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Research paper

Numerical integration of third-order singular boundary-value problems of Emden–Fowler type using hybrid block techniques

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ABSTRACT

In this article, a pair of hybrid block techniques is constructed and successfully applied to integrate Emden–Fowler third-order singular boundary problems. One of the proposed one-step hybrid block techniques is obtained by considering two intermediate points. The obtained method is then paired with a hybrid block strategy of order three to bypass the singularity at the left endpoint of the integration interval. Some third-order Emden–Fowler type problems are solved numerically to show the effectiveness of the proposed technique. The numerical results are compared with other recent numerical approaches in the literature, and the numerical simulations confirm the superiority and robust performance of the proposed strategy.

1. Introduction

In this study, we provide numerical solutions to the following third-order Emden–Fowler type equations given in [1,2]

$$u'''(x) = k(x, u(x)) - g(x)u'(x) - \frac{\lambda}{x} u''(x) = f(x, u(x), u'(x), u''(x)), \quad x_0 = 0 \le x \le x_N, \tag{1}$$

together with any of the set of boundary conditions

$$u(x_0) = u_0, \ u'(x_0) = u'_0, \ u(x_N) = u_N, \tag{2}$$

$$u(x_0) = u_0, \ u'(x_0) = u'_0, \ u'(x_N) = u'_N, \tag{3}$$

$$u(x_0) = u_0, \ u(x_N) = u_N, \ u'(x_N) = u'_N, \tag{4}$$

where $\lambda \ge 1$, u_0, u'_0, u_N, u'_N are known real-values, and $g(x)$, $k(x, u(x))$ are continuous real functions.

Because of several practical applications of the Emden–Fowler type equations in real-life modeling problems in various disciplines of applied sciences and engineering, the quest to solve the class of (1) analytically or numerically has been of great concern to researchers in this field.

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Owing to the singularity of the third-order singular boundary value problems (TSBVP) (1) at $x = 0$ and its nonlinear features, it becomes one of the complex problems to solve theoretically. Hence, numerical methods have played a vital role in providing meaningful and reasonable approximate solutions.

There are many existing numerical methods for integrating the Lane–Emden–Fowler type equation given in (1).

One of the many numerical techniques to provide solutions to (1) is the series solutions using the Adomian decomposition method (ADM). The ADM or the modified Adomian decomposition method (MADM) are semi-analytical techniques dependent on a decomposition approach to form the approximate solutions with proper and suitable initial data for nonlinear systems. It is worth noting that ADM or MADM can give the solution to the Emden–Fowler type equation (1) and related problems in the form of a series using Adomian polynomials.

This strategy has numerous merits. It is effortless to apply and can solve broad classes of complex problems. Besides, it circumvents the cumbersome integrations of the Picard method. An essential advantage of the ADM and its modifications is that they can generate theoretical approximations to nonlinear differential equations without linearization, discretization, or perturbation. The primary demerit of ADM or MADM is that they give only locally convergent results. For further details about ADM and MADM methods, we refer the reader to [3] - [4].

There are other approaches in the accessible literature for solving the TSBVP in (1). Those methods include various types of spline methods, the variational iteration techniques, neuro-swarming-based heuristic approach, spectral methods or asymptotic numerical strategies (see [1,2,5–20]).

We remark that the research of the Emden–Fowler equations arises from theories involving gaseous dynamics in mathematical physics and astrophysics. Additionally, the problem of reactant concentration in a chemical reactor, reaction–diffusion processes inside a porous catalyst, the conduction of heat in the human head, the distribution of oxygen in a spherical shell, and so on, can be modeled using Emden–Fowler equations. For more details about the application of Emden–Fowler equations in nonlinear sciences see [21,22]. Motivated by the various applications of the Emden–Fowler equations in real-world modeling problems in nonlinear physical and applied sciences mentioned above, in the present paper, we apply a pair of one-step hybrid block Nyström type methods (OHBNTM) to give a numerical solution to the nonlinear third-order Lame–Emden–Fowler equation in (1).

2. Development of the OHBNTM method

Here, we present the mathematical formulation of the proposed OHBNTM strategy.

2.1. Main formulas

Consider a partition of the form $x_j = x_0 + jh, j = 0, 1, \dots, N$ with $h = x_{j+1} - x_j$. We assume that the following polynomial on a generic subinterval $[x_n, x_{n+1}]$ can approximate the exact solution of the TSBVP given in (1)

$$u(x) \simeq w(x) = \sum_{j=0}^{6} c_j x^j, \tag{5}$$

being the approximations of the first, second, and third derivatives as follows

$$u'(x) \simeq w'(x) = \sum_{j=1}^{6} c_j j x^{j-1}, \tag{6}$$

$$u''(x) \simeq w''(x) = \sum_{j=2}^{6} c_j j(j-1) x^{j-2}, \tag{7}$$

$$u'''(x) \simeq w'''(x) = \sum_{j=3}^{6} c_j j(j-1)(j-2) x^{j-3}, \tag{8}$$

with the $c_j \in \mathbb{R}$ unknown coefficients that would be obtained imposing some collocation conditions at specified nodes.

We consider the following intermediate nodes: $x_{n+r} = x_n + rh$ and $x_{n+s} = x_n + sh$ with $0 < r < s < 1$, on $[x_n, x_{n+1}]$. Consider the approximations in (5), (6) and (7) evaluated at x_n, and the one in (8) evaluated at $x_n, x_{n+r}, x_{n+s}, x_{n+1}$. We obtain the following system, with unknowns c_n, $n = 0(1)6$,

$$\begin{aligned} &w(x_n) = u_n \,,\; w'(x_n) = u'_n \,,\; w''(x_n) = u''_n \,,\; w'''(x_n) = f_n, \\ &w'''(x_{n+r}) = f_{n+r} \,,\; w'''(x_{n+s}) = f_{n+s} \,,\; w'''(x_{n+1}) = f_{n+1}, \end{aligned}$$

where u_{n+j}, u'_{n+j}, u''_{n+j} and f_{n+j} denote approximations to $u(x_{n+j})$, $u'(x_{n+j})$, $u''(x_{n+j})$ and $u'''(x_{n+j})$, respectively. After getting the values of c_n, $n = 0(1)6$, and using the substitution, $x = x_n + zh$, the function $w(x)$ in (5) may be expressed as

$$w(x_n + zh) = \alpha_0(z) u_n + h\alpha_1(z) u'_n + h^2 \alpha_2(z) u''_n + h^3(\beta_0(z) f_n + \beta_r(z) f_{n+r} + \beta_s(z) f_{n+s} + \beta_1(z) f_{n+1}), \tag{9}$$

where the coefficients $\alpha_0(z) = 1$, $\alpha_1(z)$, $\alpha_2(z)$, $\{\beta_i(z)\}_{i=0,r,s,1}$ depend on r, s.

Evaluating the formula in (9) and its first and second derivatives at $z = 1$ we obtain the approximations of $u(x_{n+1})$, $u'(x_{n+1})$ and $u''(x_{n+1})$, which are given respectively by

$$
\begin{aligned}
u_{n+1} &= u_n + \frac{h}{2}\left(hu''_n + 2u'_n\right) + \frac{h^3\left(21\left(\sqrt{21}+12\right)f_{n+r} - 21\left(\sqrt{21}-12\right)f_{n+s} - 186f_n - 118f_{n+1}\right)}{1200},\\
u'_{n+1} &= u'_n + hu''_n + \frac{h^2}{600}\left(21\left(\sqrt{21}+20\right)f_{n+r} - 21\left(\sqrt{21}-20\right)f_{n+s} - 304f_n - 236f_{n+1}\right),\\
u''_{n+1} &= u''_n + \frac{h}{10}\left(14\left(f_{n+r}+f_{n+s}\right) - 9f_n - 9f_{n+1}\right).
\end{aligned}
\tag{10}
$$

Now, evaluating $w(x)$, $w'(x)$ and $w''(x)$ at x_{n+r}, x_{n+s}, we get the following hybrid Nyström-type formulas

$$
\begin{aligned}
u_{n+r} &= u_n - \frac{1}{168}h\left(\left(8\sqrt{21}-37\right)hu''_n + 4\left(4\sqrt{21}-21\right)u'_n\right) + \frac{\left(28788\sqrt{21}-131857\right)h^3f_n}{2116800}\\
&\quad + \frac{h^3\left(9\left(92512-20183\sqrt{21}\right)f_{n+r} + \left(2018016-440369\sqrt{21}\right)f_{n+s} + 8\left(17364\sqrt{21}-79571\right)f_{n+1}\right)}{16934400},\\
u_{n+s} &= u_n + \frac{h}{168}\left(\left(8\sqrt{21}+37\right)hu''_n + 4\left(4\sqrt{21}+21\right)u'_n\right) - \frac{\left(28788\sqrt{21}+131857\right)h^3f_n}{2116800}\\
&\quad + \frac{h^3\left(\left(440369\sqrt{21}+2018016\right)f_{n+r} + 9\left(20183\sqrt{21}+92512\right)f_{n+s} - 8\left(17364\sqrt{21}+79571\right)f_{n+1}\right)}{16934400}.
\end{aligned}
\tag{11}
$$

$$
\begin{aligned}
u'_{n+r} &= u'_n + \frac{h}{42}\left(21-4\sqrt{21}\right)u''_n + \frac{\left(7436\sqrt{21}-33879\right)h^2f_n}{151200}\\
&\quad + \frac{h^2\left(6\left(5696-1239\sqrt{21}\right)f_{n+r} + 6\left(9719-2121\sqrt{21}\right)f_{n+s} + \left(5524\sqrt{21}-25311\right)f_{n+1}\right)}{151200},\\
u'_{n+s} &= u'_n + \frac{h}{42}\left(21+4\sqrt{21}\right)u''_n - \frac{\left(7436\sqrt{21}+33879\right)h^2f_n}{151200}\\
&\quad + \frac{h^2\left(6\left(2121\sqrt{21}+9719\right)f_{n+r} + 6\left(1239\sqrt{21}+5696\right)f_{n+s} - \left(5524\sqrt{21}+25311\right)f_{n+1}\right)}{151200}.
\end{aligned}
\tag{12}
$$

$$
\begin{aligned}
u''_{n+r} &= u''_n + \frac{\left(512\sqrt{21}-2193\right)hf_n}{5040}\\
&\quad + \frac{h\left(7\left(2016-419\sqrt{21}\right)f_{n+r} + \left(14112-3083\sqrt{21}\right)f_{n+s} + 4\left(512\sqrt{21}-2343\right)f_{n+1}\right)}{20160},\\
u''_{n+s} &= u''_n - \frac{\left(512\sqrt{21}+2193\right)hf_n}{5040}\\
&\quad + \frac{h\left(\left(3083\sqrt{21}+14112\right)f_{n+r} + 7\left(419\sqrt{21}+2016\right)f_{n+s} - 4\left(512\sqrt{21}+2343\right)f_{n+1}\right)}{20160}.
\end{aligned}
\tag{13}
$$

2.2. Strategy to avoid the singularity

The above main formulas cannot be used directly for solving (1) because it is not possible to get $f_0 = f(x_0, u_0, u'_0, u''_0)$, since there is a singularity at $x_0 = 0$. We have obtained a similar approach as in 2.1 to overcome this shortcoming, explicitly designed for the subinterval $[x_0, x_1]$, where the value f_0 is not reflected. The following formulas are obtained

$$
\begin{aligned}
u_1 &= u_0 + hu'_0 + \frac{h^2}{2}u''_0 + \frac{h^3}{240}\left(63f_{\bar{r}} - 36f_{\bar{s}} + 13f_1\right),\\
u'_1 &= u'_0 + hu''_0 + \frac{1}{8}h^2\left(5f_{\bar{r}} - 2f_{\bar{s}} + f_1\right),\\
u''_1 &= u''_0 + \frac{h}{4}\left(3f_{\bar{r}} + f_1\right).
\end{aligned}
\tag{14}
$$

and the remaining formulas are given by

$$
\begin{aligned}
u_{\bar{r}} &= u_0 + \frac{h}{3}u_0' + \frac{h^2}{18}u_0'' + \frac{h^3\left(97f_{\bar{r}} - 84f_{\bar{s}} + 27f_1\right)}{6480}, \\
u_{\bar{s}} &= \frac{2}{9}h\left(hu_0'' + 3u_0'\right) + \frac{1}{405}h^3\left(39f_{\bar{r}} - 28f_{\bar{s}} + 9f_1\right).
\end{aligned} \tag{15}
$$

$$
\begin{aligned}
u_{\bar{r}}' &= u_0' + \frac{h}{3}u_0'' + \frac{1}{216}h^2\left(27f_{\bar{r}} - 22f_{\bar{s}} + 7f_1\right), \\
u_{\bar{s}}' &= u_0' + \frac{2h}{3}u_0'' + \frac{2}{27}h^2\left(5f_{\bar{r}} - 3f_{\bar{s}} + f_1\right).
\end{aligned} \tag{16}
$$

$$
\begin{aligned}
u_{\bar{r}}'' &= u_0'' + \frac{1}{36}h\left(23f_{\bar{r}} - 16f_{\bar{s}} + 5f_1\right), \\
u_{\bar{s}}'' &= u_0'' + \frac{1}{9}h\left(7f_{\bar{r}} - 2f_{\bar{s}} + f_1\right).
\end{aligned} \tag{17}
$$

We take a small step-size h, $r = \frac{1}{2} - \frac{2}{\sqrt{21}}$, $s = \frac{1}{2} + \frac{2}{\sqrt{21}}$, $\bar{r} = \frac{1}{3}, \bar{s} = \frac{2}{3}$ and using all the formulas in (10)–(13) for $n = 1, 2, \ldots, N-1$, along with the ones derived in (14)–(17) for the initial step, we obtain a global method that can give good approximations to problem (1) on the integration interval $[0, x_N]$.

3. Analysis of the proposed OHBNTM

The main characteristics of the proposed technique OHBNTM are analyzed here.

3.1. Consistency and order of the formulas

The formulas in (10)–(13) may be written as

$$
\bar{A}_1 V_n = h\bar{A}_2 V_n' + h^2\bar{A}_3 V_n'' + h^3\bar{A}_4 F_n, \tag{18}
$$

with $\bar{A}_1, \bar{A}_2, \bar{A}_3, \bar{A}_4$ constant matrices containing the coefficients of the formulas (10)–(13), and

$$
\begin{aligned}
V_n &= \left(u_n, u_{n+r}, u_{n+s}, u_{n+1}\right)^T, \\
V_n' &= \left(u_n', u_{n+r}', u_{n+s}', u_{n+1}'\right)^T, \\
V_n'' &= \left(u_n'', u_{n+r}'', u_{n+s}'', u_{n+1}'\right)^T, \\
F_n &= \left(f_n, f_{n+r}, f_{n+s}, f_{n+1}\right)^T.
\end{aligned}
$$

Assuming that $u(x)$ is sufficiently differentiable, we define the operator $\mathcal{L}$ associated to the formulas in (10)–(13):

$$
\mathcal{L}(u(x_{n+1}); h) = \sum_{j\in I}\left[\alpha_j u\left(x_n + jh\right) - h\beta_j u'\left(x_n + jh\right) - h^2\gamma_j u''\left(x_n + jh\right) - h^3\nu_j u'''\left(x_n + jh\right)\right], \tag{19}
$$

where α_j, β_j, γ_j and ν_j are respectively vector columns of $\bar{A}_1, \bar{A}_2, \bar{A}_3, \bar{A}_4$, and I denotes the set of indices, $I = \{0, r, s, 1\}$. Expanding in Taylor series about x_n we get

$$
\mathcal{L}(u(x_{n+1}); h) = C_0 u(x_n) + C_1 h u'(x_n) + C_2 h^2 u''(x_n) + C_3 h^3 u'''(x_n) + \cdots + C_q h^q u^q(x_n) + \ldots, \tag{20}
$$

where

$$
C_q = \frac{1}{q!}\left[\sum_{j\in I}^{k} j^q\alpha_j - q\sum_{j\in I}^{k} j^{q-1}\beta_j - q(q-1)\sum_{j\in I}^{k} j^{q-2}\gamma_j - q(q-1)(q-2)\sum_{j\in I}^{k} j^{q-3}\gamma_j\right], \tag{21}
$$

and $q = 0, 1, 2, 3, 4, \ldots$.

The above operator and the associated formulas are said to be of order p if $C_0 = C_1 = \cdots = C_{p+1} = C_{p+2} = 0$, $C_{p+3} \neq 0$. The C_{p+3} contains the coefficients of the principal terms of the local truncation errors (LTE). Applying this to the formulas in (10), we get the order (p) and LTEs of the main formulas as follows

$$
\begin{aligned}
p = 4, &\qquad \mathcal{L}(u(x_{n+1}); h) = \frac{11h^7 u^{(7)}\left(x_n\right)}{80640} + \mathcal{O}(h^8) \\
p = 4, &\qquad \mathcal{L}(u'(x_{n+1}); h) = \frac{59h^6 u^{(7)}\left(x_n\right)}{120960} + \mathcal{O}(h^7)
\end{aligned}
$$

$$p=4,\qquad \mathcal{L}(u''(x_{n+1});h)=\frac{59h^5u^{(7)}(x_n)}{60480}+\mathcal{O}(h^6). \tag{22}$$

We also obtain the principal LTE and order (p) of the ad-hoc formulas by applying the same procedure presented above to formulas in (14); along this line, we get

$$\begin{aligned} p=3,\qquad &\mathcal{L}(u(x_1);h)=-\frac{1}{540}\left(h^6u^{(6)}(x_0)\right)+\mathcal{O}(h^7)\\ p=3,\qquad &\mathcal{L}(u'(x_1);h)=-\frac{13h^5u^{(6)}(x_0)}{3240}+\mathcal{O}(h^6)\\ p=3,\qquad &\mathcal{L}(u''(x_1);h)=-\frac{1}{216}\left(h^4u^{(6)}(x_0)\right)+\mathcal{O}(h^5). \end{aligned} \tag{23}$$

As the order of the formulas is greater than one, then they are consistent.

3.2. Convergence analysis

We define convergence and show that the proposed method is convergent by writing it in an appropriate matrix–vector notation.

Definition 3.1. Let $u(x)$ denote the exact solution of the given singular boundary value problem and let $\{u_j\}_{j=0}^N$ be the approximations obtained with the developed numerical strategy. The method is said to be convergent of order p if for sufficiently small h, there exists a constant C independent of h such that

$$\max_{0\le j\le N}|u(x_j)-u_j|\le Ch^p.$$

Note that in this situation we get that $\max_{0\le j\le N}|u(x_j)-u_j|\to 0$ as $h\to 0$.

Theorem 3.1 (*Convergence Theorem*)**.** *[13] Let $u(x)$ denote the true solution of the SBVP in (1) along with the boundary conditions in (2), and $\{u_j\}_{j=0}^N$ the discrete solution provided by the proposed global method. Then the proposed method is convergent of order four.*

Proof. Following the ideas in [13–15], we consider the matrix D of dimension $9N\times 9N$ given by

$$D=\begin{bmatrix} D_{1,1} & D_{1,2} & \dots & D_{1,3N}\\ \vdots & \vdots & & \vdots\\ D_{3N,1} & D_{3N,2} & \dots & D_{3N,3N}\end{bmatrix},$$

where the elements $D_{i,j}$ are 3×3 sub-matrices, except the $D_{i,N}, i=1,\dots,3N$ which are of size 3×2, and the $D_{i,2N+1}, i=1,\dots,3N$ which are of size 3×4. These sub-matrices are

$D_{i,i}=I, i=N+1,\dots,2N; 2N+2,\dots,3N,$ being I the identity matrix of order three,

$$D_{N,N}=\begin{bmatrix}1&0\\0&1\\0&0\end{bmatrix};\ D_{i,i-1}=\begin{bmatrix}0&0&-1\\0&0&-1\\0&0&-1\end{bmatrix},\ \ i=2,\dots,N;N+2,\dots 2N;2N+3,\dots,3N;$$

$$D_{2N+1,2N+1}=\begin{bmatrix}-1&1&0&0\\-1&0&1&0\\-1&0&0&1\end{bmatrix};\qquad D_{2N+2,2N+1}=\begin{bmatrix}0&0&0&-1\\0&0&0&-1\\0&0&0&-1\end{bmatrix};D_{N+1,2N+1}=h\begin{bmatrix}-\frac{1}{3}&0&0&0\\-\frac{2}{3}&0&0&0\\-1&0&0&0\end{bmatrix};$$

$$D_{N+2,2N+1}=h\begin{bmatrix}0&0&0&-\left(\frac{1}{2}-\frac{2}{\sqrt{21}}\right)\\0&0&0&-\left(\frac{1}{2}+\frac{2}{\sqrt{21}}\right)\\0&0&0&-1\end{bmatrix};D_{i,N+i-1}=h\begin{bmatrix}0&0&-\left(\frac{1}{2}-\frac{2}{\sqrt{21}}\right)\\0&0&-\left(\frac{1}{2}+\frac{2}{\sqrt{21}}\right)\\0&0&-1\end{bmatrix},\ \ i=2\dots,N;N+3,\dots,2N;$$

$$D_{1,2N+1}=h^2\begin{bmatrix}-\frac{1}{18}&0&0&0\\-\frac{2}{9}&0&0&0\\-\frac{1}{2}&0&0&0\end{bmatrix};D_{2,2N+1}=h^2\begin{bmatrix}0&0&0&-\frac{\left(37-8\sqrt{21}\right)}{168}\\0&0&0&-\frac{\left(37+8\sqrt{21}\right)}{168}\\0&0&0&-\frac{1}{2}\end{bmatrix};$$

$$D_{i,2N+i-1} = h^2 \begin{bmatrix} 0 & 0 & -\frac{\left(37-8\sqrt{21}\right)}{168} \\ 0 & 0 & -\frac{\left(37+8\sqrt{21}\right)}{168} \\ 0 & 0 & -\frac{1}{2} \end{bmatrix}, \ i = 3 \ldots, N.$$

The rest of the submatrices not given above are null matrices. We also define the matrix U of dimension $9N \times (3N+1)$

$$U = \begin{bmatrix} U_{1,1} & U_{1,2} & \ldots & U_{1,N} \\ \vdots & \vdots & & \vdots \\ U_{3N,1} & U_{3N,2} & \ldots & U_{3N,N} \end{bmatrix},$$

with the $U_{i,j}$ submatrices of dimension 3×3, except the $U_{i,1}$, $i = 1, \ldots, 3N$, which are of size 3×4. These submatrices are

$$U_{1,1} = h^2 \begin{bmatrix} 0 & -\frac{97}{6480} & \frac{7}{540} & -\frac{1}{240} \\ 0 & -\frac{13}{135} & \frac{28}{405} & -\frac{1}{45} \\ 0 & -\frac{21}{80} & \frac{3}{20} & -\frac{13}{240} \end{bmatrix};$$

$$U_{i,i} = h^2 \begin{bmatrix} -\frac{59}{1200} + \frac{20183}{89600\sqrt{21}} & -\frac{143}{1200} + \frac{440369}{806400\sqrt{21}} & \frac{79571-17364\sqrt{21}}{2116800} \\ -\frac{143}{1200} - \frac{440369}{806400\sqrt{21}} & -\frac{59}{1200} - \frac{20183}{89600\sqrt{21}} & \frac{79571+17364\sqrt{21}}{2116800} \\ -\frac{7}{400}\left(12+\sqrt{21}\right) & \frac{7}{400}\left(-12+\sqrt{21}\right) & \frac{59}{600} \end{bmatrix}, i = 2 \ldots, N;$$

$$U_{i,i-1} = h^2 \begin{bmatrix} 0 & 0 & \frac{131857-28788\sqrt{21}}{2116800} \\ 0 & 0 & \frac{131857+28788\sqrt{21}}{2116800} \\ 0 & 0 & \frac{31}{200} \end{bmatrix}, i = 3, \ldots, N; \qquad U_{2,1} = h^2 \begin{bmatrix} 0 & 0 & 0 & \frac{131857-28788\sqrt{21}}{2116800} \\ 0 & 0 & 0 & \frac{131857+28788\sqrt{21}}{2116800} \\ 0 & 0 & 0 & \frac{31}{200} \end{bmatrix};$$

$$U_{N+1,1} = h \begin{bmatrix} 0 & -\frac{1}{8} & \frac{11}{108} & -\frac{7}{216} \\ 0 & -\frac{10}{27} & \frac{2}{9} & -\frac{2}{27} \\ 0 & -\frac{5}{8} & \frac{1}{4} & -\frac{1}{8} \end{bmatrix};$$

$$U_{N+j,j} = h \begin{bmatrix} -\frac{356}{1575} + \frac{59\sqrt{\frac{7}{3}}}{400} & \frac{-9719+2121\sqrt{21}}{25200} & \frac{25311-5524\sqrt{21}}{151200} \\ \frac{-9719-2121\sqrt{21}}{25200} & -\frac{356}{1575} - \frac{59\sqrt{\frac{7}{3}}}{400} & \frac{25311+5524\sqrt{21}}{151200} \\ -\frac{7}{200}\left(20+\sqrt{21}\right) & \frac{7}{200}\left(-20+\sqrt{21}\right) & \frac{59}{150} \end{bmatrix}, j = 2, \ldots, N;$$

$$U_{N+j,j-1} = h \begin{bmatrix} 0 & 0 & \frac{33879-7436\sqrt{21}}{151200} \\ 0 & 0 & \frac{33879+7436\sqrt{21}}{151200} \\ 0 & 0 & \frac{38}{75} \end{bmatrix}, j = 3, \ldots, N; \qquad U_{N+2,1} = h \begin{bmatrix} 0 & 0 & 0 & \frac{33879-7436\sqrt{21}}{151200} \\ 0 & 0 & 0 & \frac{33879+7436\sqrt{21}}{151200} \\ 0 & 0 & 0 & \frac{38}{75} \end{bmatrix};$$

$$U_{2N+1,1} = \begin{bmatrix} 0 & -\frac{23}{36} & \frac{4}{9} & -\frac{5}{36} \\ 0 & -\frac{7}{9} & \frac{2}{9} & -\frac{1}{9} \\ 0 & -\frac{3}{4} & 0 & -\frac{1}{4} \end{bmatrix}; U_{2N+j,j} = \begin{bmatrix} -\frac{7}{10} + \frac{419\sqrt{\frac{7}{3}}}{960} & -\frac{7}{10} + \frac{3083}{960\sqrt{21}} & \frac{2343-512\sqrt{21}}{5040} \\ -\frac{7}{10} - \frac{3083}{960\sqrt{21}} & -\frac{7}{10} - \frac{419\sqrt{\frac{7}{3}}}{960} & \frac{2343+512\sqrt{21}}{5040} \\ -\frac{7}{5} & -\frac{7}{5} & \frac{9}{10} \end{bmatrix}, j = 2, \ldots, N;$$

$$U_{2N+j,j-1} = \begin{bmatrix} 0 & 0 & \frac{2193-512\sqrt{21}}{5040} \\ 0 & 0 & \frac{2193+512\sqrt{21}}{5040} \\ 0 & 0 & \frac{9}{10} \end{bmatrix}, j = 3, \ldots, N; \qquad U_{2N+2,1} = \begin{bmatrix} 0 & 0 & 0 & \frac{2193-512\sqrt{21}}{5040} \\ 0 & 0 & 0 & \frac{2193+512\sqrt{21}}{5040} \\ 0 & 0 & 0 & \frac{9}{10} \end{bmatrix}.$$

The remaining submatrices not included above are null matrices.

We note that the submatrices $D_{i,j}$ and $U_{i,j}$ contain the coefficients of the formulas in (14)–(17) and those of the formulas in (10)–(13), for $n = 1, 2, \ldots, N-1$.

Let us denote the vectors of exact values as

$$Y = \left(u(x_{\bar{r}}), u(x_{\bar{s}}), u(x_1), \ldots, u(x_{N-1+s}), u'(x_{\bar{r}}), \ldots, u'(x_N), u''(x_0), u''(x_{\bar{r}}), \ldots, u''(x_N)\right)^T,$$

$$F = \left(f(x_0, u(x_0), u'(x_0), u''(x_0)), f(x_{\bar{r}}, u(x_{\bar{r}}), u'(x_{\bar{r}}, u''(x_{\bar{r}})), \ldots, f(x_N, u(x_N), u'(x_N), u''(x_N))\right).$$

Note that Y has $9N$ components, while F has $(3N+1)$ components, because due to the boundary conditions in (2) $u(x_0)$, $u(x_N)$, and $u'(x_0)$ are known values. The exact form of the discretized formulas to approximate the boundary value problem can be written as

$$D_{9N\times 9N}Y_{9N} + hU_{9N\times(3N+1)}F_{3N+1} + C_{9N} = \mathcal{L}(h)_{9N}, \tag{24}$$

We use the subscripts in (24) to indicate the dimensions of matrices and vectors. The vector C_{9N} collects the known values given through the conditions in (2), that is,

$$C_{9N} = (-u_0 - \bar{r}hu'_0, -u_0 - h\bar{s}u'_0, -u_0 - hu'_0, 0, \ldots, 0, u_N, -u'_0, -u'_0, -u'_0, 0, \ldots, 0)^T,$$

while $L(h)_{9N}$ collects the LTEs of the formulas, as follows

$$L(h)_{9N} \simeq \begin{bmatrix} \mathcal{L}[u(x_{\bar{r}}), h] \\ \mathcal{L}[u(x_{\bar{s}}), h] \\ \mathcal{L}[u(x_1), h] \\ \mathcal{L}[u(x_{1+r}), h] \\ \mathcal{L}[u(x_{1+s}), h] \\ \mathcal{L}[u(x_2), h] \\ \cdots \\ \mathcal{L}[u(x_N), h] \\ \mathcal{L}[u'(x_{\bar{r}}), h] \\ \mathcal{L}[u'(x_{\bar{s}}), h] \\ \mathcal{L}[u'(x_1), h] \\ \mathcal{L}[u'(x_{1+r}), h] \\ \mathcal{L}[u'(x_{1+s}), h] \\ \mathcal{L}[u'(x_2), h] \\ \cdots \\ \mathcal{L}[u'(x_N), h] \\ \mathcal{L}[u''(x_{\bar{r}}), h] \\ \mathcal{L}[u''(x_{\bar{s}}), h] \\ \mathcal{L}[u''(x_1), h] \\ \mathcal{L}[u''(x_{1+r}), h] \\ \mathcal{L}[u''(x_{1+s}), h] \\ \mathcal{L}[u''(x_2), h] \\ \cdots \\ \mathcal{L}[u''(x_N), h] \end{bmatrix}.$$

Concerning the approximate values, they are provided by the system

$$D_{9N\times 9N}\bar{Y}_{9N} + hU_{9N\times(3N+1)}\bar{F}_{3N+1} + C_{9N} = 0\,, \tag{25}$$

where $\bar{Y}_{9N}$ approximates the vector Y_{9N}, that is,

$$\bar{Y}_{9N} = \left(u_{\bar{r}}, u_{\bar{s}}, u_1, \ldots, u_{N-1+s}, u'_{\bar{r}}, \ldots, u'_N, u''_0, u''_{\bar{r}}, \ldots, u''_N\right)^T,$$

and

$$\bar{F}_{3N+1} = (f_0, f_{\bar{r}}, f_{\bar{s}}, f_1, \ldots, f_N)^T\,.$$

We subtract (25) from (24) to get

$$D_{9N\times 9N}E_{9N} + hU_{9N\times(3N+1)}\left(F - \bar{F}\right)_{3N+1} = L(h)_{9N}, \tag{26}$$

where $E_{9N} = Y_{9N} - \bar{Y}_{9N} = \left(e_{\bar{r}}, e_{\bar{s}}, \ldots, e_{N-1+s}, e'_{\bar{r}}, \ldots, e'_N, e''_0, e''_{\bar{r}}, \ldots, e''_N\right)^T$ is a vector of errors at the discrete points.

Through the Mean-Value Theorem, we can put for any convenient subindex i as

$$f(x_i, u(x_i), u'(x_i), u''(x_i)) - f(x_i, u_i, u'_i, u''_i) = (u(x_i) - u_i)\,\frac{\partial f}{\partial u}(\xi_i) + \left(u'(x_i) - u'_i\right)\frac{\partial f}{\partial u'}(\xi_i) + \left(u''(x_i) - u''_i\right)\frac{\partial f}{\partial u''}(\xi_i),$$

where ξ_i denotes an intermediate point in the line between $(x_i, u(x_i), u'(x_i), u''(x_i))$ and (x_i, u_i, u'_i, u''_i). Thus, we have that

$$(F-\bar{F})_{3N+1}=\begin{pmatrix} \frac{\partial f}{\partial u}(\xi_0) & \dots & 0 & \frac{\partial f}{\partial u'}(\xi_0) & \dots & 0 & \frac{\partial f}{\partial u''}(\xi_0) & \dots & 0 \\ 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ . & \dots & . & . & \dots & . & . & \dots & . \\ . & \dots & . & . & \dots & . & . & \dots & . \\ . & \dots & . & . & \dots & . & . & \dots & . \\ 0 & \dots & \frac{\partial f}{\partial u}(\xi_N) & 0 & \dots & \frac{\partial f}{\partial u'}(\xi_N) & 0 & \dots & \frac{\partial f}{\partial u''}(\xi_N) \end{pmatrix} \begin{pmatrix} e_0 \\ e_{\bar{r}} \\ \vdots \\ e_N \\ e'_0 \\ e'_{\bar{r}} \\ \vdots \\ e'_N \\ e''_0 \\ e''_{\bar{r}} \\ \vdots \\ e''_N \end{pmatrix}$$

$$=\begin{pmatrix} 0 & \dots & 0 & 0 & \dots & 0 & \frac{\partial f}{\partial u''}(\xi_0) & \dots & 0 \\ \frac{\partial f}{\partial u}(\xi_r) & \dots & 0 & \frac{\partial f}{\partial u'}(\xi_r) & \dots & 0 & 0 & \dots & 0 \\ . & \dots & . & . & \dots & . & . & . & \\ . & \dots & . & . & \dots & . & . & . & \\ . & \dots & . & . & \dots & . & . & . & \\ 0 & \dots & \frac{\partial f}{\partial u}(\xi_{N-1+s}) & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & \dots & \frac{\partial f}{\partial u'}(\xi_N) & 0 & \dots & \frac{\partial f}{\partial u''}(\xi_N) \end{pmatrix} \begin{pmatrix} e_{\bar{r}} \\ \vdots \\ e_{N-1+s} \\ e'_{\bar{r}} \\ \vdots \\ e'_N \\ e''_0 \\ e''_{\bar{r}} \\ \vdots \\ e''_N \end{pmatrix}$$

$$=J_{(3N+1)\times 9N}\, E_{9N}\,.$$

Note that in the second identity we have used that $e_0 = u(x_0) - u_0 = 0$, $e'_0 = u'(x_0) - u'_0 = 0$ and $e_N = u(x_N) - u_0 = 0$. With the use of the above equation, (26) may be arranged as

$$\left(D_{9N\times 9N} + hU_{9N\times(3N+1)}J_{(3N+1)\times 9N}\right) E_{9N} = L(h)_{9N}\,, \tag{27}$$

and setting $M = D + hUJ$ we simply get that

$$M_{9N\times 9N}E_{9N} = L(h)_{9N}\,. \tag{28}$$

Let us prove that except for a few selected values of $h > 0$, matrix M is invertible. If we use the abbreviate notation $D_N = D_{9N\times 9N}$, given the form of this matrix where the submatrices have many zeros, it is easy to verify that for $N = 2$, the determinant is $|D_2| = -2h^2$. By induction, it can be proved that $|D_N| = -\dfrac{(-1)^{(N+1)}(N^2h^2)}{2}$, and thus D_N is invertible provided that $h > 0$. Now the matrix M may be rewritten as

$$M = D + hUJ = (Id - B)D$$

where Id is the identity matrix of order $9N$, and $B = -hUJD^{-1}$. Thus, we have that $|M| = |Id - B|\,|D|$.

As $|\lambda Id - B| = \prod_{i=1}^{9N}(\lambda - \lambda_i)$ is the characteristic polynomial of B, in order to have $|Id - B| \neq 0$, if we take $\lambda = 1$, it is sufficient to choose h such that

$$h \notin \left\{1/\bar{\lambda}_i : \bar{\lambda}_i \text{ is an eigenvalue of } UJD^{-1}\right\}.$$

For such values of h the equation in (28) may be rewritten as

$$E = \left(M^{-1}\right) L(h)\,. \tag{29}$$

The maximum norm in $\mathbb{R}$ and the corresponding matrix-induced norm in $\mathbb{R}^{9N\times 9N}$ are considered. If we expand the terms of M^{-1} in powers of h it can be shown that $\|M^{-1}\| = \mathcal{O}(h^{-2})$.

Assuming that $u(x)$ has in $[0, x_N]$ bounded derivatives up to the necessary order, from (29) we obtain that

$$\begin{aligned} \|E\| &\leq \|M^{-1}\|\,\|L(h)\| \\ &= \mathcal{O}(h^{-2})\,\mathcal{O}(h^6) \\ &\leq K\,h^4. \end{aligned}$$

Therefore, the proposed method is convergent, providing fourth-order approximations. □

4. Implementation details

The current method is implemented as a global method, providing approximate solutions at the grid points at once. The system in (25) can be formulated as $\boldsymbol{F(u)=0}$ where the unknowns are

$$\tilde{\mathbf{U}} = \left\{u_{\bar{r}}, u_{\bar{s}}, u'_{\bar{r}}, u'_{\bar{s}}, u''_{\bar{r}}, u''_{\bar{s}}\right\} \bigcup \left\{u_j\right\}_{j=1,\dots,N-1} \bigcup \left\{u'_j\right\}_{j=1,\dots,N} \bigcup \left\{u''_j\right\}_{j=0,\dots,N} \bigcup \left\{u_{j+r}, u_{j+s}, u'_{j+r}, u'_{j+s}, u''_{j+r}, u''_{j+s}\right\}_{j=1,\dots,N-1}.$$

Since the proposed technique is an implicit scheme, we use a Modified Newton's method (MNM) to solve the above system. The MNM can be formulated as

$$\tilde{\mathbf{U}}^{i+1} = \tilde{\mathbf{U}}^{i} - \left(\mathbf{J}^{i}\right)^{-1}\mathbf{F}^{i},$$

where $\mathbf{J}$ is the jacobian matrix of $\mathbf{F}$. Note that the only known information are the boundary values. Thus, the starting guesses in the Newton's approach are taken based on this information. For example, in case one has the values in (2), $u(x_0)=u_0$, $u'(x_0)=u'_0$, $u(x_N)=u_N$, then we consider the mixed interpolating polynomial, $H(x)$, verifying these conditions. The values of $H(x)$, $H'(x)$, $H''(x)$ at the grid points provide the necessary starting guesses for the Newton's approach. We adopted a stopping criterion considering a maximum number of 50 iterations, while the error between two successive approximations should be less than 10^{-16}.

4.1. Pseudocode for the algorithm

A pseudocode of the proposed method is as follows:

Data: Take $N>0\in\mathbb{N}$, and define $h=\frac{x_N-x_0}{N}$
Total number of steps in the main method: $N-1$;
Starting and end-point of the integration intervals: $[x_0, x_N]$
Result: Approximations of the problem in (1) at selected points.
Take equations in (14)–(17), and equations in (10)–(13) for $n=1,2,\dots,N-1$;
Solve the system of equations in (10)–(17) for the above values of n to obtain the u_i;
Save the obtained approximate solution $\{(x_i,u_i)\}_{i=0,1,2,3,\dots,N}$
end

---

5. Numerical results

This section reports the obtained approximate solutions to the TSBVP in (1) using the newly introduced OHBNTM technique. The efficiency and accuracy of the proposed scheme are tested and calculated by using the following formulas:

$$AE = \|u(x_j)-u_j\|_\infty, \quad MAXAE = \max_{j=0,\dots,N}\|u(x_j)-u_j\|_\infty, \quad ROC(h) \simeq -\log_2\left(\frac{MAXAE_h}{MAXAE_{2h}}\right),$$

where AE stands for the absolute error at the considered node, $MAXAE$ is the maximum absolute error along with the considered interval, $ROC(h)$ denotes numerical rates of convergence, $u(x_j)$ is the exact solution, and u_j is the approximate solution at the point x_j.

5.1. Numerical example 1

We firstly consider the following TSBVP [1,10]

$$u'''(x) - \frac{2}{x}u''(x) - u(x)^2 - u(x) = -x^6\exp(2x) + 7x^2\exp(x) + 6x\exp(x) - 6\exp(x), \qquad (30)$$
$$u(0)=0, \quad u'(0)=0, \quad u(1)=\mathrm{e}, \quad x\in[0,1].$$

The exact solution of this problem is $u(x)=x^3\exp(x)$.

We solved (30) by the proposed OHBNTM. We report the comparison of the obtained solutions with the cubic B-spline method (CBSM) in [1], the quintic B-spline method (QBSM) in [10], and the exact solution to measure the accuracy of the present technique. The data in Tables $1-2$ show that the proposed OHBNTM is more accurate than CBSM and QBSM. Besides, the plot of the absolute errors displayed to the right of Fig. 1 indicates that the MAXAE is 2.5×10^{-9}. At the same time, the reported MAXAE for the CBSM in [1] with $h=\frac{1}{40}$ is 1.15000×10^{-6}, evincing the better performance of the OHBNTM.

Table 1
Order of convergence and comparison for test problem (30).

h	Method	*MAXAE*	*ROC*(h)
$\frac{1}{25}$	OHBNTM	1.57366×10^{-8}	
$\frac{1}{25}$	QBSM [10]	1.56259×10^{-7}	
$\frac{1}{50}$	OHBNTM	1.08169×10^{-9}	3.863
$\frac{1}{50}$	QBSM [10]	1.01767×10^{-8}	3.912
$\frac{1}{100}$	OHBNTM	7.0491×10^{-11}	3.940
$\frac{1}{100}$	QBSM [10]	6.12965×10^{-10}	3.970

Table 2
Comparison of approximate solutions on test problem (30) for $h = \frac{1}{30}$.

x	Exact	OHBNTM	CBSM [1]
0.1	0.001105170918075648	0.0011051706827446236	0.0011094283
0.2	0.009771222065281361	0.009771221867922246	0.0097757339
0.3	0.03644618780455208	0.03644618780455208	0.0364507031
0.4	0.09547678064904133	0.09547678064904133	0.0954809351
0.5	0.20609015883751602	0.20609015883751602	0.2060935138
0.6	0.3935776608843499	0.3935776608843499	0.3935797567
0.7	0.6907171786623734	0.6907171786623734	0.6907176118
0.8	1.1394769553881439	1.1394769553881439	1.1394754575
0.9	1.7930506680334166	1.7930506680334166	1.7930472271
1.0	2.718281828459045	2.718281828459045	2.7182767565
MAXAE	− − −−	7.83230×10^{-9}	3.98000×10^{-6}

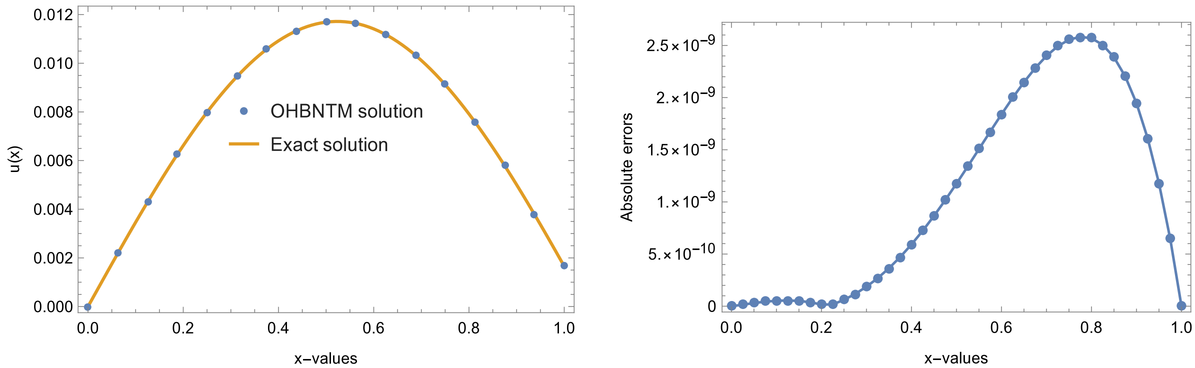

Fig. 1. Plots of exact and OHBNTM solutions (left), and absolute errors (right) for (30) with $h = \frac{1}{40}$.

5.2. Numerical example 2

For the second test problem, we consider the following TSBVP [1,23]

$$u'''(x) - \frac{3}{x}u''(x) - u(x)^3 = 24\exp(x) + 12x^2\exp(x) + 36x\exp(x) + x^3\exp(x) - x^9\exp(3x), \quad (31)$$
$$u(0) = 0, \quad u'(0) = 0, \quad u(1) = \mathrm{e}, \quad x \in [0, 1].$$

The exact solution of this problem is $u(x) = x^3\exp(x)$.

The results reported in Table 3 indicate that our method performs significantly better than the methods CBSM in [1] and SM in [23]. The left plot of Fig. 2 depicts a good agreement of the numerical solution with the exact solution. Again the plot of the absolute errors to the right of Fig. 2 indicates that the MAXAE is 8.85×10^{-9}, while the presented MAXAE for the CBSM in [1] with $h = \frac{1}{30}$ is 3.98×10^{-5}, showing the more reliable and accurate performance of the OHBNTM.

5.3. Numerical example 3

In the next test problem, we consider the following TSBVP

$$u'''(x) + \frac{6}{x}u''(x) + \frac{6}{x^2}u'(x) = \frac{6x^6 + 12x^3 + 60}{\exp(3u(x))}, \quad (32)$$
$$u(0) = 0, \quad u'(0) = 0, \quad u(1) = \log(2), \quad x \in [0, 1].$$

Table 3
Comparison of approximate solutions on test problem (31) for $h = \frac{1}{50}$.

x	Exact	OHBNTM	CBSM [1]	SM [23]
0.1	0.001105170918075648	0.001105171165652863	0.0011050461	0.0010868904
0.2	0.009771222065281361	0.009771222569744891	0.0097709476	0.0097345516
0.3	0.03644618780455208	0.03644618854494258	0.0364457990	0.0363925859
0.4	0.09547678064904133	0.0954767815890795	0.0954762717	0.0954086907
0.5	0.20609015883751602	0.20609015992230556	0.2060895778	0.2060113496
0.6	0.3935776608843499	0.39357766203615413	0.3935770360	0.3934935957
0.7	0.6907171786623734	0.6907171797756839	0.6907165753	0.6906354473
0.8	1.1394769553881439	1.1394769563241363	1.1394764444	1.1394077724
0.9	1.7930506680334166	1.7930506686138648	1.7930503509	1.7930074373
1.0	2.718281828459045	2.7182818285	2.7182767565	2.7182818285
MAXAE	– – ––	1.15353×10^{-9}	6.25000×10^{-6}	8.43000×10^{-5}

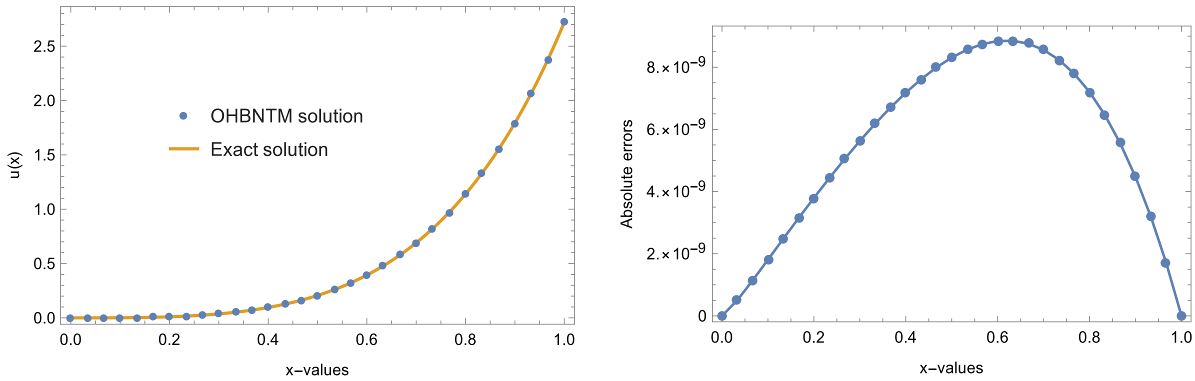

Fig. 2. Plots of analytical and OHBNTM solutions (left), and absolute errors (right) for (31) with $h = \frac{1}{30}$.

Table 4
Comparison of OHBNTM and Exact solutions on test (32) with $h = \frac{1}{10}$.

x	OHBNTM	Exact solution	AE
0.0	0	0	0
0.1	0.0010007107537268378	0.0009995003330834232	1.21042×10^{-6}
0.2	0.007969947137280598	0.007968169649176881	1.77749×10^{-6}
0.3	0.026644225320534738	0.026641930946421092	2.29437×10^{-6}
0.4	0.06203778629920728	0.0620353909194527	2.39538×10^{-6}
0.5	0.11778510483060821	0.11778303565638346	2.06917×10^{-6}
0.6	0.19556823220513123	0.19556678354397541	1.44866×10^{-6}
0.7	0.2949066931015952	0.2949059175411005	7.75560×10^{-7}
0.8	0.41343355231614093	0.4134332777573413	2.74559×10^{-7}
0.9	0.5475432406809645	0.5475432067011535	3.39798×10^{-8}
1.0	0.6931471805599453	0.6931471805599453	0.00000

Table 5
Order of convergence and CPU time in seconds for test problem (32).

h	Method	*MAXAE*	*CPU*	*ROC*(h)
$\frac{1}{20}$	OHBNTM	9.80042×10^{-8}	1.796875	
$\frac{1}{40}$	OHBNTM	5.53262×10^{-9}	9.531250	4.03247
$\frac{1}{80}$	OHBNTM	3.38094×10^{-10}	59.953125	4.14681

The exact solution of this problem is $u(x) = \log\left(1 + x^3\right)$.

Table 4 collects the approximate and exact solutions and MAXAE for $h = \frac{1}{10}$. The MAXAE, CPU time and $ROC(h)$ are recorded in Table 5. We see that the obtained $ROC(h)$ is consistent with the theoretical analysis of the proposed method given in Section 3. Also, Fig. 3 shows the good results attained with the OHBNTM when solving (32).

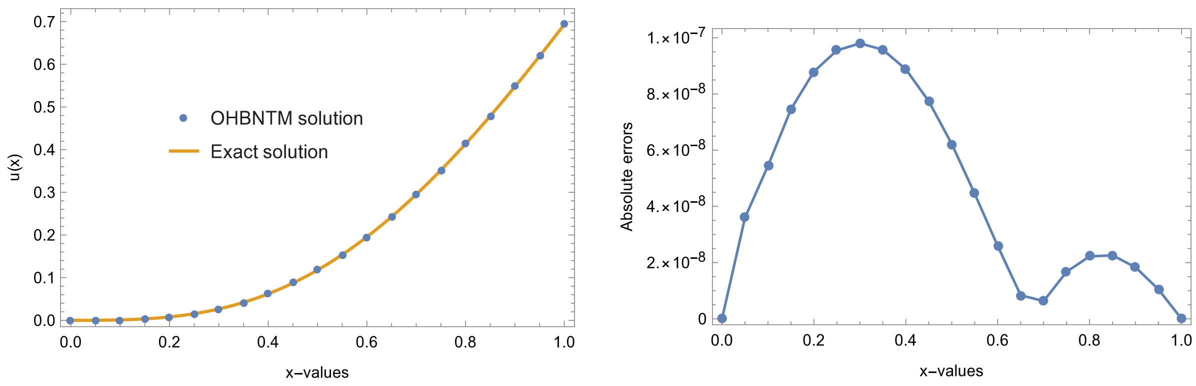

Fig. 3. Plots of exact and OHBNTM solutions (left), and absolute errors (right) for (32) with $h = \frac{1}{20}$.

Table 6
Comparison of the maximum absolute errors (MAXAE) for problem (33).

h	ϵ	Methods	MAXAE
$\frac{1}{16}$	2^{-4}	OHBNTM	9.10582×10^{-15}
$\frac{1}{16}$	2^{-4}	CBSM [1]	1.15000×10^{-6}
$\frac{1}{16}$	2^{-4}	QBSM [10]	1.130000×10^{-8}
$\frac{1}{32}$	2^{-20}	OHBNTM	1.01743×10^{-16}
$\frac{1}{32}$	2^{-20}	CBSM [1]	8.60000×10^{-12}
$\frac{1}{32}$	2^{-20}	QBSM [10]	1.01000×10^{-15}

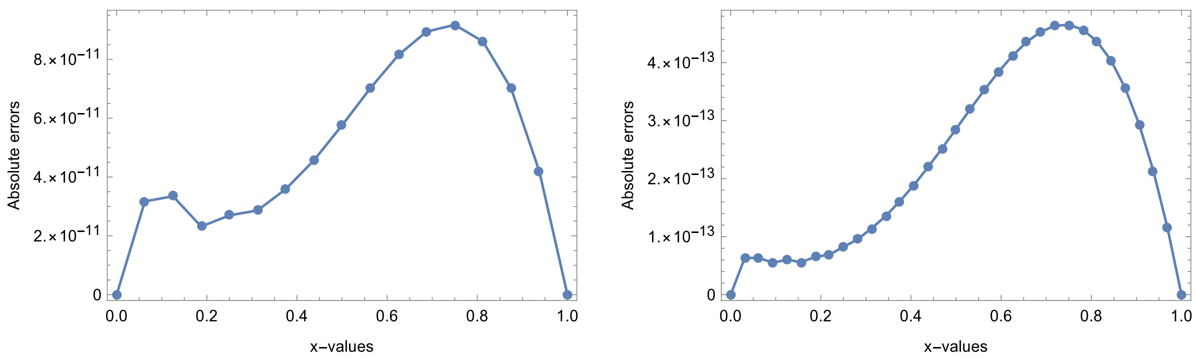

Fig. 4. Plots of absolute errors for $\epsilon = 2^{-8}, h = \frac{1}{16}$ (left) and $\epsilon = 2^{-10}, h = \frac{1}{32}$ (right) for (33).

5.4. Numerical example 4

In the last test problem, we consider following self-adjoint TSBVP [1,10], which is one of the model in nonlinear science and engineering

$$\epsilon u'''(x) + \frac{1}{x}u''(x) + u(x) = 3\epsilon\left(-27\epsilon\cos(3x) - \frac{9\sin(3x)}{x} + \sin(3x)\right), \tag{33}$$
$$u(0) = 0, \quad u'(0) = 9\epsilon, \quad u(1) = 3\epsilon\sin(3), \quad x \in [0, 1].$$

The exact solution of this problem is $u(x) = 3\epsilon\sin(3x)$.

The obtained results with our technique and those in [1,10] are reported in Table 6. We can see that the proposed method provides closer approximations to the known exact solution than those in [1,10]. In addition, the plots in Fig. 4 show the absolute errors for different values of ϵ and h, which are smaller than the ones reported in [1,10]. Figures 5 – 6 also show that the OHBNTM solutions approximate the exact solution to test problem (33) pretty well.

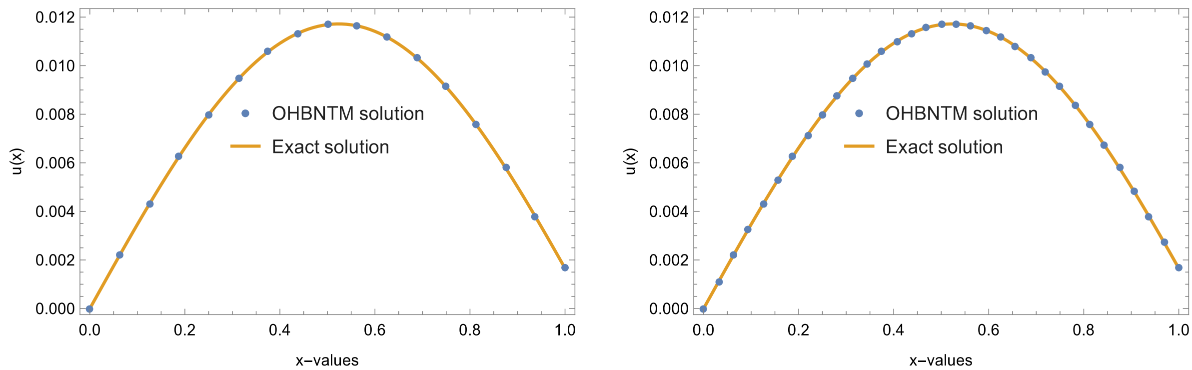

Fig. 5. Plots of exact and OHBNTM solutions for $\epsilon = 2^{-8}, h = \frac{1}{16}$ (left), and $\epsilon = 2^{-10}, h = \frac{1}{32}$ (right) for (33).

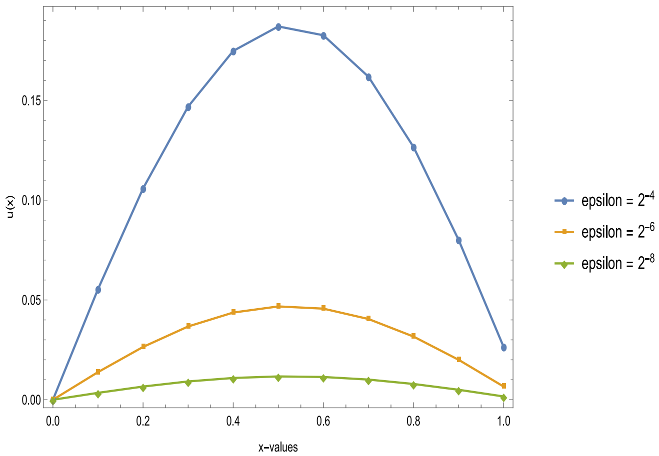

Fig. 6. Numerical solutions of OHBNTM for (33) with $h = \frac{1}{10}$ for different values of ϵ.

6. Conclusions

In this article, a new one-step hybrid block Nyström-type method (OHBNTM) is proposed and efficiently applied to obtain reliable approximate solutions to Emden–Fowler type equations given in (1). The proposed OHBNTM is zero-stable, consistent, and the convergence analysis of the current approach is established to be fourth-order convergent. Furthermore, numerical results in Tables 1–6 and Figs. 1–6 confirm that the proposed strategy is more realistic and efficient than other existing numerical techniques used for comparisons and found to be in good agreement with known exact solutions.

CRediT authorship contribution statement

Mufutau Ajani Rufai: Conceptualization, Methodology, Investigation, Writing – original draft. **Higinio Ramos:** Formal analysis, Visualization, Investigation, Supervision, Writing – review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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