

Polynomial-Time Data Reduction for Weighted Problems Beyond Additive Goal Functions

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Abstract

Dealing with NP-hard problems, kernelization is a fundamental notion for polynomial-time data reduction with performance guarantees: in polynomial time, a problem instance is reduced to an equivalent instance with size upper-bounded by a function of a parameter chosen in advance. Kernelization for weighted problems particularly requires to also shrink weights. Marx and Végé [ACM Trans. Algorithms 2015] and Etscheid et al. [J. Comput. Syst. Sci. 2017] used a technique of Frank and Tardos [Combinatorica 1987] to obtain polynomial-size kernels for weighted problems, mostly with additive goal functions. We characterize the function types that the technique is applicable to, which turns out to contain many non-additive functions. Using this insight, we systematically obtain kernelization results for natural problems in graph partitioning, network design, facility location, scheduling, vehicle routing, and computational social choice, thereby improving and generalizing results from the literature.

Keywords: NP-hard problems, problem kernelization, weight reduction, routing, scheduling, computational social choice, partitioning

1. Introduction

In the early eighties, Grötschel et al. [29] employed the famous ellipsoid method by Khachiyan [34, 35] for solving the WEIGHTED INDEPENDENT SET (WIS) problem: Given an undirected graph $G = (V, E)$ with vertex weights $w: V \rightarrow \mathbb{Q}_+$, find a set $U \subseteq V$ such that U is an independent set and maximizes $\sum_{v \in U} w(v)$. Grötschel et al. [29] proved WIS to be solvable in polynomial time on perfect graphs. The running time of their algorithm, however, was only *weakly* polynomial, which led to the question whether WIS on perfect

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¹The results obtained in this article are unrelated to the author's work at Huawei.

graphs is solvable in *strongly* polynomial time.² In their seminal work, Frank and Tardos [28] affirmatively answered this question by developing a (what we call) *losing-weight technique*. Their technique employs a preprocessing algorithm that, exemplified for WIS, does the following:

Example 1.1 (WEIGHTED INDEPENDENT SET). In strongly polynomial time, compute vertex weights \hat{w} such that

- (a) the encoding length of the maximum value of \hat{w} is upper-bounded by a polynomial in the number of graph vertices,

while preserving the relative quality of all solutions and non-solutions, that is,

- (b) for every two (independent) sets $U, U' \subseteq V$, it holds that $\sum_{v \in U} w(v) \geq \sum_{v \in U'} w(v)$ if and only if $\sum_{v \in U} \hat{w}(v) \geq \sum_{v \in U'} \hat{w}(v)$.

Thus, WIS can be solved in strongly polynomial time on perfect graphs by first applying the losing-weight technique and then the algorithm of Grötschel et al. [29]. To the best of our knowledge, the technique was used the first time in the context of parameterized algorithmics by Fellows et al. [25], where it was used to obtain fixed-parameter algorithms running in polynomial space. Marx and Vég h [42] first observed the connection of the losing-weight technique to polynomial-time data reduction, namely kernelization: intuitively, in polynomial time, a problem instance is reduced to an equivalent instance with size upper-bounded by a function of a problem-specific parameter. Notably, their kernelization first increases the size of the instance and then introduces additional edge weights. Marx and Vég h [42] stated that “[...] this technique seems to be an essential tool for kernelization of problems involving costs.” Subsequently, Etscheid et al. [22] and Knop and Koutecký [36] used the technique to prove polynomial kernels for several weighted problems, supporting Marx and Vég h’s statement.

In almost all problems studied by the four papers mentioned above, the goal functions are additive set functions (that is, functions f satisfying $f(A \uplus B) = f(A) + f(B)$ for sets A and B). In the two cases where they are not, ad-hoc adaptations of Frank and Tardos’ theorem [28] are used. We present a method of systematically recognizing non-additive functions (which are not necessarily set functions) to which the losing-weight technique applies.

Our Contributions and Structure of this Work. In Section 2, we introduce basic notation and give a brief introduction to the losing-weight technique. In Section 3, we show how to apply the losing-weight technique to two problems with non-additive goal functions in graph partitioning and network design. In Section 4, we characterize what we call α -linearizable functions to which Frank and Tardos’ [28] losing-weight technique applies. Intuitively, the

²Not to be confused with pseudo-polynomial and polynomial running time, see, e.g., Schrijver [46, Section 4.12] or Grötschel et al. [30, Section 1.3]. For strongly polynomial time, one requires from the algorithm to have space polynomial in the input size and that the number of elementary arithmetic and other operations executed by the algorithm does not depend on the sizes of the numbers in the input.

parameter α associated with a linearizable function specifies how “far” the function is from an everywhere-linear function. We additionally provide some tools that allow for a convenient computation of a linearizable function’s α -value. In [Section 5](#), we exemplify the versatility of these tools using problems from network design, facility location, scheduling, vehicle routing, and computational social choice.

We complement or improve several results in the literature: In [Section 4.2](#), we settle an open problem on the kernelizability of the MIN-POWER SYMMETRIC CONNECTIVITY problem [8]. In [Section 5.1](#), we show a problem kernel for the UNCAPACITATED FACILITY LOCATION problem whose size is polynomially upper-bounded in the number of the vertices of the input graph. Previously, only problem kernels with size exponentially upper-bounded in the optimal solution cost (which is usually larger than the number of vertices) were known [23]. In [Section 5.2](#), we shrink weights in several classical scheduling problems. Polynomial problem kernels for scheduling problems are rare [14, 36, 44] and shrinking weights will necessarily be an ingredient in kernels for weighted scheduling problems. In [Section 5.3](#), we generalize a kernelization result for the RURAL POSTMAN PROBLEM [10] to the MIN-MAX k -RURAL POSTMAN PROBLEM. In [Section 5.4](#), we prove a theorem on polynomial kernelization for the POWER VERTEX COVER problem that has been stated without proof in the literature [2].

2. Preliminaries and the Losing-Weight Technique

2.1. Basic Notation and Definitions

An n -dimensional vector $x \in S^n$ for some set S is interpreted as a column vector, and we denote by x^\top its transpose. For two vectors $x = (x_1, \dots, x_n) \in S^n$ and $y = (y_1, \dots, y_m) \in T^m$, we denote by $x \circ y := (x_1, \dots, x_n, y_1, \dots, y_m) \in (S \cup T)^{n+m}$ the concatenation of x and y . The ℓ_1 -norm of a vector $x \in \mathbb{R}^n$ is $\|x\|_1 := \sum_{i=1}^n |x_i|$. The ℓ_∞ -norm (also known as max-norm) of x is $\|x\|_\infty := \max_{i \in \{1, \dots, n\}} |x_i|$. For a number $x \in \mathbb{R}$, the signum of x is defined by $\text{sign}(x) := 1$ if $x > 0$, $\text{sign}(x) := 0$ if $x = 0$, and $\text{sign}(x) := -1$ if $x < 0$.

Let Σ be a finite alphabet. A set $P \subseteq \Sigma^* \times \mathbb{N}$ is called a *parameterized problem*. In an *instance* $(x, k) \in \Sigma^* \times \mathbb{N}$, we call x the *input* and k the *parameter*.

Definition 2.1. A *problem kernelization* for a parameterized problem $P \subseteq \Sigma^* \times \mathbb{N}$ is an algorithm that, given an instance (x, k) , computes in polynomial time an instance (x', k') such that

- (i) $(x, k) \in P$ if and only if $(x', k') \in P$, and
- (ii) $|x'| + k' \leq f(k)$ for some computable function f only depending on k .

We call f the *size of the problem kernel* (x', k') . If $f \in k^{O(1)}$, then we call the problem kernel *polynomial*.

2.2. A Useful and Central Equivalence Relation

In this section, with the goal in mind to replace any given weight vector w by a “representative” weight vector \hat{w} with upper-bounded $\|\hat{w}\|_1$, we define an equivalence relation on vectors over $\mathbb{K} \in \{\mathbb{Z}, \mathbb{Q}\}$. Its equivalence classes will be formed by partitioning the space using hyperplanes with coefficients from

$$\mathbb{Z}_r := \{\pm p \in \mathbb{Z} \mid p \in \{0, \dots, r\}\} \subseteq \mathbb{Z} \text{ or} \quad (2.1)$$

$$\mathbb{Q}_r := \left\{ \pm \frac{p}{q} \mid p \in \{0, \dots, r\}, q \in \{1, \dots, r\} \right\} \subseteq \mathbb{Q}. \quad (2.2)$$

Specifically, we will say that two vectors u and v are equivalent if and only if for all vectors β from some specific subset of \mathbb{K}_r , their dot products $\beta^\top u$ and $\beta^\top v$ have the same signum. Geometrically speaking, u and v are equivalent if and only if for all vectors β from some specific subset of \mathbb{K}_r , there is no hyperplane $\{x \mid \beta^\top x = 0\}$ separating u and v . Formally:

Definition 2.2. Let $\mathbb{K} \in \{\mathbb{Z}, \mathbb{Q}\}$ and $r, d \in \mathbb{N}$. Then, the binary relation $\sim_r^{\mathbb{K}^d}$ on \mathbb{Q}^d is given by

$$w \sim_r^{\mathbb{K}^d} w' \iff \forall \beta \in \mathbb{K}_r^d \text{ with } \|\beta\|_1 \leq r \text{ it holds that } \text{sign}(\beta^\top w) = \text{sign}(\beta^\top w').$$

For every $w \in \mathbb{Q}^d$, let $[w]_r^{\mathbb{K}^d} := \{w' \in \mathbb{Q}^d \mid w' \sim_r^{\mathbb{K}^d} w\} \subseteq \mathbb{Q}^d$ be the class of w under $\sim_r^{\mathbb{K}^d}$.

Example 2.3. Consider \mathbb{Q}^2 and $r = 2$. Any two vectors fall into the same class under $\sim_2^{\mathbb{Z}^2}$ if and only if they cannot be separated by vectors from \mathbb{Z}^2 with entries in $\{0, \pm 1\}$ (see [Figure 1](#) for an illustration).

We prove next that the relation from [Definition 2.2](#) is an equivalence relation.

Observation 2.4. For every $\mathbb{K} \in \{\mathbb{Z}, \mathbb{Q}\}$ and $r, d \in \mathbb{N}$, the relation $\sim_r^{\mathbb{K}^d}$ on \mathbb{Q}^d is an equivalence relation.

Proof. Let $\mathbb{K} \in \{\mathbb{Z}, \mathbb{Q}\}$ and $r, d \in \mathbb{N}$. Clearly, $w \sim_r^{\mathbb{K}^d} w$ (reflexivity) and $w \sim_r^{\mathbb{K}^d} w' \iff w' \sim_r^{\mathbb{K}^d} w$ (symmetry). Moreover, if $w \sim_r^{\mathbb{K}^d} w'$ and $w' \sim_r^{\mathbb{K}^d} w''$, then $w \sim_r^{\mathbb{K}^d} w''$ (transitivity): For every $\beta \in \mathbb{K}_r^d$ with $\|\beta\|_1 \leq r$, one has $\text{sign}(\beta^\top w) = \text{sign}(\beta^\top w') = \text{sign}(\beta^\top w'')$. \square

Next, we prove some properties of $\sim_r^{\mathbb{K}^d}$ and $[\cdot]_r^{\mathbb{K}^d}$. The first property is the following:

Observation 2.5. Let $\mathbb{K} \in \{\mathbb{Z}, \mathbb{Q}\}$, $d \in \mathbb{N}$, and $w \in \mathbb{K}^d$. For every $r, r' \in \mathbb{N}$ with $r \leq r'$ it holds that $[w]_r^{\mathbb{K}^d} \supseteq [w]_{r'}^{\mathbb{K}^d}$.

Proof. Let $\mathbb{K} \in \{\mathbb{Z}, \mathbb{Q}\}$, $d \in \mathbb{N}$, $w \in \mathbb{K}^d$, and $r, r' \in \mathbb{N}$ with $r \leq r'$. Let $w' \in [w]_{r'}^{\mathbb{K}^d}$. We prove that $w' \in [w]_r^{\mathbb{K}^d}$. To this end, let $\beta \in \mathbb{K}_r^d$ with $\|\beta\|_1 \leq r$. Note that $\beta \in \mathbb{K}_{r'}^d \supseteq \mathbb{K}_r^d$ and $\|\beta\|_1 \leq r \leq r'$. Hence, $\text{sign}(\beta^\top w) = \text{sign}(\beta^\top w')$, and, therefore, $w \sim_r^{\mathbb{K}^d} w'$. \square

Reconsider [Example 2.3](#) to exemplify [Observation 2.5](#). If we change r to one, then, for instance, the union of the equivalence classes C_2 , C_3 , and C_4 forms an equivalence class, say C , under $\sim_1^{\mathbb{Z}^2}$. Since $w \in C_4$, we have that $w \in C$.

The second property is the following:

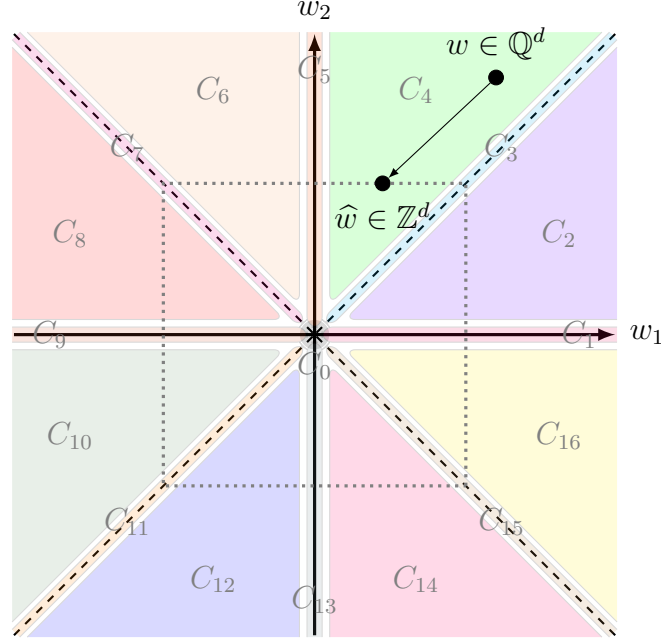


Figure 1: Illustration of the equivalence classes C_0, C_1, \dots, C_{16} regarding $\sim_r^{\mathbb{Z}^d}$ partitioning \mathbb{Q}^d with $d = 2$ and $r = 2$. [Theorem 2.7](#) is exemplified with some w and \hat{w} , each of which belonging to the equivalence class C_4 , where the dotted rectangle illustratively frames all vectors fulfilling [Theorem 2.7\(i\)](#).

Observation 2.6. Let $w = (w_1, \dots, w_d) \in \mathbb{Q}^d$, $d \in \mathbb{N}$, and $\mathbb{K} \in \{\mathbb{Z}, \mathbb{Q}\}$. Then,

- (i) for every $r \geq 1$ and $w' = (w'_1, \dots, w'_d) \in [w]_r^{\mathbb{K}^d}$ it holds that $\text{sign}(w_i) = \text{sign}(w'_i)$ for all $i \in \{1, \dots, d\}$;
- (ii) for every $r \geq 2$ and $w' = (w'_1, \dots, w'_d) \in [w]_r^{\mathbb{K}^d}$ it holds that $\text{sign}(w_i - w_j) = \text{sign}(w'_i - w'_j)$ for all $i, j \in \{1, \dots, d\}$.

Reconsider [Example 2.3](#) to illustrate [Observation 2.6](#). For instance, we have that for every vector $w = (w_1, w_2) \in C_2$ it holds that $w_1, w_2 > 0$ and $w_2 < w_1$, for every vector $w = (w_1, w_2) \in C_3$ it holds that $w_1, w_2 > 0$ and $w_1 = w_2$, and for every vector $w = (w_1, w_2) \in C_4$ it holds that $w_1, w_2 > 0$ and $w_1 < w_2$.

2.3. Losing-Weight Technique

Our work heavily relies on the following seminal result:

Theorem 2.7 (Frank and Tardos [[28](#), Section 3]). On inputs $w \in \mathbb{Q}^d$ and integer N , one can compute in time polynomial in the encoding length of w and N a vector $\hat{w} \in \mathbb{Z}^d$ with

- (i) $\|\hat{w}\|_\infty \leq 2^{4d^3} (N + 1)^{d(d+2)}$ such that
- (ii) $\text{sign}(w^\top b) = \text{sign}(\hat{w}^\top b)$ for all $b \in \mathbb{Z}^d$ with $\|b\|_1 \leq N$.

We briefly explain how [Theorem 2.7](#) relates to [Example 1.1](#): The vertex weights in WEIGHTED INDEPENDENT SET can be interpreted as a vector $w \in \mathbb{Q}^d$ with $d := |V|$. Any two vertex subsets $U, U' \subseteq V$ can be interpreted as vectors $u, u' \in \{0, 1\}^d$, where $u_v = 1$ if and only if $v \in U$, and $u'_v = 1$ if and only if $v \in U'$. Then, $\sum_{v \in U} w_v = u^\top w$ and $\sum_{v \in U'} w_v = u'^\top w$. With $b := u - u'$, the statement of [Example 1.1\(b\)](#) can thus be rewritten as $b^\top w \geq 0 \iff b^\top \hat{w} \geq 0$. Since $\|b\|_1 \leq |V|$, applying [Theorem 2.7](#) to w with $N := |V|$ yields a new weight vector \hat{w} satisfying [Example 1.1\(a\)](#) and [Example 1.1\(b\)](#).

[Theorem 2.7](#) also works for decision rather than optimization problems. Indeed, the application to decision problems is a direct corollary, first stated by Marx and Véggh [42, Remark 3.15] and then formalized by Etscheid et al. [22], thereby observing that the value given additionally along the description of the decision problem can be “attached” to the weight vector.

Corollary 2.8. *Given $w \in \mathbb{Q}^d$, $k \in \mathbb{Q}$, and $N \in \mathbb{N}$, in time polynomial in the encoding length of w , k , and N , one can compute a vector $\hat{w} \in \mathbb{Z}^d$ and an integer $\hat{k} \in \mathbb{Z}$ such that*

- (i) $\|\hat{w}\|_\infty, |\hat{k}| \leq 2^{4(d+1)^3} (N+1)^{(d+1)(d+3)}$ and
- (ii) $w^\top b \leq k \iff \hat{w}^\top b \leq \hat{k}$ for all $b \in \mathbb{Z}^d$ with $\|b\|_1 \leq N-1$.

Whenever we are facing a weighted problem with an additive goal function, that is, for example finding some set S such that $\sum_{s \in S} w(s)$ is minimized (or maximized), the application of [Theorem 2.7](#) is immediate. So it is for the well-known KNAPSACK problem, as proven by Etscheid et al. [22], giving the affirmative answer to the open question [20, 24] of whether KNAPSACK admits a kernel of size polynomial in the number of items:

Example 2.9. Recall the KNAPSACK problem: Given a set $X = \{1, \dots, n\}$ of items with weights $w: X \rightarrow \mathbb{Q}$ and values $v: X \rightarrow \mathbb{Q}$, and rational numbers $k, \ell \in \mathbb{Q}$, the question is whether there is a subset $S \subseteq X$ of items such that $\sum_{i \in S} w(i) \leq k$ and $\sum_{i \in S} v(i) \geq \ell$. Let w and v be interpreted as n -dimensional vectors with $w_i := w(i)$ and $v_i := v(i)$. Applying [Corollary 2.8](#) once with input w , k , and $N := n+1$, and once with input v , ℓ , and N , (where $d = n$ in each application) yields an equivalent instance of KNAPSACK where the weights and values are of encoding-length polynomial in n . Hence, this yields a problem kernel of size polynomial in n .

2.4. Losing-Weight Technique and our Equivalence Relation Combined

Note that [Theorem 2.7\(ii\)](#) is equivalent to \hat{w} being contained in the equivalence class $[w]_N^{\mathbb{Z}^d}$ of w . Hence, [Theorem 2.7](#), given a d -dimensional vector w and any positive integer N , efficiently computes an integral representative \hat{w} from w 's equivalence class where each entry can be upper-bounded by some number only depending on d and N (see [Figure 1](#) for an illustrative example). Consequently, we can rephrase [Theorem 2.7](#) with respect to our equivalence relation as follows:

Theorem 2.10 ([Theorem 2.7](#) rephrased). *On inputs $w \in \mathbb{Q}^d$ and integer N , in time polynomial in the encoding length of w and N one can compute a vector $\hat{w} \in \mathbb{Z}^d \cap [w]_N^{\mathbb{Z}^d}$ with $\|\hat{w}\|_\infty \leq 2^{4d^3} (N+1)^{d(d+2)}$.*

For convenience, we will refer to [Theorem 2.10](#) (instead of [Theorem 2.7](#)) in the remainder of this work.

3. Two Case Studies with Non-Additive Goal Functions

In this section, we show two applications of [Theorem 2.10](#) to optimization problems with non-additive goal functions. In [Example 1.1](#) (WEIGHTED INDEPENDENT SET) and [Example 2.9](#) (KNAPSACK) the used representation of the vectors has a one-to-one correspondence to solution candidates: Any solution candidate to WIS or KNAPSACK is a set of vertices or items, respectively. Such a set can clearly be represented with a (binary) vector and *every* (binary) vector represents a solution candidate. Yet, is the second requirement needed? In several of the applications that we are going to present, this is in fact not the case. Our core idea is hence as follows: We still require that every solution candidate can be represented as a vector, however, we do *not* require every vector to represent a solution candidate. Note that this is fine since \sim holds for all vectors b from the vector space containing vectors representing solution candidates, and thus, also for all vectors that do represent solution candidates. We next exemplify our idea using two problems with non-additive goal functions and formalize them later in [Section 4](#).

3.1. The Case of SMALL SET EXPANSION

Consider the following graph partitioning problem, which was studied in the context of bicriteria approximation [\[4\]](#) and the unique games conjecture [\[45\]](#).

SMALL SET EXPANSION (SSE)

Input: An undirected graph $G = (V, E)$ with edge weights $w: E \rightarrow \mathbb{Q}_+$.

Task: Find a non-empty subset $S \subseteq V$ of size at most $|S| \leq n/2$ that minimizes

$$\frac{1}{|S|} \sum_{e \in (S, V \setminus S)} w(e), \quad (3.1)$$

where $(S, V \setminus S)$ denotes the set of all edges with exactly one endpoint in S .

The goal function's value for a vertex set S can be represented by $w^\top s$ for a *fractional* vector $s \in \{0, 1/|S|\}^{|E|}$, where w is interpreted as vector and an entry of s is non-zero if and only if the corresponding edge is in the edge cut $(S, V \setminus S)$. Fractional numbers, however, are not captured by [Theorem 2.10](#). Yet, with a scaling argument we can derive the following analog to [Theorem 2.10](#) dealing with fractional numbers:

Proposition 3.1. *On input $w \in \mathbb{Q}^d$ and integer $r \in \mathbb{N}$, one can compute in time polynomial in the encoding length of w and r a vector $\hat{w} \in \mathbb{Z}^d \cap [w]_r^{\mathbb{Q}^d}$ with $\|\hat{w}\|_\infty \leq 2^{4d^3} (r^2 + 1)^{r \cdot d(d+2)}$.*

Proof. Apply [Theorem 2.10](#) with $N = r! \cdot r$ to obtain a vector $\widehat{w} \in \mathbb{Z}^d \cap [w]_{r!r}^{\mathbb{Z}^d}$ with

$$\|\widehat{w}\|_\infty \leq 2^{4d^3} (N+1)^{d(d+2)} = 2^{4d^3} (r! \cdot r + 1)^{d(d+2)} \leq 2^{4d^3} (r^2 + 1)^{r \cdot d(d+2)}.$$

It remains to prove that $\widehat{w} \in \mathbb{Z}^d \cap [w]_r^{\mathbb{Q}^d}$. Let $b^* \in \mathbb{Q}_r^d$ with $\|b^*\|_1 \leq r$, and let $b' := r! \cdot b^* \in \mathbb{Z}_{r!r}^d$. Note that $\|b'\|_1 \leq r! \cdot r = N$. Thus, due to [Theorem 2.10](#), we have that

$$\begin{aligned} \text{sign}(w^\top b^*) = \text{sign}(\widehat{w}^\top b^*) &\iff \text{sign}(r! \cdot w^\top b^*) = \text{sign}(r! \cdot \widehat{w}^\top b^*) \\ &\iff \text{sign}(w^\top (r! \cdot b^*)) = \text{sign}(\widehat{w}^\top (r! \cdot b^*)) \\ &\iff \text{sign}(w^\top b') = \text{sign}(\widehat{w}^\top b'). \end{aligned} \quad \square$$

From [Proposition 3.1](#), we get the following.

Lemma 3.2. *For an input instance $(G = (V, E), w)$ of SMALL SET EXPANSION with $n := |V|$ and $m := |E|$, in time polynomial in $|(G, w)|$ one can compute an instance (G, \widehat{w}) of SMALL SET EXPANSION such that*

(i) $\|\widehat{w}\|_\infty \leq 2^{4m^3} \cdot (n^4 \cdot m^2 + 1)^{n^2 m^2 (m+2)}$ and

(ii) a solution $S \subseteq V$ for (G, w) is optimal if and only if it is optimal for (G, \widehat{w}) .

Proof. Denote the edges of G as $E = \{e_1, \dots, e_m\}$ and the weight functions w and \widehat{w} as vectors in \mathbb{N}^m such that $w_i = w(e_i)$ and $\widehat{w}_i = \widehat{w}(e_i)$ for all $i \in \{1, \dots, m\}$. Apply [Proposition 3.1](#) with $d = m$ and $r = n^2 m$. Let $S \subseteq V$ and let $s \in \{0, 1/|S|\}^m$ be the vector such that $s_i \neq 0$ if and only if $e_i \in (S, V \setminus S)$. Let $S' \subseteq V$ be another set, and let $s' \in \{0, 1/|S'|\}^m$ with $s'_i \neq 0$ if and only if $e_i \in (S', V \setminus S')$. Let $b := s - s'$. Note that for each $i \in \{1, \dots, m\}$ it holds that

$$|s_i - s'_i| = \left| \frac{|S'|s_i}{|S'|} - \frac{|S|s'_i}{|S|} \right| \in \left\{ 0, \left| \frac{|S'| - |S|}{|S| \cdot |S'|} \right|, \frac{1}{|S|}, \frac{1}{|S'|} \right\},$$

and hence $b \in \mathbb{Q}_{n^2}^m$ and $\|b\|_1 \leq n^2 m$. We thus get

$$\begin{aligned} s^\top w - (s')^\top w \leq 0 &\iff (s - s')^\top w \leq 0 \\ &\stackrel{\text{Prop. 3.1}}{\iff} (s - s')^\top \widehat{w} \leq 0 \iff s^\top \widehat{w} - (s')^\top \widehat{w} \leq 0. \end{aligned} \quad \square$$

3.2. The Case of MIN-POWER SYMMETRIC CONNECTIVITY

The previous case of SMALL SET EXPANSION showed how [Theorem 2.10](#) can be applied to weighted sums. Next we show how to deal with a non-additive functions involving maxima. To this end, consider the following NP-hard optimization problem from survivable network design [\[1, 18\]](#), which has also been studied in the context of parameterized complexity with practical results [\[6, 8\]](#) (same for the asymmetric case [\[7\]](#)).

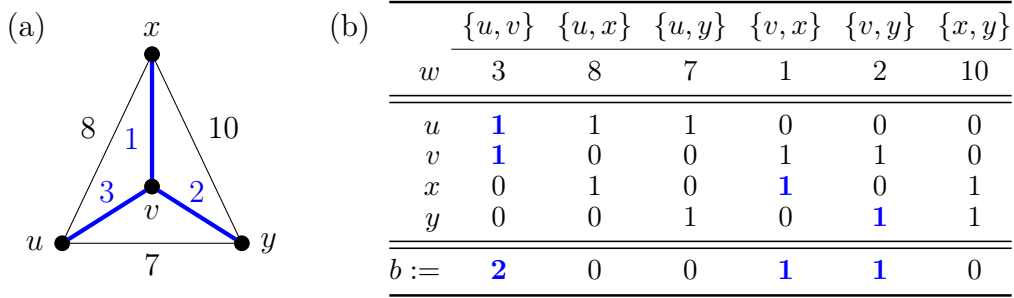


Figure 2: Illustrative example for MPSC and the application of [Theorem 2.10](#). (a) depicts an edge-weighted undirected example graph with a connected spanning subgraph (indicated by thick edges) of edge-weight six, and (b) shows the incidence matrix of the graph in (a), the vector w of edge-weights, and the vector b representing the solution from (a) with goal function value $w^\top b = 9$.

MIN-POWER SYMMETRIC CONNECTIVITY (MPSC)

Input: A connected undirected graph $G = (V, E)$ and edge weights $w: E \rightarrow \mathbb{N}$.

Task: Find a connected spanning subgraph $T = (V, F)$ of G that minimizes

$$\sum_{v \in V} \max_{\{u, v\} \in F} w(\{u, v\}). \quad (3.2)$$

Applying [Theorem 2.10](#) to the goal function (3.2) is not obvious: Let $E = \{e_1, \dots, e_m\}$ and the weight function w be represented as a vector in \mathbb{N}^m such that $w_i = w(e_i)$. Let $b \in \{0, 1\}^m$ be the vector representing the edge set F of a solution $T = (V, F)$, that is, $b_i = 1$ if and only if $e_i \in F$. Then, the value $w^\top b$ is *not* equal to $\sum_{v \in V} \max_{\{u, v\} \in F} w(\{u, v\})$. See [Figure 2\(a\)](#) for an example.

However, we can circumvent this issue (arising from the max-function in the goal function) and still apply [Theorem 2.10](#). To this end, observe that we only need to find a correct representation of a solution. An edge $e \in F$ contributes its weight to (3.2) each time it appears in the (maximum in the) sum, that is, either zero, one, or two times. Hence, a solution can be represented as vector $b \in \{0, 1, 2\}^m$ such that the term $w(e_i)$ appears $b_i \in \{0, 1, 2\}$ times in the sum of the cost function regarding $T = (V, F)$. See [Figure 2\(b\)](#) for an example. This change of the representation of a solution only changes the domain of the vector b , and hence increases the value of N in the application of [Theorem 2.10](#) by a factor of two. Eventually, we obtain:

Lemma 3.3. *For an input instance $(G = (V, E), w)$ of MPSC with $m := |E|$, in time polynomial in $|(G, w)|$ one can compute an instance (G, \hat{w}) of MPSC such that*

(i) $\|\hat{w}\|_\infty \leq 2^{4m^3} \cdot (2m + 1)^{m(m+2)}$ and

(ii) a connected subgraph $T = (V, F)$ of G is an optimal solution for (G, w) if and only if T is an optimal solution for (G, \hat{w}) .

Proof. Denote the edges of G as $E = \{e_1, \dots, e_m\}$ and the weight functions w and \widehat{w} as (column) vectors in \mathbb{N}^m such that $w_i = w(e_i)$ and $\widehat{w}_i = \widehat{w}(e_i)$ for all $i \in \{1, \dots, m\}$. We apply [Theorem 2.10](#) with $d = m$ and $N = 2m$ to the weight vector w and obtain the weight vector \widehat{w} . [Theorem 2.10](#) immediately implies statement (i). Moreover, recall that $\widehat{w}_i \geq 0$ for all $i \in \{1, \dots, m\}$ due to [Observation 2.6\(i\)](#), and hence (G, \widehat{w}) is well-defined.

Next, we prove statement (ii). Let $T = (V, F)$ be a connected subgraph of G and let $s \in \{0, 1, 2\}^m$ be an m -dimensional vector such that the term $w(e_i)$ appears s_i times in [\(3.2\)](#). Then, $\sum_{v \in V} \max_{\{u, v\} \in F} w(\{u, v\}) = s^\top w$. For a connected subgraph $T' = (V, F')$ of G , let $s' \in \{0, 1, 2\}^m$ be derived analogously so that the cost of T' is $(s')^\top w$. Define $b := s - s'$. Note that $-2 \leq b_i \leq 2$ for each $i \in \{1, \dots, m\}$. Hence, $\|b\|_1 \leq 2m = N$. Moreover, due to [Theorem 2.10](#), we have that $\widehat{w} \in [w]_{2m}^{\mathbb{Z}^m}$, and hence

$$\begin{aligned} s^\top w - (s')^\top w \leq 0 &\iff (s - s')^\top w \leq 0 \\ &\stackrel{\|b\|_1 \leq 2m}{\iff} (s - s')^\top \widehat{w} \leq 0 \iff s^\top \widehat{w} - (s')^\top \widehat{w} \leq 0. \end{aligned}$$

Finally, note that due to [Observation 2.6\(ii\)](#), the goal function's values for both T and T' with respect to \widehat{w} are still correctly represented by s and s' , that is,

$$\sum_{v \in V} \max_{\{u, v\} \in F} \widehat{w}(\{u, v\}) = s^\top \widehat{w} \quad \text{and} \quad \sum_{v \in V} \max_{\{u, v\} \in F'} \widehat{w}(\{u, v\}) = (s')^\top \widehat{w}. \quad \square$$

4. Linearizable Functions

In this section, we provide our central framework formalizing our key idea from the previous section. Our framework bases on our notion of *linearizable* functions. Before presenting the formal definition (see [Definition 4.1](#) below), we recap the central insights from the previous section.

Our case studies for SMALL SET EXPANSION and MIN-POWER SYMMETRIC CONNECTIVITY show that problems with non-additive goal functions still allow for an application of the losing-weight technique. A natural question is what characterizes these goal functions. Both of our cases have in common that, for any weight vector w , the goal function's value for every solution s can be represented as $b_s^\top w$ with b_s being a vector associated with s . Moreover, to apply the losing-weight technique, we also need that if we change the weight vector to a "smaller" weight vector \widehat{w} , then the goal function's value is still represented for solution s as $b_s^\top \widehat{w}$ and vice versa (for this we used [Observation 2.6\(ii\)](#) in the proof of [Lemma 3.3](#)). That is, we want that the value of solution s with respect to w is $b_s^\top w$ if and only if the value of solution s with respect to \widehat{w} is $b_s^\top \widehat{w}$. Formally, this is captured by the following. (Let \mathbb{K}_r with $\mathbb{K} \in \{\mathbb{Z}, \mathbb{Q}\}$ be as defined in [\(2.1\)](#) and [\(2.2\)](#).)

Definition 4.1. Let $f: L \times \mathbb{Q}^d \rightarrow \mathbb{Q}$, where L (here and in the following) is some arbitrary domain. We say that f is α - \mathbb{K} -linearizable for some $\alpha \in \mathbb{N}$ if for all $w \in \mathbb{Q}^d$ and for all $x \in L$ there exists a vector $b_{x,w} \in \mathbb{K}_\alpha^d$ with $\|b_{x,w}\|_1 \leq \alpha$ such that $f(x, w') = b_{x,w}^\top w'$ for all $w' \in [w]_\alpha^{\mathbb{K}^d}$.

Intuitively, an α - \mathbb{K} -linearizable function maps a solution (contained in the set L) together with a weight vector to a number. For a fixed weight vector this number can be expressed for *every* solution as the product of some vector representing the solution and the weight vector. Moreover, this representation of the solution is robust against exchanging weight vectors with any representative from its class.

We start with three basic properties of linearizable functions.

Observation 4.2. *For any set $X = \{x_1, x_2, \dots, x_d\}$ and any weight vector $w = (w_1, w_2, \dots, w_d) \in \mathbb{Q}^d$, the function $f: X \times \mathbb{Q}^d \rightarrow \mathbb{Q}$, $(x_i, w) \mapsto w_i$ is 1- \mathbb{K} -linearizable for every $\mathbb{K} \in \{\mathbb{Z}, \mathbb{Q}\}$.*

Observation 4.3. *Let $\mathbb{K} \in \{\mathbb{Z}, \mathbb{Q}\}$. If f is α - \mathbb{K} -linearizable, then f is α' - \mathbb{K} -linearizable for all $\alpha' \in \mathbb{N}$ with $\alpha' \geq \alpha$.*

Proof. Let $w \in \mathbb{Q}^d$. We know that, for all $x \in L$, there exists a vector $b_x \in \mathbb{K}_\alpha^d \subseteq \mathbb{K}_{\alpha'}^d$ with $\|b_x\|_1 \leq \alpha \leq \alpha'$ such that $f(x, w') = b_x^\top w'$ for all $w' \in [w]_\alpha^{\mathbb{K}^d} \supseteq [w]_{\alpha'}^{\mathbb{K}^d}$ (recall [Observation 2.5](#) for the latter inclusion). \square

Lemma 4.4. *Let $\mathbb{K} \in \{\mathbb{Z}, \mathbb{Q}\}$, $f, f^*: L \times \mathbb{Q}^d \rightarrow \mathbb{Q}$, and $c: L \rightarrow \mathbb{K}_n \setminus \{0\}$, where $n \in \mathbb{N}$. If f is α - \mathbb{K} -linearizable, then $f^*(x, w) = c(x) \cdot f(x, w)$ is $n\alpha$ - \mathbb{K} -linearizable.*

Proof. Let $w \in \mathbb{Q}^d$. Since f is α - \mathbb{K} -linearizable, we know that for every $x \in L$ there exists a vector $b_{x,w} \in \mathbb{K}_\alpha^d$ with $\|b_{x,w}\|_1 \leq \alpha$ such that $f(x, w') = b_{x,w}^\top w'$ for all $w' \in [w]_\alpha^{\mathbb{K}^d}$. Let $b_{x,w}^* := c(x) \cdot b_{x,w}$. We have that $b_{x,w}^* \in \mathbb{K}_{n\alpha}^d$ and $\|b_{x,w}^*\|_1 \leq n\alpha$. Thus, for any $w' \in [w]_{n\alpha}^{\mathbb{K}^d} \subseteq [w]_\alpha^{\mathbb{K}^d}$, it holds that

$$f^*(x, w') = c(x) \cdot f(x, w') = c(x) \cdot b_{x,w}^\top w' = (c(x) \cdot b_{x,w})^\top w' = (b_{x,w}^*)^\top w'. \quad \square$$

Next, we prove next that the losing-weight technique applies to linearizable functions. We first discuss \mathbb{Z} -linearizable functions, and afterwards \mathbb{Q} -linearizable functions.

4.1. \mathbb{Z} -linearizable Functions

The losing-weight technique applies to \mathbb{Z} -linearizable functions as follows.

Theorem 4.5. *Let $f: L \times \mathbb{Q}^d \rightarrow \mathbb{Q}$ be an α - \mathbb{Z} -linearizable function, and let $w \in \mathbb{Q}^d$, $k \in \mathbb{Q}$. Then in time polynomial in the encoding length of w , k , and α , one can compute a vector $\hat{w} \in \mathbb{Z}^d$ and an integer $\hat{k} \in \mathbb{Z}$ such that*

- (i) $\|\hat{w}\|_\infty, |\hat{k}| \leq 2^{4(d+1)^3} (2\alpha + 1)^{(d+1)(d+3)}$,
- (ii) $f(x, w) \geq f(y, w) \iff f(x, \hat{w}) \geq f(y, \hat{w})$ for all $x, y \in L$, and
- (iii) $f(x, w) \geq k \iff f(x, \hat{w}) \geq \hat{k}$ for all $x \in L$.

Proof. Apply [Theorem 2.10](#) with $N = 2\alpha$ to the vector $w \circ k$ to obtain the concatenated vector $\widehat{w} \circ \widehat{k}$ with

$$\widehat{w} \circ \widehat{k} \in [w \circ k]_{2\alpha}^{\mathbb{Z}^{d+1}} \quad (4.1)$$

and $\|\widehat{w} \circ \widehat{k}\|_\infty \leq 2^{4(d+1)^3} (2\alpha + 1)^{(d+1)(d+3)}$. Thus, \widehat{w} and \widehat{k} fulfill statement (i). Since f is α - \mathbb{Z} -linearizable, by [Definition 4.1](#), for every $x, y \in L$ there are $b_x, b_y \in \mathbb{Z}_\alpha^d$ with $\|b_x\|_1, \|b_y\|_1 \leq \alpha$ such that $f(x, w') = b_x^\top w'$ and $f(y, w') = b_y^\top w'$ for all $w' \in [w]_\alpha^{\mathbb{Z}^d}$.

For statement (ii), let $b := b_x - b_y$. We have $b \in \mathbb{Z}_{2\alpha}^d$ and $\|b\|_1 \leq 2\alpha$. Moreover

$$\begin{aligned} \text{sign}(f(x, w) - f(y, w)) &= \text{sign}((b_x - b_y)^\top w) \\ &\stackrel{(4.1)}{=} \text{sign}((b_x - b_y)^\top \widehat{w}) = \text{sign}(f(x, \widehat{w}) - f(y, \widehat{w})), \end{aligned}$$

and hence

$$f(x, w) - f(y, w) \geq 0 \iff f(x, \widehat{w}) - f(y, \widehat{w}) \geq 0.$$

For statement (iii), let $b := b_x \circ (-1)$. We have $b \in \mathbb{Z}_\alpha^{d+1} \subseteq \mathbb{Z}_{2\alpha}^{d+1}$ and $\|b\|_1 \leq \alpha + 1 \leq 2\alpha$. Moreover,

$$\begin{aligned} \text{sign}(f(x, w) - k) &= \text{sign}(b_x^\top w - k) = \text{sign}(b^\top (w \circ k)) \\ &\stackrel{(4.1)}{=} \text{sign}(b^\top (\widehat{w} \circ \widehat{k})) = \text{sign}(b_x^\top \widehat{w} - \widehat{k}) = \text{sign}(f(x, \widehat{w}) - \widehat{k}) \end{aligned}$$

and hence

$$f(x, w) \geq k \iff f(x, \widehat{w}) \geq \widehat{k}. \quad \square$$

Using [Theorem 4.5](#), we can shrink the weights in α - \mathbb{Z} -linearizable functions so that their encoding length is polynomially upper-bounded in α and the dimension d . For easy application of [Theorem 4.5](#), we need to easily recognize α - \mathbb{Z} -linearizable functions and, in particular, to determine α . To this end, we show how to recognize an α - \mathbb{Z} -linearizable function by simply looking at the functions it is composed of. We subsequently demonstrate this using the example of MIN-POWER SYMMETRIC CONNECTIVITY (MPSC).

Lemma 4.6. *Let $f: L \times \mathbb{Q}^d \rightarrow \mathbb{Q}$ be a function. If f is α - \mathbb{Z} -linearizable, then the function $f': \{X \subseteq L \mid |X| \leq n\} \times \mathbb{Q}^d \rightarrow \mathbb{Q}$ with $n \in \mathbb{N}$ and*

(i) $f'(X, w) = \sum_{x \in X} f(x, w)$ *is $n\alpha$ - \mathbb{Z} -linearizable;*

(ii) $f'(X, w) = \max_{x \in X} f(x, w)$ *is 2α - \mathbb{Z} -linearizable;*

(iii) $f'(X, w) = \min_{x \in X} f(x, w)$ *is 2α - \mathbb{Z} -linearizable.*

Proof. (i): Let $w \in \mathbb{Q}^d$ and $X \subseteq L$ with $|X| \leq n$. Since f is α - \mathbb{Z} -linearizable, we know that, for all $x \in X \subseteq L$ there is a $b_{x,w} \in \mathbb{Z}_\alpha^d$ with $\|b_{x,w}\|_1 \leq \alpha$ such that $f(x, w') = b_{x,w}^\top w'$ for all $w' \in [w]_\alpha^{\mathbb{Z}^d}$. Let $b_X := (\sum_{x \in X} b_{x,w}) \in \mathbb{Z}_{n\alpha}^d$. Let $w' \in [w]_{n\alpha}^{\mathbb{Z}^d} \subseteq [w]_\alpha^{\mathbb{Z}^d}$. We have that $f'(X, w') = \sum_{x \in X} f(x, w') = \sum_{x \in X} b_{x,w}^\top w' = b_X^\top w'$.

(ii): Let $w \in \mathbb{Q}^d$ and $X \subseteq L$ with $|X| \leq n$. We know that, for all $x \in X \subseteq L$, there is a vector $b_{x,w} \in \mathbb{Z}_\alpha^d$ such that $f(x, w') = b_{x,w}^\top w'$ for all $w' \in [w]_\alpha^{\mathbb{Z}^d}$. Let $z \in \arg \max_{x \in X} b_{x,w}^\top w$ and $b_X := b_{z,w} \in \mathbb{Z}_\alpha^d$. Let $w' \in [w]_{2\alpha}^{\mathbb{Z}^d} \subseteq [w]_\alpha^{\mathbb{Z}^d}$. For any $y \in X$, let $b := b_z - b_y$. Note that $b \in \mathbb{Z}_{2\alpha}^d$ and $\|b\|_1 \leq 2\alpha$, and hence $\text{sign}(b^\top w) = \text{sign}(b^\top w')$. Thus, we have that

$$\begin{aligned} \text{sign}(f(z, w) - f(y, w)) &= \text{sign}(b_z^\top w - b_y^\top w) = \text{sign}(b^\top w) \\ &= \text{sign}(b^\top w') = \text{sign}(f(z, w') - f(y, w')) \end{aligned}$$

and hence it holds that $f(z, w) \geq f(y, w) \iff f(z, w') \geq f(y, w')$. Finally, it follows that $f'(X, w') = \max_{x \in X} f(x, w') = b_{z,w}^\top w' = b_X^\top w'$.

(iii): Works analogously to (ii). \square

Revisiting the Case of MIN-POWER SYMMETRIC CONNECTIVITY. The goal function in MPSC is composed of a sum over maxima. We proved that such a composition preserves linearizability. We explain the use of our machinery for MPSC. To this end, rewrite the goal function as follows. Let $F_v := \{e \in F \mid v \in e\}$ and $\mathcal{F} := \{F_v \mid v \in V\}$. Then, the goal function becomes

$$h(\mathcal{F}, w) = \sum_{F_v \in \mathcal{F}} g(F_v, w) \quad \text{with} \quad g(F, w) = \max_{e \in F} w(e).$$

Due to [Observation 4.2](#), $f(e, w) = w(e)$ is 1- \mathbb{Z} -linearizable. Due to [Lemma 4.6\(ii\)](#), $g(F, w) = \max_{e \in F} f(e, w)$ is 2- \mathbb{Z} -linearizable. Finally, due to [Lemma 4.6\(i\)](#) (with $L = 2^E$ and $n = |V|$), $h(\mathcal{F}, w) = \sum_{F_v \in \mathcal{F}} g(F_v, w)$ is $2n$ - \mathbb{Z} -linearizable. Employing [Theorem 4.5](#), we get in polynomial time a vector $\hat{w} \in \mathbb{Q}^m$ such that $\|\hat{w}\|_\infty \in 2^{O(m^3 \log(n))}$, and for any two connected subgraphs $T = (V, F)$ and $T' = (V, F')$ of G , we have that

$$\sum_{v \in V} \max_{e \in F} w(e) \geq \sum_{v \in V} \max_{e \in F'} w(e) \iff \sum_{v \in V} \max_{e \in F} \hat{w}(e) \geq \sum_{v \in V} \max_{e \in F'} \hat{w}(e).$$

We have thus reproven [Lemma 3.3](#). Moreover, for the decision variant of MINPSC, which asks whether there is a solution of at most a given cost k , with [Theorem 4.9](#) on input (G, w, k) , we immediately obtain a polynomial kernel.

Proposition 4.7. *MIN-POWER SYMMETRIC CONNECTIVITY admits a polynomial kernel with respect to the number of vertices.*

In previous work [\[6, 8\]](#), we developed a partial kernel, that is, an algorithm that maps any instance of MPSC to an equivalent instance where the number of vertices and edges, yet not necessarily the edge weights, are polynomially upper-bounded in the feedback edge number.³ Finding a polynomial kernel regarding the feedback edge number was an open problem. Given the partial kernel [\[6, 8\]](#), [Proposition 4.7](#) yields the following affirmative answer.

³The smallest number of edges to be removed to transform a graph into a forest.

Corollary 4.8. MIN-POWER SYMMETRIC CONNECTIVITY admits a polynomial kernel with respect to the feedback edge number of the input graph.

We will revisit the case of SMALL SET EXPANSION (SSE) in the next section, using (an analog of) [Theorem 4.5](#) for \mathbb{Q} -linearizable functions.

4.2. \mathbb{Q} -linearizable Functions

In this section, we give an analog of [Theorem 4.5](#) for \mathbb{Q} -linearizable functions and revisit the case of SMALL SET EXPANSION. The analog of [Theorem 4.5](#) for \mathbb{Q} -linearizable functions is as follows.

Theorem 4.9. Let $f: L \times \mathbb{Q}^d \rightarrow \mathbb{Q}$ be an α - \mathbb{Q} -linearizable function, and let $w \in \mathbb{Q}^d$, $k \in \mathbb{Q}$. Then, in time polynomial in the encoding length of w , k , and α , one can compute a vector $\hat{w} \in \mathbb{Z}^d$ and an integer $\hat{k} \in \mathbb{Z}$ such that

$$(i) \quad \|\hat{w}\|_\infty, |\hat{k}| \leq 2^{4(d+1)^3} (4\alpha^4 + 1)^{2\alpha^2 \cdot (d+1)(d+3)},$$

$$(ii) \quad f(x, w) \geq f(y, w) \iff f(x, \hat{w}) \geq f(y, \hat{w}) \text{ for all } x, y \in L, \text{ and}$$

$$(iii) \quad f(x, w) \geq k \iff f(x, \hat{w}) \geq \hat{k} \text{ for all } x \in L.$$

Proof. Apply [Proposition 3.1](#) with $r = 2\alpha^2$ to the concatenated vector $w \circ k$ to obtain the concatenated vector

$$\hat{w} \circ \hat{k} \in [w \circ k]_{2\alpha^2}^{\mathbb{Q}^{d+1}}. \quad (4.2)$$

with $\|\hat{w} \circ \hat{k}\|_\infty \leq 2^{4(d+1)^3} (4\alpha^4 + 1)^{2\alpha^2 \cdot (d+1)(d+3)}$. Hence, \hat{w} and \hat{k} fulfill statement (i). Since f is α - \mathbb{Q} -linearizable, by [Definition 4.1](#), for every $x, y \in L$ there are $b_{x,w}, b_{y,w} \in \mathbb{Q}_\alpha^d$ with $\|b_{x,w}\|_1 \leq \alpha$ and $\|b_{y,w}\|_1 \leq \alpha$ such that $f(x, w) = b_{x,w}^\top w$ and $f(y, w) = b_{y,w}^\top w$ for all $w' \in [w]_\alpha^{\mathbb{Q}^d} \supseteq [w]_{2\alpha^2}^{\mathbb{Q}^d}$.

For statement (ii), let $b := b_{x,w} - b_{y,w}$. We have that $b \in \mathbb{Q}_{2\alpha^2}^d$ and $\|b\|_1 \leq 2\alpha \leq 2\alpha^2$. Moreover,

$$\begin{aligned} \text{sign}(f(x, w) - f(y, w)) &= \text{sign}((b_x - b_y)^\top w) \\ &\stackrel{(4.2)}{=} \text{sign}((b_x - b_y)^\top \hat{w}) = \text{sign}(f(x, \hat{w}) - f(y, \hat{w})), \end{aligned}$$

and hence

$$f(x, w) \geq f(y, w) \iff f(x, \hat{w}) \geq f(y, \hat{w}).$$

For statement (iii), let $b := b_x \circ (-1)$. We have that $b \in \mathbb{Q}_\alpha^{d+1} \subseteq \mathbb{Q}_{2\alpha^2}^{d+1}$ and $\|b\|_1 \leq \alpha + 1 \leq 2\alpha^2$. Moreover,

$$\begin{aligned} \text{sign}(f(x, w) - k) &= \text{sign}(b_x^\top w - k) = \text{sign}(b^\top (w \circ k)) \\ &\stackrel{(4.2)}{=} \text{sign}(b^\top (\hat{w} \circ \hat{k})) = \text{sign}(b_x^\top \hat{w} - \hat{k}) = \text{sign}(f(x, \hat{w}) - \hat{k}), \end{aligned}$$

and hence

$$f(x, w) \geq k \iff f(x, \hat{w}) \geq \hat{k}. \quad \square$$

Next, we present an analog of [Lemma 4.6](#) for \mathbb{Q} -linearizable functions. It turns out that composing \mathbb{Q} -linearizable functions introduces larger α -values compared to \mathbb{Z} -linearizable functions.

Lemma 4.10. *Let $f : L \times \mathbb{Q}^d \rightarrow \mathbb{Q}$ be a function. If f is α - \mathbb{Q} -linearizable, then the function $f' : \{X \subseteq L \mid |X| \leq n\} \times \mathbb{Q}^d \rightarrow \mathbb{Q}$ with $n \in \mathbb{N}$ and*

(i) $f'(X, w) = \sum_{x \in X} f(x, w)$ *is $\alpha!n\alpha$ - \mathbb{Q} -linearizable;*

(ii) $f'(X, w) = \max_{x \in X} f(x, w)$ *is $2\alpha^2$ - \mathbb{Q} -linearizable;*

(iii) $f'(X, w) = \min_{x \in X} f(x, w)$ *is $2\alpha^2$ - \mathbb{Q} -linearizable.*

Proof. (i): Let $w \in \mathbb{Q}^d$ and $X \subseteq L$ with $|X| \leq n$. Since f is α - \mathbb{Q} -linearizable, we know that for all $x \in X \subseteq L$ there is a vector $b_{x,w} \in \mathbb{Q}_\alpha^d$ with $\|b_{x,w}\|_1 \leq \alpha$ such that $f(x, w) = b_{x,w}^\top w'$ for all $w' \in [w]_\alpha^{\mathbb{Q}^d}$. Let $b_X := (\sum_{x \in X} b_{x,w}) \in \mathbb{Q}_{\alpha!n\alpha}^d$. Let $w' \in [w]_{\alpha!n\alpha}^{\mathbb{Q}^d} \subseteq [w]_\alpha^{\mathbb{Q}^d}$. We have that $f'(X, w') = \sum_{x \in X} f(x, w') = \sum_{x \in X} b_{x,w}^\top w' = b_X^\top w'$.

(ii): Let $w \in \mathbb{Q}^d$ and $X \subseteq L$ with $|X| \leq n$. Since f is α - \mathbb{Q} -linearizable, we know that for all $x \in X \subseteq L$ there is a vector $b_{x,w} \in \mathbb{Q}_\alpha^d$ such that $f(x, w) = b_{x,w}^\top w'$ for all $w' \in [w]_\alpha^{\mathbb{Q}^d}$. Let $z \in \arg \max_{x \in X} b_{x,w}^\top w$ and $b_X := b_{z,w} \in \mathbb{Q}_\alpha^d$. Let $w' \in [w]_{2\alpha^2}^{\mathbb{Q}^d} \subseteq [w]_\alpha^{\mathbb{Q}^d}$. For any $y \in X$, let $b := b_z - b_y$. Note that $b \in \mathbb{Q}_{2\alpha^2}^d$ and $\|b\|_1 \leq 2\alpha \leq 2\alpha^2$, and hence $\text{sign}(b^\top w) = \text{sign}(b^\top w')$. Thus,

$$\begin{aligned} \text{sign}(f(z, w) - f(y, w)) &= \text{sign}(b_z^\top w - b_y^\top w) = \text{sign}(b^\top w) \\ &= \text{sign}(b^\top w') = \text{sign}(f(z, w') - f(y, w')) \end{aligned}$$

and thus it holds that $f(z, w) \geq f(y, w) \iff f(z, w') \geq f(y, w')$. It follows that $f'(X, w') = \max_{x \in X} f(x, w') = b_X^\top w' = b_X^\top w$.

(iii): Works analogously to (ii). \square

The framework for \mathbb{Z} -linearizable functions allows for “chaining up sums” while keeping α polynomially bounded. Note that this is in general not the case for \mathbb{Q} -linearizable functions when applying [Lemma 4.10](#). Although more restrictive, however, the framework for \mathbb{Z} -linearizable functions is sufficient for MPSC and all upcoming examples except for the following.

Revisiting the Case of SMALL SET EXPANSION. The goal function in SSE is a multiplication of a number and a sum. By [Lemma 4.4](#), we know that multiplication preserves linearizability. Moreover, by [Lemma 4.10\(i\)](#), we know that the sum preserves linearizability. So, we are set to use our machinery for SSE.

Let $E_S := (S, V \setminus S)$ for all $S \subseteq V$. Let $L := \{(S, E_S) \mid S \subseteq V, 1 \leq |S| \leq n/2\}$. Let $c : L \rightarrow \mathbb{Q}_n \setminus \{0\}$, $(S, E_S) \mapsto \frac{1}{|S|}$. Then, the goal function of SSE becomes $h((S, E_S), w) = \frac{1}{|S|} \cdot g((S, E_S), w)$ with $g((S, E_S), w) = \sum_{e \in E_S} w(e)$. By [Observation 4.2](#), $f(e, w) = w(e)$ is 1- \mathbb{Q} -linearizable. Moreover, by [Lemma 4.10\(i\)](#), g is m - \mathbb{Q} -linearizable. Finally, due to [Lemma 4.4](#), h is $n \cdot m$ - \mathbb{Q} -linearizable. Finally, employing [Theorem 4.9](#), we can reprove [Lemma 3.2](#) and additionally obtain the following kernel.

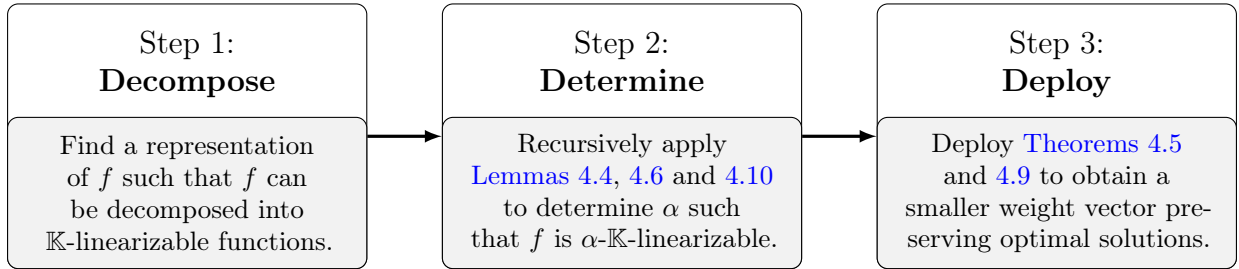


Figure 3: Recipe DDD for applying our framework, illustrated for a weighted problem seeking for a set that maximizes or minimizes some function f .

Proposition 4.11. SMALL SET EXPANSION admits a polynomial kernel with respect to the number of vertices.

Summary of our Framework. We introduced α - \mathbb{K} -linearizable functions (Definition 4.1) for $\mathbb{K} \in \{\mathbb{Z}, \mathbb{Q}\}$. Due to Lemmas 4.4, 4.6 and 4.10, we can easily recognize special types of α - \mathbb{K} -linearizable functions by simply looking at their composition. Further, we proved that the losing-weight technique applies to α - \mathbb{K} -linearizable functions (Theorems 4.5 and 4.9). Thus, for applying our framework, we offer the recipe in Figure 3.

We showed that any combination of sums, maxima, minima, and multiplication with (rational) numbers preserves linearizability. In the next section, we show that we can also compose functions using case distinctions on linearizable constraints (Lemma 5.4). Finding compositions of further functions preserving linearizability remains a task for future work.

5. Further Applications of the Losing-Weight Technique

In this section, we provide further problems with linearizable goal functions and demonstrate how our framework applies to them via the recipe DDD. The further problems stem from network design, facility location, scheduling, vehicle routing, and computational social choice.

5.1. Uncapacitated Facility Location.

The UNCAPACITATED FACILITY LOCATION problem is one of the most fundamental and well-studied problems in operations research [38, Section 3.4]. It has also been studied in the context of parameterized complexity and data reduction [15, 23].

UNCAPACITATED FACILITY LOCATION (UFL)

Input: A set \mathcal{C} of n clients, a set \mathcal{F} of m facilities, facility opening costs $f: \mathcal{F} \rightarrow \mathbb{Q}_{\geq 0}$, and client service costs $c: \mathcal{F} \times \mathcal{C} \rightarrow \mathbb{Q}_{\geq 0}$.

Task: Find a subset $\mathcal{F}' \subseteq \mathcal{F}$ that minimizes

$$\sum_{i \in \mathcal{F}'} f(i) + \sum_{j \in \mathcal{C}} \min_{i \in \mathcal{F}'} c(i, j). \quad (5.1)$$

When the cost function is a metric, then the problem is also called METRIC UNCAPACITATED FACILITY LOCATION (MUFL). By showing that the goal function (5.1) is linearizable, we can prove:

Lemma 5.1. *There is an algorithm that, on an input consisting of an instance $(\mathcal{C}, \mathcal{F}, f, c)$ of UFL and $k \in \mathbb{Q}$, in time polynomial in $|(\mathcal{C}, \mathcal{F}, f, c, k)|$ computes an instance $(\mathcal{C}, \mathcal{F}, \bar{f}, \bar{c})$ of UFL and $\bar{k} \in \mathbb{Z}$ such that*

$$(i) \quad \|\bar{f}\|_\infty + \|\bar{c}\|_\infty, |\bar{k}| \leq 2^{4(nm+m+1)^3} (4(2n+m) + 1)^{(nm+m+1)(nm+m+3)},$$

(ii) *any subset $\mathcal{F}' \subseteq \mathcal{F}$ forms an optimal solution for $(\mathcal{C}, \mathcal{F}, f, c)$ if and only if \mathcal{F}' forms an optimal solution for $(\mathcal{C}, \mathcal{F}, \bar{f}, \bar{c})$, and*

$$(iii) \quad \text{for any subset } \mathcal{F}' \subseteq \mathcal{F} \text{ we have that } \sum_{i \in \mathcal{F}'} f(i) + \sum_{j \in \mathcal{C}} \min_{i \in \mathcal{F}'} c(i, j) \geq k \iff \sum_{i \in \mathcal{F}'} \bar{f}(i) + \sum_{j \in \mathcal{C}} \min_{i \in \mathcal{F}'} \bar{c}(i, j) \geq \bar{k}.$$

Proof. First, observe that f and c are $1\text{-}\mathbb{Z}$ -linearizable as they can be represented as $e_i^\top w$, where e_i denotes the unit vector with the i th entry being one and $w = (f(1), \dots, f(m), c(1, 1), \dots, c(m, n))$ denotes a weight vector that contains all possible opening and serving costs. Since the goal function is composed of a sum of two sums, we will first analyze each of the sums individually and then analyze the outer sum. Observe that $\sum_{i \in \mathcal{F}'} f(i)$ is $m\text{-}\mathbb{Z}$ -linearizable by Lemma 4.6(i) as $|\mathcal{F}'| \leq m$. Similarly, since $\min_{i \in \mathcal{F}'} c(i, j)$ is $2\text{-}\mathbb{Z}$ -linearizable by Lemma 4.6(iii), it follows from Lemma 4.6(i) that $\sum_{j \in \mathcal{C}} \min_{i \in \mathcal{F}'} c(i, j)$ is $2n\text{-}\mathbb{Z}$ -linearizable as $|\mathcal{C}| = n$. Next, we define

$$f'(\ell, \mathcal{C}, \mathcal{F}') := \begin{cases} \sum_{i \in \mathcal{F}'} f(i) & \text{if } \ell = 1, \\ \sum_{j \in \mathcal{C}} \min_{i \in \mathcal{F}'} c(i, j) & \text{if } \ell = 2. \end{cases}$$

Observe that f' is $(2n+m)\text{-}\mathbb{Z}$ -linearizable as it is $(2n+m)\text{-}\mathbb{Z}$ -linearizable in each of the two cases by Observation 4.3. Moreover, note that the goal function can be represented as $\sum_{\ell \in \{1, 2\}} f'(\ell, \mathcal{C}, \mathcal{F}')$. Due to Lemma 4.6(i), it follows that the goal function is $2 \cdot (2n+m)\text{-}\mathbb{Z}$ -linearizable. Finally, Theorem 4.5 yields the desired statement with $\alpha = 2(2n+m)$ and $d = nm + m$. \square

We can apply Lemma 5.1 also for the MUFL, which requires the cost function to satisfy the triangle inequality $c(i, j) \leq c(i, j') + c(i', j') + c(i', j)$ for all $i, i' \in \mathcal{C}$ and $j, j' \in \mathcal{F}$. This easily follows from the following:

Observation 5.2. *Let $w \in \mathbb{Q}^d$, $d \in \mathbb{N}$, and $\mathbb{K} \in \{\mathbb{Z}, \mathbb{Q}\}$. Then for every $r \geq 4$ and $\hat{w} \in [w]_r^{\mathbb{K}^d}$ it holds that $\text{sign}(w^\top (\vec{e}_i + \vec{e}_j + \vec{e}_k - \vec{e}_l)) = \text{sign}(\hat{w}^\top (\vec{e}_i + \vec{e}_j + \vec{e}_k - \vec{e}_l))$ for each $i, j, k, l \in \{1, \dots, d\}$.*

The consequence of Observation 5.2 is that if Theorem 2.10 is applied to a vector w encoding a metric (i. e., the entries of w are pairwise distances of some points), then the resulting vector \hat{w} also encodes a metric. This property carries over to Theorems 4.5 and 4.9. Overall, we obtain the following result.

Proposition 5.3. *Each of UFL and MUFL admits a problem kernel of size $(n + m)^{O(1)}$.*

This complements a result of Fellows and Fernau [23] who showed a problem kernel with size exponential in a given upper bound on the optimum (which is unbounded in $n + m$).

5.2. Scheduling with Tardy Jobs

The parameterized complexity of scheduling problems recently gained increased interest [44]. In the following, we demonstrate our framework on two single-machine scheduling problems where the goal functions are functions not of sets, but of permutations, so that the notions of linearity or additivity do clearly not apply to them.

In the first problem, we minimize the weighted number of tardy jobs. Interestingly, we are going to shrink not only the weights of the jobs, but also their processing times and due dates, where the goal function contains products of terms depending on these numbers:

SINGLE-MACHINE MINIMUM WEIGHTED TARDY JOBS ($1||\Sigma w_j U_j$)

Input: A set $J := \{1, \dots, n\}$ of jobs, for each job $j \in J$ a processing time $p_j \in \mathbb{N}$, a due date $d_j \in \mathbb{N}$, and a weight $w_j \in \mathbb{N}$.

Task: Find a total order \preceq on J that minimizes the *weighted number of tardy jobs*

$$\sum_{j \in J} w_j U_j, \quad \text{where} \quad U_j := \begin{cases} 1 & \text{if } d_j < C_j \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad C_j := \sum_{i \preceq j} p_i.$$

In other words, U_j is 1 if job j is *tardy*, that is, its *completion time* C_j is after its due date d_j . The problem is weakly NP-hard [33], solvable in pseudo-polynomial time [40], and is well-studied in terms of parameterized complexity [31, 32, 44], yet there are no known kernelization results.

To apply our framework to $1||\Sigma w_j U_j$, we show that we can also compose functions via case distinctions (like the one used to define U_j) with linearizable constraints.

Lemma 5.4. *Let $f_1, f_2, g: L \times \mathbb{Q}^d \rightarrow \mathbb{Q}$. If f_1, f_2 , and g are α - \mathbb{K} -linearizable, then the following function is α - \mathbb{K} -linearizable:*

$$h(x, w) = \begin{cases} f_1(x, w) & \text{if } g(x, w) \leq 0, \\ f_2(x, w) & \text{otherwise.} \end{cases}$$

Proof. Let $w \in \mathbb{Q}^d$, $x \in L$, and $\hat{w} \in [w]_{\alpha}^{\mathbb{K}^d}$. We know that there exists a vector $b_{x,w} \in \mathbb{K}_{\alpha}^d$ with $\|b_{x,w}\|_1 \leq \alpha$ such that $g(x, w') = b_{x,w}^{\top} w'$ for all $w' \in [w]_{\alpha}^{\mathbb{K}^d}$. It follows that

$$\text{sign}(g(x, w)) = \text{sign}(b_{x,w}^{\top} w) = \text{sign}(b_{x,w}^{\top} \hat{w}) = \text{sign}(g(x, \hat{w})).$$

Thus, we have that $h(x, w) = f_1(x, w) \iff h(x, \hat{w}) = f_1(x, \hat{w})$ and $h(x, w) = f_2(x, w) \iff h(x, \hat{w}) = f_2(x, \hat{w})$. Since both f_1 and f_2 are α - \mathbb{K} -linearizable, the statement follows. \square

Using [Lemma 5.4](#) and [Lemma 4.6\(i\)](#), one can decompose the goal function $\sum_{j \in J} w_j U_j$ for an order \preceq into simple linearizable functions. Our framework then yields the following:

Lemma 5.5. *There is an algorithm that, on an instance I of $1||\Sigma w_j U_j$ and $k \in \mathbb{N}$, computes in polynomial time an instance I' of $1||\Sigma w_j U_j$ and a k' such that*

- (i) *each processing time, due date, weight, and k' is at most $2^{4(3n+1)^3}(2n^2 + 1)^{(3n+1)(3n+3)}$,*
- (ii) *any solution \preceq is optimal for I if and only if it is optimal for I' , and*
- (iii) *I has a solution of cost at most k if and only if I' has a solution of cost at most k' .*

Proof. We shrink the entries in the vector $u = (w_1, \dots, w_n, p_1, \dots, p_n, d_1, \dots, d_n)^\top \in \mathbb{N}^{3n}$. We can rewrite the goal function value for a solution \preceq as

$$f(\preceq, u) = \sum_{j=1}^n h(\preceq, j, u) \quad \text{with} \quad h(\preceq, j, u) = \begin{cases} 0 & \text{if } d_j \geq C_j = \sum_{i \preceq j} p_i, \\ w_j & \text{otherwise } (d_j < C_j). \end{cases}$$

To this end, define $g(\preceq, j, u) = -d_j + \sum_{i \preceq j} p_i$. Note that by [Observation 4.2](#) and [Lemma 4.6\(i\)](#) we have that g is n - \mathbb{Z} -linearizable. Hence, applying [Lemma 5.4](#) with the 0- \mathbb{Z} -linearizable $f_1 \equiv 0$ and the 1- \mathbb{Z} -linearizable $f_2(j, u) = w_j$ (see [Observation 4.2](#)) shows that h is n - \mathbb{Z} -linearizable. Thus, by [Lemma 4.6\(i\)](#), f is n^2 - \mathbb{Z} -linearizable. The statement of the lemma now follows from [Theorem 4.5](#). \square

We point out that a more careful and direct analysis shows that the goal function is even 1- \mathbb{Z} -linearizable. However, we skip this here since it is more tedious. Using [Lemma 5.5\(iii\)](#), one gets the following.

Proposition 5.6. *$1||\Sigma w_j U_j$ admits a problem kernel of size polynomial in n .*

In the next problem, one minimizes the total tardiness of jobs on a single machine.

SINGLE-MACHINE MINIMUM TOTAL TARDINESS ($1||\Sigma T_j$)

Input: A set $J := \{1, \dots, n\}$ of jobs, for each job $j \in J$ a processing time $p_j \in \mathbb{N}$ and a due date $d_j \in \mathbb{N}$.

Task: Find a total order \preceq on J that minimizes the *total tardiness*

$$\sum_{j \in J} T_j, \quad \text{where} \quad T_j := \max\{0, C_j - d_j\} \quad \text{and} \quad C_j := \sum_{i \preceq j} p_i.$$

Minimizing the total tardiness is motivated by its equivalence to minimizing the average tardiness (just divide the goal function by n). The problem is fixed-parameter tractable parameterized by the maximum processing time [\[39\]](#) (this result was very recently strengthened

by Knop et al. [37], who showed fixed-parameter tractability even for the version with parallel unrelated machines, jobs with release dates and weights, where jobs and machines are given in a high-multiplicity encoding that encodes the numbers of jobs and machines of each type in binary).

It is easy to see that the goal function is a composition of sums and maxima. Hence, using Lemma 4.6 one can show that the goal function is linearizable and thus prove:

Lemma 5.7. *There is an algorithm that, on any input instance I of $1||\Sigma T_j$ and $k \in \mathbb{N}$, in polynomial time computes an instance I' of $1||\Sigma T_j$ and k' such that*

- (i) *each processing time, due date, and k' is at most $2^{4(2n+1)^3}(4n^2 + 1)^{(2n+1)(2n+3)}$,*
- (ii) *any solution \preceq is optimal for I if and only if it is optimal for I' , and*
- (iii) *I has a solution of cost at most k if and only if I' has a solution of cost at most k' .*

Proof. We want to shrink weights in the vector $u = (p_1, \dots, p_n, d_1, \dots, d_n)^\top \in \mathbb{N}^{2n}$. To show that the goal function is linearizable, we express its value for a solution \preceq as

$$f(\preceq, u) := \sum_{j \in J} f_T(\preceq, j, u) \quad \text{with} \quad f_T(\preceq, j, u) := \max\{0, \sum_{i \preceq j} p_i - d_j\}.$$

By Observation 4.2 and Lemma 4.6(i), we have that $g(\preceq, j, u) := \sum_{i \preceq j} p_i - d_j$ is $n\mathbb{Z}$ -linearizable. Observe that $h \equiv 0$ is $0\mathbb{Z}$ -linearizable. Thus, by Observation 4.3 and Lemma 4.6(ii), we have that $f_T(\preceq, j, u)$ is $2n\mathbb{Z}$ -linearizable. Using again Lemma 4.6(i), we obtain that $f(\preceq, u)$ is $2n^2\mathbb{Z}$ -linearizable. The statement of the lemma now follows from Theorem 4.5. \square

From Lemma 5.7(iii) we get the following.

Proposition 5.8. *$1||\Sigma T_j$ admits a problem kernel of size polynomial in n .*

Possible Generalizations. The results in this section can easily be generalized to scheduling problems with parallel machines (even with machine-dependent processing times, so-called unrelated machines) since the completion time C_j of job j can still be represented as the sum of processing times of predecessors of j on the same machine. Variants with precedence constraints (whose parameterized complexity is also well-studied [11, 16, 26]) can be handled since the completion time C_j of a job j can be expressed as the sum p_j and the completion time C_i of a direct predecessor i of j in the precedence order (possibly on a different machine).

Other goal functions can be handled as follows: the total completion time $\sum_{j \in J} C_j$ is just the special case with $d_j = 0$ for all jobs $j \in J$. Moreover, one can replace the outer sums by maxima to minimize the makespan or maximum tardiness. We leave open whether Proposition 5.8 can be proven for the weighted variant $1||\Sigma w_j T_j$, where each job j has a weight w_j and one minimizes $\sum_{j \in J} w_j T_j$: in this case, the goal function contains products of the weights we want to shrink.

5.3. Arc Routing Problems with Min-Max Objective.

Arc routing problems have applications in garbage collection, mail delivery, meter reading, drilling, and plotting [19]. Their parameterized complexity is intensively studied [10], which led to promising results on real-world instances [9, 13]. Of particular interest are problem variants with multiple vehicles with tours of balanced length [3, 5], for example:

MIN-MAX k -RURAL POSTMAN PROBLEM (MM k -RPP)

Input: An undirected graph $G = (V, E)$, edge lengths $c: E \rightarrow \mathbb{N}$, and a subset $R \subseteq E$ of *required edges*.

Task: Find closed walks w_1, \dots, w_k in G such that $R \subseteq \bigcup_{i=1}^k E(w_k)$ that minimize $\max\{c(w_i) \mid 1 \leq i \leq k\}$, where $E(w_i)$ is the set of edges and $c(w_i)$ is the total length of edges on w_i .

A key feature of the $k = 1$ case (known as the RURAL POSTMAN PROBLEM) is that one can simply enforce the triangle inequality [12] and thus get an equivalent instance with $2|R|$ vertices [10]. For MM k -RPP, we *partly* enforce the triangle inequality to generalize this:

Lemma 5.9. *In polynomial time, one can turn an instance (G, R, c) of MM k -RPP into an instance (G', R, c^∇) on $3|R|$ vertices such that any solution for (G, R, c) can be turned in polynomial time into a solution of at most the same cost for (G', R, c^∇) , and vice versa.*

Proof. We first turn G into a complete graph $G^* = (V, E')$ with edge lengths

$$c^\nabla: E' \rightarrow \mathbb{N}, \{u, v\} \mapsto \begin{cases} c(\{u, v\}) & \text{if } \{u, v\} \in R, \\ \text{dist}_c(u, v) & \text{otherwise,} \end{cases}$$

where $\text{dist}_c(u, v)$ is the length of a shortest u - v -path in G according to c . Any feasible solution for (G, R, c) is feasible for (G^*, R, c^∇) and has at most the same cost. In the other direction, one can replace non-required edges in a feasible solution for (G^*, R, c^∇) by shortest paths in G in polynomial time to get a feasible solution of at most the same cost for (G, R, c) .

Let $\bar{V}(R) \subseteq V$ be so that for each edge $\{u, v\} \in R$, it contains the vertices of at least one shortest u - v -path p (including u and v). Observe that if p contains a vertex x not incident to any edge in R , then (u, x, v) is a u - v -path of at most the same length. Thus, we can easily compute the set $\bar{V}(R)$ so that $|\bar{V}(R)| \leq 3|R|$: it contains the end points u and v for each edge $\{u, v\} \in R$ and at most one vertex on a shortest u - v -path not incident to edges in R .

The key observation is now that any shortest closed walk in G^* containing a subset $R' \subseteq R$ of required edges can be shortcut (in polynomial time) so as to only contain vertices of $\bar{V}(R)$. Thus, we can simply take $G' = G^*[\bar{V}(R)]$. \square

One can prove a problem kernel by shrinking the weights.

Lemma 5.10. *There is an algorithm that, on an input instance I of MM k -RPP with m edges and $\kappa \in \mathbb{N}$, computes in polynomial time an instance I' and κ' such that*

- (i) *each edge cost is upper-bounded by $2^{4(m+1)^3} \cdot (8m+1)^{(m+1)(m+3)}$,*
- (ii) *a set of walks is an optimal solution for I if and only if it is optimal for I' , and*
- (iii) *I has a solution of cost at most κ if and only if I' has a solution of cost at most κ' .*

Proof. First observe that, without loss of generality, each walk w_i in a solution contains each edge of G at most two times: if it contains an edge e three times, then two occurrences of e can be removed, the walk gets shorter by $2c(e)$, yet the edge e remains covered. Thus, one can write $f(w_i, c) := c(w_i) = x^\top c$, where the vector $c \in \mathbb{N}^m$ contains the edge costs and $x \in \{0, 1, 2\}^m$ indicates how often each edge is on walk w_i . Since for *any* alternative edge weight vector c' we have $f(w_i, c') = x^\top c'$, it follows that f is $2m$ - \mathbb{Z} -linearizable.

By Lemma 4.6, it follows that the goal function $\max_{1 \leq i \leq k} f(w_i, c)$ of MM k -RPP is $4m$ - \mathbb{Z} -linearizable and the lemma follows from Theorem 4.5. \square

Proposition 5.11. *MIN-MAX k -RURAL POSTMAN PROBLEM has a $3|R|$ -vertex kernel of size polynomial in $|R|$.*

It is straightforward to transfer Lemma 5.10 to other vehicle routing problems that minimize maximum tour length.

5.4. Power Vertex Cover

Angel et al. [2] claimed a polynomial-size problem kernel for the following problem parameterized by the number of vertices that are assigned non-zero values in a solution:

POWER VERTEX COVER (PVC)

Input: An undirected graph $G = (V, E)$ with edge weights $w: E \rightarrow \mathbb{Q}_{\geq 0}$.

Task: Find an assignment $\mu: V \rightarrow \mathbb{Q}_{\geq 0}$ minimizing $\sum_{v \in V} \mu(v)$ such that, for each edge $e = \{u, v\} \in E$, one has $\max\{\mu(u), \mu(v)\} \geq w(e)$.

In fact, Angel et al. [2] only proved a partial kernel, since the edge weights in the kernel can be arbitrarily large. Using Theorem 4.5, we prove that we can shrink the weights. To this end, our application of the losing-weight technique for PVC relies on the following.

Observation 5.12. *If μ is an optimal solution, then for every $v \in V$, we have $\mu(v) \in \{w(e) \mid e \in E\} \cup \{0\}$.*

This leads to the following equivalent problem formulation of PVC.

POWER VERTEX COVER 2 (PVC2)

Input: An undirected graph $G = (V, E)$ with edge weights $w: E \cup \{\emptyset\} \rightarrow \mathbb{Q}_{\geq 0}$ with $w(\emptyset) = 0$.

Task: Find an assignment $\mu: V \rightarrow E \cup \{\emptyset\}$ such that for each edge $e = \{u, v\} \in E$ it holds true that $\max\{w(\mu(u)), w(\mu(v))\} \geq w(e)$ and μ minimizes $\sum_{v \in V} w(\mu(v))$.

Lemma 5.13. *There is an algorithm that, on an instance $I = (G = (V, E), w)$ of PVC2 with $n := |V|$ and $m := |E|$, and $k \in \mathbb{Q}$, in time polynomial in $|I, k|$ computes an instance $I' = (G = (V, E), \hat{w})$ of PVC2 and $\hat{k} \in \mathbb{Z}$ such that*

$$(i) \quad \|\hat{w}\|_{\infty}, |\hat{k}| \leq 2^{4(m+1)^3} \cdot (2n+1)^{(m+1)(m+3)},$$

(ii) *any assignment $\mu: V \rightarrow E \cup \{\emptyset\}$ forms an optimal solution for I if and only if μ forms an optimal solution for I' , and*

(iii) *for any assignment $\mu: V \rightarrow E \cup \{\emptyset\}$ it holds that $\sum_{v \in V} w(\mu(v)) \leq k \iff \sum_{v \in V} \hat{w}(\mu(v)) \leq \hat{k}$.*

Proof. Let $f(v, w) = w(e)$ if $\mu(v) = e$, and 0 if $\mu(v) = \emptyset$. Due to [Observation 4.2](#), f is 1- \mathbb{Z} -linearizable. Hence, $g(V, w) = \sum_{v \in V} f(v, w)$ is n - \mathbb{Z} -linearizable. [Theorem 4.5](#) now yields the desired statement with $\alpha = n$ and $d = m$. \square

Using [Lemma 5.13](#) and the partial kernel of Angel et al. [2], we obtain [Proposition 5.14](#).

Proposition 5.14. *POWER VERTEX COVER admits a polynomial kernel with respect to the number of non-zero values in a solution.*

5.5. Chamberlin-Courant Committee with Cardinal Utilities

Another exemplary application is the following problem from computational social choice. It deals with the Chamberlin-Courant voting rule [17], which already has been studied from a parameterized complexity point of view [27, 43, 47].

CHAMBERLIN-COURANT COMMITTEE WITH CARDINAL UTILITIES (C⁴U)

Input: A set V of voters, a set A of alternatives, a function $u: V \times A \rightarrow \mathbb{Q}_{\geq 0}$, and $k \in \mathbb{N}$.

Task: Find a subset $A' \subseteq A$ of size at most k that maximizes

$$\sum_{v \in V} \max_{a \in A'} u(v, a). \tag{5.2}$$

We will show that the goal function (5.2) is linearizable.

Lemma 5.15. *There is an algorithm that, on an input consisting of an instance (V, A, u, k) of C^4U with $n := |V|$ and $m := |A|$, and $p \in \mathbb{Q}$, computes in time polynomial in $|(V, A, u, k, p)|$ an instance (V, A, \bar{u}, k) of C^4U and $\bar{p} \in \mathbb{Z}$ such that*

$$(i) \quad \|\bar{u}\|_\infty, |\bar{p}| \leq 2^{4(nm+1)^3} \cdot (4n+1)^{(nm+1)(nm+3)},$$

(ii) *any subset $A' \subseteq A$ forms an optimal solution for (V, A, u, k) if and only if A' forms an optimal solution for (V, A, \bar{u}, k) , and*

(iii) *for any subset $A' \subseteq A$ we have that $\sum_{v \in V} \max_{a \in A'} u(v, a) \geq p \iff \sum_{v \in V} \max_{a \in A'} \bar{u}(v, a) \geq \bar{p}$.*

Proof. Observe that the goal function can be restated as follows:

$$\sum_{v \in V} \max_{a \in A'} u(v, a) = \sum_{v \in V} \max_{(v, a) \in \{v\} \times A'} u(v, a).$$

Due to [Observation 4.2](#), $u: V \times A \rightarrow \mathbb{Q}_{\geq 0}$ is 1- \mathbb{Z} -linearizable. Note that the weight vector representing u is of dimension $d = nm$. By [Lemma 4.6\(ii\)](#), we know that $g(v, A') = \max_{(v, a) \in \{v\} \times A'} u(v, a)$ is 2- \mathbb{Z} -linearizable. Finally, by [Lemma 4.6\(i\)](#), $h(V, A') = \sum_{v \in V} g(v, A')$ is $2n$ - \mathbb{Z} -linearizable. By [Theorem 4.5](#) with $\alpha = 2n$ and $d = nm$, the claim follows. \square

[Lemma 5.15](#) yields [Proposition 5.16](#).

Proposition 5.16. CHAMBERLIN-COURANT COMMITTEE WITH CARDINAL UTILITIES *admits a problem kernel of size polynomial in the combined parameter number of voters and alternatives.*

6. Concluding Remarks

The losing-weight technique due to Frank and Tardos [[28](#)] is a key tool to obtain polynomial problem kernels for weighted parameterized problems. While Marx and Végé [[42](#)] and Etscheid et al. [[22](#)] proved the usefulness of the technique for several problems with additive goal functions, we demonstrated its applicability for a larger class of functions (linearizable functions) containing next to additive also non-additive functions. In addition, in [Section 5](#) we displayed our recipe DDD to be a neat manual for applying the losing-weight technique to the class of (linearizable) functions.

As Etscheid et al. [[22](#)] pointed out, one direction for future work is to improve the upper bound in [Theorem 2.10](#) on the maximum norm of the output vector. In this direction, Eisenbrand et al. [[21](#)] recently proved a stronger upper bound, yet non-constructively. Another direction, seemingly not addressed so far, aims for a better running time: Frank and Tardos [[28](#)] state no explicit running time of their algorithm, and Lenstra et al. [[41](#), Proposition 1.26] state that their simultaneous Diophantine approximation algorithm, which forms a subroutine in Frank and Tardos' technique, runs in $d^6 \cdot \log(\|w\|_\infty)^{O(1)}$ time. This is clearly a bottleneck for the practical applicability of the techniques we discussed. Hence, we put forward the following: Can [Theorem 2.10](#) be executed in quadratic, or even linear time? We point out that for approximate kernelizations, there is an analog to [Theorem 2.10](#) executable in linear time [[9](#)].

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