Analysis of the L1 scheme for fractional wave equations with nonsmooth data *

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Abstract

This paper analyzes the well-known L1 scheme for fractional wave equations with nonsmooth data. A new stability estimate is obtained, and the temporal accuracy $\mathcal{O}(\tau^{3-\alpha})$ is derived for the nonsmooth data. In addition, a modified L1 scheme is proposed, stability and temporal accuracy $\mathcal{O}(\tau^2)$ are derived for this scheme with nonsmooth data. The convergence of these schemes in inhomogeneous case are also established. Finally, numerical experiments are performed to verify the theoretical results.

Keywords: fractional wave equation, L1 scheme, stability, convergence, nonsmooth data.

1 Introduction

Let $1 < \alpha < 2$ and $\Omega \subset \mathbb{R}^d$ $(d = 1, 2, 3)$ be a convex d-polytope. We consider the following fractional wave equation:

$$\mathrm{D}_{0+}^{\alpha-1}(u' - u_1)(t) - \Delta u(t) = f(t), \quad t > 0, \tag{1}$$

subjected to the initial value condition $u(0) = u_0$, where $u(t) \in H_0^1(\Omega)$ for all $t > 0$, u_0, u_1 and f are given functions, and $\mathrm{D}_{0+}^{\alpha-1}$ is a Riemann-Liouville fractional differential operator of order $\alpha - 1$.

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As a extension of integer order equation; the fractional diffusion and wave equations are widely used to model some processes with non-local effect, see [31, 3, 4, 10]. We also refer readers to [13] for more background of fractional differential equations. By now there is an extensive literature on the numerical treatment of fractional diffusion and wave equations. Some of these researches give the convergence result under the condition that the solution is a C^2- or C^3- function in time. However, it is well known that the solution of a fractional diffusion (or wave) equation generally has singularity in time despite how smooth the inital data is [12]. In fact, the main challenge is to design stable numerical scheme and to derive convergence result, without regularity restrictions on the solution, especially for the case with nonsmooth data.

Let us give a brief introduction of two kinds of numerical methods for solving fractional diffusion equations with nonsmooth data: the L1-type method [14, 19, 9, 32, 18], discontinuous Galerkin method [27, 30, 29, 1]. The L1-type method use L1 scheme to approximate the fractional derivative, these methods are very popular due to their ease of implementation. Jin et al. [11] proved that the L1 scheme is of temporal accuracy $\mathcal{O}(\tau)$ for fractional diffusion equations with smooth and nonsmooth initial data. Yan et al. [36] proposed a modified L1 scheme for fractional diffusion equations, which possesses temporal accuracy $\mathcal{O}(\tau^{2-\alpha})$ for smooth and nonsmooth initial data. The discontinuous Galerkin method use the finite element method to approximate the fractional derivative. McLean and Mustapha [25] showed that the piecewise constant discontinuous Galerkin method is of temporal accuracy $\mathcal{O}(\tau)$ for fractional diffusion equations with nonsmooth initial data. Li et al. [16] investigate the regularity of fractional diffusion equations with nonsmooth data and they proved that discontinuous Galerkin method possesses optimal convergence rates in $L^2(0,T;L^2(\Omega))$ and $L^2(0,T;H^1(\Omega))$ norm, with respect to the regularity of the solution. For more related works, we refer reader to [37, 17, 5, 15].

Next, let us first briefly summarize some works on a variant of fractional wave equation:

$$u'(t) - \Delta(\mathrm{D}_{0+}^{1-\alpha}\, u)(t) = u_1 + \mathrm{D}_{0+}^{1-\alpha}\, f(t), \quad t > 0,$$

which is obtained by applying $\mathrm{D}_{0+}^{1-\alpha}$ to both sides of (1). For this equation, McLean et al. [23, 24] proposed two positive definite quadratures for the time fractional integral operator. Combing the convolution quadratures in [20] and the backward difference methods in time, Lubich et al. [21, 6] proposed the first- and second-order time-stepping schemes and derived optimal error estimates with nonsmooth inital data. Applying the famous discontinuous Galerkin method, Mustapha and McLean [28] proposed a new class of algorithms. We note that the low-order algorithm in [28] is identical to the low-order algorithm proposed in [24]. For more related works, we refer the reader to [7, 26].

The study on fractional wave equation is limitied. Using the convolution quadratures in [20] and techniques in [21], Jin et al. [12] developed first- and second-order time-stepping methods for fractional wave equations and derived optimal error estimates with nonsmooth inital data. In [22], the convergence in the $H_0^1(\Omega)$-norm has been derived for a low-order Petrov-Galerkin method with nonsmooth source term. We note that the low-order Petrov-Galerkin method in [22] is identical to the L1 scheme.

As far as we know, the convergence in the $L^2(\Omega)$-norm of the L1 scheme for fractional wave equations with nonsmooth data has not been established. In this paper, for a full discretization using the L1 scheme in time and the standard P_1-element in space, we derive a new stability estimate and obtain the temporal accuracy $\mathcal{O}(\tau^{3-\alpha})$ in the $L^2(\Omega)$-norm at positive times, with nonsmooth initial data. For another full discretization using a modified L1 scheme in time and the P_1-element in space, we obtain the temporal accuracy $\mathcal{O}(\tau^2)$ for nonsmooth initial data. We also establish the convergence of the two discretizations in inhomogeneous case (i.e., $f \not\equiv 0$). The derived error estimates require that the temporal grid is uniform and that $\tau^\alpha/h_{\min}^2$ is uniformly bounded, where $h_{\min}$ is the minimum diameter of the elements in the spatial triangulation. Our analysis implies that for nonzero initial value u_0 large ratio $\tau^\alpha/h_{\min}^2$ will significantly worsen the temporal accuracy of the L1 scheme, and this is confirmed by the numerical result. To our knowledge, this interesting phenomenon is firstly reported in this paper.

The rest of this paper is organized as follows. Section 3 establishes the stability and convergence of the L1 scheme and a modified L1 scheme for a fractional ordinary equation. Section 4 derives the stability and convergence of two full discretizations for problem (1), which use the L1 scheme and a modified L1 scheme in time, respectively. Section 5 performs several numerical experiments to verify the theoretical results. Finally, Section 6 provides some concluding remarks.

2 Preliminaries

Let $-\infty \leqslant a < b \leqslant \infty$ and assume that X is a separable Hilbert space X with inner product $(\cdot,\cdot)_X$. For any $-\infty < \gamma < 0$, define

$$
\begin{aligned}
(\mathrm{D}_{a+}^\gamma v)(t) &:= \frac{1}{\Gamma(-\gamma)} \int_a^t (t-s)^{-\gamma-1} v(s)\,\mathrm{d}s, \quad a<t<b,\\
(\mathrm{D}_{b-}^\gamma v)(t) &:= \frac{1}{\Gamma(-\gamma)} \int_t^b (s-t)^{-\gamma-1} v(s)\,\mathrm{d}s, \quad a<t<b,
\end{aligned}
$$

for all $v \in L^1(a,b;X)$, where $\Gamma(\cdot)$ is the gamma function. For any $m \leqslant \gamma < m+1$ with $m \in \mathbb{N}$, define

$$
\begin{aligned}
\mathrm{D}_{a+}^\gamma v &:= \mathrm{D}^{m+1}\,\mathrm{D}_{a+}^{\gamma-m-1} v,\\
\mathrm{D}_{b-}^\gamma v &:= (-1)^{m+1}\,\mathrm{D}^{m+1}\,\mathrm{D}_{b-}^{\gamma-m-1} v,
\end{aligned}
$$

for all $v \in L^1(a,b;X)$, where D is the first order differential operator in the distribution sense.

Then we introduce some properties of fractional calculus operators used in this paper. Define

$$
\begin{aligned}
{}_0H^1(a,b;X) &:= \big\{ v \in L^2(a,b;X) : v' \in L^2(a,b;X),\ \lim_{t\to a+} v(t) = 0 \big\},\\
{}^0H^1(a,b;X) &:= \big\{ v \in L^2(a,b;X) : v' \in L^2(a,b;X),\ \lim_{t\to b-} v(t) = 0 \big\}.
\end{aligned}
$$

Assume that $0<\gamma<1$. Define

$$
\begin{aligned}
{}_0H^\gamma(a,b;X) &:= [L^2(a,b;X), {}_0H^1(a,b;X)]_{\gamma,2},\\
{}^0H^\gamma(a,b;X) &:= [L^2(a,b;X), {}^0H^1(a,b;X)]_{\gamma,2},
\end{aligned}
$$

where $[\cdot,\cdot]_{\gamma,2}$ means the interpolation space defined by the famous K-method [34]. We use ${}_0H^{-\gamma}(a,b;X)$ and ${}^0H^{-\gamma}(a,b;X)$ to denote the dual spaces of ${}^0H^{\gamma}(a,b;X)$ and ${}_0H^{\gamma}(a,b;X)$, respectively. By [22, Lemma 3.3] we have that, for any $v \in L^2(a,b;X)$,

$$\|\mathrm{D}_{b-}^{-\gamma} v\|_{{}^0H^{\gamma}(a,b;X)} \leqslant C\|v\|_{L^2(a,b;X)},$$

where C is a positive constant depending only on γ. Therefore, we can define $\mathrm{D}_{a+}^{-\gamma} : {}_0H^{-\gamma}(a,b;X) \to L^2(a,b;X)$ by that

$$\int_a^b \big(\mathrm{D}_{a+}^{-\gamma} v(t), w(t)\big)_X \,\mathrm{d}t = \langle v, \mathrm{D}_{b-}^{-\gamma} w\rangle_{{}^0H^{\gamma}(a,b;X)}$$

for all $v \in {}_0H^{-\gamma}(a,b;X)$ and $w \in L^2(a,b;X)$, where $\langle\cdot,\cdot\rangle_{{}^0H^{\gamma}(a,b;X)}$ means the duality pairing between ${}_0H^{-\gamma}(a,b;X)$ and ${}^0H^{\gamma}(a,b;X)$. Moreover, it is evident that

$$\|\mathrm{D}_{a+}^{-\gamma} v\|_{L^2(a,b;X)} \leqslant C\|v\|_{{}_0H^{-\gamma}(a,b;X)}, \quad \forall v \in {}_0H^{-\gamma}(a,b;X), \tag{2}$$

where C is a positive constant depending only on γ.

Lemma 2.1. *Assume that $v \in {}_0H^{\gamma}(a,b;X)$ and $w \in {}^0H^{\gamma}(a,b;X)$, with $0 < \gamma < 1/2$. Then*

$$C_1\|\mathrm{D}_{a+}^{\gamma} v\|_{L^2(a,b;X)}^2 \leqslant \int_a^b (\mathrm{D}_{a+}^{\gamma} v(t), \mathrm{D}_{b-}^{\gamma} v(t))_X \,\mathrm{d}t \leqslant C_2\|\mathrm{D}_{a+}^{\gamma} v\|_{L^2(a,b;X)}^2,$$
$$C_1\|\mathrm{D}_{b-}^{\gamma} v\|_{L^2(a,b;X)}^2 \leqslant \int_a^b (\mathrm{D}_{a+}^{\gamma} v(t), \mathrm{D}_{b-}^{\gamma} v(t))_X \,\mathrm{d}t \leqslant C_2\|\mathrm{D}_{b-}^{\gamma} v\|_{L^2(a,b;X)}^2,$$
$$\langle \mathrm{D}_{a+}^{2\gamma} v, w\rangle_{{}^0H^{\gamma}(a,b;X)} = \int_a^b (\mathrm{D}_{a+}^{\gamma} v(t), \mathrm{D}_{b-}^{\gamma} w(t))_X \,\mathrm{d}t = \langle \mathrm{D}_{b-}^{2\gamma} w, v\rangle_{{}_0H^{\gamma}(a,b;X)},$$

where C_1 and C_2 are two positive constants depending only on γ.

Remark 2.1. *For the proof of Lemma 2.1, we refer the reader to [8]. Assume that $0 < \gamma < 1/2$. If $v \in {}_0H^{\gamma}(a,b;X)$ and $w \in {}^0H^{\gamma}(a,b;X)$ satisfy that $\mathrm{D}_{a+}^{2\gamma} v \in L^p(a,b;X)$ and $w \in L^{p/(p-1)}(a,b;X)$ for some $1 < p < \infty$, then*

$$\langle \mathrm{D}_{a+}^{2\gamma} v, w\rangle_{{}^0H^{\gamma}(a,b;X)} = \int_a^b (\mathrm{D}_{a+}^{2\gamma} v(t), w(t))_X \,\mathrm{d}t.$$

Finally, we introduce some conventions as follows: $H_0^1(\Omega)$ denotes the usual Sobolev space, and $H^{-1}(\Omega)$ is its dual space; the spaces ${}_0H^{\gamma}(a,b;\mathbb{R})$ and ${}^0H^{\gamma}(a,b;\mathbb{R})$ are abbreviated to ${}_0H^{\gamma}(a,b)$ and ${}^0H^{\gamma}(a,b)$, respectively; $C_\times$ means a generic positive constant depending only on its subscript(s), and its value may differ at each occurrence; for an interval $\omega \subset \mathbb{R}$, the notation $\langle p,q\rangle_\omega$ denotes $\int_\omega pq$ whenever $pq \in L^1(\omega)$.

3 Two discretizations of a fractional ordinary equation

This section considers two discretizations of the following fractional ordinary equation:

$$\mathrm{D}_{0+}^{\alpha-1}(y' - y_1)(t) + \lambda y(t) = f(t), \quad t > 0, \tag{3}$$

subjected to the initial value condition $y(0) = y_0$, where $y_0, y_1 \in \mathbb{R}$, $f \in L^1(0,\infty) \cap {}_0H^{(1-\alpha)/2}(0,\infty)$, and $\lambda \geqslant 1$ is a positive constant. Let $\mu := \lambda\tau^\alpha/2$ and define $t_j := j\tau$ for each $j \in \mathbb{N}$, where τ is a positive constant. Applying the L1-scheme proposed in [33], we obtain the first discretization of equation (3).

Remark 3.1. *In order to obtain the error estimates of PDE (1), λ will be chosen as one of the eigen values of the discrete Laplace operator $-\Delta_h$ in the next section.*

Discretization 1. *Let $Y_0 = y_0$; for each $k \in \mathbb{N}$, the value of Y_{k+1} is determined by*

$$\begin{aligned}(Y_1 - Y_0)(b_{k+1} - b_k) + \sum_{j=1}^{k}(Y_{j+1} - 2Y_j + Y_{j-1})(b_{k-j+1} - b_{k-j}) \\ + \mu(Y_k + Y_{k+1}) = \tau^{\alpha-1}\int_{t_k}^{t_{k+1}} f(t)\,\mathrm{d}t + \tau y_1(b_{k+1} - b_k),\end{aligned} \tag{4}$$

where $b_j := j^{2-\alpha}/\Gamma(3-\alpha)$, $j \in \mathbb{N}$.

Remark 3.2. *The above discretization is actually an variant of the temporal discretization in [33], but it is identical to a low-order Petrov-Galerkin method analyzed in [22].*

The second discretization is a simple modification of the first one.

Discretization 2. *Let $\mathcal{Y}_0 = y_0$; for each $k \in \mathbb{N}$, the value of $\mathcal{Y}_{k+1}$ is determined by*

$$\begin{aligned}(\mathcal{Y}_1 - \mathcal{Y}_0)(\beta_{k+1} - \beta_k) + \sum_{j=1}^{k}(\mathcal{Y}_{j+1} - 2\mathcal{Y}_j + \mathcal{Y}_{j-1})(\beta_{k-j+1} - \beta_{k-j}) \\ + \mu(\mathcal{Y}_k + \mathcal{Y}_{k+1}) = \tau^{\alpha-1}\int_{t_k}^{t_{k+1}} f(t)\,\mathrm{d}t + \tau y_1(\beta_{k+1} - \beta_k),\end{aligned} \tag{5}$$

where $\beta_1 = b_1 + 2\sin(\alpha\pi/2)\sum_{k=1}^\infty (2k\pi)^{\alpha-3}$ and $\beta_k := b_k$ for all $k \in \mathbb{N} \setminus \{1\}$.

Remark 3.3. *In the numerical analysis of Discretization 1 (cf. Remark 3.8 and Remark 3.9), we found that $\big(\widehat{b}(z) - z^{\alpha-3}\big)(0) \neq 0$ caused $(3-\alpha)$-order accuracy of the first discretization, where $\widehat{b}(z)$ is the discrete Laplace transform of $(b_k)_{k=0}^\infty$. This is the motivation for the second discretization. Let $\widehat{\beta}(z)$ be the discrete Laplace transform of $(\beta_k)_{k=0}^\infty$. The definition of the sequence $(\beta_k)_{k=0}^\infty$ implies*

$$\begin{aligned}\widehat{\beta}(z) &= \widehat{b}(z) + 2\sin(\alpha\pi/2)\sum_{k=1}^{\infty}(2k\pi)^{\alpha-3} \\ &= \widehat{b}(z) - \big(\widehat{b}(z) - z^{\alpha-3}\big)(0) \quad \textit{(by (8))}.\end{aligned}$$

Hence, $\big(\widehat{\beta}(z) - z^{\alpha-3}\big)(0) = 0$.

In the rest of Section 3, we shall use the well-known Laplace transform technique to analyze Discretizations 1 and 2. Firstly, we prove that the discrete Laplace transform of numerical solutions are well defined (i.e. they will not blow up in some places). Secondly, we give the integral representations of the exact and numerical solutions. Finally, we establish the error estimates by comparing the differences between the above two integrals.

3.1 Stability of the two discretizations

By an energy argument, it is easy to derive the following stability estimate of Discretization 1.

Lemma 3.1. *For each $m \in \mathbb{N}_{>0}$,*

$$|Y_m| \leqslant C_\alpha \Big(|y_0| + \lambda^{-1/2} \big(t_m^{1-\alpha/2} |y_1| + \|f\|_{{}_0H^{(1-\alpha)/2}(0,t_m)} \big) \Big). \tag{6}$$

Proof. Multiplying both sides of (4) by $\tau^{1-\alpha}(Y_{k+1} - Y_k)$ and summing over k from 0 to $m-1$, we obtain

$$\langle \mathrm{D}_{0+}^{\alpha-1} Y', Y' \rangle_{(0,t_m)} + \lambda \langle Y, Y' \rangle_{(0,t_m)} = \langle f, Y' \rangle_{(0,t_m)} + y_1 \langle \mathrm{D}_{0+}^{(\alpha-1)} 1, Y' \rangle_{(0,t_m)},$$

where, for each $k \in \mathbb{N}$, Y is linear on the interval $[t_k, t_{k+1}]$ and $Y(t_k) = Y_k$. A straightforward computation then gives

$$\begin{aligned}
& \langle \mathrm{D}_{0+}^{\alpha-1} Y', Y' \rangle_{(0,t_m)} + \lambda \langle Y, Y' \rangle_{(0,t_m)} \\
& \leqslant |\langle f, Y' \rangle_{(0,t_m)} + y_1 \langle \mathrm{D}_{0+}^{\alpha-1} 1, Y' \rangle_{(0,t_m)}| \\
& = |\langle \mathrm{D}_{0+}^{(\alpha-1)/2} \mathrm{D}_{0+}^{(1-\alpha)/2} f, Y' \rangle_{(0,t_m)}| + |y_1 \langle \mathrm{D}_{0+}^{(\alpha-1)/2} \mathrm{D}_{0+}^{(\alpha-1)/2} 1, Y' \rangle_{(0,t_m)}| \\
& = |\langle \mathrm{D}_{0+}^{(1-\alpha)/2} f, \mathrm{D}_{t_m-}^{(\alpha-1)/2} Y' \rangle_{(0,t_m)}| + |y_1 \langle \mathrm{D}_{0+}^{(\alpha-1)/2} 1, \mathrm{D}_{t_m-}^{(\alpha-1)/2} Y' \rangle_{(0,t_m)}| \\
& \leqslant \Big(\|\mathrm{D}_{0+}^{(1-\alpha)/2} f\|_{L^2(0,t_m)} + |y_1| \|\mathrm{D}_{0+}^{(\alpha-1)/2} 1\|_{L^2(0,t_m)} \Big) \|\mathrm{D}_{t_m-}^{(\alpha-1)/2} Y'\|_{L^2(0,t_m)} \\
& \leqslant C_\alpha \Big(\|\mathrm{D}_{0+}^{(1-\alpha)/2} f\|_{L^2(0,t_m)} + t_m^{1-\alpha/2} |y_1| \Big) \|\mathrm{D}_{t_m-}^{(\alpha-1)/2} Y'\|_{L^2(0,t_m)}.
\end{aligned}$$

Using integration by parts yields

$$\langle Y, Y' \rangle_{(0,t_m)} = (Y_m^2 - Y_0^2)/2,$$

and by Lemma 2.1 we have

$$C_1 \|\mathrm{D}_{t_m-}^{(\alpha-1)/2} Y'\|_{L^2(0,t_m)}^2 \leqslant C_2 \|\mathrm{D}_{0+}^{(\alpha-1)/2} Y'\|_{L^2(0,t_m)}^2 \leqslant \langle \mathrm{D}_{0+}^{\alpha-1} Y', Y' \rangle_{(0,t_m)},$$

where C_1 and C_2 are two positive constants depending only on α. By the above three estimates and the Young's inequality with ϵ, a simple calculation gives

$$|Y_m| \leqslant C_\alpha \left(|y_0| + \lambda^{-1/2} \Big(t_m^{1-\alpha/2} |y_1| + \|\mathrm{D}_{0+}^{(1-\alpha)/2} f\|_{L^2(0,t_m)} \Big) \right).$$

Therefore, (2) implies (6) and thus concludes the proof. ■

To derive the stability of Discretization 2, for $z \in \mathbb{C}_+ := \{w \in \mathbb{C} : \operatorname{Re} w > 0\}$, we introduce the discrete Laplace transform of $(b_k)_{k=0}^\infty$ by that

$$\widehat{b}(z) := \sum_{k=0}^\infty b_k e^{-kz}.$$

By the routine analytic continuation technique, $\widehat{b}$ has a Hankel integral representation (see [35])

$$\widehat{b}(z) = \frac{1}{2\pi i} \int_{-\infty}^{(0+)} \frac{w^{\alpha-3}}{e^{z-w} - 1} \, \mathrm{d}w, \quad z \in \mathbb{C} \setminus (-\infty, 0], \tag{7}$$

where $\int_{-\infty}^{(0+)}$ means an integral on a piecewise smooth and non-self-intersecting path enclosing the negative real axis and orienting counterclockwise, 0 and $\{z + 2k\pi i \neq 0 : k \in \mathbb{Z}\}$ lie on the different sides of this path, and $w^{\alpha-3}$ is evaluated in the sense that

$$w^{\alpha-3} = e^{(\alpha-3)\,\mathrm{Log}\,w}.$$

Therefore, by Cauchy's integral theorem and Cauchy's integral formula, we have (see [35, (13.1)])

$$\widehat{b}(z) = \sum_{k=-\infty}^{\infty} (z + 2k\pi i)^{\alpha-3} \tag{8}$$

for all $z \in \mathbb{C} \setminus (-\infty, 0]$ satisfying $-2\pi < \mathrm{Im}\, z < 2\pi$. From (7) it follows that

$$\overline{\widehat{b}(z)} = \widehat{b}(\overline{z}) \text{ for all } z \in \mathbb{C} \setminus (-\infty, 0].$$

From (8) it follows that

$$\widehat{b}(z) - z^{\alpha-3} \text{ is analytic on } \{w \in \mathbb{C} : |\mathrm{Im}\, w| < 2\pi\}.$$

Remark 3.4. *The $\widehat{b}(z)$ also has another representation [35],*

$$\widehat{b}(z) = \frac{Li_{\alpha-2}(e^{-z})}{\Gamma(3-\alpha)},$$

where the polylogarithm is defined by

$$Li_p(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^p}, \quad \text{for } |z| < 1 \text{ and } p \in \mathbb{C}.$$

Lemma 3.2. *For each $m \in \mathbb{N}_{>0}$,*

$$|\mathcal{Y}_m| \leqslant C_\alpha \Big(|y_0| + \lambda^{-1/2} \big(t_m^{1-\alpha/2} |y_1| + \|f\|_{{}_0H^{(1-\alpha)/2}(0,t_m)} \big) \Big). \tag{9}$$

Proof. In virtue of the proof of Lemma 3.1, it suffices to prove

$$\sum_{k=0}^{m} Z_k \delta_k \leqslant C_\alpha \sum_{k=0}^{m} \mathcal{Z}_k \delta_k, \tag{10}$$

where

$$\delta_j := \begin{cases} \mathcal{Y}_{j+1} - \mathcal{Y}_j, & 0 \leqslant j < m, \\ 0, & m \leqslant j < \infty, \end{cases}$$

$$Z_k := (b_{k+1} - b_k)\delta_0 + \sum_{j=1}^{k} (b_{k-j+1} - b_{k-j})(\delta_j - \delta_{j-1}),$$

$$\mathcal{Z}_k := (\beta_{k+1} - \beta_k)\delta_0 + \sum_{j=1}^{k} (\beta_{k-j+1} - \beta_{k-j})(\delta_j - \delta_{j-1}).$$

To this end, we proceed as follows. For $z \in \mathbb{C}_+$, let $\widehat{\beta}(z)$, $\widehat{\delta}(z)$, $\widehat{Z}(z)$ and $\widehat{\mathcal{Z}}(z)$ be the discrete Laplace transforms of $(\beta_k)_{k=0}^{\infty}$, $(\delta_k)_{k=0}^{\infty}$, $(Z_k)_{k=0}^{\infty}$ and $(\mathcal{Z}_k)_{k=0}^{\infty}$, respectively. It is easy to verify that $\widehat{\beta}$, $\widehat{\delta}$, $\widehat{Z}$ and $\widehat{\mathcal{Z}}$ are analytic on $\mathbb{C}_+$. A straightforward computation gives that, for $z \in \mathbb{C}_+$,

$$\widehat{Z}(z) = e^{-z}(e^z - 1)^2 \widehat{b}(z)\widehat{\delta}(z),$$
$$\widehat{\mathcal{Z}}(z) = e^{-z}(e^z - 1)^2 \widehat{\beta}(z)\widehat{\delta}(z),$$

and hence, by (8) and the fact

$$\widehat{\beta}(z) = \widehat{b}(z) + (\beta_1 - b_1)e^{-z},$$

we obtain

$$\begin{aligned}
&\sup_{0<x<1} \int_{-\pi}^{\pi} |\widehat{Z}(x+iy)|^2 \, \mathrm{d}y < \infty, \\
&\sup_{0<x<1} \int_{-\pi}^{\pi} |\widehat{\mathcal{Z}}(x+iy)|^2 \, \mathrm{d}y < \infty, \\
&\lim_{x\to 0+} \int_{-\pi}^{\pi} |\widehat{Z}(x+iy) - e^{-iy}(e^{iy}-1)^2 \widehat{b}(iy)\widehat{\delta}(iy)|^2 \, \mathrm{d}y = 0, \\
&\lim_{x\to 0+} \int_{-\pi}^{\pi} |\widehat{\mathcal{Z}}(x+iy) - e^{-iy}(e^{iy}-1)^2 \widehat{\beta}(iy)\widehat{\delta}(iy)|^2 \, \mathrm{d}y = 0.
\end{aligned}$$

Following the proof of the well-known Paley-Wiener Theorem [2, Theorem 1.8.3]), we easily conclude that

$$\begin{aligned}
&\sum_{k=0}^{\infty} Z_k^2 < \infty, \quad \sum_{k=0}^{\infty} \mathcal{Z}_k^2 < \infty, \\
&\sum_{k=0}^{\infty} Z_k e^{-iy} = e^{-iy}(e^{iy}-1)^2 \widehat{b}(iy)\widehat{\delta}(iy) \quad \text{in } L^2(-\pi,\pi;\mathrm{d}y), \\
&\sum_{k=0}^{\infty} \mathcal{Z}_k e^{-iy} = e^{-iy}(e^{iy}-1)^2 \widehat{\beta}(iy)\widehat{\delta}(iy) \quad \text{in } L^2(-\pi,\pi;\mathrm{d}y).
\end{aligned}$$

Therefore, by the famous Parseval's theorem,

$$\begin{aligned}
\sum_{k=0}^{m} Z_k\delta_k = \sum_{k=0}^{\infty} Z_k\delta_k &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-iy}(e^{iy}-1)^2 \widehat{b}(iy)|\widehat{\delta}(iy)|^2 \, \mathrm{d}y \\
&= \frac{1}{\pi}\int_{0}^{\pi} \operatorname{Re}\left(e^{-iy}(e^{iy}-1)^2 \widehat{b}(iy)\right)|\widehat{\delta}(iy)|^2 \, \mathrm{d}y. \qquad (11)
\end{aligned}$$

Similarly,

$$\sum_{k=0}^{m} \mathcal{Z}_k\delta_k = \frac{1}{\pi}\int_{0}^{\pi} \operatorname{Re}\left(e^{-iy}(e^{iy}-1)^2 \widehat{\beta}(iy)\right)|\widehat{\delta}(iy)|^2 \, \mathrm{d}y. \qquad (12)$$

In addition, a straightforward calculation gives, by (8), that

$$\begin{aligned}
&\operatorname{Re}\left(e^{-iy}(e^{iy}-1)^2 \widehat{\beta}(iy)\right) \\
&= 2(1-\cos y)\sin\left(\frac{\alpha\pi}{2}\right)\sum_{k=1}^{\infty}\left((2k\pi - y)^{\alpha-3} + (2k\pi + y - 2\pi)^{\alpha-3} - 2\cos y(2k\pi)^{\alpha-3}\right) \\
&> C_\alpha(1-\cos y)\sum_{k=1}^{\infty}\left((2k\pi - y)^{\alpha-3} + (2k\pi + y - 2\pi)^{\alpha-3}\right) \\
&> C_\alpha \operatorname{Re}\left(e^{-iy}(e^{iy}-1)^2 \widehat{b}(iy)\right), \qquad (13)
\end{aligned}$$

for all $y \in [-\pi,\pi]\setminus\{0\}$. Finally, combining (11), (12) and (13) yields (10) and thus concludes the proof. ■

3.2 Convergence of Discretization 1

3.2.1 Integral representation of Y_k

For any $z \in \mathbb{C}_+$, let $\widehat{Y}(z)$ be the discrete Laplace transform of $(Y_k)_{k=0}^{\infty}$. In virtue of Lemma 3.1, $\widehat{Y}$ is analytic on $\mathbb{C}_+$. Multiplying both sides of (4) by e^{-kz} and summing over k from 0 to ∞, we obtain

$$\begin{aligned}(\psi(z) + \mu(e^z+1))\widehat{Y}(z) = ((e^z-1)^2\widehat{b}(z) + \mu e^z)y_0 + \tau(e^z-1)\widehat{b}(z)y_1 +& \\ + \tau^{\alpha-1}\sum_{k=0}^{\infty}\int_{t_k}^{t_{k+1}} f(t)\,\mathrm{d}te^{-kz}, \quad \forall z \in \mathbb{C}_+,&\end{aligned} \tag{14}$$

where

$$\psi(z) := e^{-z}(e^z-1)^3\widehat{b}(z). \tag{15}$$

By the properties of the function $\widehat{b}$ in the previous subsection, ψ has an analytic continuation as follows:

$$\psi(z) = e^{-z}(e^z-1)^3\sum_{k=-\infty}^{\infty}(z+2k\pi i)^{\alpha-3} \tag{16}$$

for all $z \in \mathbb{C}\setminus(-\infty,0]$ satisfying $-2\pi < \operatorname{Im} z < 2\pi$. Moreover,

$$\overline{\psi(z)} = \psi(\overline{z}) \text{ for all } z \in \mathbb{C}\setminus(-\infty,0] \text{ with } -2\pi < \operatorname{Im} z < 2\pi, \tag{17}$$

$$\psi(z) - e^{-z}(e^z-1)^3z^{\alpha-3} \text{ is analytic on } \{w \in \mathbb{C} : |\operatorname{Im} w| < 2\pi\}, \tag{18}$$

and

$$\lim_{r\to0+}\frac{\psi(re^{i\theta})}{r^\alpha(\cos(\alpha\theta)+i\sin(\alpha\theta))} = 1 \quad \text{uniformly for all } -\pi<\theta<\pi. \tag{19}$$

In the rest of Section 3, we assume that $\mu \leqslant \mu_0$, where μ_0 is a given positive constant.

Remark 3.5. *Let λ be any eigen value of discrete Laplace operator $-\Delta_h$, then $\mu \leqslant \mu_0$ implies that τ^α/h^2 is bounded. The L1 scheme in Discretization 1 reduces to the second order central difference scheme when $\alpha = 2$, and this scheme require that τ/h is bounded (stability), which is consistent with the condition that $\mu \leqslant \mu_0$.*

Lemma 3.3. *There exists $\frac{\pi}{2} < \theta_{\alpha,\mu_0} \leqslant \frac{\alpha+2}{4\alpha}\pi$ depending only on α and μ_0 such that*

$$\begin{aligned}\psi(z)+\mu(1+e^z) \neq 0 \quad &\textit{for all } 0<\mu\leqslant\mu_0 \textit{ and} \\ z \in \{w\in\mathbb{C}:\ 0<|\operatorname{Im} w| \leqslant \pi,\ &\frac{\pi}{2} \leqslant |\operatorname{Arg} w| \leqslant \theta_{\alpha,\mu_0}\}.\end{aligned} \tag{20}$$

Proof. By (19), there exists $0 < r_\alpha < \pi$, depending only on α, such that $\operatorname{Im}\big((1+e^z)^{-1}\psi(z)\big) > 0$ and hence

$$\begin{aligned}\psi(z)+\mu(1+e^z) \neq 0 \quad &\text{for all } 0<\mu\leqslant\mu_0 \text{ and} \\ z \in \Big\{w\in\mathbb{C}: \frac{\pi}{2} \leqslant \operatorname{Arg} w \leqslant &\frac{\alpha+2}{4\alpha}\pi, 0<\operatorname{Im} w\leqslant r_\alpha\Big\}.\end{aligned} \tag{21}$$

From (17) and (21), it remains therefore to show that there exists $\frac{\pi}{2} < \theta_{\alpha,\mu_0} \leqslant \frac{\alpha+2}{4\alpha}\pi$ such that

$$\begin{aligned}\psi(z)+\mu(1+e^z) \neq 0 \quad &\text{for all } \ 0<\mu\leqslant\mu_0 \text{ and} \\ z\in\Big\{w\in\mathbb{C}: \frac{\pi}{2}\leqslant \operatorname{Arg} w \leqslant \theta_{\alpha,\mu_0}, r_\alpha &< \operatorname{Im} w \leqslant \pi\Big\}.\end{aligned} \tag{22}$$

To this end, we proceed as follows. For $0<y\leqslant\pi$, by (16) we have

$$\begin{aligned}
\psi(iy) &= e^{-iy}(e^{iy}-1)^3 \sum_{k=-\infty}^{\infty}(iy+2k\pi i)^{\alpha-3} \\
&= e^{-iy}(e^{iy}-1)^3\Big(\sum_{k=-\infty}^{-1}(-2k\pi-y)^{\alpha-3}(-i)^{\alpha-3}+\sum_{k=0}^{\infty}(2k\pi+y)^{\alpha-3}i^{\alpha-3}\Big) \\
&= e^{-iy}(e^{iy}-1)^3\Big(\sum_{k=1}^{\infty}(2k\pi-y)^{\alpha-3}e^{i(3-\alpha)\pi/2}+\sum_{k=0}^{\infty}(2k\pi+y)^{\alpha-3}e^{-i(3-\alpha)\pi/2}\Big) \\
&= e^{-iy}(e^{iy}-1)^3\Big(-\sum_{k=1}^{\infty}(2k\pi-y)^{\alpha-3}e^{i(1-\alpha)\pi/2}-\sum_{k=0}^{\infty}(2k\pi+y)^{\alpha-3}e^{-i(1-\alpha)\pi/2}\Big) \\
&= e^{-iy}(e^{iy}-1)^3(A(y)+iB(y)),
\end{aligned} \tag{23}$$

where

$$\begin{aligned}
A(y) &:= -\cos((\alpha-1)\pi/2)\sum_{k=0}^{\infty}(2k\pi+2\pi-y)^{\alpha-3}+(2k\pi+y)^{\alpha-3}, \\
B(y) &:= \sin((\alpha-1)\pi/2)\sum_{k=0}^{\infty}(2k\pi+2\pi-y)^{\alpha-3}-(2k\pi+y)^{\alpha-3}.
\end{aligned}$$

Moreover,

$$\operatorname{Im}\Big((1+e^{iy})^{-1}\psi(iy)\Big) = 4A(y)\frac{(\cos y-1)\sin y}{|e^{iy}+e^{2iy}|^2} > 0, \quad \forall 0<y<\pi. \tag{24}$$

Inserting $y=\pi$ into (23) yields

$$\psi(\pi i) = 8A(\pi) < 0 = \mu(1+e^{\pi i}),$$

so that by the continuity of ψ, there exists $0<r^1_{\alpha,\mu_0}\leqslant r_\alpha\tan((2-\alpha)/(4\alpha)\pi)$ and $0<r^2_{\alpha,\mu_0}<\pi$, depending only on α and μ_0, such that

$$\begin{aligned}\psi(z)+\mu(1+e^z)\neq 0 \quad &\text{for all } \ 0<\mu\leqslant\mu_0 \text{ and} \\ z\in\{w\in\mathbb{C}: -r^1_{\alpha,\mu_0}\leqslant \operatorname{Re} w &\leqslant 0, r^2_{\alpha,\mu_0}\leqslant \operatorname{Im} w\leqslant \pi\}.\end{aligned} \tag{25}$$

For the case of $r_\alpha\leqslant \operatorname{Im} w\leqslant r^2_{\alpha,\mu_0}$, by (24) and the continuity of ψ, it follows that there exists $0<r^3_{\alpha,\mu_0}\leqslant r^1_{\alpha,\mu_0}$, depending only on α and μ_0, such that $\operatorname{Im}\big((1+e^z)^{-1}\psi(z)\big)>0$ and hence

$$\begin{aligned}\psi(z)+\mu(1+e^z)\neq 0 \quad &\text{for all } 0<\mu\leqslant\mu_0 \text{ and} \\ z\in\{w\in\mathbb{C}: -r^3_{\alpha,\mu_0}\leqslant \operatorname{Re} w &\leqslant 0,\ r_\alpha\leqslant \operatorname{Im} w\leqslant r^2_{\alpha,\mu_0}\}.\end{aligned} \tag{26}$$

Finally, letting $\theta_{\alpha,\mu_0} := \pi/2+\arctan(r^3_{\alpha,\mu_0}/\pi)$ yields (22), by (25) and (26). This completes the proof. ■

Remark 3.6. *The* r^1_{α,μ_0} *in the above proof will approximate 0, when* $\mu_0 \to \infty$. *Hence,* $\theta_{\alpha,\mu_0} \to (\pi/2)+$ *as* $\mu_0 \to \infty$.

Lemma 3.4. *For each* $z \in \mathbb{C}_+$ *and* $\mu > 0$,

$$\psi(z) + \mu(e^z + 1) \neq 0. \tag{27}$$

Proof. Assume that $z \in \mathbb{C}_+$ satisfies that

$$\psi(z) + \mu(e^z + 1) = 0. \tag{28}$$

It follows that

$$\widehat{b}(z) = -\mu e^z (e^z + 1)(e^z - 1)^{-3},$$

and hence

$$(e^z - 1)^2 \widehat{b}(z) + \mu e^z = -2\mu e^z (e^z - 1)^{-1}.$$

In the case that $y_0 = 1$, $y_1 = 0$ and $f \equiv 0$, from (14) and (28) we obtain

$$(e^z - 1)^2 \widehat{b}(z) + \mu e^z = 0.$$

Since the above two equations are contradictory, this proves the lemma. ■

Remark 3.7. *The above two lemmas indicate that* $\psi(z) + \mu(e^z + 1) \neq 0$ *in some places. Hence, by (14),* $\widehat{Y}(z)$ *will not blow up in these places. Then it is reasonable to give the integral representation of the numerical solution* Y.

For the sake of simplicity, in the rest of this subsection (i.e., Subsection 3.2) we use the following conventions: μ_0 is a positive constant and $\mu \leqslant \mu_0$; θ_{α,μ_0} defined in Lemma 3.3 is abbreviated to θ. Define

$$\Upsilon := (\infty, 0]e^{-i\theta} \cup [0, \infty)e^{i\theta},$$
$$\Upsilon_1 := \{z \in \Upsilon : \ |\operatorname{Im} z| \leqslant \pi\},$$

where Υ is oriented so that $\operatorname{Im} z$ increases along Υ and Υ_1 inherit the orientation of Υ. In addition, if the integral over Υ/Υ_1 is divergent, caused by the singularity of the underlying integrand near the origin, then Υ/Υ_1 should be deformed so that the origin lies at its left side; for example,

$$\Upsilon := (\infty, \epsilon]e^{-i\theta} \cup \{\epsilon e^{i\varphi} : -\theta \leqslant \varphi \leqslant \theta\} \cup [\epsilon, \infty)e^{i\theta},$$

where ϵ is an arbitrary positive constant.

Lemma 3.5 ([12])**.** *For any* $t > 0$,

$$\begin{aligned} y(t) = \frac{1}{2\pi i} \int_{\Upsilon} e^{(t/\tau)z} \frac{y_0 z^{\alpha-1} + \tau y_1 z^{\alpha-2}}{z^\alpha + 2\mu} \,\mathrm{d}z \\ + \int_0^t E(t-s) f(s) \,\mathrm{d}s, \end{aligned} \tag{29}$$

where

$$E(t) := \frac{\tau^{\alpha-1}}{2\pi i} \int_{\Upsilon} e^{(t/\tau)z} (z^\alpha + 2\mu)^{-1} \,\mathrm{d}z. \tag{30}$$

Lemma 3.6. *For each $k \in \mathbb{N}_{>0}$,*

$$
\begin{aligned}
Y_k = \frac{1}{2\pi i} \int_{\Upsilon_1} e^{kz} \frac{\big((e^z-1)^2\widehat{b}(z) - \psi(z)/2 + \mu(e^z-1)/2\big)y_0 + \tau(e^z-1)\widehat{b}(z)y_1}{\psi(z)+\mu(e^z+1)} \, \mathrm{d}z \\
+ \int_0^{t_k} \widetilde{E}(t_k - t) f(t) \, \mathrm{d}t,
\end{aligned}
\tag{31}
$$

where

$$
\widetilde{E}(t) := \tau^{\alpha-1} E_{\lceil t/\tau \rceil}, \quad t > 0, \tag{32}
$$

with $\lceil \cdot \rceil$ being the ceiling function and

$$
E_j := \frac{1}{2\pi i} \int_{\Upsilon_1} e^{jz} (\psi(z) + \mu(e^z+1))^{-1} \, \mathrm{d}z, \quad \textit{for } j \in \mathbb{Z}. \tag{33}
$$

Proof. A straightforward computation yields, by (14) and Lemma 3.4, that

$$
\widehat{Y}(z) = \frac{\big((e^z-1)^2\widehat{b}(z) + \mu e^z\big)y_0 + \tau(e^z-1)\widehat{b}(z)y_1 + \tau^{\alpha-1}\sum_{j=0}^{\infty}\int_{t_j}^{t_{j+1}} f(t) \mathrm{d}t e^{-jz}}{\psi(z)+\mu(e^z+1)}, \tag{34}
$$

for all $z \in \mathbb{C}_+$. Hence,

$$
Y_k = \frac{1}{2\pi i} \int_{a-i\pi}^{a+i\pi} e^{kz} \widehat{Y}(z) \, \mathrm{d}z = \mathbb{I}_1 + \mathbb{I}_2 + \mathbb{I}_3, \quad \textit{for } 0 < a < \infty,
$$

where

$$
\begin{aligned}
\mathbb{I}_1 &:= \frac{y_0}{2\pi i} \int_{a-i\pi}^{a+i\pi} e^{kz} \frac{(e^z-1)^2\widehat{b}(z) + \mu e^z}{\psi(z)+\mu(e^z+1)} \, \mathrm{d}z, \\
\mathbb{I}_2 &:= \frac{\tau y_1}{2\pi i} \int_{a-i\pi}^{a+i\pi} e^{kz} \frac{(e^z-1)\widehat{b}(z)}{\psi(z)+\mu(e^z+1)} \, \mathrm{d}z, \\
\mathbb{I}_3 &:= \frac{\tau^{\alpha-1}}{2\pi i} \int_{a-i\pi}^{a+i\pi} e^{kz} \frac{\sum_{j=0}^{\infty}\int_{t_j}^{t_{j+1}} f(t) \, \mathrm{d}t e^{-jz}}{\psi(z)+\mu(e^z+1)} \, \mathrm{d}z.
\end{aligned}
$$

Here, by Lemma 3.4 and Cauchy's integral theorem we have

$$
\begin{aligned}
\mathbb{I}_1 &= \frac{1}{2\pi i} \int_{a-i\pi}^{a+i\pi} e^{kz} \frac{(e^z-1)^2\widehat{b}(z) + \mu e^z}{\psi(z)+\mu(e^z+1)} \, \mathrm{d}z \\
&= \frac{1}{2\pi i} \int_{a-i\pi}^{a+i\pi} e^{kz} \Big(\frac{(e^z-1)^2\widehat{b}(z) + \mu e^z}{\psi(z)+\mu(e^z+1)} - \frac{1}{2} \Big) \, \mathrm{d}z \\
&= \frac{1}{2\pi i} \int_{a-i\pi}^{a+i\pi} e^{kz} \frac{(e^z-1)^2\widehat{b}(z) - \psi(z)/2 + \mu(e^z-1)/2}{\psi(z)+\mu(e^z+1)} \, \mathrm{d}z \\
&= \frac{y_0}{2\pi i} \int_{\Upsilon_1} e^{kz} \frac{(e^z-1)^2\widehat{b}(z) - \psi(z)/2 + \mu(e^z-1)/2}{\psi(z)+\mu(e^z+1)} \, \mathrm{d}z,
\end{aligned}
$$

where the latter equality follows from Lemma 3.3 and the fact that

$$
\begin{aligned}
&e^{kz} \frac{(e^z-1)^2\widehat{b}(z) - \psi(z)/2 + \mu(e^z-1)/2}{\psi(z)+\mu(e^z+1)} \\
&= e^{k(z+2\pi i)} \frac{(e^{z+2\pi i}-1)^2\widehat{b}(z+2\pi i) - \psi(z+2\pi i)/2 - \mu(e^{z+2\pi i}-1)/2}{\psi(z+2\pi i)+\mu(e^{z+2\pi i}+1)}
\end{aligned}
$$

for $\operatorname{Re} z \geqslant -\pi \cot(\theta)$ and $\operatorname{Im} z = -\pi$.

A similar argument gives

$$\mathbb{I}_2 = \frac{y_1 \tau}{2\pi i} \int_{\Upsilon_1} e^{kz} \frac{(e^z - 1)\widehat{b}(z)}{\psi(z) + \mu(e^z + 1)} \,\mathrm{d}z. \tag{35}$$

We now turn to $\mathbb{I}_3$. Using Fubini's theorem and Cauchy's integral theorem, we have

$$\begin{aligned}
\mathbb{I}_3 &= \frac{\tau^{\alpha-1}}{2\pi i} \int_{a-i\pi}^{a+i\pi} e^{kz} \frac{\sum_{j=0}^{\infty} \int_{t_j}^{t_{j+1}} f(t) \,\mathrm{d}t e^{-jz}}{\psi(z) + \mu(e^z + 1)} \,\mathrm{d}z \\
&= \sum_{j=0}^{\infty} \int_{t_j}^{t_{j+1}} f(t) \,\mathrm{d}t \frac{\tau^{\alpha-1}}{2\pi i} \int_{a-i\pi}^{a+i\pi} e^{(k-j)z} (\psi(z) + \mu(e^z + 1))^{-1} \,\mathrm{d}z \\
&= \sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} f(t) \,\mathrm{d}t \frac{\tau^{\alpha-1}}{2\pi i} \int_{a-i\pi}^{a+i\pi} e^{(k-j)z} (\psi(z) + \mu(e^z + 1))^{-1} \,\mathrm{d}z \\
&= \sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} f(t) \,\mathrm{d}t \tau^{\alpha-1} E_{k-j} \\
&= \int_0^{t_k} f(t) \widetilde{E}(t_k - t) \,\mathrm{d}t.
\end{aligned}$$

Combining the estimates of $\mathbb{I}_1$, $\mathbb{I}_2$ and $\mathbb{I}_3$ proves (31) and hence the lemma. ∎

3.2.2 Convergence for $f \equiv 0$

Lemma 3.7. *For each $z \in \Upsilon_1 \setminus \{0\}$,*

$$|\psi(z) + \mu(1 + e^z)| > C_{\alpha,\mu_0}(\mu + |z|^\alpha). \tag{36}$$

Proof. By (16) there exists a continuous function g on $[0, \pi/\sin\theta]$ such that

$$(1 + re^{i\theta})^{-1} \psi(re^{i\theta}) = r^\alpha e^{i\alpha\theta}/2 + r^{\alpha+1} g(r).$$

It follows that

$$\begin{aligned}
&|\mu + (1 + re^{i\theta})^{-1} \psi(re^{i\theta})|^2 \\
&= |\mu + r^\alpha e^{i\alpha\theta}/2 + r^{\alpha+1} g(r)|^2 \\
&\geqslant |\mu + r^\alpha e^{i\alpha\theta}/2|^2/2 - r^{2(\alpha+1)} |g(r)|^2 \\
&= (\mu + r^\alpha \cos(\alpha\theta)/2)^2/2 + r^{2\alpha} \sin(\alpha\theta)^2/8 - r^{2(\alpha+1)} |g(r)|^2,
\end{aligned} \tag{37}$$

and hence there exists $0 < r_{\alpha,\mu_0} < \pi/\sin\theta$, depending only on α and μ_0, such that

$$|\mu + (1 + re^{i\theta})^{-1} \psi(re^{i\theta})| > C_{\alpha,\mu_0}(\mu + r^\alpha) \quad \text{for all } 0 < r \leqslant r_{\alpha,\mu_0}.$$

Therefore,

$$\inf_{0 < r \leqslant r_{\alpha,\mu_0}} \frac{|\mu + (1 + re^{i\theta})^{-1} \psi(re^{i\theta})|}{\mu + r^\alpha} > C_{\alpha,\mu_0}.$$

Using this estimate and

$$|1 + re^{i\theta}| > C_{\alpha,\mu_0}, \quad \text{for all } 0 \leqslant r \leqslant \pi/\sin\theta,$$

we have

$$\inf_{0<r\leqslant r_{\alpha,\mu_0}} \frac{|\psi(re^{i\theta}) + \mu(1 + r^{i\theta})|}{\mu + r^\alpha} > C_{\alpha,\mu_0}.$$

In addition, applying the extreme value theorem yields, by (20), that

$$\inf_{r_{\alpha,\mu_0}\leqslant r\leqslant \pi/\sin\theta} \frac{|\psi(re^{i\theta}) + \mu(re^{i\theta} + 1)|}{\mu + r^\alpha} > C_{\alpha,\mu_0}.$$

Together, the above two estimates show

$$\inf_{0<r\leqslant \pi/\sin\theta} \frac{|\psi(re^{i\theta}) + \mu(re^{i\theta} + 1)|}{\mu + r^\alpha} > C_{\alpha,\mu_0},$$

which completes the proof. ■

Lemma 3.8. *For each* $z \in \Upsilon_1 \setminus \{0\}$,

$$|z + 2\mu z^{1-\alpha}| > C_\alpha(|z| + \mu|z|^{1-\alpha}). \tag{38}$$

Proof. A simple calculation yields

$$\begin{aligned}
|z + 2\mu z^{1-\alpha}| &= |z||1 + 2\mu z^{-\alpha}| \\
&= r|1 + 2\mu r^{-\alpha}\cos(-\alpha\theta) + 2i\mu r^{-\alpha}\sin(-\alpha\theta)| \\
&\geqslant C_\alpha \mu r^{1-\alpha}.
\end{aligned}$$

Analogously, we have

$$\begin{aligned}
|z + 2\mu z^{1-\alpha}| &= |z|^{1-\alpha}|z^\alpha + 2\mu| \\
&= r^{1-\alpha}|2\mu + r^\alpha\cos(\alpha\theta) + ir^\alpha\sin(\alpha\theta)| \\
&\geqslant C_\alpha r.
\end{aligned}$$

Combining above two estimates proves (38) and hence the lemma. ■

Theorem 3.1. *For each* $k \in \mathbb{N}_{>0}$,

$$|y(t_k) - Y_k| \leqslant C_{\alpha,\mu_0}\tau^{3-\alpha}\big(t_k^{\alpha-3}|y_0| + t_k^{\alpha-2}|y_1|\big). \tag{39}$$

Proof. From (29) and (31), it follows that

$$y(t_k) - Y_k = \mathbb{I}_1 + \mathbb{I}_2 + \mathbb{I}_3,$$

where

$$\begin{aligned}
\mathbb{I}_1 &:= \frac{1}{2\pi i}\int_{\Upsilon\setminus\Upsilon_1} e^{kz}\frac{y_0 z^{\alpha-1} + \tau y_1 z^{\alpha-2}}{z^\alpha + 2\mu}\,\mathrm{d}z, \\
\mathbb{I}_2 &:= \frac{y_0}{2\pi i}\int_{\Upsilon_1} e^{kz}\Big(\frac{z^{\alpha-1}}{z^\alpha + 2\mu} - \frac{(e^z - 1)^2\widehat{b}(z) - \psi(z)/2 + \mu(e^z - 1)/2}{\psi(z) + \mu(e^z + 1)}\Big)\,\mathrm{d}z, \\
\mathbb{I}_3 &:= \frac{\tau y_1}{2\pi i}\int_{\Upsilon_1} e^{kz}\Big(\frac{z^{\alpha-2}}{z^\alpha + 2\mu} - \frac{\widehat{b}(z)(e^z - 1)}{\psi(z) + \mu(e^z + 1)}\Big)\,\mathrm{d}z.
\end{aligned}$$

Let us first estimate $\mathbb{I}_1$. A simple calculation gives

$$\mathbb{I}_1 = \frac{1}{\pi}\operatorname{Im}\int_{\pi/\sin\theta}^{\infty} e^{kre^{i\theta}}\frac{y_0(re^{i\theta})^{\alpha-1}+\tau y_1(re^{i\theta})^{\alpha-2}}{(re^{i\theta})^\alpha+2\mu}e^{i\theta}\,\mathrm{d}r,$$

and the fact $\pi/2<\theta<(\alpha+2)/(4\alpha)\pi$ implies

$$\begin{aligned}
&|\frac{y_0(re^{i\theta})^{\alpha-1}+\tau y_1(re^{i\theta})^{\alpha-2}}{(re^{i\theta})^\alpha+2\mu}e^{i\theta}| \\
\leqslant{}& \frac{|y_0|r^{\alpha-1}+\tau|y_1|r^{\alpha-2}}{|r^\alpha\cos(\alpha\theta)+2\mu+ir^\alpha\sin(\alpha\theta)|} \\
\leqslant{}& C_{\alpha,\mu_0}\big(|y_0|r^{-1}+\tau|y_1|r^{-2}\big).
\end{aligned}$$

Hence,

$$\begin{aligned}
|\mathbb{I}_1| &\leqslant C_{\alpha,\mu_0}\int_{\pi/\sin\theta}^{\infty} e^{kr\cos\theta}\big(|y_0|r^{-1}+\tau|y_1|r^{-2}\big)\,\mathrm{d}r \\
&\leqslant C_{\alpha,\mu_0}\big(|y_0|k^{-1}+\tau|y_1|k^{-1}\big)e^{k\pi\cot\theta}.
\end{aligned}\tag{40}$$

Then let us estimate $\mathbb{I}_2$. For $z\in\Upsilon_1\setminus\{0\}$, a straightforward calculation gives

$$\begin{aligned}
&|\psi(z)+\mu(1+e^z)-(z+2\mu z^{1-\alpha})\Big((e^z-1)^2\widehat{b}(z)-\psi(z)/2+\mu(e^z-1)/2\Big)| \\
<{}&C_\alpha\big(|z|^{\alpha+2}+\mu|z|^{3-\alpha}+\mu^2|z|^{2-\alpha}\big),
\end{aligned}$$

and so Lemmas 3.7 and 3.8 imply

$$\begin{aligned}
&|\frac{1}{z+2\mu z^{1-\alpha}}-\frac{(e^z-1)^2\widehat{b}(z)-e^{-z}(e^z-1)^3\widehat{b}(z)/2+\mu(e^z-1)/2}{\psi(z)+\mu(1+e^z)}| \\
<{}& C_{\alpha,\mu_0}\frac{|z|^{\alpha+2}+\mu|z|^{3-\alpha}+\mu^2|z|^{2-\alpha}}{(|z|+\mu|z|^{1-\alpha})(|z|^\alpha+\mu)}.
\end{aligned}$$

It follows that

$$|\mathbb{I}_2|\leqslant C_{\alpha,\mu_0}|y_0|\int_0^{\pi/\sin\theta} e^{kr\cos\theta}\frac{r^{\alpha+2}+\mu r^{3-\alpha}+\mu^2r^{2-\alpha}}{(r+\mu r^{1-\alpha})(r^\alpha+\mu)}\,\mathrm{d}r.$$

If $0<r<\mu^{1/\alpha}$ then

$$\begin{aligned}
&\frac{r^{\alpha+2}+\mu r^{3-\alpha}+\mu^2r^{2-\alpha}}{(r+\mu r^{1-\alpha})(r^\alpha+\mu)} \\
<{}&\mu^{-2}r^{\alpha-1}\big(r^{\alpha+2}+\mu r^{3-\alpha}+\mu^2r^{2-\alpha}\big) \\
={}&\mu^{-2}r^{2\alpha+1}+\mu^{-1}r^2+r<2r+r^{2-\alpha},
\end{aligned}$$

and if $\mu^{1/\alpha}<r$ then

$$\begin{aligned}
&\frac{r^{\alpha+2}+\mu r^{3-\alpha}+\mu^2r^{2-\alpha}}{(r+\mu r^{1-\alpha})(r^\alpha+\mu)} \\
<{}&r^{-\alpha-1}\big(r^{\alpha+2}+\mu r^{3-\alpha}+\mu^2r^{2-\alpha}\big) \\
={}&r+\mu r^{2-2\alpha}+\mu^2r^{1-2\alpha}<2r+r^{2-\alpha}.
\end{aligned}$$

Therefore,

$$|\mathbb{I}_2| \leqslant C_{\alpha,\mu_0}|y_0| \int_0^{\pi/\sin\theta} e^{kr\cos\theta} r^{2-\alpha}\,\mathrm{d}r \leqslant C_{\alpha,\mu_0}|y_0|k^{\alpha-3}. \tag{41}$$

Finally, a similar argument as that to derive (41) yields

$$|\mathbb{I}_3| \leqslant C_{\alpha,\mu_0}\tau k^{\alpha-2}|y_1|, \tag{42}$$

and then combining (40), (41) and (42) gives

$$\begin{aligned} |y(t_k) - Y_k| &\leqslant C_{\alpha,\mu_0}\big(k^{\alpha-3}|y_0| + \tau k^{\alpha-2}|y_1|\big) \\ &= C_{\alpha,\mu_0}\tau^{3-\alpha}\big(t_k^{\alpha-3}|y_0| + t_k^{\alpha-2}|y_1|\big), \end{aligned}$$

which proves (39) and hence this theorem. ∎

Remark 3.8. *In the above proof,*

$$\begin{aligned} &|\psi(z) + \mu(1+e^z) - (z + 2\mu z^{1-\alpha})\Big((e^z-1)^2\widehat{b}(z) - \psi(z)/2 + \mu(e^z-1)/2\Big)| \\ <&C_\alpha\big(|z|^{\alpha+2} + \mu|z|^{3-\alpha} + \mu^2|z|^{2-\alpha}\big), \end{aligned}$$

and the term $\mu|z|^{3-\alpha}$ leads to $(3-\alpha)$-order accuracy. If we choose a β such that (i.e. $(\widehat{\beta}(z) - z^{\alpha-3}) = 0$)

$$\begin{aligned} &|\Psi(z) + \mu(1+e^z) - (z + 2\mu z^{1-\alpha})\Big((e^z-1)^2\widehat{\beta}(z) - \Psi(z)/2 + \mu(e^z-1)/2\Big)| \\ <&C_\alpha\big(|z|^{\alpha+2} + \mu|z|^2 + \mu^2|z|^{2-\alpha}\big), \end{aligned}$$

then we can obtain 2-order accuracy, where $\Psi(z) = e^{-z}(e^z-1)^3\widehat{\beta}(z)$. This is the motivation of the second discretization.

3.2.3 Convergence for $y_0 = y_1 = 0$

Define

$$\mathcal{E}(t) := \int_0^t (E - \widetilde{E})(s)\,\mathrm{d}s, \quad t > 0,$$

where E and $\widetilde{E}$ are defined by (30) and (32), respectively.

Lemma 3.9. *For any $t_k < t \leqslant t_{k+1}$ with $k \in \mathbb{N}$,*

$$|\mathcal{E}(t)| < C_{\alpha,\mu_0}\varepsilon(\alpha,\tau,k)\tau^{3-\alpha}, \tag{43}$$

where

$$\varepsilon(\alpha,\tau,k) := \begin{cases} t_{k+1}^{2\alpha-3} & \text{if } 1 < \alpha < 3/2, \\ 1 + |\ln\tau| & \text{if } \alpha = 3/2, \\ 1 & \text{if } 3/2 < \alpha < 2. \end{cases} \tag{44}$$

Proof. Since the proof of the case $k = 0$ is simpler, we only prove the case $k \geqslant 1$. By Lemmas 3.3 and 3.4 and the fact that

$$(1 - e^{-z})(\psi(z) + \mu(e^z+1)) = (1 - e^{-(z+2\pi i)})(\psi(z+2\pi i) + \mu(e^{z+2\pi i}+1))$$

for all $z = x - i\pi$ with $x \geqslant \pi \cot\theta$, applying Cauchy integral theorem yields that

$$\int_{\Upsilon_1} \frac{1}{(1-e^{-z})(\psi(z)+\mu(e^z+1))} \,\mathrm{d}z = 0,$$

and using Cauchy integral theorem again gives

$$\int_{\Upsilon} \frac{1}{z(z^\alpha+2\mu)} \,\mathrm{d}z = 0.$$

Therefore, from (30) and (32) we have

$$\begin{aligned}
\mathcal{E}(t) &= \int_0^{t_k} E(s)\,\mathrm{d}s - \sum_{j=1}^{k} \tau^\alpha E_j + \int_{t_k}^{t} E(s)\,\mathrm{d}s - (t-t_k)E_{k+1} \\
&= \frac{\tau^\alpha}{2\pi i}\int_{\Upsilon} \frac{e^{kz}}{z(z^\alpha+2\mu)}\,\mathrm{d}z - \frac{\tau^\alpha}{2\pi i}\int_{\Upsilon_1} \frac{e^{kz}}{(1-e^{-z})(\psi(z)+\mu(e^z+1))}\,\mathrm{d}z \\
&\quad + \frac{\tau^\alpha}{2\pi i}\int_{\Upsilon} \frac{e^{(t/\tau)z}-e^{kz}}{z(z^\alpha+2\mu)}\,\mathrm{d}z - \frac{\tau^\alpha}{2\pi i}\int_{\Upsilon_1} \frac{(t/\tau-k)e^{(k+1)z}}{\psi(z)+\mu(e^z+1)}\,\mathrm{d}z.
\end{aligned}$$

Inserting $t = t_k$ into above equation yields

$$\mathcal{E}(t_k) = \mathbb{I}_1 + \mathbb{I}_2 + \mathbb{I}_3, \tag{45}$$

where

$$\begin{aligned}
\mathbb{I}_1 &:= \frac{\tau^\alpha}{2\pi i}\int_{\Upsilon\setminus\Upsilon_1} \frac{e^{t/\tau z}}{z(z^\alpha+2\mu)}\,\mathrm{d}z, \\
\mathbb{I}_2 &:= \frac{\tau^\alpha}{2\pi i}\int_{\Upsilon_1} \frac{e^{kz}}{z(z^\alpha+2\mu)} - \frac{e^{kz}(1-e^{-z})^{-1}}{\psi(z)+\mu(e^z+1)}\,\mathrm{d}z, \\
\mathbb{I}_3 &:= \frac{\tau^\alpha}{2\pi i}\int_{\Upsilon_1} e^{kz}\left(\frac{e^{(t/\tau-k)z}-1}{z(z^\alpha+2\mu)} - \frac{(t/\tau-k)e^z}{\psi(z)+\mu(e^z+1)}\right)\mathrm{d}z.
\end{aligned}$$

It is clear that

$$|\mathbb{I}_1| < C_{\alpha,\mu_0}\tau^\alpha \int_{\pi/\sin\theta}^{\infty} e^{t/\tau r\cos\theta} r^{-1-\alpha}\,\mathrm{d}r < C_{\alpha,\mu_0}\tau^{\alpha+1}t^{-1}e^{k\pi\cot\theta}. \tag{46}$$

Let us proceed to estimate $\mathbb{I}_2$. For $z \in \Upsilon_1 \setminus \{0\}$, a simple calculation yields

$$|\psi(z)+\mu(e^z+1) - z(z^\alpha+2\mu)(1-e^{-z})^{-1}| < C_{\alpha,\mu_0}(\mu|z|^2+|z|^3),$$

and Lemmas 3.7 and 3.8 imply

$$|z(z^\alpha+2\mu)(\psi(z)+\mu(e^z+1))| > C_{\alpha,\mu_0}|z|(|z|^{2\alpha}+\mu^2). \tag{47}$$

Hence, if $\mu^{1/\alpha} < \pi/\sin\theta$ then

$$\begin{aligned}
|\mathbb{I}_2| &< C_{\alpha,\mu_0}\tau^\alpha\bigg(\int_0^{\mu^{1/\alpha}} e^{kr\cos\theta}(\mu^{-1}r+\mu^{-2}r^2)\,\mathrm{d}r + \\
&\qquad\qquad \int_{\mu^{1/\alpha}}^{\pi/\sin\theta} e^{kr\cos\theta}(\mu r^{1-2\alpha}+r^{2-2\alpha})\,\mathrm{d}r\bigg) \\
&< C_{\alpha,\mu_0}\tau^\alpha \begin{cases} \int_0^{\pi/\sin\theta} e^{kr\cos\theta} r^{2-2\alpha}\,\mathrm{d}r & \text{if } 1<\alpha<3/2, \\ \int_0^{\mu^{1/\alpha}} \mu^{-1}r+\mu^{-2}r^2\,\mathrm{d}r + \int_{\mu^{1/\alpha}}^{\pi/\sin\theta} r^{1-\alpha}+r^{2-2\alpha}\,\mathrm{d}r & \text{if } 3/2\leqslant\alpha<2, \end{cases} \\
&< C_{\alpha,\mu_0}\tau^\alpha \begin{cases} k^{2\alpha-3} & \text{if } 1<\alpha<3/2, \\ 1+|\ln\tau| & \text{if } \alpha=3/2, \\ \tau^{3-2\alpha} & \text{if } 3/2<\alpha<2, \end{cases}
\end{aligned}$$

and if $\mu^{1/\alpha} \geqslant \pi/\sin\theta$ then

$$
\begin{aligned}
|\mathbb{I}_2| &< C_{\alpha,\mu_0}\tau^\alpha \int_0^{\pi/\sin\theta} e^{kr\cos\theta}(\mu^{-1}r + \mu^{-2}r^2)\,\mathrm{d}r \\
&< C_{\alpha,\mu_0}\tau^\alpha \int_0^{\pi/\sin\theta} e^{kr\cos\theta} r\,\mathrm{d}r < C_{\alpha,\mu_0}\tau^\alpha k^{-2}.
\end{aligned}
$$

Consequently,

$$
|\mathbb{I}_2| < C_{\alpha,\mu_0}\varepsilon(\alpha,\tau,k)\tau^{3-\alpha}. \tag{48}
$$

Now, let us estimate $\mathbb{I}_3$. For $z \in \Upsilon_1 \setminus \{0\}$, a routine calculation yields

$$
\begin{aligned}
&|(e^{(t/\tau-k)z} - 1)(\psi(z) + \mu(e^z+1)) - z(z^\alpha + 2\mu)(t/\tau - k)e^z| \\
&< C_{\alpha,\mu_0}(t/\tau - k)\big(|z|^{\alpha+2} + \mu|z|^2\big),
\end{aligned}
$$

so that by (47) we obtain

$$
\begin{aligned}
|\mathbb{I}_3| &< C_{\alpha,\mu_0}\tau^\alpha(t/\tau - k)\bigg(\int_0^{\min\{\mu^{1/\alpha},\pi/\sin\theta\}} e^{kr\cos\theta}(\mu^{-2}r^{\alpha+1} + \mu^{-1}r)\,\mathrm{d}r + \\
&\qquad \int_{\min\{\mu^{1/\alpha},\pi/\sin\theta\}}^{\pi/\sin\theta} e^{kr\cos\theta}(r^{1-\alpha} + \mu r^{1-2\alpha})\,\mathrm{d}r\bigg) \\
&< C_{\alpha,\mu_0}\tau^\alpha(t/\tau - k)\int_0^{\pi/\sin\theta} e^{kr\cos\theta} r^{1-\alpha}\,\mathrm{d}r \\
&< C_{\alpha,\mu_0}\tau^\alpha(t/\tau - k)k^{\alpha-2}.
\end{aligned} \tag{49}
$$

Finally, combining (45), (46), (48) and (49) proves (43) and thus concludes the proof. ■

Remark 3.9. *In the above proof,*

$$
|\psi(z) + \mu(e^z+1) - z(z^\alpha + 2\mu)(1 - e^{-z})^{-1}| < C_{\alpha,\mu_0}(\mu|z|^2 + |z|^3),
$$

and the term $|z|^3$ leads to (3-α)-order accuracy. If we choose a β such that (i.e. $(\widehat{\beta}(z) - z^{\alpha-3}) = 0$)

$$
|\Psi(z) + \mu(e^z+1) - z(z^\alpha + 2\mu)(1 - e^{-z})^{-1}| < C_{\alpha,\mu_0}(\mu|z|^2 + |z|^{2+\alpha}),
$$

then we can obtain 2-order accuracy, where $\Psi(z) = e^{-z}(e^z - 1)^3\widehat{\beta}(z)$.

Theorem 3.2. *For each $k \in \mathbb{N}_{>0}$, if $f' \in L^1(0,t_k)$ then*

$$
\begin{aligned}
&|y(t_k) - Y_k| \leqslant C_{\alpha,\mu_0}\tau^{3-\alpha}\varepsilon(\alpha,\tau,k)|f(0)| + \\
&\qquad C_{\alpha,\mu_0}\tau^{3-\alpha}\begin{cases} \int_0^{t_k}(t_{k+1} - t)^{2\alpha-3}|f'(t)|\,\mathrm{d}t & \text{if } 1 < \alpha < 3/2, \\ (1 + |\ln\tau|)\|f'\|_{L^1(0,t_k)} & \text{if } \alpha = 3/2, \\ \|f'\|_{L^1(0,t_k)} & \text{if } 3/2 < \alpha < 2, \end{cases}
\end{aligned} \tag{50}
$$

where $\varepsilon(\alpha,\tau,k)$ is defined by (44).

Proof. By (29) and (31), a straightforward computation yields that

$$\begin{aligned} y(t_k) - Y_k &= \int_0^{t_k} (E - \widetilde{E})(t_k - t) f(t) \, \mathrm{d}s \\ &= \int_0^{t_k} (E - \widetilde{E})(t_k - t) \left(f(0) + \int_0^t f'(s) \, \mathrm{d}s \right) \, \mathrm{d}t \\ &= f(0)\mathcal{E}(t_k) + \int_0^{t_k} \mathcal{E}(t_k - t) f'(t) \, \mathrm{d}t \end{aligned}$$

for each $k \in \mathbb{N}_{>0}$. Therefore, by Lemma 3.9 we obtain the theorem. ∎

Remark 3.10. *As pointed out in Remark 3.6, $\theta \to (\pi/2)+$ as $\mu_0 \to \infty$. Hence, (40) and (46) imply that the C_{α,μ_0} in (39) and the C_{α,μ_0} in (50) will both approach infinity as $\theta \to (\pi/2)+$. Analogously, the C_{α,μ_0} in (51) will approach infinity as $\tau^\alpha/h^2 \to \infty$.*

3.3 Convergence of the second discretization

From the proofs of Theorems 3.1 and 3.2, it is easily perceived that the fact (cf. Remark 3.8 and Remark 3.9)

$$\big(\widehat{b}(z) - z^{\alpha-3}\big)(0) \neq 0$$

caused $(3-\alpha)$-order accuracy of the first discretization. This is the inspiration for the second discretization. Let $\widehat{\beta}(z)$ be the discrete Laplace transform of $(\beta_k)_{k=0}^\infty$. The definition of the sequence $(\beta_k)_{k=0}^\infty$ implies

$$\begin{aligned} \widehat{\beta}(z) &= \widehat{b}(z) + 2\sin(\alpha\pi/2) \sum_{k=1}^\infty (2k\pi)^{\alpha-3} \\ &= \widehat{b}(z) - \big(\widehat{b}(z) - z^{\alpha-3}\big)(0) \quad \text{(by (8))}. \end{aligned}$$

Hence, $\big(\widehat{\beta}(z) - z^{\alpha-3}\big)(0) = 0$. Finally, by a simple modification of the proofs of Theorems 3.1 and 3.2, we readily obtain the following error estimate.

Theorem 3.3. *For $k \in \mathbb{N}_{>0}$,*

$$|y(t_k) - \mathcal{Y}_k| \leqslant C_{\alpha,\mu_0}\tau^2 \left(t_k^{-2}|y_0| + t_k^{-1}|y_1| + t_k^{\alpha-2}|f(0)| + \int_0^{t_k} (t_{k+1} - t)^{\alpha-2} |f'(t)| \mathrm{d}t \right). \tag{51}$$

4 Two full discretizations

Let $\mathcal{K}_h$ be a quasi-uniform and shape-regular triangulation of Ω consisting of d-simplexes, and we use h to denote the maximum diameter of the elements in $\mathcal{K}_h$. Define

$$S_h := \left\{ v_h \in H_0^1(\Omega) : \; v_h|_K \in P_1(K) \quad \forall\, K \in \mathcal{K}_h \right\},$$

where $P_1(K)$ is the set of all linear functions defined on K. Let $\Delta_h : S_h \to S_h$ be the usual discrete Laplace operator, namely,

$$\langle -\Delta_h v_h, w_h \rangle_\Omega = \langle \nabla v_h, \nabla w_h \rangle_\Omega$$

for all $v_h, w_h \in S_h$. In addition, let P_h be the L^2-orthogonal projection onto S_h.

Assume that $u_0, u_1 \in L^2(\Omega)$ and

$$f \in L^1(0,\infty; L^2(\Omega)) \cap {}_0H^{(1-\alpha)/2}(0,\infty; H^{-1}(\Omega)).$$

Using Discretizations 1 and 2 in time and using $-\Delta_h$ as the discretization of $-\Delta$, we obtain two full discretizations of problem (1) as follows.

Discretization 3. *Let $U_0 = P_h u_0$; for each $k \in \mathbb{N}$, the value of U_{k+1} is determined by*

$$\begin{aligned} &(b_{k+1} - b_k)(U_1 - U_0) + \sum_{j=1}^{k} (b_{k-j+1} - b_{k-j})(U_{j+1} - 2U_j + U_{j-1}) \\ &\quad - \frac{\tau^\alpha}{2}\Delta_h(U_k + U_{k+1}) = \tau^{\alpha-1} P_h \int_{t_k}^{t_{k+1}} f(t)\,\mathrm{d}t + \tau(b_{k+1} - b_k) P_h u_1. \end{aligned} \tag{52}$$

Discretization 4. *Let $\mathcal{U}_0 = P_h u_0$; for each $k \in \mathbb{N}$, the value of $\mathcal{U}_{k+1}$ is determined by*

$$\begin{aligned} &(\beta_{k+1} - \beta_k)(\mathcal{U}_1 - \mathcal{U}_0) + \sum_{j=1}^{k} (\beta_{k-j+1} - \beta_{k-j})(\mathcal{U}_{j+1} - 2\mathcal{U}_j + \mathcal{U}_{j-1}) \\ &\quad - \frac{\tau^\alpha}{2}\Delta_h(\mathcal{U}_k + \mathcal{U}_{k+1}) = \tau^{\alpha-1} P_h \int_{t_k}^{t_{k+1}} f(t)\,\mathrm{d}t + \tau(\beta_{k+1} - \beta_k) P_h u_1. \end{aligned} \tag{53}$$

Remark 4.1. *We note that Discretization 3 has already been analyzed in [22], and the following error estimate has been established in the case $u_0 = u_1 = 0$:*

$$\|u(t_k) - U_k\|_{H_0^1(\Omega)} \lesssim \|f\|_{L^2(0,t_k;L^2(\Omega))} \begin{cases} \tau^{(\alpha-1)/2} + h^{1-1/\alpha} & \text{if } 1 < \alpha \leqslant 3/2, \\ \tau^{(\alpha-1)/2} + \tau^{-1/2}h & \text{if } 3/2 < \alpha < 2, \end{cases}$$

where $h \leqslant \tau^{\alpha/2}$ if $3/2 < \alpha < 2$. This error estimate is optimal with respect to the regularity of u.

By Lemmas 3.1 and 3.2, we easily obtain the following stability estimates of Discretizations 3 and 4.

Theorem 4.1. *For each $k \in \mathbb{N}_{>0}$,*

$$\begin{aligned} \|U_k\|_{L^2(\Omega)} &\leqslant C_\alpha \big(\|u_0\|_{L^2(\Omega)} + t_k^{1-\alpha/2} \|u_1\|_{\dot{H}^{-1}(\Omega)} + \|f\|_{{}_0H^{(1-\alpha)/2}(0,t_k;H^{-1}(\Omega))} \big), \\ \|\mathcal{U}_k\|_{L^2(\Omega)} &\leqslant C_\alpha \big(\|u_0\|_{L^2(\Omega)} + t_k^{1-\alpha/2} \|u_1\|_{\dot{H}^{-1}(\Omega)} + \|f\|_{{}_0H^{(1-\alpha)/2}(0,t_k;H^{-1}(\Omega))} \big). \end{aligned}$$

Remark 4.2. *Since we do not use Laplace transform technique in the proof of Lemma 3.1, the first stability estimate in the above theorem does not require the temporal grid to be uniform. We also note that the stability estimate in [33, Theorem 3.2] essentially requires the initial value to be continuously differentiable.*

The main task of the rest of this section is to establish the convergence of Discretizations 3 and 4. To this end, we first introduce the following conventions: $a \lesssim b$ means that there exists a positive constant C depending only on α, Ω, the shape regularity of $\mathcal{K}_h$ or $h_{\min}^{-2}\tau^\alpha$, such that $a \leqslant Cb$, where $h_{\min}$ is the minimum diameter of the elements in $\mathcal{K}_h$. Then let us consider the error estimate of the following spatial semidiscretization of problem (1):

$$\mathrm{D}_{0+}^{\alpha-1}(u_h' - P_h u_1)(t) - \Delta_h u_h(t) = P_h f(t), \quad t > 0, \tag{54}$$

subjected to the initial value condition $u_h(0) = P_h u_0$.

Lemma 4.1. *If $u_0, u_1 \in L^2(\Omega)$ and $f \in L^\infty(0, \infty, L^2(\Omega))$, then*

$$\begin{aligned} \|(u-u_h)(t)\|_{L^2(\Omega)} \lesssim h^2 \Big(& t^{-\alpha}\|u_0\|_{L^2(\Omega)} + t^{1-\alpha}\|u_1\|_{L^2(\Omega)} \\ & + (1+|\ln h|)\|f\|_{L^\infty(0,t;L^2(\Omega))} \Big) \end{aligned} \tag{55}$$

for each $t > 0$.

Proof. For $f = 0$, [12, Theorem 3.2] implies

$$\|(u-u_h)(t)\|_{L^2(\Omega)} \lesssim h^2 \big(t^{-\alpha}\|u_0\|_{L^2(\Omega)} + t^{1-\alpha}\|u_1\|_{L^2(\Omega)} \big),$$

it suffices to prove, for $u_0 = u_1 = 0$, that

$$\|(u-u_h)(t)\|_{L^2(\Omega)} \lesssim (1+|\ln h|)h^2\|f\|_{L^\infty(0,t;L^2(\Omega))}, \tag{56}$$

which is an improvement of [12, Theorem 3.3]. To this end, we proceed as follows. Similar to [21, Equation (25)], we have

$$(u-u_h)(t) = \int_0^t \frac{1}{2\pi i}\int_\Upsilon e^{sz}\big((z^\alpha-\Delta)^{-1} - (z^\alpha-\Delta_h)^{-1}P_h\big)\,\mathrm{d}z\, f(t-s)\,\mathrm{d}s,$$

where Υ is defined in Section 3. The proof of [21, Theorem 2.1] proves that

$$\|(z^\alpha-\Delta)^{-1} - (z^\alpha-\Delta_h)^{-1}P_h\|_{\mathcal{L}(L^2(\Omega))} \lesssim h^2, \quad \forall z \in \Upsilon \setminus \{0\},$$

and hence

$$\Big\| \int_\Upsilon e^{sz}\big((z^\alpha-\Delta)^{-1} - (z^\alpha-\Delta_h)^{-1}P_h\big)\,\mathrm{d}z \Big\|_{\mathcal{L}(L^2(\Omega))} \lesssim s^{-1}h^2.$$

We also have

$$\Big\| \int_\Upsilon e^{sz}\big((z^\alpha-\Delta)^{-1} - (z^\alpha-\Delta_h)^{-1}P_h\big)\,\mathrm{d}z \Big\|_{\mathcal{L}(L^2(\Omega))} \lesssim 1,$$

by the fact that, for $z \in \Upsilon \setminus \{0\}$,

$$\begin{aligned} \|(z^\alpha-\Delta)^{-1}\|_{\mathcal{L}(L^2(\Omega))} &\lesssim (1+|z|^\alpha)^{-1}, \\ \|(z^\alpha-\Delta_h)^{-1}\|_{\mathcal{L}(L^2(\Omega))} &\lesssim (1+|z|^\alpha)^{-1}. \end{aligned}$$

Therefore, if $h^2 < t$ then

$$\begin{aligned} \|(u-u_h)(t)\|_{L^2(\Omega)} &\lesssim \int_0^{h^2} \|f(t-s)\|_{L^2(\Omega)}\,\mathrm{d}s + \int_{h^2}^t s^{-1}h^2\|f(t-s)\|_{L^2(\Omega)}\,\mathrm{d}s \\ &\lesssim h^2(1+|\ln h|)\|f\|_{L^\infty(0,t;L^2(\Omega))}, \end{aligned}$$

and if $t \leqslant h^2$ then

$$\|(u-u_h)(t)\|_{L^2(\Omega)} \lesssim \int_0^t \|f(t-s)\|_{L^2(\Omega)}\,\mathrm{d}s \lesssim h^2\|f\|_{L^\infty(0,t;L^2(\Omega))}.$$

This proves (56) and thus concludes the proof. ∎

Remark 4.3. *Since*

$$u' - \Delta \, \mathrm{D}_{0+}^{1-\alpha} \, u = \mathrm{D}_{0+}^{1-\alpha} \, f,$$
$$u_h' - \Delta_h \, \mathrm{D}_{0+}^{1-\alpha} \, u = \mathrm{D}_{0+}^{1-\alpha} \, P_h f,$$

we have

$$\langle u'(t) - u_h'(t), v_h \rangle_\Omega + \langle \nabla \, \mathrm{D}_{0+}^{1-\alpha} (u - u_h)(t), \nabla v_h \rangle_\Omega = 0$$

for all $v_h \in S_h$. Then, by the techniques used in Lemma 3.1, a standard energy argument yields

$$\|(u - u_h)(t)\|_{L^2(\Omega)} \leqslant 2 \|(I - R_h) u'\|_{L^1(0,t;L^2(\Omega))} + \|u(t) - R_h u(t)\|_{L^2(\Omega)}, \quad t > 0,$$

where $R_h : H_0^1(\Omega) \to S_h$ is defined by that, for each $v \in H_0^1(\Omega)$,

$$\int_\Omega \nabla (v - R_h v) \cdot \nabla w_h = 0 \quad \text{for all } w_h \in S_h.$$

We can also use this estimate to analyze the convergence of (54) in $L^2(\Omega)$-norm with nonsmooth data.

Finally, let us give the error estimates of Discretization 3. By triangle inequality, we have

$$\|(u - U)(t)\|_{L^2(\Omega)} \leqslant \|(u - u_h)(t)\|_{L^2(\Omega)} + \|(u_h - U)(t)\|_{L^2(\Omega)}, \quad \text{for } 0 < t \leqslant T,$$

where $U := \sum_{k=0}^\infty U_k \varphi_k$ and φ_k is the hat function at node t_k. The estimate of $\|(u - u_h)(t)\|_{L^2(\Omega)}$ already exists (cf. Lemma 4.1), and hence we only need to give the estimate of $\|(u_h - U)(t)\|_{L^2(\Omega)}$. For $i = 1, 2, \cdots, N$, let (ϕ_i, λ_i) be the eigen-pair of the operator $-\Delta_h$. We have

$$u_h = \sum_{i=1}^N \langle u_h, \phi_i \rangle_\Omega \phi_i, \quad U = \sum_{i=1}^N \langle U, \phi_i \rangle_\Omega \phi_i.$$

It is easy to verify that $u_h^i := \langle u_h, \phi_i \rangle_\Omega$ with $u_h^i(0) = \langle u_0, \phi_i \rangle_\Omega$ satisfies that

$$\mathrm{D}_{0+}^{\alpha-1} ((u_h^i)' - u_1^i) + \lambda_i u_h^i = f_i, \quad \text{for } i = 1, 2, \cdots, N,$$

where $f_i = \langle f, \phi_i \rangle_\Omega$ and $u_1^i = \langle u_1, \phi_i \rangle_\Omega$. Letting $U^i := \langle U, \phi_i \rangle_\Omega$, by Theorems 3.1 and 3.2 we can obtain the error estimates between u_h^i and U^i, and hence the error estimates between u_h and U. By Theorem 3.3, the error estimates of Discretization 4 follows similarly. By the above procedure, we have the following two theorems.

Theorem 4.2. *For $k \in \mathbb{N}_{>0}$, if $f' \in L^1(0, t_k; L^2(\Omega))$ then*

$$\begin{aligned}
& \|u(t_k) - U_k\|_{L^2(\Omega)} \\
\lesssim & \left(t_k^{\alpha-3} \tau^{3-\alpha} + t_k^{-\alpha} h^2\right) \|u_0\|_{L^2(\Omega)} + \left(t_k^{\alpha-2} \tau^{3-\alpha} + t_k^{1-\alpha} h^2\right) \|u_1\|_{L^2(\Omega)} \\
& + \tau^{3-\alpha} \varepsilon(\alpha, \tau, k) \|f(0)\|_{L^2(\Omega)} + (1 + |\ln h|) h^2 \|f\|_{L^\infty(0,t_k;L^2(\Omega))} \\
& + \tau^{3-\alpha} \begin{cases} \int_0^{t_k} (t_{k+1} - t)^{2\alpha-3} \|f'(t)\|_{L^2(\Omega)} \, \mathrm{d}t & \text{if } 1 < \alpha < 3/2, \\ (1 + |\ln \tau|) \|f'\|_{L^1(0,t_k;L^2(\Omega))} & \text{if } \alpha = 3/2, \\ \|f'\|_{L^1(0,t_k;L^2(\Omega))} & \text{if } 3/2 < \alpha < 2, \end{cases}
\end{aligned} \tag{57}$$

where $\varepsilon(\alpha, \tau, k)$ is defined by (44).

Theorem 4.3. *For $k \in \mathbb{N}_{>0}$, if $f' \in L^1(0, t_k; L^2(\Omega))$ then*

$$\begin{aligned} \|u(t_k) - \mathcal{U}_k\|_{L^2(\Omega)} \lesssim \big(\tau^2 t_k^{-2} + t_k^{-\alpha} h^2\big)\|u_0\|_{L^2(\Omega)} + \big(\tau^2 t_k^{-1} + t_k^{1-\alpha} h^2\big)\|u_1\|_{L^2(\Omega)} \\ + \tau^2 t_k^{\alpha-2}\|f(0)\|_{L^2(\Omega)} + (1 + |\ln h|)h^2\|f\|_{L^\infty(0,t_k;L^2(\Omega))} \\ + \tau^2 \int_0^{t_k} (t_{k+1} - t)^{\alpha-2}\|f'(t)\|_{L^2(\Omega)} \,\mathrm{d}t. \end{aligned} \tag{58}$$

Remark 4.4. *From Remark 3.10 it follows that the implicit constants in (57) and (58) will approach infinity as $\tau^\alpha/h^2 \to \infty$.*

5 Numerical experiments

5.1 Discretizations 1 and 2

For equation (3), we set $\lambda = 1$ and consider the following three problems:

(a). $y_0 := 1$, $y_1 := 0$, and $f(t) := 0$;

(b). $y_0 := 0$, $y_1 := 1$, and $f(t) := 0$;

(c). $y_0 := 0$, $y_1 := 0$, and $f(t) := 1 + t^{0.2}$.

In this subsection, "Error" means the error of the numerical solution at $t = 1$, where the reference solution is the numerical solution of Discretization 2 with $\tau = 2^{-18}$. The numerical results in Tables 1, 2 and 3 demonstrate that the accuracies of Discretizations 1 and 2 are close to $\mathcal{O}(\tau^{3-\alpha})$ and $\mathcal{O}(\tau^2)$, respectively, which agrees well with Theorems 3.1, 3.2 and 3.3.

	Discretization 1						Discretization 2					
	$\alpha = 1.2$		$\alpha = 1.4$		$\alpha = 1.8$		$\alpha = 1.2$		$\alpha = 1.4$		$\alpha = 1.8$	
τ	Error	Order	Error	Order	Error	Order	Error	Order	Error	Order	Error	Order
2^{-10}	2.05e-7	–	8.40e-7	–	4.97e-5	–	6.15e-8	–	4.19e-08	–	8.47e-8	–
2^{-11}	6.12e-8	1.75	2.81e-7	1.58	2.16e-5	1.20	1.54e-8	2.00	1.07e-08	1.98	1.97e-8	2.10
2^{-12}	1.81e-8	1.75	9.35e-8	1.59	9.42e-6	1.20	3.84e-9	2.00	2.70e-09	1.98	4.62e-9	2.09
2^{-13}	5.35e-9	1.76	3.11e-8	1.59	4.10e-6	1.20	9.61e-10	2.00	6.79e-10	1.99	1.09e-9	2.09
2^{-14}	1.57e-9	1.77	1.03e-8	1.59	1.78e-6	1.20	2.42e-10	1.99	1.71e-10	1.99	2.57e-10	2.08

Table 1: *Convergence history of Discretizations 1 and 2 for problem* (a)

	Discretization 1						Discretization 2					
	$\alpha = 1.2$		$\alpha = 1.5$		$\alpha = 1.9$		$\alpha = 1.2$		$\alpha = 1.5$		$\alpha = 1.9$	
τ	Error	Order	Error	Order	Error	Order	Error	Order	Error	Order	Error	Order
2^{-7}	2.22e-6	–	7.90e-5	–	9.98e-4	–	2.65e-6	–	3.67e-6	–	4.58e-6	–
2^{-8}	7.32e-7	1.60	2.83e-5	1.48	4.67e-4	1.10	6.53e-7	2.02	9.13e-7	2.01	9.95e-7	2.20
2^{-9}	2.34e-7	1.65	1.01e-5	1.49	2.18e-4	1.10	1.62e-7	2.01	2.27e-7	2.01	2.14e-7	2.22
2^{-10}	7.31e-8	1.68	3.60e-6	1.49	1.02e-4	1.10	4.03e-8	2.01	5.66e-8	2.00	4.54e-8	2.24
2^{-11}	2.25e-8	1.70	1.28e-6	1.49	4.75e-5	1.10	1.01e-8	2.00	1.41e-8	2.00	9.52e-9	2.26

Table 2: *Convergence history of Discretizations 1 and 2 for problem* (b)

	Discretization 1						Discretization 2					
	$\alpha = 1.2$		$\alpha = 1.4$		$\alpha = 1.9$		$\alpha = 1.2$		$\alpha = 1.4$		$\alpha = 1.9$	
τ	Error	Order	Error	Order	Error	Order	Error	Order	Error	Order	Error	Order
2^{-7}	1.36e-5	–	2.73e-5	–	2.61e-3	–	5.31e-6	–	2.35e-6	–	9.39e-6	–
2^{-8}	4.11e-6	1.72	9.31e-6	1.55	1.22e-3	1.10	1.32e-6	2.01	6.62e-7	1.83	2.32e-6	2.02
2^{-9}	1.23e-6	1.74	3.14e-6	1.57	5.70e-4	1.10	3.27e-7	2.01	1.77e-7	1.90	5.71e-7	2.02
2^{-10}	3.67e-7	1.75	1.05e-6	1.58	2.66e-4	1.10	8.10e-8	2.01	4.61e-8	1.94	1.40e-7	2.03
2^{-11}	1.08e-7	1.76	3.52e-7	1.58	1.24e-4	1.10	2.01e-8	2.01	1.18e-8	1.96	3.44e-8	2.03

Table 3: *Convergence history of Discretizations 1 and 2 for problem* (c)

5.2 Discretizations 3 and 4

For equation (1) in the case $\Omega = (0, 1)$, we consider the following three problems:

(d). $u_0(x) := x^{-0.49}$, $u_1(x) := 0$, and $f(x, t) := 0$;

(e). $u_0(x) := 0$, $u_1(x) := x^{-0.49}$, and $f(x, t) := 0$;

(f). $u_0(x) := 0$, $u_1(x) := 0$, and $f(x, t) := x^{-0.49}(1 + t^{0.2})$.

Throughout this subsection, we will use uniform spatial grids, and "Error1" and "Error2" denote the errors (in $L^2(\Omega)$-norm) of the numerical solutions of Discretizations 3 and 4 at $t = 1$, respectively, where the reference solution is the numerical solution of Discretization 4 with $h = 2^{-11}$ and $\tau = 2^{-16}$.

Experiment 1. This experiment verifies the spatial accuracies of Discretizations 3 and 4. Table 4 demonstrates that the spatial accuracy of Discretization 3 is close to $\mathcal{O}(h^2)$, which is in good agreement with Theorem 4.2. Since the numerical results of Discretization 4 are almost identical to that of Discretization 3, they are omitted here.

		$\alpha = 1.2$		$\alpha = 1.4$		$\alpha = 1.8$	
	h	Error1	Order	Error1	Order	Error1	Order
Problem (d)	2^{-3}	1.07e-3	–	4.43e-3	–	4.74e-2	–
	2^{-4}	2.73e-4	1.97	1.12e-3	1.99	1.57e-2	1.59
	2^{-5}	6.94e-5	1.98	2.80e-4	2.00	4.46e-3	1.82
	2^{-6}	1.76e-5	1.98	7.00e-5	2.00	1.15e-3	1.96
	2^{-7}	4.45e-6	1.98	1.75e-5	2.00	2.85e-4	2.01
Problem (e)	2^{-3}	2.71e-3	–	2.33e-3	–	7.76e-3	–
	2^{-4}	7.21e-4	1.91	6.15e-4	1.92	2.04e-3	1.93
	2^{-5}	1.90e-4	1.92	1.61e-4	1.93	5.10e-4	2.00
	2^{-6}	4.97e-5	1.93	4.19e-5	1.94	1.27e-4	2.00
	2^{-7}	1.29e-5	1.94	1.08e-5	1.95	3.16e-5	2.00
Problem (f)	2^{-3}	6.12e-3	–	6.64e-3	–	8.77e-3	–
	2^{-4}	1.63e-3	1.91	1.75e-3	1.92	2.27e-3	1.95
	2^{-5}	4.31e-4	1.92	4.60e-4	1.93	5.86e-4	1.96
	2^{-6}	1.13e-4	1.93	1.20e-4	1.94	1.51e-4	1.96
	2^{-7}	2.95e-5	1.94	3.12e-5	1.95	3.89e-5	1.95

Table 4: *Convergence history of Discretization 3 for problems* (d), (e) *and* (f) *with* $\tau = 2^{-16}$

Experiment 2. To obtain the temporal accuracies $\mathcal{O}(\tau^{3-\alpha})$ and $\mathcal{O}(\tau^2)$ of Discretizations 3 and 4, respectively, Theorems 4.2 and 4.3 require the ratio

τ^α/h^2 to be uniformly bounded. Hence, this experiment verifies the temporal accuracies of Discretizations 3 and 4 in an indirect way. In this experiment, we set $\tau^\alpha = h^2$. Theorem 4.2 predicts that "Error1" is close to $\mathcal{O}(h^2)$ for $1 < \alpha \leqslant 3/2$ and close to $\mathcal{O}(h^{6/\alpha-2})$ for $3/2 < \alpha < 2$. Theorem 4.3 predicts that "Error2" is close to $\mathcal{O}(h^2)$ for all $1 < \alpha < 2$. The above two predictions are confirmed by the numerical results in Tables 5, 6 and 7.

	$\alpha = 1.2$				$\alpha = 1.5$				$\alpha = 1.8$			
h	Error1	Order	Error2	Order	Error1	Order	Error2	Order	Error1	Order	Error2	Order
2^{-5}	6.87e-5	–	6.99e-5	–	4.83e-4	–	5.00e-4	–	4.19e-2	–	7.12e-3	–
2^{-6}	1.75e-5	1.97	1.76e-5	1.99	1.25e-4	1.95	1.25e-4	2.00	1.98e-2	1.08	1.27e-3	2.48
2^{-7}	4.44e-6	1.98	4.45e-6	1.99	3.16e-5	1.98	3.10e-5	2.01	8.57e-3	1.21	2.59e-4	2.29
2^{-8}	1.11e-6	1.99	1.12e-6	2.00	8.08e-6	1.97	7.65e-6	2.02	3.54e-3	1.27	5.88e-5	2.14

Table 5: *Convergence history of Discretizations 3 and 4 for problem* (d)

	$\alpha = 1.2$				$\alpha = 1.5$				$\alpha = 1.8$			
h	Error1	Order	Error2	Order	Error1	Order	Error2	Order	Error1	Order	Error2	Order
2^{-5}	1.90e-4	–	1.90e-4	–	1.17e-4	–	1.71e-4	–	6.12e-3	–	6.06e-4	–
2^{-6}	4.98e-5	1.93	4.97e-5	1.93	3.14e-5	1.90	4.23e-5	2.01	2.60e-3	1.24	1.08e-4	2.48
2^{-7}	1.29e-5	1.94	1.29e-5	1.94	8.25e-6	1.93	1.06e-5	1.99	1.06e-3	1.29	1.96e-5	2.46
2^{-8}	3.33e-6	1.96	3.33e-5	1.96	2.15e-6	1.94	2.68e-6	1.99	4.26e-4	1.31	4.44e-6	2.15

Table 6: *Convergence history of Discretizations 3 and 4 for problem* (e)

	$\alpha = 1.2$				$\alpha = 1.5$				$\alpha = 1.9$			
h	Error1	Order	Error2	Order	Error1	Order	Error2	Order	Error1	Order	Error2	Order
2^{-4}	1.63e-3	–	1.63e-3	–	1.53e-3	–	1.87e-3	–	2.22e-2	–	2.61e-3	–
2^{-5}	4.31e-4	1.92	4.31e-4	1.92	4.06e-4	1.92	4.91e-4	1.93	1.03e-2	1.10	7.35e-4	1.83
2^{-6}	1.13e-4	1.93	1.13e-4	1.93	1.07e-4	1.92	1.28e-4	1.94	4.75e-3	1.12	1.79e-4	2.03
2^{-7}	2.95e-5	1.94	2.95e-5	1.94	2.80e-5	1.93	3.30e-5	1.95	2.15e-3	1.15	4.17e-5	2.11

Table 7: *Convergence history of Discretizations 3 and 4 for problem* (f)

Experiment 3. This experiment investigates the effect of large ration τ^α/h^2 on the accuracy of Discretizations 3 and 4 for problem (d). The numerical results in Table 8 illustrate that, with fixed τ, the accuracy of Discretizations 3 and 4 will deteriorate as $h \to 0+$, which confirms Remark 4.4.

	$\alpha = 1.2$		$\alpha = 1.4$		$\alpha = 1.8$	
h	Error1	Error2	Error1	Error2	Error1	Error2
2^{-4}	3.20e-3	5.90e-4	1.29e-3	1.52e-3	5.82e-2	1.12e-2
2^{-5}	1.04e-1	6.00e-2	1.72e-2	4.05e-3	6.13e-2	2.25e-2
2^{-6}	3.81e-1	3.04e-1	1.87e-1	1.17e-1	6.29e-2	2.60e-2
2^{-7}	7.04e-1	6.26e-1	4.94e-1	4.00e-1	1.59e-1	6.81e-2
2^{-8}	9.97e-1	9.28e-1	8.09e-1	7.20e-1	4.44e-1	3.09e-1
2^{-9}	1.25e-0	1.19e-0	8.09e-1	1.01e-0	7.58e-1	6.27e-1

Table 8: *Convergence history of Discretizations 3 and 4 for problem* (d) *with* $\tau = 2^{-5}$

6 Conclusion

The well-known L1 scheme for fractional wave equations is analyzed in this paper. New stability estimate is established, and temporal accuracy $\mathcal{O}(\tau^{3-\alpha})$ is derived for nonsmooth initial values u_0 and u_1. A modified L1 scheme is also proposed, which possesses temporal accuracy $\mathcal{O}(\tau^2)$. The theoretical results reveal that $\tau^\alpha/h_{\min}^2$ should be uniformly bounded, where $h_{\min}$ is the minimum diameter of the elements in $\mathcal{K}_h$; otherwise, the temporal accuracy will deteriorate. Numerical experiments are performed to verify the theoretical results.

If the temporal grid is nonuniform or the governing equation is of the form

$$\mathrm{D}_{0+}^{\alpha-1}(u' - u_1)(t) - \operatorname{div}(a(x,t)\nabla u(t)) = f(t), \quad t > 0,$$

then the techniques used in this paper can not be applied. Hence, an interesting question is, on the nonuniform temporal grid or for the above equation, how to derive sharp error estimates for the L1 scheme with nonsmooth data. This will be our future work.

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