

# Some tight bounds on the minimum and maximum forcing numbers of graphs<sup>1</sup>

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**Abstract:** Let  $G$  be a simple graph with  $2n$  vertices and a perfect matching. We denote by  $f(G)$  and  $F(G)$  the minimum and maximum forcing number of  $G$ , respectively.

Hetyei obtained that the maximum number of edges of graphs  $G$  with a unique perfect matching is  $n^2$ . We know that  $G$  has a unique perfect matching if and only if  $f(G) = 0$ . Along this line, we generalize the classical result to all graphs  $G$  with  $f(G) = k$  for  $0 \leq k \leq n - 1$ , and characterize corresponding extremal graphs as well. Hence we get a non-trivial lower bound of  $f(G)$  in terms of the order and size. For bipartite graphs, we gain corresponding stronger results. Further, we obtain a new upper bound of  $F(G)$ . For bipartite graphs  $G$ , Che and Chen (2013) obtained that  $f(G) = n - 1$  if and only if  $G$  is complete bipartite graph  $K_{n,n}$ . We completely characterize all bipartite graphs  $G$  with  $f(G) = n - 2$ .

*Keywords:* Perfect matching; Minimum forcing number; Maximum forcing number; Bipartite graph

## 1 Introduction

We consider only finite and simple graphs. Let  $G$  be a graph with vertex set  $V(G)$  and edge set  $E(G)$ . The *order* of  $G$  is the number of vertices in  $G$ , and the *size* of  $G$ , written  $e(G)$ , is the number of edges in  $G$ .

A *perfect matching*  $M$  of a graph  $G$  is a set of disjoint edges covering all vertices of  $G$ . A subset  $S \subseteq M$  is called a *forcing set* of  $M$  if  $S$  is not contained in any other perfect matching of  $G$ . The smallest cardinality of a forcing set of  $M$  is called the *forcing number* of  $M$ , denoted by  $f(G, M)$ . The concept was originally introduced by Harary et al. [10] and by Klein and Randić [12], which plays an important role in resonance theory.

For a perfect matching  $M$  of  $G$ , a cycle of  $G$  is  *$M$ -alternating* if its edges appear alternately in  $M$  and  $E(G) \setminus M$ . Clearly,  $M$  is a unique perfect matching of  $G$  if and only

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if  $G$  contains no  $M$ -alternating cycles.

**Lemma 1.1.** [21] *Let  $G$  be a graph with a perfect matching  $M$ . Then  $S \subseteq M$  is a forcing set of  $M$  if and only if  $S$  contains at least one edge of every  $M$ -alternating cycle.*

Let  $C(G, M)$  denote the maximum number of disjoint  $M$ -alternating cycles in  $G$ . Then  $f(G, M) \geq C(G, M)$  by Lemma 1.1. For plane bipartite graphs, Pachter and Kim pointed out the following minimax theorem.

**Theorem 1.2.** [21] *Let  $G$  be a plane bipartite graph. Then  $f(G, M) = C(G, M)$  for any perfect matching  $M$  of  $G$ .*

For a vertex subset  $T$  of  $G$ , we write  $G - T$  for the subgraph of  $G$  obtained by deleting all vertices in  $T$  and their incident edges. Sometimes, we write  $G[V(G) \setminus T]$  for the subgraph  $G - T$ , induced by  $V(G) \setminus T$ . If  $T = \{v\}$ , we write  $G - v$  rather than  $G - \{v\}$ .

Let  $G$  and  $H$  be bipartite graphs. We say  $G$  contains  $H$  if  $G$  has a subgraph  $L$  such that  $G - V(L)$  has a perfect matching and  $L$  is isomorphic to an even subdivision of  $H$ . In [20] and some articles related to matching theory,  $G$  contains  $H$  is also called  $H$  is a *conformal minor* of  $G$ . Guenin and Thomas obtained the following general minimax result in somewhat different manner (see Corollary 5.8 in [9]).

**Theorem 1.3.** [9] *Let  $G$  be a bipartite graph with a perfect matching  $M$ . Then  $G$  has no  $K_{3,3}$  or the Heawood graph as a conformal minor if and only if  $f(G', M') = C(G', M')$  for each subgraph  $G'$  of  $G$  such that  $M' = M \cap E(G')$  is a perfect matching in  $G'$ .*

The *minimum* and *maximum forcing number* of  $G$  are the minimum and maximum values of  $f(G, M)$  over all perfect matchings  $M$  of  $G$ , denoted by  $f(G)$  and  $F(G)$ , respectively. The *degree* of a vertex  $v$  in  $G$ , written  $d_G(v)$ , is the number of edges incident to  $v$ . A *pendant vertex* of  $G$  is a vertex of degree 1. We denote by  $\delta(G)$  and  $\Delta(G)$  the minimum and maximum degrees of the vertices of  $G$ . The problem of finding the minimum forcing number of bipartite graphs with the maximum degree 4 is NP-complete [3].

The path and cycle with  $n$  vertices are denoted by  $P_n$  and  $C_n$ , respectively. The *cartesian product* of graphs  $G$  and  $H$ , written  $G \times H$ . Pachter and Kim [21] showed that  $f(P_{2n} \times P_{2n}) = n$  and  $F(P_{2n} \times P_{2n}) = n^2$ . Riddle [22] got that  $f(C_{2m} \times C_{2n}) = 2\min\{m, n\}$ , and Kleinerman [13] obtained that  $F(C_{2m} \times C_{2n}) = mn$ . Afshani et al. [3] obtained that  $F(P_{2k} \times C_{2n}) = kn$  and  $F(P_{2k+1} \times C_{2n}) = kn + 1$ , and they [3] proposed a problem: what is the maximum forcing number of non-bipartite graph  $P_{2m} \times C_{2n+1}$ ? Jiang and Zhang [11] solved the problem and obtained that  $F(P_{2m} \times C_{2n+1}) = m(n + 1)$ . For any  $k$ -regular bipartite graph  $G$  with  $n$  vertices in each partite set, Adams et al. [2] showed

that  $F(G) \geq (1 - \frac{\log(2e)}{\log k})n$ , where  $e$  is the base of the natural logarithm. Hence, for hypercube  $Q_k$  where  $k \geq 2$ ,  $F(Q_k) > c2^{k-1}$  for any constant  $0 < c < 1$  and sufficient large  $k$  (see [22]). Diwan [8] proved that  $f(Q_k) = 2^{k-2}$  by linear algebra for  $k \geq 2$ , which solved a conjecture proposed by Pachter and Kim [21]. For hexagonal systems, Xu et al. [28] proved that the maximum forcing number is equal to its resonant number. For polyomino graphs [15, 30] and BN-fullerene graphs [23], the same result also holds. For more researches on the minimum and maximum forcing numbers, see [5, 11, 26, 29].

For graphs with a unique perfect matching, there are some classical results. To describe these results, we define a bipartite graph  $H_{n,k}$  of order  $2n$  as follows, where  $n$  and  $k$  are integers with  $0 \leq k \leq n-1$ : The bipartition of  $H_{n,k}$  is  $U \cup V$ , where  $U = \{u_1, u_2, \dots, u_n\}$  and  $V = \{v_1, v_2, \dots, v_n\}$ , such that  $u_i v_j \notin E(H_{n,k})$  if and only if  $1 \leq i < j \leq n-k$  (see  $H_{6,2}$  in Fig. 1). It is clear that

$$d_{H_{n,k}}(u_l) = k + l \text{ and } d_{H_{n,k}}(v_l) = n - l + 1 \text{ for } l = 1, 2, \dots, n - k$$

and the other vertices have degree  $n$ . So  $H_{n,n-1}$  is isomorphic to  $K_{n,n}$ , which is the complete bipartite graph with each partite set having  $n$  vertices.

Let  $\hat{H}_{n,0}$  be the graph obtained by adding all possible edges in  $V$  to  $H_{n,0}$  (see  $\hat{H}_{5,0}$  in Fig. 1). Obviously,  $\{u_i v_i | i = 1, 2, \dots, n\}$  is the unique perfect matching of  $H_{n,0}$  and  $\hat{H}_{n,0}$ . A graph is *split* if its vertex set can be partitioned into a clique and an independent set. Since  $U$  is an independent set and  $V$  is a clique of  $\hat{H}_{n,0}$ ,  $\hat{H}_{n,0}$  is a split graph.

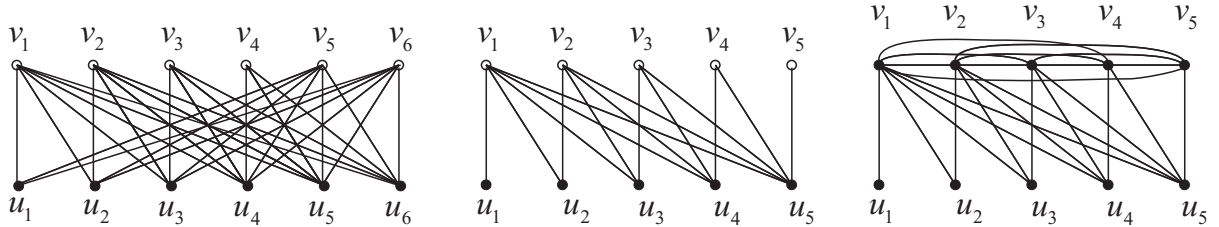


Fig. 1. The graphs  $H_{6,2}$ ,  $H_{5,0}$  and  $\hat{H}_{5,0}$ .

set. Since  $U$  is an independent set and  $V$  is a clique of  $\hat{H}_{n,0}$ ,  $\hat{H}_{n,0}$  is a split graph. A graph is called a *cograph* if it is either a singleton or it can be obtained by the disjoint union or join of two cographs, where the *join* of two graphs  $G$  and  $H$ , written  $G \vee H$ , is formed by taking the disjoint union of these two graphs and additionally adding the edges  $\{xy | x \in V(G), y \in V(H)\}$ .

For (bipartite) graphs with a unique perfect matching, there are some classical results (see Lemma 4.3.2 in [19] for bipartite graphs, and Corollary 1.6 in [17] or Corollary 5.3.14 in [19] for general graphs).

**Theorem 1.4.** [19] *Let  $G$  be a bipartite graph of order  $2n$  and with a unique perfect matching. Then  $G$  has two pendant vertices lying in different partite sets and  $e(G) \leq \frac{n(n+1)}{2}$ . Moreover, equality holds if and only if  $G$  is  $H_{n,0}$ .*

**Theorem 1.5.** [17, 19] *Let  $G$  be a graph of order  $2n$  and with a unique perfect matching. Then  $e(G) \leq n^2$ , and equality holds if and only if  $G$  is  $\hat{H}_{n,0}$ .*

We assume that the graphs  $G$  in question have  $2n$  vertices and a perfect matching. Then  $0 \leq f(G) \leq F(G) \leq n - 1$ . If we use the terminology of forcing number, then  $G$  has a unique perfect matching if and only if  $f(G) = 0$ . Along this line, we generalize Theorems 1.4 and 1.5 to all graphs  $G$  with  $f(G) = k$  for  $0 \leq k \leq n - 1$  in Section 2. In detail, we show that  $e(G) \leq n^2 + 2nk - k^2 - k$  and characterize corresponding extremal graphs. In turn, we obtain that  $f(G) \geq n - \frac{1}{2} - \sqrt{2n^2 - n - e(G) + \frac{1}{4}}$ . For bipartite graphs, both bounds can be improved to  $\frac{(n-k)(n+k+1)}{2} + nk$  and  $n - \frac{1}{2} - \sqrt{2n^2 - 2e(G) + \frac{1}{4}}$ , respectively. For some special graphs, we give another lower bound of  $f(G)$  in terms of  $\delta(G)$ . Precisely, if  $G$  is a bipartite graph then  $f(G) \geq \delta(G) - 1$ , and if  $G$  is a split graph or a cograph then  $f(G) \geq \frac{\delta(G)-1}{2}$ . In Section 3, we consider all graphs  $G$  with  $F(G) = k$  for  $0 \leq k \leq n - 1$  and get that  $e(G) \geq \frac{n(n+1)}{n-k} - k - 1$ . As a result, we obtain a new upper bound of  $F(G)$  and compare it with two known bounds derived from the maximum anti-forcing numbers. A bipartite graph  $G$  has  $f(G) = n - 1$  if and only if  $G$  is  $K_{n,n}$ . In Section 4, we determined all bipartite graphs  $G$  with  $f(G) = n - 2$ .

## 2 Some lower bounds of the minimum forcing number

In this section, we generalize Theorems 1.4 and 1.5 to all bipartite and general graphs  $G$  of order  $2n$  and with  $f(G) = k$  for  $0 \leq k \leq n - 1$ , respectively. By these results, we obtain two non-trivial lower bounds of  $f(G)$  with respect to the order and size. For some special classes of graphs  $G$ , we also give a lower bound of  $f(G)$  by using  $\delta(G)$ . For a subset  $S$  of  $E(G)$ , we use  $V(S)$  to denote the set of all end-vertices of edges in  $S$ .

**Theorem 2.1.** *Let  $G$  be a graph of order  $2n$  and with  $f(G) = k$  for  $0 \leq k \leq n - 1$ . Then*

$$e(G) \leq n^2 + 2nk - k^2 - k, \tag{2.1}$$

*and equality holds if and only if  $G$  is  $\hat{H}_{n-k,0} \vee K_{2k}$  where  $K_{2k}$  denotes the complete graph of order  $2k$ .*

*Proof.* Suppose to the contrary that  $e(G) \geq n^2 + 2nk - k^2 - k + 1$ . Let  $M$  be any perfect matching of  $G$  and  $S$  be any subset of  $M$  with size no less than  $n - k$ . We are to prove

that  $M \setminus S$  is not a forcing set of  $M$ . If we have done, then  $f(G, M) \geq k + 1$ . By the arbitrariness of  $M$ , we acquire that  $f(G) \geq k + 1$ , a contradiction.

Since  $e(K_{2n}) = 2n^2 - n$ , we have

$$n^2 + 2nk - k^2 - k + 1 = e(K_{2n}) - (n - k)(n - k - 1) + 1, \quad (2.2)$$

$$e(G[V(S)]) \geq \binom{2|S|}{2} - [(n - k)(n - k - 1) - 1] = 2|S|^2 - |S| - (n - k)(n - k - 1) + 1.$$

So  $e(G[V(S)]) - (|S|^2 + 1) \geq |S|(|S| - 1) - (n - k)(n - k - 1) \geq 0$  for  $x^2 - x$  is monotonically increasing in  $[\frac{1}{2}, +\infty)$  and  $|S| \geq n - k \geq 1$ . Thus,  $e(G[V(S)]) \geq |S|^2 + 1$ . By Theorem 1.5,  $G[V(S)]$  has at least two perfect matchings. That is,  $M \setminus S$  is not a forcing set of  $M$ .

Suppose that  $G$  is the join of  $\hat{H}_{n-k,0}$  and  $K_{2k}$ . By Theorem 1.5,  $e(\hat{H}_{n-k,0}) = (n - k)^2$ . Since exactly two vertices in  $\hat{H}_{n-k,0}$  may be not adjacent in  $G$ , we get that

$$e(G) = \binom{2n}{2} - \left[ \binom{2(n-k)}{2} - (n-k)^2 \right] = n^2 + 2nk - k^2 - k.$$

Conversely, suppose that equality in (2.1) holds. Since  $f(G) = k$ , there exists a perfect matching  $M$  of  $G$  and a minimum forcing set  $S$  of  $M$  such that  $|S| = f(G, M) = f(G)$ . By Lemma 1.1,  $G[V(M \setminus S)]$  contains no  $M$ -alternating cycles. Since (2.2) holds, we have

$$e(G[V(M \setminus S)]) \geq \binom{2(n-k)}{2} - (n-k)(n-k-1) = (n-k)^2.$$

By Theorem 1.5,  $e(G[V(M \setminus S)]) = (n-k)^2$  and  $G[V(M \setminus S)]$  is  $\hat{H}_{n-k,0}$ . Furthermore, each vertex in  $V(S)$  is adjacent to all other vertices in  $G$ . So we have  $G = \hat{H}_{n-k,0} \vee K_{2k}$ .  $\square$

By inverting (2.1), we obtain a general lower bound on  $f(G)$ .

**Corollary 2.2.** *Let  $G$  be a graph of order  $2n$  and with a perfect matching. Then*

$$f(G) \geq n - \frac{1}{2} - \sqrt{2n^2 - n - e(G) + \frac{1}{4}}, \quad (2.3)$$

and equality holds if and only if  $G$  is  $\hat{H}_{n-k,0} \vee K_{2k}$ .

*Proof.* Let  $f(G) = k$ . Then  $0 \leq k \leq n - 1$ . By Theorem 2.1,  $e(G) \leq n^2 + 2nk - k^2 - k$ . That is,  $k^2 - (2n - 1)k - n^2 + e(G) \leq 0$ . So

$$n - \frac{1}{2} - \sqrt{2n^2 - n - e(G) + \frac{1}{4}} \leq k \leq n - \frac{1}{2} + \sqrt{2n^2 - n - e(G) + \frac{1}{4}}. \quad (2.4)$$

Hence (2.3) holds.

Since  $n - \frac{1}{2} + \sqrt{2n^2 - n - e(G) + \frac{1}{4}} \geq n$  and  $n - 1$  is a trivial upper bound of  $f(G)$ , the second inequality in (2.4) holds. So equality in (2.3) holds if and only if equality in (2.1) holds. Hence these graphs such that two equalities in (2.1) and (2.3) hold are the same.  $\square$

For bipartite graphs, we can obtain corresponding stronger results than Theorem 2.1 and Corollary 2.2.

**Theorem 2.3.** *Let  $G = (U, V)$  be a bipartite graph of order  $2n$  and with  $f(G) = k$  for  $0 \leq k \leq n - 1$ . Then*

$$e(G) \leq \frac{(n-k)(n+k+1)}{2} + nk, \quad (2.5)$$

and equality holds if and only if  $G$  is  $H_{n,k}$ .

*Proof.* Suppose to the contrary that  $e(G) \geq \frac{(n-k)(n+k+1)}{2} + nk + 1$ . Let  $M$  and  $S$  be defined as that in the proof of Theorem 2.1. By the same arguments, we will prove that  $M \setminus S$  is not a forcing set of  $M$ . Since  $e(K_{n,n}) = n^2$  and

$$\frac{(n-k)(n+k+1)}{2} + nk + 1 = e(K_{n,n}) - \frac{(n-k)(n-k-1)}{2} + 1, \quad (2.6)$$

we have  $e(G[V(S)]) \geq |S|^2 - [\frac{(n-k)(n-k-1)}{2} - 1]$ . So

$$e(G[V(S)]) - [\frac{|S|(|S|+1)}{2} + 1] \geq \frac{1}{2}[|S|^2 - |S| - (n-k)^2 + n - k] \geq 0$$

for  $x^2 - x$  is strictly monotonic increasing in  $[\frac{1}{2}, +\infty)$  and  $|S| \geq n - k \geq 1$ . Therefore,  $e(G[V(S)]) \geq \frac{|S|(|S|+1)}{2} + 1$ . By Theorem 1.4,  $G[V(S)]$  has at least two perfect matchings. That is,  $M \setminus S$  is not a forcing set of  $M$ .

Suppose that  $G$  is  $H_{n,k}$ . Let  $G' = G[\{u_i, v_i | i = 1, 2, \dots, n-k\}]$ . Then  $G'$  is isomorphic to  $H_{n-k,0}$ . By Theorem 1.4,  $e(G') = \frac{(n-k)(n-k+1)}{2}$ . Since each vertex of  $V(G) \setminus V(G')$  has vertex  $n$ ,  $u_i v_j \notin E(G)$  if and only if  $1 \leq i < j \leq n - k$  if and only if  $u_i v_j \notin E(G')$ . Thus,

$$e(G) = n^2 - [(n-k)^2 - e(G')] = \frac{(n-k)(n+k+1)}{2} + nk.$$

Conversely, suppose that equality in (2.5) holds. Since  $f(G) = k$ , there exists a perfect matching  $M$  of  $G$  and a minimum forcing set  $S$  of  $M$  such that  $|S| = f(G, M) = f(G)$ . By Lemma 1.1,  $G[V(M \setminus S)]$  has a unique perfect matching. Since (2.6) holds, we have

$$e(G[V(M \setminus S)]) \geq (n-k)^2 - \frac{(n-k)(n-k-1)}{2} = \frac{(n-k)(n-k+1)}{2}.$$

By Theorem 1.4, we obtain that  $e(G[V(M \setminus S)]) = \frac{(n-k)(n-k+1)}{2}$  and  $G[V(M \setminus S)]$  is  $H_{n-k,0}$ . Let  $u_i$  and  $v_j$  be two vertices of  $U \cap V(S)$  and  $V \cap V(S)$ , respectively. Then  $u_i$  is adjacent to all vertices of  $V$  and  $v_j$  is adjacent to all vertices of  $U$ . So  $G$  is  $H_{n,k}$ .  $\square$

By inverting (2.5), we obtain a lower bound on  $f(G)$  for bipartite graphs.

**Corollary 2.4.** *Let  $G$  be a bipartite graph of order  $2n$  and with a perfect matching. Then*

$$f(G) \geq n - \frac{1}{2} - \sqrt{2n^2 - 2e(G) + \frac{1}{4}}, \quad (2.7)$$

*and equality holds if and only if  $G$  is  $H_{n,k}$ .*

*Proof.* Let  $f(G) = k$ . Then  $0 \leq k \leq n - 1$ . By Theorem 2.3,  $e(G) \leq \frac{(n-k)(n+k+1)}{2} + nk$ . That is to say,  $k^2 - (2n - 1)k - n^2 - n + 2e(G) \leq 0$ . So

$$n - \frac{1}{2} - \sqrt{2n^2 - 2e(G) + \frac{1}{4}} \leq k \leq n - \frac{1}{2} + \sqrt{2n^2 - 2e(G) + \frac{1}{4}}. \quad (2.8)$$

Consequently, (2.7) holds.

Since  $n - \frac{1}{2} + \sqrt{2n^2 - 2e(G) + \frac{1}{4}} \geq n$  and  $n - 1$  is a trivial upper bound of  $f(G)$ , the second inequality in (2.8) holds. So equality in (2.7) holds if and only if equality in (2.5) holds. Hence the graphs such that two equalities in (2.5) and (2.7) hold are the same.  $\square$

**Remark 2.5.** The right sides in (2.3) and (2.7) are strictly monotonic increasing about  $e(G)$ . Hence the bounds in (2.3) and (2.7) are effective respectively for graphs  $G$  with  $e(G) \geq n^2$  and  $e(G) \geq \frac{1}{2}(n^2 + n)$ .

In the sequel, we will give some lower bounds of  $f(G)$  in terms of  $\delta(G)$ .

**Theorem 2.6.** *If  $G$  is a bipartite graph with a perfect matching, then  $f(G) \geq \delta(G) - 1$ . Moreover, the bound is tight.*

*Proof.* Let  $M$  be a perfect matching of  $G$  and  $S$  be a minimum forcing set of  $M$  such that  $|S| = f(G, M) = f(G)$ . By Lemma 1.1,  $G - V(S)$  has a unique perfect matching. By Theorem 1.4,  $G - V(S)$  has a pendant vertex, say  $u$ . Then all but one of the neighbors of  $u$  are incident with edges in  $S$ . Combining that  $G$  is a bipartite graph, we obtain that  $f(G) = |S| \geq d_G(u) - 1 \geq \delta(G) - 1$ .

Note that  $H_{n,k}$  is a bipartite graph with  $\delta(H_{n,k}) = k + 1$ . Since equality in (2.5) holds for  $H_{n,k}$ , we have  $f(H_{n,k}) = k = \delta(H_{n,k}) - 1$ . Thus the bound is tight.  $\square$

For a graph  $G$  of order  $2n$ , we say a set  $U = \{u_1, u_2, \dots, u_n\}$  forces a unique perfect matching in  $G$  if  $u_i$  is a pendant vertex of  $G_i$  whose only neighbor is  $v_i$  for every  $1 \leq i \leq n$ , where  $G_1 = G$ ,  $G_i = G_{i-1} - \{u_{i-1}, v_{i-1}\}$  for  $2 \leq i \leq n$ . Clearly, if  $U$  forces a unique perfect matching in  $G$ , then  $\{u_i v_i | i = 1, 2, \dots, n\}$  is a unique perfect matching of  $G$ .

For cographs and split graphs, Chaplick et al. [4] obtained the following result.

**Lemma 2.7.** [4] *If  $G$  is a cograph or a split graph, then  $G$  has a unique perfect matching if and only if some set forces a unique perfect matching in  $G$ .*

Lemma 2.7 guarantees the following result.

**Theorem 2.8.** *If  $G$  is a split graph or a cograph with a perfect matching, then  $f(G) \geq \frac{\delta(G)-1}{2}$ . Moreover, the bound is tight.*

*Proof.* Let  $M$  and  $S$  be defined as that in the proof of Theorem 2.6. By Lemma 1.1,  $G - V(S)$  has a unique perfect matching. Since  $G - V(S)$  is still a split graph or a cograph,  $G - V(S)$  has a pendant vertex by Lemma 2.7, say  $u$ . Then all but one of the neighbors are incident with edges in  $S$ . Hence we have  $2|S| \geq d_G(u) - 1 \geq \delta(G) - 1$ . So  $f(G) = |S| \geq \frac{\delta(G)-1}{2}$ .

Next we will show that this bound is tight. Let  $G_1 = \hat{H}_{n-k,0} \vee K_{2k}$  where  $0 \leq k \leq n-1$ . Since  $V(G_1)$  can be partitioned into an independent set  $I = \{u_1, u_2, \dots, u_{n-k}\}$  and a clique  $V(G_1) \setminus I$ ,  $G_1$  is a split graph with  $\delta(G_1) = 2k+1$ . Combining that equality in (2.1) holds for  $G_1$ , we obtain that  $f(G_1) = k = \frac{\delta(G_1)-1}{2}$ .

Let  $G_2 = (n-k)K_2 \vee K_{2k}$  where  $0 \leq k \leq n-1$  and  $(n-k)K_2$  denotes  $(n-k)$  disjoint copies of  $K_2$ . Since  $(n-k)K_2$  and  $K_{2k}$  are two cographs,  $G_2$  is a cograph with  $\delta(G_2) = 2k+1$ . By Theorem 2.8, we have  $f(G_2) \geq \frac{\delta(G_2)-1}{2} = k$ . Let  $M$  be a perfect matching of  $G_2$  consisting of  $(n-k)K_2$  and a perfect matching  $M_1$  of  $K_{2k}$ . Then  $M_1$  is a forcing set of  $M$ . So  $f(G_2) \leq f(G_2, M) \leq |M_1| = k$ . Thus,  $f(G_2) = k = \frac{\delta(G_2)-1}{2}$ .  $\square$

**Remark 2.9.** Theorem 2.8 is not necessarily true for general graphs.

Suppose that  $G_3 = H \vee K_{2(n-4)}$  where  $H$  is shown in Fig. 2 and  $n \geq 4$ . Assume that the vertices of  $K_{2(n-4)}$  is  $\{u_i, v_i | i = 5, 6, \dots, n\}$ .

Let  $M = M_1 \cup M_2$  be a perfect matching of  $G_3$  where  $M_1 = \{u_1v_1, u_2v_2, u_3v_3, u_4v_4\}$  is a perfect matching of  $H$  and  $M_2$  is that of  $K_{2(n-4)}$ . Then  $M_2 \cup \{u_4v_4\}$  is a forcing set of  $M$  since  $H - \{u_4, v_4\}$  has a unique perfect matching. So  $f(G_3) \leq f(G_3, M) \leq n-3$ . But  $\delta(G_3) = 2(n-4) + 4 = 2n-4$  and  $\frac{\delta(G_3)-1}{2} = n - \frac{5}{2} > n-3 \geq f(G_3)$ .

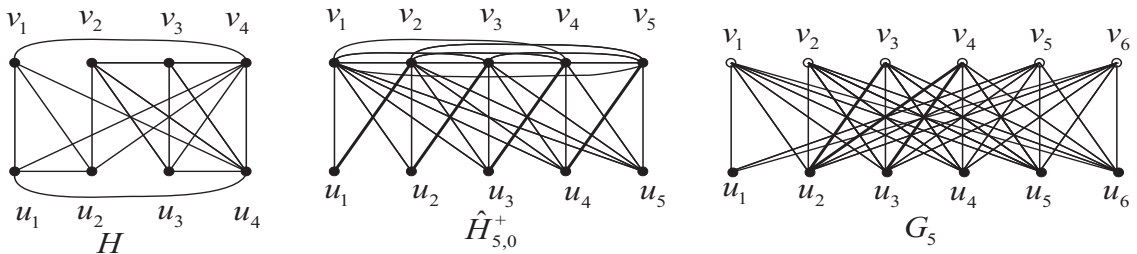


Fig. 2. Graph  $H$ ,  $\hat{H}_{5,0}^+$  where  $n = 5, k = 0$ , and  $G_5$  where  $n = 6$  and  $k = i = 2$ .

Using these lower bounds obtained, we can calculate the minimum forcing numbers of some graphs which are not extremal graphs of corresponding minimum forcing numbers.



**Example 2.10.** Let  $G_4 = \hat{H}_{n-k,0}^+ \vee K_{2k}$  where  $\hat{H}_{n-k,0}^+$  is a graph obtained from  $\hat{H}_{n-k,0}$  by adding a set of edges  $T = \{u_i v_{i+1} | i = 1, 2, \dots, n-k-1\}$  for some  $0 \leq k \leq n-2$  (see  $\hat{H}_{5,0}^+$  in Fig. 2). Then  $f(G_4) = k+1$ .

*Proof.* By Remark 2.5,  $f$  is strictly monotonic increasing about the number of edges. Combining Corollary 2.2 and  $|T| \geq 1$ , we have

$$f(G_4) \geq n - \frac{1}{2} - \sqrt{2n^2 - n - e(G_4) + \frac{1}{4}} > n - \frac{1}{2} - \sqrt{2n^2 - n - e(\hat{H}_{n-k,0} \vee K_{2k}) + \frac{1}{4}} = k$$

as  $\hat{H}_{n-k,0} \vee K_{2k}$  is the extremal graph of Theorem 2.1. So  $f(G_4) \geq k+1$ .

On the other hand, let  $M = T \cup \{u_{n-k} v_1\} \cup \{u_i v_i | i = n-k+1, n-k+2, \dots, n\}$  be a perfect matching of  $G_4$ . Since  $\{u_1, u_2, \dots, u_{n-k-1}\}$  forces a unique perfect matching in  $G_4[V(T)]$ ,  $G_4[V(T)]$  has a unique perfect matching. So  $M \setminus T$  is a forcing set of  $M$  and  $f(G_4) \leq f(G_4, M) \leq |M \setminus T| = k+1$ .  $\square$

**Example 2.11.** Let  $G_5 = (U, V)$  be a bipartite graph obtained from  $H_{n,k}$  by adding a set of edges  $\{u_i v_{n-k-1}, u_i v_{n-k}, u_{i+1} v_{n-k}\}$  for some  $1 \leq i \leq n-k-2$  (an example  $G_5$  shown in Fig. 2). Then  $f(G_5) = k+2$ .

*Proof.* On one hand, let  $M_0 = \{u_i v_i | i = 1, 2, \dots, n\}$  be a perfect matching of  $G_5$  and  $S_0 = \{u_i v_i | i = n-k-1, n-k, \dots, n\}$ . Since  $\{u_i | i = 1, 2, \dots, n-k-2\}$  forces a unique perfect matching in  $G_5 - V(S_0)$ ,  $G_5 - V(S_0)$  has a unique perfect matching. So  $S_0$  is a forcing set of  $M_0$  and  $f(G_5) \leq f(G_5, M_0) \leq |S_0| = k+2$ .

On the other hand, if  $i = 1$ , then  $\delta(G_5) = k+3$ . By Theorem 2.6,  $f(G_5) \geq k+2$ . Suppose that  $i \geq 2$  and  $M$  is a perfect matching of  $G_5$ . Let  $L = \{u_j v_{l_j} | j = 1, 2, \dots, i-1\}$  be a subset of  $M$ . Then  $1 \leq l_j \leq j$  or  $l_j \geq n-k+1$ .

Let  $G'_5 = G_5 - V(L)$  and  $M' = M \setminus L$ . Then  $G'_5$  is a bipartite graph with bipartition  $U' \cup V'$ , where  $U' = U \cap V(G'_5)$  and  $V' = V \cap V(G'_5)$ . Next we will prove that  $\delta(G'_5) \geq k+3$ . Since  $d_{G_5}(u_i) = d_{G_5}(u_{i+1}) = k+i+2$ ,  $d_{G_5}(v_{n-k-1}) = d_{G_5}(v_{n-k}) = k+3$  and other vertices have same degree as in  $H_{n,k}$ . Combining that  $|V \setminus V'| = i-1$  we have

$$d_{G'_5}(u_j) \geq d_{G_5}(u_i) - (i-1) = k+i+2 - (i-1) \geq k+3 \text{ for } j = i, i+1, \dots, n \text{ and}$$

$$d_{G'_5}(v_j) = d_{G_5}(v_j) \geq d_{G_5}(v_{n-k}) = k+3 \text{ for } j = i, i+1, \dots, n-k.$$

For  $1 \leq j \leq i-1$  and  $v_j \in V'$ , we have

$$d_{G'_5}(v_j) = d_{G_5}(v_j) - (i-j) = (n-j+1) - (i-j) = n-i+1 \geq k+3$$

and other vertices of  $G'_5$  have degree  $n - (i-1) \geq k+3$ . Thus  $\delta(G'_5) \geq k+3$ .

By Theorem 2.6,  $f(G'_5, M') \geq f(G'_5) \geq k+2$ . By definition of forcing sets, we have  $f(G_5, M) \geq f(G'_5, M') \geq k+2$ . By the arbitrariness of  $M$ ,  $f(G_5) \geq k+2$ .  $\square$

### 3 Some upper bounds of the maximum forcing number

Let  $G$  be a graph with a perfect matching. Lei et al. [14] obtained that  $F(G)$  is no more than the maximum anti-forcing number of  $G$ . Hence, we can derive two upper bounds of  $F(G)$  from those of the maximum anti-forcing number.

The anti-forcing number of a graph was introduced by Vukičević and Trinajstić [25] as the smallest number of edges whose removal results in a subgraph with a unique perfect matching. Recently, Lei et al. [14] defined the *anti-forcing number* of a perfect matching  $M$  of  $G$  as the minimal number of edges not in  $M$  whose removal to make  $M$  as a single perfect matching of the resulting graph. The *maximum anti-forcing number* of  $G$ , denoted by  $Af(G)$ , is the maximum value of anti-forcing numbers over all perfect matchings of  $G$ .

For a connected graph  $G$ , the *cyclomatic number* of it is defined as  $r(G) = |E(G)| - |V(G)| + 1$ . Deng and Zhang [7] obtained that  $Af(G) \leq r(G)$ . Afterwards, Shi and Zhang [24] gave a new bound  $Af(G) \leq \frac{2|E(G)| - |V(G)|}{4}$ . By these, we obtain the following result.

**Corollary 3.1.** *Let  $G$  be a connected graph of order  $2n$  and with a perfect matching. Then*

$$F(G) \leq \begin{cases} \frac{e(G)-n}{2}, & \text{if } e(G) \geq 3n - 2; \\ e(G) - 2n + 1, & \text{otherwise.} \end{cases}$$

In this section, we will characterize all graphs  $G$  with  $F(G) = \frac{e(G)-n}{2}$ . But we have not been able to characterize the other yet. Furthermore, we would give a new upper bound on  $F(G)$  and obtain that the new bound is better than Corollary 3.1 for graphs  $G$  with a larger number of edges.

Given  $S, T \subseteq V(G)$ , we write  $E(S, T)$  for the set of edges having one end-vertex in  $S$  and the other in  $T$  and  $e(S, T)$  for the number of edges in  $E(S, T)$ .

**Proposition 3.2.** *Let  $G$  be a graph of order  $2n$  and with a perfect matching. Then  $F(G) \leq \frac{e(G)-n}{2}$ , and equality holds if and only if  $G$  consists of  $\frac{e(G)-n}{2}$  cycles of length 4 and  $2n - e(G) \geq 0$  independent edges.*

*Proof.* It suffices to prove the second part. If  $G$  consists of  $\frac{e(G)-n}{2}$  cycles of length 4 and  $2n - e(G)$  independent edges, then  $G$  is a plane bipartite graph and has exactly  $\frac{e(G)-n}{2}$   $M$ -alternating cycles for any perfect matching  $M$  of  $G$ . By Theorem 1.2,  $f(G, M) = \frac{e(G)-n}{2}$ . So  $F(G) = \frac{e(G)-n}{2}$ .

Conversely, if  $F(G) = \frac{e(G)-n}{2}$ , then there exists a perfect matching  $M$  of  $G$  and a minimum forcing set  $S$  of  $M$  such that  $|S| = f(G, M) = \frac{e(G)-n}{2}$ . By Lemma 1.1, we have  $G[V(M \setminus S)]$  contains no  $M$ -alternating cycles. But  $S \setminus \{e\}$  is not a forcing set of  $M$  for

any edge  $e$  of  $S$  by the minimality of  $S$ . By Lemma 1.1,  $G[V((M \setminus S) \cup \{e\})]$  contains an  $M$ -alternating cycle  $C_e$ . So  $e$  is contained in  $C_e$  and

$$e(V(M \setminus S), V(S)) = \sum_{e \in S} e(V(M \setminus S), V(e)) \geq 2|S|. \quad (3.1)$$

Since  $e(G) \geq e(V(M \setminus S), V(S)) + |M| \geq 2|S| + n = e(G)$ , we obtain that all equalities hold. Thus  $e(V(M \setminus S), V(S)) = 2|S|$ , and both  $G[V(S)]$  and  $G[V(M \setminus S)]$  consist of independent edges. By equality (3.1),  $e(V(M \setminus S), V(e)) = 2$  for each edge  $e$  of  $S$ . So  $C_e$  is an  $M$ -alternating 4-cycle.

Moreover,  $C_{e_1} \cap C_{e_2} = \emptyset$  for any pair of distinct edges  $e_1$  and  $e_2$  of  $S$ . Otherwise, there exist two edges  $e_1$  and  $e_2$  of  $S$  so that  $e' \in E(C_{e_1}) \cap E(C_{e_2})$  for some edge  $e'$  of  $M \setminus S$ . Then  $E(V(M \setminus (S \cup \{e'\})), V(\{e_1, e_2\})) = \emptyset$ . Thus  $(S \setminus \{e_1, e_2\}) \cup \{e'\}$  is a forcing set of  $M$  with size less than  $S$ , a contradiction. Hence  $|M \setminus S| \geq |S|$ , which implies  $e(G) \leq 2n$ . Therefore,  $G$  consists of  $\frac{e(G)-n}{2}$  cycles of length 4 and  $2n - e(G)$  independent edges.  $\square$

Next we will give a new upper bound of  $F(G)$  and we need a lemma as follows.

**Lemma 3.3.** *Let  $G$  be a graph of order  $2n$  and with  $f(G, M) = k$  for  $0 \leq k \leq n - 1$ . Then there exists an edge  $uv \in M$  such that  $d_G(u) + d_G(v) \geq \frac{2n}{n-k}$ . If equality holds, then  $(n - k) \mid n$ .*

*Proof.* Let  $S$ ,  $e$  and  $C_e$  be defined as in the proof of necessity of Proposition 3.2. Then  $e$  is contained in an  $M$ -alternating cycle  $C_e$  and (3.1) holds.

Let  $d_G(u) + d_G(v) = \max\{d_G(x) + d_G(y) \mid xy \in M \setminus S\}$ . Then

$$(n - k)[d_G(u) + d_G(v)] \geq \sum_{xy \in M \setminus S} [d_G(x) + d_G(y)] \quad (3.2)$$

$$\begin{aligned} &= 2e(G[V(M \setminus S)]) + e(V(M \setminus S), V(S)) \\ &\geq 2(n - k) + 2k \\ &= 2n. \end{aligned} \quad (3.3)$$

So we obtain the required result.

If  $d_G(u) + d_G(v) = \frac{2n}{n-k}$ , then all equalities in (3.1)-(3.3) hold. So  $e(V(M \setminus S), V(e)) = 2$  for each edge  $e$  of  $S$ ,  $d_G(u) + d_G(v) = d_G(x) + d_G(y)$  for each edge  $xy$  of  $M \setminus S$ , and  $G[V(M \setminus S)]$  consists of  $n - k$  independent edges. Thus  $C_e$  is a cycle of length 4, and  $d_G(x) = d_G(y)$  for each edge  $xy$  of  $M \setminus S$ . So  $2n = (n - k)[d_G(u) + d_G(v)] = 2(n - k)d_G(u)$  and  $(n - k) \mid n$ .  $\square$

**Theorem 3.4.** *Let  $G$  be a graph of order  $2n$  and with  $F(G) = k$  for  $0 \leq k \leq n - 1$ . Then*

$$e(G) \geq \frac{n(n+1)}{n-k} - k - 1. \quad (3.4)$$

*Proof.* We proceed by induction on  $n$ . For  $n = 1$ , we have  $F(G) = k = 0$  and  $e(G) = 1$ . So (3.4) holds. Suppose that  $n \geq 2$ . If  $k = 0$ , then  $G$  has a unique perfect matching and  $e(G) \geq n$ . Next we suppose that  $1 \leq k \leq n - 1$ .

Since  $F(G) = k$ , there exists a perfect matching  $M$  of  $G$  such that  $f(G, M) = k$ . By Lemma 3.3, there exists an edge  $uv \in M$  such that  $d_G(u) + d_G(v) \geq \frac{2n}{n-k}$ . Let  $G' = G - \{u, v\}$ . Then  $F(G') \geq k - 1$ . Suppose to the contrary that  $F(G') \leq k - 2$ . Then  $M' = M \setminus \{uv\}$  is a perfect matching of  $G'$  and  $f(G', M') \leq k - 2$ . Let  $S'$  be a minimum forcing set of  $M'$ . Then  $|S'| = f(G', M')$ . By Lemma 1.1,  $G' - V(S')$  has a unique perfect matching. Combining that  $G - V(S' \cup \{uv\}) = G - \{u, v\} - V(S') = G' - V(S')$ , we obtain that  $S' \cup \{uv\}$  is a forcing set of  $M$ . So  $f(G, M) \leq |S' \cup \{uv\}| \leq k - 1$ , which is a contradiction. Therefore,  $k - 1 \leq F(G') \leq k$ .

If  $F(G') = k - 1 \leq n - 2$ , then  $e(G') \geq \frac{(n-1)n}{n-1-(k-1)} - (k-1) - 1 = \frac{(n-1)n}{n-k} - k$  by the induction hypothesis. By Lemma 3.3, we get that

$$e(G) = e(G') + d_G(u) + d_G(v) - 1 \geq \frac{(n-1)n}{n-k} - k + \frac{2n}{n-k} - 1 = \frac{n^2 + n}{n-k} - k - 1.$$

Otherwise, we have  $F(G') = k$ . Since  $G'$  has  $2(n-1)$  vertices, we have  $F(G') \leq n - 2$ . By the induction hypothesis,  $e(G') \geq \frac{(n-1)n}{n-1-k} - k - 1$ . By Lemma 3.3 and  $1 \leq k \leq n - 1 < 2n - 1$ ,

$$\begin{aligned} e(G) - \left(\frac{n^2 + n}{n-k} - k - 1\right) &= [e(G') + d_G(u) + d_G(v) - 1] - \left(\frac{n^2 + n}{n-k} - k - 1\right) \\ &\geq \left[\frac{(n-1)n}{n-1-k} - k - 1 + \frac{2n}{n-k} - 1\right] - \left(\frac{n^2 + n}{n-k} - k - 1\right) \\ &= \frac{(n-1)n}{n-1-k} + \frac{k - n^2}{n-k} \\ &= \frac{k(2n - 1 - k)}{(n-1-k)(n-k)} > 0. \end{aligned}$$

Hence (3.4) holds and we complete the proof.  $\square$

By inverting (3.4), we obtain an upper bound of  $F(G)$ .

**Corollary 3.5.** *Let  $G$  be a graph of order  $2n$  and with a perfect matching. Then*

$$F(G) \leq \frac{\sqrt{e^2(G) + 2(n+1)e(G) - 3n^2 - 2n + 1}}{2} - \frac{e(G) + 1 - n}{2}. \quad (3.5)$$

*Proof.* Let  $F(G) = k$ . Then  $0 \leq k \leq n - 1$ . By Theorem 3.4,  $e(G) \geq \frac{n^2+n}{n-k} - k - 1$ . That is,  $k^2 + k(e(G) + 1 - n) - ne(G) + n^2 \leq 0$ . By solving the quadratic inequality of  $k$ , we obtain that (3.5) holds.  $\square$

Note that  $nK_2$  and  $K_{n,n}$  are two graphs such that equalities in (3.4) and (3.5) hold. So the bounds in Theorem 3.4 and Corollary 3.5 are tight.

**Remark 3.6.** By a simple calculation, we obtain that the upper bound in Corollary 3.5 is less than  $\frac{e(G)-n}{2}$  when  $e(G) > \frac{7n-2}{3}$  and less than  $r(G)$  when  $e(G) > 2n - 1 + \frac{\sqrt{2n^2-2n}}{2}$ . Hence for connected graphs  $G$  of order  $2n$  ( $n \geq 2$ ), the upper bound in Corollary 3.5 is less than that of Corollary 3.1 when  $e(G) > 2n - 1 + \frac{\sqrt{2n^2-2n}}{2}$ .

## 4 Characterization of bipartite graphs $G$ of order $2n$ and with $f(G) = n - 2$

Che and Chen [5] asked a question: how to characterize the graphs  $G$  of order  $2n$  and with  $f(G) = n - 1$ . For bipartite graphs, they obtained the following result.

**Theorem 4.1.** [6] *Let  $G$  be a bipartite graph of order  $2n$ . Then  $f(G) = n - 1$  if and only if  $G$  is complete bipartite graph  $K_{n,n}$ .*

The present authors have obtained the following result for general graphs.

**Theorem 4.2.** [16] *Let  $G$  be a graph of order  $2n$ . Then  $f(G) = n - 1$  if and only if  $G$  is a complete multipartite graph with each partite set having size no more than  $n$  or  $G$  is a graph obtained by adding arbitrary additional edges in the same partite set to  $K_{n,n}$ .*

In this section, we will determine all bipartite graphs  $G$  of order  $2n$  and with  $f(G) = n - 2$  for  $n \geq 2$ . For an edge subset  $S$  of  $G$ , we write  $G - S$  for the subgraph of  $G$  obtained by deleting the edges in  $S$ . Let  $F_0$  be a bipartite graph which contains exactly one edge and each partite set has exactly two vertices. A bipartite graph  $G$  is  $F_0$ -free (resp.  $P_4$ -free) if it contains no induced subgraph isomorphic to  $F_0$  (resp.  $P_4$ ), where the two partite sets of the induced subgraph have the same sizes.

**Lemma 4.3.** *Let  $G = (U, V)$  be a bipartite graph. Then  $G$  is  $F_0$ -free if and only if  $G$  can be obtained from  $K_{|U|,|V|}$  by deleting all edges of some disjoint complete bipartite subgraphs.*

*Proof.* Sufficiency. For a pair of vertices  $u \in U$  and  $v \in V$ , we have  $uv \notin E(G)$  if and only if  $u$  and  $v$  lie in the same complete bipartite subgraph deleted edges of  $K_{|U|,|V|}$ . Suppose to the contrary that  $G$  contains an induced subgraph  $H$  isomorphic to  $F_0$ . Without loss of generality, we may suppose that  $V(H) = \{u_1, u_2, v_1, v_2\}$  and  $u_1v_1$  is the edge of  $H$ . Then these three pairs of vertices  $\{u_1, v_2\}$ ,  $\{u_2, v_1\}$  and  $\{u_2, v_2\}$  are in the same complete bipartite subgraphs deleted edges of  $K_{|U|,|V|}$ , respectively. Hence the four vertices  $u_1, u_2, v_1$  and  $v_2$  lie in the same complete bipartite subgraph deleted edges of  $K_{|U|,|V|}$ , which contradicts that  $u_1v_1$  is an edge of  $G$ .

Necessity. Let  $G' = K_{|U|,|V|} - E(G)$ . Then  $G$  and  $G'$  are bipartite spanning subgraphs of  $K_{|U|,|V|}$ . It is obvious that  $G$  is  $F_0$ -free if and only if  $G'$  is  $P_4$ -free. It suffices to prove

that every component of  $G'$  with at least two vertices is a complete bipartite graph, and let  $C'$  be such a component with bipartition  $\{u_1, u_2, \dots, u_i\} \cup \{v_1, v_2, \dots, v_j\}$ .

We will proceed by induction on  $|V(C')|$ . If  $i = 1$  or  $j = 1$ , then we have done. So let  $i \geq 2$  and  $j \geq 2$ . Then there exists a vertex  $x$  of  $C'$  such that  $C' - x$  is connected. This is verified by choosing  $x$  as an end-vertex of a longest path of  $C'$ . Without loss of generality, we may assume that  $x = u_i$ . Since  $C' - u_i$  is  $P_4$ -free,  $C' - u_i$  is isomorphic to  $K_{i-1, j}$  by the induction hypothesis. Since  $C'$  is connected, there exists  $1 \leq k \leq j$  such that  $u_i v_k \in E(C')$ . Since  $\{u_1 v_k, u_1 v_l\} \subseteq E(C')$  for any  $1 \leq l \leq j$  and  $l \neq k$  and  $C'$  is  $P_4$ -free, we obtain that  $u_i v_l \in E(C')$ . So  $C'$  is a complete bipartite graph  $K_{i, j}$ .  $\square$

If  $G$  is a graph obtained from  $K_{n, n}$  by deleting all edges of some disjoint complete bipartite subgraphs, then we call these disjoint complete bipartite subgraphs *deleted subgraphs of  $G$* . Naturally, we assume that each deleted subgraph contains at least one vertex of each partite set of  $K_{n, n}$ . Also, we say that a graph is obtained from  $K_{n, n}$  by such operations, we mean that the graph is not  $K_{n, n}$ .

The independence number of  $G$  is denoted by  $\alpha(G)$ . An equivalent condition of bipartite graphs with a perfect matching is given below. (see Exercise 3.1.40 in [27]).

**Lemma 4.4.** [27] *Let  $G$  be a bipartite graph of order  $2n$ . Then  $\alpha(G) = n$  if and only if  $G$  has a perfect matching.*

An edge  $e$  of  $G$  is *allowed* if it lies in some perfect matching of  $G$  and *forbidden* otherwise. A graph is said to be *elementary* if its allowed edges form a connected subgraph. Hetyei obtained the following result (see Theorem 1 in [18]).

**Lemma 4.5.** [18] *A bipartite graph is elementary if and only if it is connected and every edge is allowed.*

Let  $\mathcal{G}_1$  be the set of all graphs obtained from  $K_{n, n}$  by deleting all edges of some disjoint complete bipartite subgraphs and the orders of its deleted subgraphs are no more than  $n$ , and  $\mathcal{G}_2$  be the set of all bipartite graphs of order  $2n$  consisting of two complete bipartite graphs with perfect matchings and some forbidden edges between them (see Fig. 3).

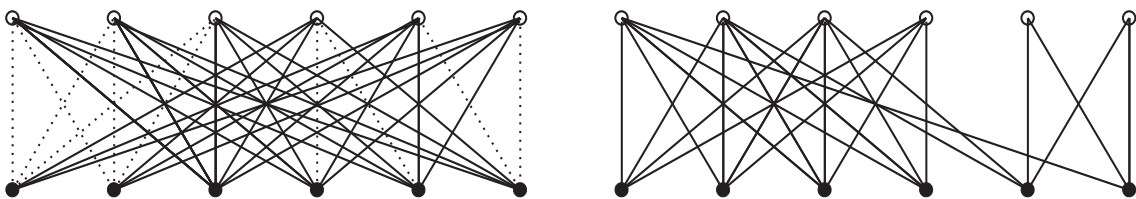


Fig. 3. A graph in  $\mathcal{G}_1$  and a graph in  $\mathcal{G}_2$ .

**Theorem 4.6.** *Let  $G$  be a bipartite graph of order  $2n$  for  $n \geq 2$ . Then  $f(G) = n - 2$  if and only if  $G$  is a graph in  $\mathcal{G}_1$  or  $\mathcal{G}_2$ .*

*Proof.* Sufficiency. First we prove that a graph  $G \in \mathcal{G}_1$  has a perfect matching. Since the orders of deleted subgraphs of  $G$  are no more than  $n$ , we have  $\alpha(G) = n$ . By Lemma 4.4,  $G$  has a perfect matching. Let  $G$  be a graph in  $\mathcal{G}_1$  or  $\mathcal{G}_2$ . By Theorem 4.1,  $f(G) \leq n - 2$  since  $G$  is not  $K_{n,n}$ . Next we will prove that  $f(G) \geq n - 2$ .

Suppose that  $G$  is a graph in  $\mathcal{G}_1$ . For  $n = 2$ , we have  $f(G) = 0$  and the theorem holds. Let  $n \geq 3$ . Suppose to the contrary that  $f(G) \leq n - 3$ . Then there exists a perfect matching  $M$  of  $G$  and a minimum forcing set  $S$  of  $M$  such that  $|S| = f(G, M) = f(G)$ . By Lemma 1.1,  $G - V(S)$  has a unique perfect matching. So there are three distinct edges  $\{e_1, e_2, e_3\} \subseteq M \setminus S$  such that  $G[V(\{e_1, e_2, e_3\})]$  has a unique perfect matching. Set  $e_i = u_i v_i$  for  $1 \leq i \leq 3$ . By Theorem 1.4,  $G[V(\{e_1, e_2, e_3\})]$  contains two pendant vertices and we may assume such two vertices are  $u_1$  and  $v_3$ . Then  $G[\{u_1, v_2, u_2, v_3\}]$  is isomorphic to  $F_0$ , which contradicts Lemma 4.3.

Suppose that  $G$  is a graph in  $\mathcal{G}_2$ . We denote by  $G_1$  and  $G_2$  the two complete bipartite subgraphs of  $G$  with perfect matchings. Then  $M \cap E(G_i)$  is a perfect matching of  $G_i$  for any perfect matching  $M$  of  $G$  where  $i \in \{1, 2\}$ . For any subset  $S$  of  $M$  such that  $|S| \leq n - 3$ ,  $G - V(S)$  contains three edges of  $M$  and two of them lie in some complete bipartite subgraph, say  $G_1$ . Then  $G - V(S)$  contains an  $M$ -alternating cycle in  $G_1$ . By Lemma 1.1,  $S$  is not a forcing set of  $M$ . Thus,  $f(G, M) \geq n - 2$ . By the arbitrariness of  $M$ , we have  $f(G) \geq n - 2$ .

Necessity. Since  $f(G) = n - 2$ ,  $G$  has a perfect matching and each partite set has  $n$  vertices. By Theorem 4.1,  $G$  is not  $K_{n,n}$ . If  $G$  is  $F_0$ -free, then  $G$  is a graph obtained from  $K_{n,n}$  by deleting all edges of some disjoint complete bipartite subgraphs by Lemma 4.3. Since  $G$  has a perfect matching, the orders of its deleted subgraphs are no more than  $n$ . So  $G$  is a graph in  $\mathcal{G}_1$ . If  $G$  is not  $F_0$ -free, then  $G$  contains an induced subgraph  $H$  isomorphic to  $F_0$  and  $n \geq 3$ . We claim that the edge  $e$  of  $H$  is a forbidden edge in  $G$ . Otherwise, there exists a perfect matching  $M$  of  $G$  containing  $e$ . Let  $\{e, e', e''\}$  be the three distinct edges of  $M$  incident with the vertices of  $H$ . Then  $G[V(\{e, e', e''\})]$  contains no  $M$ -alternating cycles. By Lemma 1.1,  $M \setminus \{e, e', e''\}$  is a forcing set of  $M$ . So  $f(G) \leq f(G, M) \leq n - 3$ , which is a contradiction. So the claim holds, and  $G$  is not elementary by Lemma 4.5.

The subgraph of  $G$  consisting of all allowed edges in  $G$  and their end-vertices has components, say,  $L_1, L_2, \dots, L_k$  where  $k \geq 2$ . Then two end-vertices of any forbidden edge of  $G$  lie in different components. If not, there exists a forbidden edge  $e$  of  $G$  whose two end-vertices belong to some component  $L_i$ . Let  $L'_i$  be a graph obtained from  $L_i$  by

adding the edge  $e$ . Then  $e$  is also a forbidden edge of  $L'_i$ , which contradicts Lemma 4.5. Hence all edges between distinct components are precisely forbidden edges of  $G$ . Thus,

$$n - 2 = f(G) = \sum_{i=1}^k f(L_i) \leq \sum_{i=1}^k \left( \frac{|V(L_i)|}{2} - 1 \right) = \frac{1}{2} \sum_{i=1}^k |V(L_i)| - k = n - k, \quad (4.1)$$

which implies that  $k \leq 2$ . So  $k = 2$  and all equalities in (4.1) hold. So  $f(L_i) = \frac{|V(L_i)|}{2} - 1$  for  $i = 1$  and  $2$ . By Theorem 4.1,  $L_1$  and  $L_2$  are two complete bipartite graphs. Hence  $G$  is a graph in  $\mathcal{G}_2$ .  $\square$

Let  $G$  be a bipartite graph of order  $2n$  and with  $F(G) = n - 1$ . By Theorem 3.4,  $e(G) = n^2$  and  $G$  is  $K_{n,n}$ . Combining Theorem 4.1, we obtain the following result.

**Remark 4.7.** Let  $G$  be a bipartite graph of order  $2n$  for  $n \geq 2$ . Then  $f(G) = n - 2$  if and only if each perfect matching of  $G$  has the forcing number  $n - 2$ .

**Remark 4.8.** Let  $G$  be a graph in  $\mathcal{G}_1$  or  $\mathcal{G}_2$ . Then  $G$  is disconnected if and only if  $G$  is the disjoint union of two complete bipartite graphs with perfect matchings, i.e., there are exactly two deleted subgraphs and their orders are  $n$ .

It suffices to prove the necessity. Since  $G$  is disconnected,  $G$  has at least two components, say  $L_1, L_2, \dots, L_k$  where  $k \geq 2$ . By Theorem 4.6,  $f(G) = n - 2$  and all equalities in (4.1) hold. By the same arguments as the proof of Theorem 4.6, we obtain that  $k = 2$ , and  $L_1$  and  $L_2$  are two complete bipartite graphs.

Next we will determine all elementary bipartite graphs in  $\mathcal{G}_2$ .

**Proposition 4.9.** *Let  $G$  be a graph in  $\mathcal{G}_1$ . Then  $G$  is elementary if and only if each deleted subgraph of  $G$  has order less than  $n$ .*

*Proof.* Sufficiency. By Remark 4.8,  $G$  is connected. For an edge  $e$  of  $G$ , let  $G' = G - V(e)$ . Then  $G'$  is a graph obtained from  $K_{n-1, n-1}$  by deleting all edges of some disjoint complete bipartite subgraphs and the orders of its deleted subgraphs (if exists) are no more than  $n - 1$ . So  $\alpha(G') = n - 1$ . By Lemma 4.4,  $G'$  has a perfect matching  $M'$ , and  $M' \cup \{e\}$  is a perfect matching of  $G$ . Hence  $e$  is allowed. By Lemma 4.5,  $G$  is elementary.

Necessity. Since  $G$  is elementary, it has a perfect matching. So each deleted subgraph of  $G$  has order no more than  $n$ . Suppose to the contrary that  $K_{i, n-i}$  is a deleted subgraph of  $G$  with  $1 \leq i \leq n - 1$ . Since  $G$  is connected, the remaining  $n$  vertices can not form another deleted subgraph of  $G$  by Remark 4.8. So the orders of other deleted subgraphs of  $G$  (if exists) are no more than  $n - 1$ . Hence  $G - V(K_{i, n-i})$  contains at least one edge, say  $e$ . Since  $V(K_{i, n-i})$  forms an independent set of  $G$  with cardinality  $n$ ,  $e$  is not allowed, which contradicts Lemma 4.5.  $\square$



## 5 Problems and conjectures

Let  $G$  be a graph of order  $2n$  and with a perfect matching. By Theorem 3.4, we obtain that  $e(G) \geq \frac{n(n+1)}{n-F(G)} - F(G) - 1$ . But plenty of examples imply that this bound is not good enough. Since

$$\frac{n^2}{n-F(G)} - \left[ \frac{n(n+1)}{n-F(G)} - F(G) - 1 \right] = \frac{F(G)[n-1-F(G)]}{n-F(G)} \geq 0,$$

and equality holds if and only if  $F(G) = 0$  or  $n - 1$ . So we give a conjecture as follows.

**Conjecture 5.1.** *Let  $G$  be a graph of order  $2n$  and with a perfect matching. Then  $e(G) \geq \frac{n^2}{n-F(G)}$ . Equivalently,  $F(G) \leq \frac{ne(G)-n^2}{e(G)}$ .*

There are some examples showing that Conjecture 5.1 holds.

**Proposition 5.2.** *For  $F(G) \leq \frac{n}{2}$ , Conjecture 5.1 holds.*

*Proof.* Since  $F(G) \leq \frac{n}{2}$ , we have  $\frac{F(G)^2}{n-F(G)} \leq F(G)$ . So  $\lceil \frac{F(G)^2}{n-F(G)} \rceil \leq F(G)$ . By Proposition 3.2, we have  $F(G) \leq \frac{e(G)-n}{2}$ . So  $e(G) \geq n+2F(G) \geq n+F(G) + \lceil \frac{F(G)^2}{n-F(G)} \rceil = \lceil \frac{n^2}{n-F(G)} \rceil$ .  $\square$

**Proposition 5.3.** *Let  $G$  be a graph of order  $2n$ . If  $F(G) = n-1$  or  $n-2$ , then Conjecture 5.1 holds.*

*Proof.* For  $F(G) = n-1$ , two bounds in Conjecture 5.1 and Theorem 3.4 are equal. So Conjecture 5.1 holds.

For  $F(G) = n-2$ , we will proceed by induction on  $n$ . For  $n = 2$ , we have  $F(G) = 0$ . So  $G$  has a unique perfect matching and  $e(G) \geq 2$ . Suppose that  $n \geq 3$ . Since  $F(G) = n-2$ , there exists a perfect matching  $M$  of  $G$  such that  $f(G, M) = n-2$ . By Lemma 3.3, there exists an edge  $uv \in M$  such that  $d_G(u) + d_G(v) \geq n$ . Let  $G' = G - \{u, v\}$ . Then  $n-3 \leq F(G') \leq n-2$ .

If  $F(G') = n-2 = (n-1) - 1$ , then  $e(G') \geq (n-1)^2$  by Theorem 3.4. So

$$e(G) = e(G') + d_G(u) + d_G(v) - 1 \geq (n-1)^2 + n - 1 \geq \frac{n^2}{2}.$$

Otherwise, we obtain that  $F(G') = n-3 = (n-1) - 2$ . By the induction hypothesis,  $e(G') \geq \lceil \frac{(n-1)^2}{n-1-(n-3)} \rceil = \lceil \frac{(n-1)^2}{2} \rceil$ . By Lemma 3.3, we obtain that  $d_G(u) + d_G(v) \geq n+1$  when  $n$  is odd and  $d_G(u) + d_G(v) \geq n$  when  $n$  is even. Hence we have

$$e(G) = e(G') + d_G(u) + d_G(v) - 1 \geq \begin{cases} \lceil \frac{(n-1)^2}{2} \rceil + (n+1) - 1 \geq \frac{n^2+1}{2}, & \text{if } n \text{ is odd;} \\ \lceil \frac{(n-1)^2}{2} \rceil + n - 1 = \frac{n^2}{2}, & \text{otherwise.} \end{cases}$$

Here we complete the proof.  $\square$

In Theorem 4.6, we have completely characterized all bipartite graphs  $G$  of order  $2n$  and with  $f(G) = n - 2$ . Here we propose the following problem.

**Problem 5.4.** *Determine all non-bipartite graphs  $G$  of order  $2n$  and with  $f(G) = n - 2$ .*

For general 2-connected plane bipartite graphs, Abeledo and Atkinson [1] obtained that the resonant number can be computed in polynomial time. Hence the maximum forcing numbers of hexagonal systems [28], polyomino graphs [30] and BN-fullerene graphs [23] can be computed in polynomial time.

Afshani [3] proposed a problem which has not been solved yet.

**Problem 5.5.** [3] *What is the computational complexity of the maximum forcing numbers of graphs ?*

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