus \{xy\})$, and the same computation as in the proof of Claim 3.3 shows that any even number $L \geqslant 2 \cdot 2 + \epsilon + \alpha = \alpha + \epsilon + 4$ of labels is sufficient. Using $\alpha = (1+\epsilon)2^{(13+\epsilon)/2}$, and taking $\epsilon > 0$ sufficiently small, we can obtain that L = 96 labels are sufficient.

So, we obtain a group with a set S of 96 generators whose Cayley graph $Cay(\Gamma, S)$ contains infinitely many graphs of the sequence of cubic Ramanujan graphs as isometric subgraphs.

5. Conclusion

The number L of labels in Theorem 3.1 is of order $O(\Delta^{2A/\lambda+2})$, as $\Delta \to \infty$, and the remarks in the previous section improve this bound to $O(\Delta^{A/\lambda+2})$. In typical applications, A is a small constant and the bound becomes $\Delta^{O(1/\lambda)}$. We now observe that this is the right order of magnitude. If G is a Δ -regular graph of girth g, then the ball of radius g/2 centered in any vertex induces a tree, and thus for any $\lambda < 1/2$, G contains $\Omega(\Delta^{g/2+\lambda g-1})$ paths of length λg (the ball of radius g/2 centered in a vertex contains $\Omega(\Delta^{g/2})$ edges and each of them is the starting point of $\Omega(\Delta^{\lambda g-1})$ paths of length λg). By the $C'(\lambda)$ -small cancellation property, all these paths must correspond to different words. As there are at most $L^{\lambda g}$ possible words of length λg , we obtain $L^{\lambda g} = \Omega(\Delta^{g/2+\lambda g-1})$, and thus $L = \Omega(\Delta^{1/2\lambda+1-1/g})$. As the girth of the graphs in our family is unbounded, it follows that $L = \Omega(\Delta^{1/2\lambda+1})$, which shows that the bound in Theorem 3.1 is fairly close to the optimum (up to a small multiplicative factor in the exponent). It remains an interesting problem to close the gap between the upper and lower bounds, both in the case of small degree $(\Delta = 3)$ and asymptotically as $\Delta \to \infty$.

It might also be interesting to consider other cancellation properties. For an integer $k \geq 1$, a family of labellings $(\ell_n)_{n \geq 1}$ of a graph family $\mathcal{G} = (G_n)_{n \geq 1}$ satisfies the C(k+1)-small cancellation property if for any $n \geq 1$, ℓ_n is reduced and no cycle C in G_n can be divided into k paths P_1, \ldots, P_k such that for each $1 \leq i \leq k$, the ℓ_n -word associated to P_i appears on a different path in \mathcal{G} . This condition is weaker than the C'(1/k)-small cancellation property, but nevertheless allows to construct finitely generated groups with interesting properties when $k \geq 7$ [6]. A natural problem is to obtain a version of Theorem 3.1 for C(k)-small cancellation, with an improved exponent.

We conclude with some algorithmic remarks. Using the constructive proof of the Lovász Local Lemma by Moser and Tardos [9], the original proof of existence of the labelling given by Osajda [11] can be turned into an efficient algorithm computing the labels, by which we mean a randomized algorithm, running in polynomial time (in the size of G_n), and computing a $C'(\lambda)$ -small cancellation labelling for the sequence of graphs $(G_i)_{1 \leq i \leq n}$. As our main goal was to obtain a simple, self-contained proof of the existence of the labels, we chose to use counting rather than constructive techniques such as the entropy compression method (see [4]). It turns out that our result can also be obtained with this type of techniques, at the cost of a longer and more technical analysis.

Acknowledgements. We thank Goulnara Arzhantseva for her comments on a previous version of this manuscript.

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