Mean Field Production Output Control with Sticky Prices: Nash and Social Solutions *

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Abstract

This paper presents an application of mean field control to dynamic production optimization. Both noncooperative and cooperative solutions are considered. We first introduce a market of a large number of agents (firms) with sticky prices and adjustment costs. By solving auxiliary limiting optimal control problems subject to consistent mean field approximations, two sets of decentralized strategies are obtained and further shown to asymptotically attain Nash equilibria and social optima, respectively. The performance estimate of the social optimum strategies exploits a passivity property of the underlying model. A numerical example is given to compare market prices, firms' outputs and costs under two two solution frameworks.

Key words: Mean field game; social optimum; Nash equilibrium; production output adjustment; sticky price.

1 Introduction

Mean field game theory is effective to design decentralized strategies in a system of many players which are individually negligible but collectively affect a particular player (see e.g., [18], [19], [22], [31]). By identifying a consistency relationship between the individual's best response and the mass (population macroscopic) behavior, one may obtain a fixed-point equation to specify the mean field. This procedure leads to a set of decentralized strategies as an ε-Nash equilibrium for the actual model with a large but finite population. By now, mean field games have been intensively studied in the LQG (linear-quadratic-Gaussian) framework [18], [19], [11], [39], [12]; there is also a large body of works on nonlinear models [22], [32], [3], [14]. For further literature, readers are referred to [17], [43], [44] for mean field models with a major player, [46] for oblivious equilibria proposed for large-scale Markov decision processes of industry dynamics, [42] for mean field games with Markov jump parameters. For a survey on mean field game theory, see [3], [14], and [4]. Besides noncooperative games, social optima in mean field control have been investigated in some literature [20], [45]. Mean field games and control have found wide applications, including smart grids [34], [7], [26], finance, economics [46], [15], [6], [23], operation research [28], [1], [33], [36], and social sciences [29], [21], [2], etc.

This paper aims to present an application of mean field control to production output adjustment in a large market with many firms and sticky prices. Under the stickiness assumption, the price of the underlying product does not adjust instantaneously according to its demand function, but evolves slowly and smoothly. Dynamic game models for duopolistic competition with sticky prices were initially proposed by Simaan and Takayama [38], and then extended to investigate asymptotically stable steady-state equilibrium prices in [13]. In [5], [47], the authors considered open and closed-loop Nash equilibria for dynamic oligopoly with N firms and compared prices' behavior in and outside the steady-state levels, respectively. Adjustment costs in production models have been addressed in the economic literature (see e.g. [35]) and they have been taken into account in the study of dynamic oligopoly [10], [37], [24]. The work [10] introduces a duopoly where each firm has output level subject to control according to a first-order integrator dynamics. However, when the number of firms is large (e.g. in a perfectly competitive market) and the adjustment cost is considered, the computational complexity of output adjustment is high. In the mean field control framework, one can effectively address the complexity issue.

Within our model, a large number of producers supply a certain product with sticky prices, and the output adjustment incurs a cost. The cost function of a firm is based on product

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cost, price, and adjustment cost. In [41], we combined the price and firm's output as a 2-dimensional system. Thus, the cost function has indefinite state weights, which differs from many existing LQG models of mean field games in the literature [19], [42]. In this paper, the price in the mean field limit model is taken as an exogenous signal without the need of state space augmentation. This contributes to deriving a simple condition that ensures the solvability of the resulting equation system.

The Nash equilibrium and the social optimum are two fundamental solution notions to competitive markets with many firms, where the former applies to the noncooperative model, and the latter is for the cooperative model. In this paper, we design Nash and social optimum strategies for the production output control model based on the mean field control methodology, respectively, and further compare two solutions numerically. The Nash solution of our model starts by solving a limiting optimal control problem and next applies the consistency requirement for the mean field approximation. We then obtain a set of decentralized strategies and show that the set of strategies is an ε-Nash equilibrium. For the social optimum solution, we first provide an auxiliary optimal control problem by a person-by-person optimality approach, and then design a set of decentralized strategies by solving the limiting auxiliary problem subject to consistent mean field approximations. The set of strategies is shown to be asymptotically socially optimal by exploiting a passivity property of the underlying model.

An illustrative numerical example is given to compare market prices, firms outputs and optimal costs under the game and social optimum frameworks. It is numerically shown that the social optimum has a lower average output level than that in the noncooperative case. This is similar to the behavior in a duopoly model [48] where cooperation of the two players results in a lower total output than in the Cournot equilibrium.

The paper is organized as follows. Section II introduces the game and social optimum problems with N players. In Section III, we first design a set of decentralized strategies by the mean field control methodology and then show its asymptotic Nash equilibrium property. In Section IV, we construct a set of decentralized strategies, which is shown to be asymptotically socially optimal. In Section V, a comparison of two solutions is demonstrated by a numerical example. Section VI concludes the paper.

Notation: $\|\cdot\|$ denotes the Euclidean vector norm or matrix spectral norm. For a matrix M, $|M|$ denotes the determinant of M. $C([0,\infty),\mathbb{R}^n)$ denotes the class of n-dimensional continuous functions on $[0,\infty)$; $C_b([0,\infty),\mathbb{R}^n)$ is the class of bounded and continuous functions; $C_\rho([0,\infty),\mathbb{R}^n) = \{f|f \in C([0,\infty),\mathbb{R}^n), \sup_{t\geq 0}|f(t)|e^{-\rho't} < \infty$ for some $\rho' \in [0,\rho)\}$. For a family of $\mathbb{R}^n$-values random variables $\{x(\lambda),\lambda \geq 0\}$, $\sigma(x(\lambda),\lambda \leq t)$ is the σ-algebra generated by the collection of random variables; $\|x\|_\rho =$ $\left[E\int_0^\infty e^{-\rho t}|x(t)|^2 dt\right]^{1/2}$; $\|x\|_\infty = \sup_{t\geq 0}|x(t)|$. For two sequences $\{a_n, n = 0, 1, \cdots\}$ and $\{b_n, n = 0, 1, \cdots\}$, $a_n = O(b_n)$ denotes $\limsup_{n\to\infty}|a_n/b_n| \leq C$, and $a_n = o(b_n)$ denotes $\limsup_{n\to\infty}|a_n/b_n| = 0$. For convenience of presentation, we use $C, C_1, C_2, \cdots$ to denote generic positive constants, which may vary from place to place.

2 Problem Description

2.1 Dynamic oligopoly with sticky prices

Dynamic game models for oligopolistic competition with sticky prices were initially proposed by Simaan and Takayama [38], and then further investigated in [13], [5], [47]. According to the model in [5], [47], the sticky price evolves by

$$\frac{dp}{dt} = \alpha(\beta - \delta\sum_{j=1}^N q_j - p), \quad p(0) \text{ given,}$$

where q_j is the output of firm j, $j = 1, \cdots, N$, and has the role of control. The payoff function of firm i is described by

$$K_i(q_1, \cdots, q_N) = \int_0^\infty e^{-\rho t}(pq_i - cq_i - \frac{1}{2}q_i^2)dt.$$

The constants α, β, δ and c are positive, and c is the cost of unit output.

2.2 Output adjustment in a mean field framework

The paper considers a large market of many firms. Based on the formulation of sticky prices in [13], [47], we assume that the price evolves by

$$\begin{aligned}\frac{dp(t)}{dt} &= \alpha[\beta - p(t) - q^{(N)}(t)] \\ &= -\alpha p(t) - \alpha q^{(N)}(t) + \alpha\beta,\end{aligned} \quad (1)$$

where $\alpha > 0$ denotes the speed of adjustment to the level on the demand function, and $q^{(N)}(t) = \frac{1}{N}\sum_{i=1}^N q_i(t)$ is the average of firms' outputs. The output of each firm is described by the stochastic differential equation (SDE)

$$dq_i(t) = -\mu q_i(t)dt + b_i u_i(t)dt + \sigma dw_i(t), \quad (2)$$

where $\{w_i(t), i = 1, \cdots, N\}$ are independent standard Brownian motions, which are also independent of initial outputs of all firms $\{q_i(0), i = 1, \cdots, N\}$. The constants α, β, μ and b_i are positive.

Adjustment costs in production models have been addressed in the economic literature (see e.g. [35]) and they have been taken into account in the study of dynamic oligopoly [10], [9], [37], [24]. The work [10] introduces a duopoly where

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each firm has output level q_i subject to control u_i according to a first order integrator dynamics. In the resulting differential game, the instantaneous payoff of each firm is determined from its net profit minus quadratic penalty terms of q_i and u_i

Remark 1 *As in [13], $\beta - q^{(N)}$ is the price on the demand function for the given level of firms' outputs. In the static case, the inverse demand function has a linear version $p = \beta - \delta q^{(N)}$; here for simplicity we set δ as 1. The scaling factor $1/N$ for $q^{(N)}$ is standard in modelling and analysis of large markets, and some closely related price modelling in a large dynamic market can be found in [30], [40]. μ is used to indicate friction in adjusting the output, and w_i is random shocks in output.*

The cost function of each firm is given by

$$J_i(u) = E \int_0^\infty e^{-\rho t} L(p, q_i, u_i) dt, \tag{3}$$

where

$$L = -p(t)q_i(t) + cq_i(t) + ru_i^2(t),$$
$$u = (u_1, \cdots, u_i, \cdots, u_N),$$

$r > 0$ and $0 < c < \beta$. Here, c denotes the production cost, and $ru_i^2(t)$ denotes the adjustment cost. The minimization of $J_i(u)$ is equivalent to maximizing the payoff

$$K_i(u) = E \int_0^\infty e^{-\rho t} [q_i(t)(p(t) - c) - ru_i^2(t)] dt.$$

We only consider the case $\beta > c$ to make the subsequent optimization problems be of practical interest. Otherwise, given a positive $q^{(N)}$, the production cost already exceeds the price, and the optimization problem is not too meaningful.

The social cost is defined as

$$J_{\text{soc}}^{(N)}(u) = \frac{1}{N} \sum_{j=1}^N J_j(u). \tag{4}$$

Based on costs (3) and (4), one may formulate a standard LQG game and an optimal control problem, respectively. A limitation of this approach is that the control strategy will be centralized. Our goal is to look for decentralized strategies for the corresponding optimization criterion.

The basic objective of this paper is to seek Nash solutions and social solutions to mean field production output control with sticky prices. Specifically, we study the following two problems:

Problem I: Find ε-Nash equilibrium strategies for agents to minimize the individual cost J_i over the set of decentralized strategies

$$\mathscr{U}_{d,i} = \left\{ u_i : u_i(t) \text{ is adapted to } \mathscr{F}_t^i, E \int_0^\infty e^{-\rho t} u_i^2(t) dt < \infty \right\},$$

where $\mathscr{F}_t^i = \sigma\{q_i(0), w_i(s), s \leq t\}$, $t \geq 0$, $i = 1, \cdots, N$.

Problem II: Find asymptotic social optimum strategies for agents to minimize $J_{\text{soc}}^{(N)}$ over the set of decentralized strategies $\mathscr{U}_{d,i}$, $i = 1, \cdots, N$.

For a large market, a natural way of modeling the sequence of parameters $b_1, \cdots, b_N$ is to view them as being sampled from a space such that this sequence exhibits certain statistical properties when $N \to \infty$. Define the associated empirical distribution function $F_N(\theta) = \frac{1}{N} \sum_{i=1}^N I_{[\theta_i \leq \theta]}$, where $I_{[\theta_i \leq \theta]} = 1$ if $\theta_i \leq \theta$ and $I_{[\theta_i \leq \theta]} = 0$ otherwise.

We introduce the assumptions.

A1) The initial price $p(0) = p_0 > 0$ is a constant. The initial outputs of all firms $\{q_i(0), i = 1, \cdots, N\}$ are independent. $Eq_i(0) = q_0 > 0$ for all $i = 1, \cdots, N$; there exists $c_0 < \infty$ independent of N such that $\max_{i=1,\cdots,N} E|q_i(0)|^2 \leq c_0$.

A2) There exists a distribution function $F(\cdot)$ such that the empirical distribution F_N converges weakly to F, where $F_N(b) = \frac{1}{N} \sum_{i=1}^N I_{[b_i \leq b]}$. Furthermore, each $b_i > 0$ and $\int_{\mathbb{R}} \theta^2 dF(\theta) > 0$.

A3) For all N, $\{b_i, i = 1, \cdots, N\}$ is contained in a fixed compact set Θ, and $\int_\Theta \theta dF(\theta) = 1$.

3 Nash Solutions to Output Adjustment

3.1 Optimal control for the limiting problem

Assume that $\bar{q} \in C_{\rho/2}([0, \infty), \mathbb{R})$ is given for approximation of $q^{(N)}$. Replacing $q^{(N)}$ in (1) by $\bar{q}$, we introduce

$$\frac{d\bar{p}(t)}{dt} = \alpha[\beta - \bar{p}(t) - \bar{q}(t)], \quad \bar{p}(0) = p_0. \tag{5}$$

Accordingly, by replacing p in (3) with $\bar{p}$ we define the cost function:

$$\bar{J}_i(u_i) = E \int_0^\infty e^{-\rho t} [(c - \bar{p})q_i + ru_i^2] dt. \tag{6}$$

The corresponding admissible control set is $\mathscr{U}_{d,i}$.

We first take $\bar{p}$ as an exogenous signal and solve the problem in (2), (5) and (6). For a general initial condition $q_i(t) = q_i$ at time t, define the value function

$$V_i(t, q_i) = \inf_{u_i \in \mathscr{U}_{d,i}} E \left[\int_t^\infty e^{-\rho(\tau - t)} L(q_i, u_i) d\tau \,\Big|\, q_i(t) = q_i \right].$$

We introduce the HJB equation:

$$\rho V_i = \inf_{u_i \in \mathbb{R}} \left\{ \frac{\partial V_i}{\partial t} + \frac{\partial V_i}{\partial q_i}(-\mu q_i + b_i u_i) \right.$$
$$\left. + \frac{\sigma}{2}\frac{\partial^2 V_i}{\partial q_i^2} + (c - \bar{p})q_i + r u_i^2 \right\}, \qquad (7)$$

where $V_i \in C_\rho([0,\infty),\mathbb{R})$. Let $V_i(t,q_i) = k_i q_i^2 + 2s_i(t)q_i + g_i(t)$. Then the optimal control law is

$$\bar{u}_i = -\frac{1}{2r}b_i\frac{\partial V_i}{\partial q_i} = -\frac{b_i}{r}(k_i q_i + s_i). \qquad (8)$$

Substituting the control (8) into (7), we obtain

$$\rho(k_i q_i^2 + 2s_i q_i + g_i) = (-2\mu k_i - \frac{b_i^2}{r}k_i^2)q_i^2$$
$$+ 2\left[\frac{ds_i}{dt} + (-\mu - \frac{b_i^2}{r}k_i)s_i + \frac{c-\bar{p}}{2}\right]q_i$$
$$+ \frac{dg_i}{dt} - \frac{b_i^2}{r}s_i^2 + \sigma k_i.$$

This yields

$$\rho k_i = -2\mu k_i - \frac{b_i^2}{r}k_i^2, \qquad (9)$$

$$\rho s_i = \frac{ds_i}{dt} + (-\mu - \frac{b_i^2}{r}k_i)s_i + \frac{c-\bar{p}}{2}, \qquad (10)$$

$$\rho g_i = \frac{dg_i}{dt} - \frac{b_i^2}{r}s_i^2 + \sigma k_i. \qquad (11)$$

Lemma 1 $k_i = 0$ is the unique solution to the algebraic Riccati equation (9) such that $-\mu - \frac{b_i^2}{r}k_i - \frac{\rho}{2} < 0$.

Proof. By solving (9), we have $k_i = 0$ or $k_i = -\frac{r}{b_i^2}(\rho + 2\mu)$. If $k_i = 0$, $-\mu - \frac{b_i^2}{r}k_i - \frac{\rho}{2} = -\mu - \frac{\rho}{2} < 0$. Otherwise, when $k_i = -\frac{r}{b_i^2}(\rho + 2\mu)$, $-\mu - \frac{b_i^2}{r}k_i - \frac{\rho}{2} = \frac{\rho}{2} + \mu > 0$. $\square$

Remark 2 *The inequality in Lemma 1 specifies a stability condition for the closed-loop system which must be satisfied by the solution of k_i.*

Theorem 1 *For the optimal control problem in (2), (5) and (6), assume that $\bar{q} \in C_{\rho/2}([0,\infty),\mathbb{R})$ is given. Then we have 1) there exists a unique solution $s_i \in C_{\rho/2}([0,\infty),\mathbb{R})$ to (10); 2) the optimal control law is uniquely given by $\bar{u}_i = -\frac{b_i}{r}s_i$; 3) there exists a unique solution $g_i \in C_\rho([0,\infty),\mathbb{R})$ to (11), and the optimal cost is given by*

$$V_i(0, q_i(0)) = 2s_i(0)q_0 + g_i(0).$$

Proof. Note that by (5), $\bar{q} \in C_{\rho/2}([0,\infty),\mathbb{R})$ implies $\bar{p} \in C_{\rho/2}([0,\infty),\mathbb{R})$. We can prove parts 1) and 3) by showing that $s_i(0)$ and $g_i(0)$ are uniquely determined from the fact $s_i \in C_{\rho/2}([0,\infty),\mathbb{R})$ and $g_i \in C_\rho([0,\infty),\mathbb{R})$, respectively (see e.g., [17], [19]). To show part 2) we first obtain a prior integral estimate of q_i (see (12)) and then use the completion of squares technique (see e.g., [20], [45]). By Lemma A.1, $E\int_0^\infty e^{-\rho t}u_i^2 dt < \infty$ implies $E\int_0^\infty e^{-\rho t}q_i^2 dt < \infty$, which further gives that $\bar{J}_i$ is well defined to be finite since $\bar{p} \in C_{\rho/2}([0,\infty),\mathbb{R})$. By Proposition A.1, $\bar{J}_i(u_i) < \infty$ leads to $E\int_0^\infty e^{-\rho t}u_i^2 dt < \infty$ which further implies

$$E\int_0^\infty e^{-\rho t}q_i^2 dt < \infty. \qquad (12)$$

$\square$

3.2 Control synthesis and analysis

Following the standard approach in mean field games [18], [19], we construct the equation system as follows:

$$\frac{d\bar{p}}{dt} = \alpha[\beta - \bar{p} - \bar{q}]. \qquad (13)$$

$$\rho s = \frac{ds}{dt} - \mu s + \frac{c - \bar{p}}{2} \qquad (14)$$

$$\frac{d\bar{q}_\theta}{dt} = -\mu \bar{q}_\theta - \frac{\theta^2}{r}s \qquad (15)$$

$$\bar{q} = \int_\mathbb{R} \bar{q}_\theta dF(\theta). \qquad (16)$$

In the above, θ is a continuum parameter. $\bar{q}_\theta$ is regarded as the expectation of the state given the parameter θ in the individual dynamics. The last equation is due to the consistency requirement for the mean field approximation. $p(0) = p_0, q_\theta(0) = q_0$ and $s(0)$ is to be determined. For further analysis, we make the following assumption.

A4) There exists a solution $(s, \bar{q}_\theta, \theta \in \mathbb{R})$ to (13)-(16) such that for each $\theta \in \mathbb{R}$, both s and $\bar{q}_\theta$ are within $C_b([0,\infty),\mathbb{R})$.

Some sufficient conditions for ensuring **A4)** may be obtained by using the fixed-point methods similar to those in [18], [19].

Proposition 1 *If*

$$\frac{1}{2r\mu(\rho + \mu)}\int_\Theta \theta^2 dF(\theta) < 1,$$

*then **A4)** holds.* $\square$

Proof. By (13)-(15), we have

$$\bar{p}(t) = \bar{p}(0)e^{-\alpha t} + \int_0^t e^{-\alpha(t-\tau)}[\alpha\beta - \alpha\bar{q}(\tau)]d\tau,$$

$$s(t) = \int_t^\infty e^{(\rho+\mu)(t-\tau)}\left[\frac{\bar{p}(\tau)-c}{2}\right]d\tau,$$

$$\bar{q}_\theta(t) = \bar{q}_\theta(0) + \int_0^t e^{-\mu(t-\tau)}\left[-\frac{\theta^2}{r}s(\tau)\right]d\tau.$$

Thus,

$$\bar{q}(t) = \int_\Theta \bar{q}_\theta(0)dF(\theta) + \int_\Theta dF(\theta)\int_0^t e^{-\mu(t-\tau_1)}$$
$$\cdot\left\{-\frac{\theta^2}{r}\int_{\tau_1}^\infty e^{(\rho+\mu)(\tau_1-\tau_2)}(\mathscr{A}\bar{q})(\tau_2)d\tau_2\right\}d\tau_1$$
$$\stackrel{\Delta}{=}(\mathscr{T}\bar{q})(t),$$

where $(\mathscr{A}\bar{q})(\tau_2) = \frac{1}{2}\bar{p}(0)e^{-\alpha\tau_2} + \frac{1}{2}\int_0^{\tau_2}e^{-\alpha(\tau_2-\tau_3)}(\alpha\beta - \alpha\bar{q}(\tau_3))d\tau_3 - \frac{c}{2}$. It can be verified that $\mathscr{T}$ is a map from the Banach space $C_b([0,\infty),\mathbb{R})$ to itself. For any $\bar{q}_1, \bar{q}_2 \in C_b([0,\infty),\mathbb{R})$,

$$|(\mathscr{T}\bar{q}_1 - \mathscr{T}\bar{q}_2)(t)|$$
$$\leq \|\bar{q}_1 - \bar{q}_2\|_\infty \int_\Theta\int_0^t e^{-\mu(t-\tau_1)}\left\{\frac{\theta^2}{r}\int_{\tau_1}^\infty e^{(\rho+\mu)(\tau_1-\tau_2)}\right.$$
$$\left.\cdot\left[\frac{1}{2}\int_0^{\tau_2}e^{-\alpha(\tau_2-\tau_3)}\alpha d\tau_3\right]d\tau_2\right\}d\tau_1 dF(\theta)$$
$$\leq \|\bar{q}_1 - \bar{q}_2\|_\infty \int_\Theta \frac{\theta^2}{2r\mu(\rho+\mu)}dF(\theta).$$

It follows that $\mathscr{T}$ is a contraction and hence has a unique fixed point $\bar{q} \in C_b([0,\infty),\mathbb{R})$. $\qquad\square$

3.2.1 The case of uniform agents

We now consider the case of uniform agents, i.e., $b_i \equiv b$, $i = 1,\cdots,N$. In this case, (13)-(16) reduce to the following equation:

$$\frac{d}{dt}\begin{bmatrix}\bar{p}\\\bar{q}\\s\end{bmatrix} = \begin{bmatrix}-\alpha & -\alpha & 0\\0 & -\mu & -\frac{b^2}{r}\\\frac{1}{2} & 0 & \rho+\mu\end{bmatrix}\begin{bmatrix}\bar{p}\\\bar{q}\\s\end{bmatrix} + \begin{bmatrix}\alpha\beta\\0\\-\frac{c}{2}\end{bmatrix}.$$

Let

$$M = \begin{bmatrix}-\alpha & -\alpha & 0\\0 & -\mu & -\frac{b^2}{r}\\\frac{1}{2} & 0 & \rho+\mu\end{bmatrix}, \quad \bar{b} = \begin{bmatrix}\alpha\beta\\0\\-\frac{c}{2}\end{bmatrix}.$$

Then

$$\frac{d}{dt}\begin{bmatrix}\bar{p}\\\bar{q}\\s\end{bmatrix} = M\begin{bmatrix}\bar{p}\\\bar{q}\\s\end{bmatrix} + \bar{b}. \tag{17}$$

By direct computations, we have

$$|M| = \alpha\mu(\rho+\mu) + \frac{\alpha b^2}{2r}$$

and the equation $Mz + \bar{b} = 0$ has the solution

$$z = \left[\frac{2r\beta\mu(\rho+\mu) + b^2c}{2r\mu(\rho+\mu) + b^2}, \frac{b^2(\beta - c)}{2r\mu(\rho+\mu) + b^2},\right.$$
$$\left.\frac{r\mu(c-\beta)}{2r\mu(\rho+\mu) + b^2}\right]^T.$$

Note that

$$|\lambda I - M| = (\lambda + \alpha)(\lambda + \mu)[\lambda - (\rho+\mu)] - \frac{\alpha b^2}{2r}$$
$$= \lambda^3 + (\alpha - \rho)\lambda^2 - (\mu^2 + \rho\mu + \alpha\rho)\lambda$$
$$- \alpha\mu(\rho+\mu) - \frac{\alpha b^2}{2r}. \tag{18}$$

In what follows, we use Routh's stability criterion [8] to determine the number of roots of $|\lambda I - M| = 0$ with negative real parts. The first column of the Routh array for $|\lambda I - M|$ is -1, $\rho - \alpha$, $\mu^2 + \rho\mu + \alpha\rho + \frac{2r\alpha\mu(\rho+\mu) + \alpha b^2}{2r(\rho-\alpha)}$, $\alpha\mu(\rho+\mu) + \frac{\alpha b^2}{2r}$. It can be verified that the first column of the Routh array always has a sign change. By Routh's stability criterion, (18) has a root with a positive real part, and two roots with negative real parts.

Let λ_1, λ_2 be two roots of (18) with negative real parts, and ξ_1, ξ_2 be the corresponding (generalized) complex eigenvectors. Let $[\tilde{p}, \tilde{q}, \tilde{s}]^T = [\bar{p}, \bar{q}, s]^T - z$. The solution to equation (17) given by $z + e^{Mt}[\tilde{p}(0), \tilde{q}(0), \tilde{s}(0)]^T$ is in $C_b([0,\infty),\mathbb{R}^3)$ if and only if there exist constants a_1, a_2 such that $[\tilde{p}(0), \tilde{q}(0), \tilde{s}(0)]^T = a_1\xi_1 + a_2\xi_2$. Indeed, suppose

$$[\tilde{p}(0), \tilde{q}(0), \tilde{s}(0)]^T = a_1\xi_1 + a_2\xi_2 + a_3\xi_3,$$

where λ_3 is a root of (18) with a positive real part and ξ_3 is the corresponding complex eigenvector. The solution

$$z + e^{Mt}[\tilde{p}(0), \tilde{q}(0), \tilde{s}(0)]^T = z + \sum_{i=1}^3 h_i(t)e^{\lambda_i t}\xi_i$$

is in $C_b([0,\infty),\mathbb{R}^3)$ if and only if $a_3 = 0$, where $h_i(t)$ are polynomials of t.

Denote $\xi_1 = [(\xi_1^\dagger)^T, \xi_1^\ddagger]^T$, $\xi_2 = [(\xi_2^\dagger)^T, \xi_2^\ddagger]^T$ and $z = [(z^\dagger)^T, z^\ddagger]^T$, where $\xi_1^\ddagger, \xi_2^\ddagger, z^\ddagger \in \mathbb{R}$. Then we have

$$a_1\xi_1^\dagger + a_2\xi_2^\dagger = [\tilde{p}(0), \tilde{q}(0)]^T. \tag{19}$$

Note that $[\tilde{p}(0), \tilde{q}(0)]^T = [p_0, q_0]^T - z^{\dagger}$ is given. There exists a unique solution (a_1, a_2) to (19) if and only if $\xi_1^{\dagger}$ and $\xi_2^{\dagger}$ are linearly independent.

From the analysis above, we have the following result.

Proposition 2 *(17) admits a unique solution $(s, \bar{q}_{\theta})$ such that s and $\bar{q}_{\theta}$ are in $C_b([0, \infty), \mathbb{R})$ if and only if $\xi_1^{\dagger}$ and $\xi_2^{\dagger}$ are linearly independent. In this case, A4) holds.* □

Example 1 *Take parameters as $[\alpha \ \beta \ \mu \ b_i \ \sigma \ \rho \ r] = [1 \ 10 \ 0.15 \ 1 \ 0.2 \ 0.6 \ 1]$. Let $p(0) = 1$, $q_i(0) \sim N(2, 0.2)$. In this case, M has only two eigenvalues with negative real parts $-0.6875 + 0.3944i$ and $-0.6875 - 0.3944i$. The corresponding eigenvectors are $[-0.8557, \ 0.2674 + 0.3375i, \ 0.2768 + 0.0759i]^T$ and $[-0.8557, \ 0.2674 - 0.3375i, \ 0.2768 - 0.0759i]^T$, respectively. By (19), we have $a_1 = 1.4429 + 7.8552i$ and $a_2 = 1.4429 - 7.8552i$. Then (17) admits a unique solution in $C_b([0, \infty), \mathbb{R}^3)$. However, $\frac{b^2}{2r\mu(\rho + \mu)} = 4.444 > 1$. The parameters in this example satisfy the condition of Proposition 2, but not of Proposition 1.*

3.3 ε-Nash equilibrium

Consider the system of N firms. Let the control strategy of firm i be given by

$$\hat{u}_i = -\frac{b_i}{r}s, \quad i = 1, \cdots, N, \tag{20}$$

where $s \in C_b([0, \infty), \mathbb{R})$ is determined by the equation system (13)-(16). After the strategy (20) is applied, the closed-loop dynamics for firm i may be written as follows:

$$\frac{d\hat{p}(t)}{dt} = -\alpha\hat{p}(t) - \alpha\hat{q}^{(N)}(t) + \alpha\beta, \tag{21}$$

$$d\hat{q}_i(t) = -\mu\hat{q}_i(t)dt - \frac{b_i^2}{r}s(t)dt + \sigma dw_i, \ i = 1, \cdots, N. \tag{22}$$

Denote $\varepsilon_N = \left| \int_{\Theta} \theta^2 dF_N(\theta) - \int_{\Theta} \theta^2 dF(\theta) \right|$.

Theorem 2 *For the system (1)-(2), if assumptions A1)-A4) hold, then the closed-loop system (21)-(22) satisfies*

$$\sup_{t \geq 0, N \geq 1} \left\{ E|\hat{p}(t)|^2 + E|\hat{q}^{(N)}(t)|^2 \right\} \leq C_0, \tag{23}$$

$$\sup_{t \geq 0} E\left\{ |\hat{p}(t) - \bar{p}(t)|^2 + |\hat{q}^{(N)}(t) - \bar{q}(t)|^2 \right\}$$
$$= O(\varepsilon_N^2 + 1/N). \tag{24}$$

Proof. By (22), it follows that

$$d\hat{q}^{(N)}(t) = \left[-\mu\hat{q}^{(N)}(t) - \frac{1}{N}\sum_{i=1}^{N}\frac{b_i^2}{r}s(t) \right]dt + \frac{1}{N}\sum_{i=1}^{N}\sigma dw_i(t).$$

From this together with (21), we have

$$\begin{bmatrix} d\hat{p} \\ d\hat{q}^{(N)} \end{bmatrix} = \begin{bmatrix} -\alpha & -\alpha \\ 0 & -\mu \end{bmatrix}\begin{bmatrix} \hat{p} \\ \hat{q}^{(N)} \end{bmatrix}dt$$
$$- \begin{bmatrix} \alpha\beta \\ \frac{1}{N}\sum_{i=1}^{N}\frac{b_i^2}{r}s \end{bmatrix}dt + \begin{bmatrix} 0 \\ \frac{1}{N}\sum_{i=1}^{N}\sigma dw_i(t) \end{bmatrix}. \tag{25}$$

By $s \in C_b([0, \infty), \mathbb{R})$ and elementary linear SDE estimates, we have

$$\sup_{t \geq 0, N \geq 1} \left\{ E|\hat{p}(t)|^2 + E|\hat{q}^{(N)}(t)|^2 \right\} \leq C_0.$$

Denote $\eta = [\hat{p} - \bar{p}, \hat{q}^{(N)} - \bar{q}]^T$. By (13), (15) and (25), we have

$$d\eta = G\eta dt + \begin{bmatrix} 0 \\ \Delta_s \end{bmatrix}dt + \begin{bmatrix} 0 \\ \frac{1}{N}\sum_{i=1}^{N}\sigma dw_i \end{bmatrix},$$

where

$$G = \begin{bmatrix} -\alpha & -\alpha \\ 0 & -\mu \end{bmatrix}, \tag{26}$$

$$\Delta_s = \int_{\Theta}\frac{\theta^2}{r}sdF(\theta) - \frac{1}{N}\sum_{i=1}^{N}\frac{b_i^2}{r}s.$$

By solving this linear SDE and using the fact that G is Hurwitz, we can show

$$\sup_{t \geq 0} E\|\eta(t)\|^2 = O(\varepsilon_N^2 + \frac{1}{N}),$$

which leads to (24). □

By the above theorem, we can obtain the next corollary.

Corollary 1 *For the system (1)-(2), if assumptions A1)-A4) hold, then the closed-loop system (21)-(22) satisfies*

$$\sup_{t \geq 0, N \geq 1} E\int_0^{\infty} e^{-\rho t}\left\{ |\hat{p}(t)|^2 + |\hat{q}^{(N)}(t)|^2 \right\}dt \leq C_0,$$

$$E\int_0^{\infty} e^{-\rho t}\left\{ |\hat{p}(t) - \bar{p}(t)|^2 + |\hat{q}^{(N)}(t) - \bar{q}(t)|^2 \right\}dt$$
$$= O(\varepsilon_N^2 + 1/N).$$

We are now in a position to show an asymptotic Nash equilibrium property. Denote

$$\mathscr{U}_c = \left\{ u_i : u_i(t) \text{ is adapted to } \sigma\{\cup_{j=1}^{N}\mathscr{F}_t^j\}, \right.$$
$$\left. E\int_0^{\infty} e^{-\rho t}u_i^2(t)dt < \infty \right\}.$$
$$\hat{u}_{-i} = (\hat{u}_1, \cdots, \hat{u}_{i-1}, \hat{u}_{i+1}, \cdots, \hat{u}_N).$$

Theorem 3 *For the problem (1)-(2), assume that **A1)-A4)** hold. Then the set of strategies $(\hat{u}_1, \cdots, \hat{u}_N)$ given by (20) is an ε-Nash equilibrium, i.e.,*

$$J_i(\hat{u}_i, \hat{u}_{-i}) - \varepsilon \leq \inf_{u_i \in \mathcal{U}_c} J_i(u_i, \hat{u}_{-i}) \leq J_i(\hat{u}_i, \hat{u}_{-i}), \qquad (27)$$

where $\varepsilon = O(\varepsilon_N + \frac{1}{\sqrt{N}})$.

Proof. See Appendix A. □

4 Social Solutions to Output Adjustment

We first construct an auxiliary optimal control problem by examining the social cost variation due to the control perturbation of a single agent. Then, by mean field approximations we design a set of decentralized strategies which is shown to have asymptotic social optimality.

4.1 An auxiliary optimal control problem

We now provide a property of the social optimum problem which implies that $J_{\text{soc}}^{(N)}$ has a minimizer.

Lemma 2 *$J_{\text{soc}}^{(N)}(u)$ is coercive with respect to $(u_1, \cdots, u_N)$, i.e., there exist constants $C_2 > 0$ and $C_3 > 0$ such that*

$$J_{\text{soc}}^{(N)}(u) \geq \frac{C_2}{N} E \int_0^\infty e^{-\rho t} \sum_{i=1}^N u_i^2 dt - C_3.$$

Proof. From Lemma B.2, we can get the lemma immediately. □

This lemma ensures the existence of a centralized optimal solution to the social optimum problem in (1)-(2) and (4) (see [27]).

We now derive an auxiliary optimal control problem from the original social optimum problem by perturbing the strategy of a fixed agent. Denote the control problem **(P1)**:

$$\frac{dp}{dt} = -\alpha p - \frac{\alpha}{N} q_i - \alpha \hat{q}_{-i}^{(N)} + \alpha\beta,$$
$$dq_i = -\mu q_i dt + b_i u_i dt + \sigma dw_i,$$
$$\frac{dv_i}{dt} = -\alpha v_i - \alpha q_i, \; v_i(0) = 0,$$
$$J_i^*(u_i) = E \int_0^\infty e^{-\rho t} \left[(c-p)q_i - v_i \hat{q}_{-i}^{(N)} + r u_i^2 \right] dt,$$

where $\hat{q}_{-i}^{(N)} = \frac{1}{N} \sum_{j=1, j\neq i}^N \hat{q}_j$ and $\hat{u}_{-i}$ is given.

Lemma 3 *If $\hat{u} = (\hat{u}_1, \ldots, \hat{u}_N)$ minimizes $J_{\text{soc}}^{(N)}$ where each $\hat{u}_i \in \mathcal{U}_c$, then $\hat{u}_i$ is necessarily the optimal strategy of Problem **(P1)**.*

Proof. It follows from (3) that

$$J_{\text{soc}}^{(N)} = \frac{1}{N} \sum_{i=1}^N J_i(\hat{u}_1, \cdots, \hat{u}_{i-1}, u_i, \hat{u}_{i+1}, \cdots, \hat{u}_N)$$
$$= \frac{1}{N} E \int_0^\infty e^{-\rho t} (Y_i + Y_i') dt,$$

where

$$Y_i = (c-p)q_i - p \sum_{j=1, j\neq i}^N \hat{q}_j + r u_i^2,$$

$$Y_i' = c \sum_{j=1, j\neq i}^N \hat{q}_j + \sum_{j=1, j\neq i}^N r \hat{u}_j^2.$$

By (1),

$$p(t) = e^{-\alpha t} p(0) + \int_0^t e^{-\alpha(t-\tau)} (\alpha\beta - \alpha \hat{q}_{-i}^{(N)}) d\tau$$
$$- \frac{\alpha}{N} \int_0^t e^{-\alpha(t-\tau)} q_i d\tau.$$

Thus, $Y_i = Z_i + Z_i'$ where

$$Z_i = (c-p)q_i + \alpha \int_0^t e^{-\alpha(t-\tau)} q_i d\tau \cdot \hat{q}_{-i}^{(N)} + r u_i^2,$$

$$Z_i' = -\left[e^{-\alpha t} p(0) + \int_0^t e^{-\alpha(t-\tau)} (\alpha\beta - \alpha \hat{q}_{-i}^{(N)}) d\tau \right] \sum_{j=1, j\neq i}^N \hat{q}_j.$$

Note that $J_{\text{soc}}^{(N)} = \frac{1}{N} E \int_0^\infty e^{-\rho t} (Z_i + Z_i' + Y_i') dt$, where neither Z_i' nor Y_i' changes with u_i. Thus, for agent i minimizing $J_{\text{soc}}^{(N)}$ is equivalent to minimizing $E \int_0^\infty e^{-\rho t} Z_i dt$, which in turn is equal to $J_i^*(u_i)$. □

4.2 Mean field approximation

To approximate Problem (P1) for large N, we construct the auxiliary limiting optimal control problem **(P2)**:

$$\begin{bmatrix} d\bar{p} \\ dq_i \\ dv_i \end{bmatrix} = \begin{bmatrix} -\alpha & 0 & 0 \\ 0 & -\mu & 0 \\ 0 & -\alpha & -\alpha \end{bmatrix} \begin{bmatrix} \bar{p} \\ q_i \\ v_i \end{bmatrix} dt + \begin{bmatrix} 0 \\ b_i \\ 0 \end{bmatrix} u_i dt$$
$$+ \begin{bmatrix} \alpha\beta - \alpha\bar{q} \\ 0 \\ 0 \end{bmatrix} dt + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} dw_i, \begin{bmatrix} \bar{p}(0) \\ q_i(0) \\ v_i(0) \end{bmatrix} = \begin{bmatrix} p_0 \\ q_i(0) \\ 0 \end{bmatrix} \quad (28)$$

with cost function

$$\bar{J}_i^*(u_i) = E \int_0^\infty e^{-\rho t} [(c-\bar{p})q_i - \bar{q}v_i + r u_i^2] dt. \qquad (29)$$

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Here $\bar{q} \in C_{\rho/2}([0,\infty),\mathbb{R})$ is a deterministic function, which is an approximation of $q^{(N)}$ for large N.

For the system (28)-(29), we take $\bar{p}$ as an exogenous signal.

Lemma 4 $\bar{J}_i^*(u_i)$ *is strictly convex and coercive.*

Proof. For the system (28)-(29), the state is (q_i, v_i); $\bar{p}$ and $\bar{q}$ are not dependent on the control u_i. We can directly show that $\bar{J}_i^*(u_i)$ is strictly convex in u_i. Following the proof of Proposition A.1, we can show that $\bar{J}_i^*(u_i)$ is coercive. $\square$

Let

$$A = \begin{bmatrix} -\mu & 0 \\ -\alpha & -\alpha \end{bmatrix}, \quad B_i = \begin{bmatrix} b_i \\ 0 \end{bmatrix}.$$

By Theorem 1, Lemma 4 and [27], Problem (P2) has the unique optimal control given by

$$u_i = -\frac{1}{r} B_i^T \check{s},$$

where $\check{s} \in C_{\rho/2}([0,\infty),\mathbb{R}^2)$ is determined from

$$\rho \check{s} = \frac{d\check{s}}{dt} + A^T \check{s} + \frac{1}{2}[c - \bar{p}, -\bar{q}]^T.$$

Let $B_\theta = [\theta, 0]^T$. When $\theta = b_i$, we have $B_\theta = B_i$, and $y_\theta = y_i$. y_θ is regarded as the expectation of $(q_\theta, v_\theta)^T$ given the parameter θ in the individual dynamics. Following the standard approach in mean field control [19], [20], we construct the equation system as follows:

$$\frac{d\bar{p}}{dt} = \alpha[-\bar{p} + \beta - \bar{q}], \tag{30}$$

$$\rho \check{s} = \frac{d\check{s}}{dt} + A^T \check{s} + \frac{1}{2}[c - \bar{p}, -\bar{q}]^T, \tag{31}$$

$$\frac{dy_\theta}{dt} = Ay_\theta - \frac{1}{r} B_\theta B_\theta^T \check{s}, \tag{32}$$

$$\bar{q} = [1,0] \int_\Theta y_\theta dF(\theta), \tag{33}$$

where $y_\theta(0) = [q_0, 0]^T$. For further analysis, we assume

A5) There exists a solution $(\check{s}, y_\theta, \theta \in \Theta)$ to (30)-(33) such that for any $\theta \in \Theta$, both $\check{s}$ and y_θ are within $C_b([0,\infty),\mathbb{R}^2)$.

For the case of uniform agents ($b_i \equiv b$), the equation system (30)-(33) reduces to

$$\frac{d\bar{p}}{dt} = \alpha[\beta - \bar{p} - \bar{q}], \tag{34}$$

$$\rho \check{s} = \frac{d\check{s}}{dt} + A^T \check{s} + \frac{1}{2}[c - \bar{p}, -\bar{q}]^T, \tag{35}$$

$$\frac{dy}{dt} = Ay - \frac{1}{r} BB^T \check{s}, \tag{36}$$

where $\bar{p}(0) = p_0$, $y(0) = [q_0, 0]^T$ and $B = [b, 0]^T$. Let

$$M_s = \begin{bmatrix} -\alpha & 0 & 0 & -\alpha & 0 \\ \frac{1}{2} & \rho + \mu & \alpha & 0 & 0 \\ 0 & 0 & \rho + \alpha & \frac{1}{2} & 0 \\ 0 & -\frac{b^2}{r} & 0 & -\mu & 0 \\ 0 & 0 & 0 & -\alpha & -\alpha \end{bmatrix}, \tag{37}$$

$$\bar{b}_s = \begin{bmatrix} \alpha\beta \\ -\frac{c}{2} \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \varphi = \begin{bmatrix} \bar{p} \\ \check{s} \\ y \end{bmatrix}.$$

Then

$$\frac{d\varphi}{dt} = M_s \varphi + \bar{b}_s. \tag{38}$$

By straightforward computation, we can show $M_s \varphi + \bar{b}_s = 0$ has a unique solution, denoted as z_s. Furthermore, we have

$$|\lambda I - M_s| = (\lambda + \alpha) \times \Big[\lambda^4 - 2\rho\lambda^3$$
$$+ (\rho^2 - (\alpha + \mu)\rho - \alpha^2 - \mu^2)\lambda^2 + \rho[(\alpha + \mu)\rho + \alpha^2 + \mu^2]\lambda$$
$$+ \alpha\mu(\rho + \alpha)(\rho + \mu) + \frac{\alpha b^2}{2r}(\rho + 2\alpha) \Big].$$

By Routh's stability criterion [8], we obtain that $|\lambda I - M_s| = 0$ has two roots with positive real parts, and three roots with negative real parts.

Let $\check{\lambda}_1, \check{\lambda}_2, \check{\lambda}_3$ be two roots of $|\lambda I - M_s| = 0$ with negative real parts, and $\zeta_1, \zeta_2, \zeta_3$ be the corresponding (generalized) complex eigenvectors. Let $\tilde{\varphi}(0) = [\tilde{p}^T(0), \tilde{s}^T(0), \tilde{y}^T(0)]^T = \varphi(0) - z_s$. The solution to equation (38) given by $z_s + e^{M_s t} \tilde{\varphi}(0)$ is in $C_b([0,\infty),\mathbb{R}^5)$ if and only if there exist constants $\check{a}_1, \check{a}_2, \check{a}_3$ such that $\tilde{\varphi}(0) = \check{a}_1 \zeta_1 + \check{a}_2 \zeta_2 + \check{a}_3 \zeta_3$.

Denote $\zeta_i = [(\zeta_i^\dagger)^T, (\zeta_i^\ddagger)^T, (\zeta_i^\S)^T]^T$ and $\vartheta_i = [(\zeta_i^\dagger)^T, (\zeta_i^\S)^T]^T$, where $\zeta_i^\dagger \in \mathbb{R}$ and $\zeta_i^\ddagger, \zeta_i^\S \in \mathbb{R}^2$, $i = 1,2,3$. Then we have

$$\check{a}_1 \vartheta_1 + \check{a}_2 \vartheta_2 + \check{a}_3 \vartheta_3 = [\tilde{p}(0), \tilde{y}(0)]^T. \tag{39}$$

Note that $[\tilde{p}(0), \tilde{y}(0)]^T$ is given. There exists a unique solution $(\check{a}_1, \check{a}_2, \check{a}_3)$ to (39) if and only if ϑ_1, ϑ_2 and ϑ_3 are linearly independent.

From the analysis above, we have the following result.

Proposition 3 *A5) holds if (39) admits a solution. In particular, (34)-(36) admits a unique solution $(\check{s}, y_\theta)$ such that $\check{s}$ and y_θ are within $C_b([0,\infty),\mathbb{R}^2)$ if and only if ϑ_1, ϑ_2 and ϑ_3 are linearly independent.* $\square$

4.3 Asymptotic optimality

Consider the system of N firms. Let the control strategy of firm i be given by

$$\check{u}_i = -\frac{1}{r}B_i^T \check{s}, \quad i = 1,\cdots,N, \tag{40}$$

where $\check{s} \in C_b([0,\infty),\mathbb{R}^2)$ is determined by the equation system (30)-(33). After the strategy (40) is applied, the closed-loop dynamics for firm i may be written as follows:

$$\frac{d\check{p}}{dt} = -\alpha\check{p} - \alpha\check{q}^{(N)} + \alpha\beta, \tag{41}$$

$$d\check{q}_i = -\mu\check{q}_i - \left[\frac{b_i^2}{r},0\right]\check{s}dt + \sigma dw_i, \tag{42}$$

$$\frac{d\check{v}_i}{dt} = -\alpha\check{q}_i - \alpha\check{v}_i, \tag{43}$$

where $\check{p}(0) = p_0$, $\check{q}_i(0) = q_i(0)$, and $\check{v}_i(0) = 0$.

Theorem 4 *Assume that A1)-A3) and A5) hold. The set of strategies $\{\check{u}_i = -\frac{1}{r}B_i^T\check{s}, i = 1,\cdots,N\}$ has asymptotic social optimality, i.e.,*

$$\left| J_{\text{soc}}^{(N)}(\check{u}) - \inf_{u_i \in \mathcal{U}_c} J_{\text{soc}}^{(N)}(u) \right| = O(\frac{1}{\sqrt{N}} + \varepsilon_N).$$

Proof. See Appendix B. □

We now give a closed-form expression of the asymptotic social cost.

Theorem 5 *Assume that A1)-A3) and A5) hold. Then the asymptotic optimal social cost is given by*

$$\lim_{N\to\infty} \inf_{u_i \in \mathcal{U}_c} J_{\text{soc}}^{(N)}(u) = 2\check{s}_1(0)q_0 + \int_{\mathbb{R}} \check{g}_\theta(0)dF(\theta)$$
$$+ \int_0^\infty e^{-\rho t}\bar{q}(t)\bar{v}(t)dt,$$

where $\check{s}_1 \overset{\Delta}{=} [1,0]\check{s}$, $\bar{v}(t) = -\alpha\int_0^t e^{-\alpha(t-\tau)}\bar{q}(\tau)d\tau$ and $\check{g}_i \in C_b([0,\infty),\mathbb{R})$ satisfies

$$\rho\check{g}_i = \frac{d\check{g}_i}{dt} - \frac{b_i^2}{r}\check{s}_1^2.$$

Proof. From Lemma B.1 and Schwarz's inequality, it follows that

$$\max_{1\le i\le N} \left| J_i(\check{u}) - E\int_0^\infty e^{-\rho t}[(c-\bar{p})\check{q}_i + r\check{u}_i^2]dt \right| = O(\varepsilon_N + \frac{1}{\sqrt{N}}).$$

Let $J_i^*(\check{u}_i)$ be the optimal cost of Problem (P2). By Theorem 1, we have

$$\frac{1}{N}\sum_{i=1}^N E\int_0^\infty e^{-\rho t}[(c-\bar{p})\check{q}_i + r\check{u}_i^2]dt$$

$$= \frac{1}{N}\sum_{i=1}^N J_i^*(\check{u}_i) + \frac{1}{N}\sum_{i=1}^N E\int_0^\infty e^{-\rho t}\bar{q}\check{v}_i dt$$

$$= \frac{1}{N}\sum_{i=1}^N \left\{ 2\check{s}(0)[q_0,0]^T + \check{g}_i(0) \right.$$

$$\left. -\alpha E\int_0^\infty e^{-\rho t}\bar{q}(t)\int_0^t e^{-\rho\tau}\check{q}_i(\tau)d\tau dt \right\}$$

$$= 2\check{s}_1(0)q_0 + \frac{1}{N}\sum_{i=1}^N \check{g}_i(0) + E\int_0^\infty e^{-\rho t}\bar{q}(t)\check{v}^{(N)}(t)dt, \tag{44}$$

where

$$\check{v}_i(t) = -\alpha\int_0^t e^{-\rho\tau}\check{q}_i(\tau)d\tau,$$

and $\check{g}_i \in C_b([0,\infty),\mathbb{R})$ satisfies

$$\rho\check{g}_i = \frac{d\check{g}_i}{dt} - \frac{b_i^2}{r}\check{s}_1^2.$$

By Schwarz's inequality and Lemma B.1,

$$\left| E\int_0^\infty e^{-\rho t}\bar{q}(t)|\check{v}^{(N)}(t) - \bar{v}(t)|dt \right|^2$$

$$\le E\int_0^\infty e^{-\rho t}|\bar{q}(t)|^2 dt \cdot E\int_0^\infty e^{-\rho t}|\check{v}^{(N)}(t) - \bar{v}(t)|^2 dt$$

$$\le C(\frac{1}{N} + \varepsilon_N^2). \tag{45}$$

Notice that $\check{g}_\theta(0)$ is continuous in θ. By the weak convergence of F_N to F, we obtain

$$\lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^N \check{g}_i(0) = \int_{\mathbb{R}} \check{g}_\theta(0)dF(\theta),$$

which together with (44) and (45) completes the proof. □

5 Numerical Simulation

In this section, we provide a numerical example to illustrate the evolution of firms' outputs and compare prices, average outputs and optimal costs under two different solution frameworks: the mean field game and social optimization.

Take $[\alpha, \beta, \mu, b_i, \sigma, \rho\ r] = [1, 10, 0.15, 1, 0.2, 0.6\ 1]$. Let $p(0) = 1$, and $q_i(0) \sim N(2,0.2)$. By (19), we have $a_1 = 1.4429 + 7.8552i$ and $a_2 = 1.4429 - 7.8552i$; by (39), we get $\check{a}_1 = 9$, $\check{a}_2 = -3.5906 - 6.5046i$ and $\check{a}_3 = -3.5906 + 6.5046i$. Thus, we can write the exact expressions of s and $\check{s}$.

It can be verified that **A1)-A5)** hold. Figs. 1 and 2 show the curves of output levels of firms within the two frameworks,

respectively. They remain positive, although random fluctuations appear and there are greater fluctuations in the social optimum framework. After the transient phase, the output levels of the firms behave similarly.
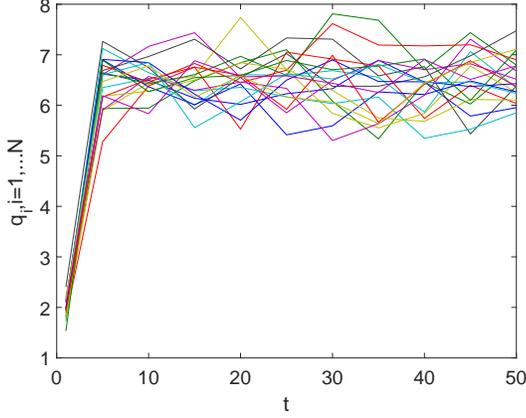

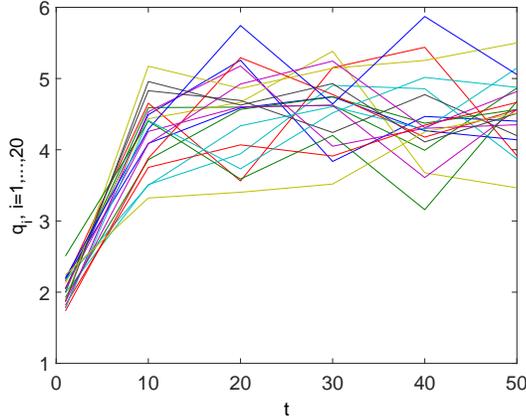Fig. 1: Curves of $q_i, i = 1, \cdots, 20$ in the game solution

Fig. 2: Curves of $q_i, i = 1, \cdots, 20$ in the social optimum solution

Fig. 3 depicts the curves of p and $\bar{p}$ within game and social frameworks, when the total number of agents is 50. Fig. 4 shows the curves of $q^{(50)}$ and $q^{(\infty)}$ within two frameworks, where $q^{(\infty)} = \bar{q}$ is the average output of firms in the infinite population case. It can be seen that the curves of p and $\bar{p}$ as well as $q^{(50)}$ and $q^{(\infty)}$ coincide well, which illustrates the accuracy of the mean field approximation. From the game framework to the social framework, the price gets a significant increase, and the average of outputs becomes lower.
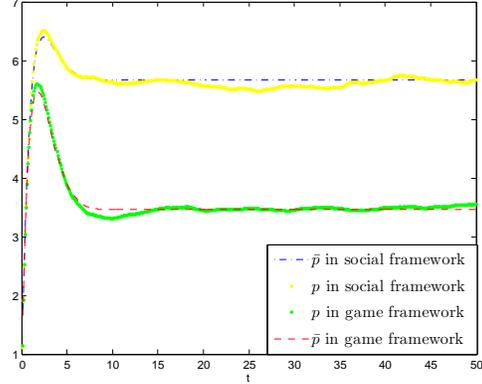

Fig. 3: Curves of p and $\bar{p}$ in the game and social optimum, respectively
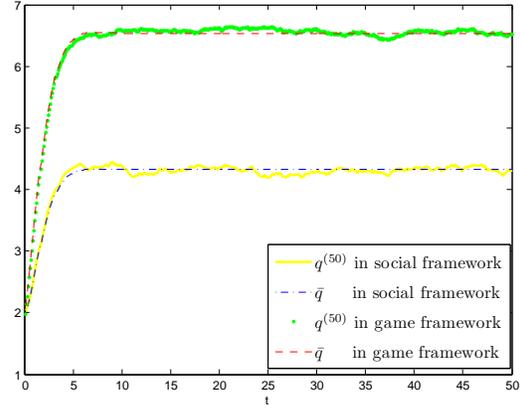

Fig. 4: Curves of $q^{(50)}$, $q^{(\infty)}$ in the game and social optimum, respectively

By Theorem 1, we get that the asymptotic Nash cost is given by

$$J_{\text{Nash}}^{(\infty)} \triangleq \lim_{N \to \infty} J_i(\hat{u}_i, \hat{u}_{-i}) = 2s(0)q_0 + g(0),$$

where $g(0) = -\int_0^\infty e^{-\rho t} s^2 dt$. Note that $s = z^{\ddagger} + a_1 e^{\lambda_1 t} \xi_1^{\ddagger} + a_2 e^{\lambda_2 t} \xi_2^{\ddagger}$. Then

$$\begin{aligned} J_{\text{Nash}}^{(\infty)} =& 2s(0)q_0 - \frac{b^2}{r} \int_0^\infty e^{-\rho t} s^2(t) dt \\ =& 2q_0(z^{\ddagger} + a_1 \xi_1^{\ddagger} + a_2 \xi_2^{\ddagger}) \\ & - \frac{b^2}{r} \int_0^\infty e^{-\rho t} (z^{\ddagger} + a_1 e^{\lambda_1 t} \xi_1^{\ddagger} + a_2 e^{\lambda_2 t} \xi_2^{\ddagger})^2 dt \\ =& 2q_0(z^{\ddagger} + a_1 \xi_1^{\ddagger} + a_2 \xi_2^{\ddagger}) - \frac{b^2}{r} \Big[\frac{(z^{\ddagger})^2}{\rho} + \frac{a_1^2(\xi_1^{\ddagger})^2}{\rho - 2\lambda_1} \\ & + \frac{a_2^2(\xi_2^{\ddagger})^2}{\rho - 2\lambda_2} + \frac{2a_1 z^{\ddagger} \xi_1^{\ddagger}}{\rho - \lambda_1} + \frac{2a_2 z^{\ddagger} \xi_2^{\ddagger}}{\rho - \lambda_2} + \frac{2a_1 a_2 \xi_1^{\ddagger} \xi_2^{\ddagger}}{\rho - \lambda_1 - \lambda_2} \Big]. \end{aligned}$$

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α	0.1	0.2	0.5	1	5
$J_{\text{Nash}}^{(\infty)}$	-0.31	-2.09	-5.47	-7.95	-11.11
$J_{\text{soc}}^{(\infty)}$	-0.78	-3.32	-8.59	-12.72	-18.02

Tab. 1: Optimal costs under game and social solutions

By Theorem 5,

$$J_{\text{soc}}^{(\infty)} \triangleq \lim_{N \to \infty} J_{\text{soc}}^{(N)}(\check{u}) = 2\check{s}_1(0)q_0 + \check{g}(0)$$
$$+ \int_0^\infty e^{-\rho t} \bar{q}(t)\bar{v}(t)dt,$$

where $\check{s}_1 = [1,0]\check{s}$ and $\check{g}(0) = -\int_0^\infty e^{-\rho t}\check{s}_1^2 dt$. Let $z_s = [z_s^{(1)}, z_s^{(2)}, z_s^{(3)}, z_s^{(4)}, z_s^{(5)}]^T$ and

$$\zeta_i = [\zeta_i^{(1)}, \zeta_i^{(2)}, \zeta_i^{(3)}, \zeta_i^{(4)}, \zeta_i^{(5)}]^T, \ i = 1, 2, 3.$$

From (37), we obtain $\check{\lambda}_1 = -\alpha$ and $\zeta_1 = [0,0,0,0,1]^T$, which implies $\zeta_1^{(2)} = 0$, $\zeta_1^{(4)} = 0$, $\zeta_1^{(5)} = 1$. Thus,

$$J_{\text{soc}}^{(\infty)}$$
$$= 2q_0(z_s^{(2)} + \check{a}_2\zeta_2^{(2)} + \check{a}_3\zeta_3^{(2)}) - \frac{b^2}{r}\left[\frac{(z_s^{(2)})^2}{\rho} + \frac{\check{a}_2^2(\zeta_2^{(2)})^2}{\rho - 2\check{\lambda}_2}\right.$$
$$+ \frac{\check{a}_3^2(\zeta_3^{(2)})^2}{\rho - 2\check{\lambda}_3} + \frac{2\check{a}_2 z_s^{(2)}\zeta_2^{(2)}}{\rho - \check{\lambda}_2} + \frac{2\check{a}_3 z_s^{(2)}\zeta_3^{(2)}}{\rho - \check{\lambda}_3} + \frac{2\check{a}_2\check{a}_3\zeta_2^{(2)}\zeta_3^{(2)}}{\rho - \check{\lambda}_2 - \check{\lambda}_3}\right]$$
$$+ \left[\frac{z_s^{(4)} z_s^{(5)}}{\rho} + \sum_{i=1}^3 \frac{\check{a}_i z_s^{(4)}\zeta_i^{(5)}}{\rho - \check{\lambda}_i} + \frac{\check{a}_2\zeta_2^{(4)} z_s^{(5)}}{\rho - \check{\lambda}_2} + \sum_{i=1}^3 \frac{\check{a}_2\check{a}_i\zeta_2^{(4)}\zeta_i^{(5)}}{\rho - \check{\lambda}_2 - \check{\lambda}_i}\right.$$
$$+ \frac{\check{a}_3\zeta_3^{(4)} z_s^{(5)}}{\rho - \check{\lambda}_3} + \sum_{i=1}^3 \frac{\check{a}_3\check{a}_i\zeta_3^{(4)}\zeta_i^{(5)}}{\rho - \check{\lambda}_3 - \check{\lambda}_i}\right].$$

The comparison of the two costs is shown in Tab. 1 and Fig. 5. It can be seen that the asymptotic Nash cost $J_{\text{Nash}}^{(\infty)}$ is greater than the optimal social cost $J_{\text{soc}}^{(\infty)}$, and more so when α increases. This illustrates that the collusion of firms leads to rise in the price and drop in outputs and costs [48, p. 467].
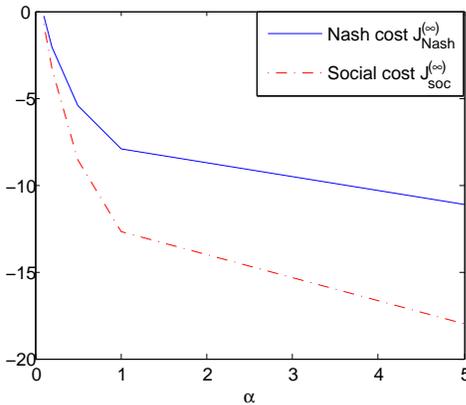

Fig. 5: Curves of $J_{\text{Nash}}^{(\infty)}$ and $J_{\text{soc}}^{(\infty)}$ with respect to α

6 Concluding Remarks

This paper studies dynamic production output optimization with sticky prices and adjustment costs based on the mean field control methodology. By consistent mean field approximations, we first present the Nash solution for noncooperative firms, and then give the social solution, where agents cooperate to optimize the social cost. The two sets of decentralized strategies are shown to approximate Nash equilibria and social optima, respectively. For future work, it is of interest to consider dynamic production output competition with noisy sticky prices.

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Acknowledgements

Appendix A: Proof of Theorem 3

Proposition A.1 *Under A1)-A3), $\bar{J}_i(u_i)$ is coercive in u_i, i.e., there exists a constant C depending on p_0, q_0 and a constant $\varepsilon_0 > 0$ such that*

$$\bar{J}_i(u_i) \geq \varepsilon_0 \|u_i\|_\rho^2 - C, \qquad (A.1)$$

where $\|u_i\|_\rho^2 = E \int_0^\infty e^{-\rho t} |u_i(t)|^2 dt$.

Proof. From (2),

$$q_i(t) = q_i(0)e^{-\mu t} + \int_0^t e^{-\mu(t-\tau)} \left[b_i u_i(\tau) d\tau + \sigma dw_i(\tau) \right].$$

Thus, by Cauchys inequality and **A3)**,

$$E \int_0^\infty e^{-\rho t} |q_i(t)| dt$$
$$\leq C + E \int_0^\infty e^{-(\rho+\mu)t} \int_0^t e^{\mu\tau} |b_i u_i(\tau)| d\tau dt$$
$$= C + E \int_0^\infty e^{\mu\tau} |b_i u_i(\tau)| \int_\tau^\infty e^{-(\rho+\mu)t} dt d\tau$$
$$= C + \frac{1}{\rho+\mu} E \int_0^\infty e^{-\rho\tau} |b_i u_i(\tau)| d\tau$$
$$\leq C + \delta_1 E \int_0^\infty e^{-\rho\tau} |u_i(\tau)|^2 d\tau, \qquad (A.2)$$

where δ_1 is a sufficiently small positive number. Note $\bar{q} \in C_b([0,\infty),\mathbb{R})$. By (5), it follows that $\bar{p} \in C_b([0,\infty),\mathbb{R})$. From this together with (A.2) and (6), we obtain that there exists a constant $\varepsilon_0 > 0$ such that (A.1) holds. $\qquad\square$

Lemma A.1 *For any $i = 1,\cdots,N$, there exists a constant C such that*

$$\|q_i\|_\rho^2 \leq C\|u_i\|_\rho^2 + C. \qquad (A.3)$$

Proof. Denote $q_{i,\rho} = e^{-\frac{\rho}{2}} q_i$ and $u_{i,\rho} = e^{-\frac{\rho}{2}} u_i$. By Ito's formula, we have

$$dq_{i,\rho} = -(\mu + \frac{\rho}{2})q_{i,\rho} dt + u_{i,\rho} dt + e^{-\frac{\rho}{2}} dw_i.$$

From this it follows that

$$E \int_0^\infty |q_{i,\rho}(t)|^2 dt \leq C + E \int_0^\infty \left| \int_0^t e^{-(\mu+\frac{\rho}{2})(t-\tau)} u_{i,\rho}(\tau) d\tau \right|^2 dt.$$

By the argument in the proof of Lemma A.1 in [17], we get (A.3). $\qquad\square$

Lemma A.2 *For the problem (1)-(2), assume that **A1)-A4)** hold. For $u_i \in \mathscr{U}_c$, if $J(u_i,\hat{u}_{-i}) \leq C_1$, then there exist an integer N_0 and a constant C_2 such that for all $N \geq N_0$, $\|u_i\|_\rho^2 \leq C_2$.*

Proof. For $u_i \in \mathscr{U}_c$, we have

$$J_i(u_i,\hat{u}_{-i}) = \bar{J}_i(u_i) - E \int_0^\infty e^{-\rho t} [p(t) - \bar{p}(t)] q_i(t) dt$$
$$\geq \bar{J}_i(u_i) - \left[E \int_0^\infty e^{-\rho t} |p(t) - \bar{p}(t)|^2 dt \right.$$
$$\left. \cdot E \int_0^\infty e^{-\rho t} |q_i(t)|^2 dt \right]^{1/2}$$
$$\overset{\Delta}{=} \bar{J}_i(u_i) - I_1. \qquad (A.4)$$

By Proposition A.1, $\bar{J}_i(u_i)$ is coercive, i.e.,

$$\bar{J}_i(u_i) \geq \varepsilon_0 \|u_i\|_\rho^2 - C. \qquad (A.5)$$

We now estimate I_1. Notice $p(t) - \bar{p}(t) = p(t) - \hat{p}(t) + \hat{p}(t) - \bar{p}(t)$. We have

$$d \begin{bmatrix} p - \hat{p} \\ q^{(N)} - \hat{q}^{(N)} \end{bmatrix} = G \begin{bmatrix} p - \hat{p} \\ q^{(N)} - \hat{q}^{(N)} \end{bmatrix} dt + \begin{bmatrix} 0 \\ \frac{b_i}{N}(u_i - \frac{b_i s}{r})dt \end{bmatrix},$$

where G is defined in (26). Noticing that G is Hurwitz, by basic estimates as in the proof of in Lemma A.1, we obtain

$$\int_0^\infty e^{-\rho t} |p(t) - \hat{p}(t)|^2 dt \leq \frac{C}{N^2} + \frac{C}{N^2} \|u_i\|_\rho^2. \qquad (A.6)$$

From this together with Theorem 2, it follows that

$$E \int_0^\infty e^{-\rho t} |p(t) - \bar{p}(t)|^2 dt \leq C + \frac{C}{N^2} \|u_i\|_\rho^2,$$

which with Lemma A.1 implies

$$|I_1| \leq C + \frac{C}{N} \|u_i\|_\rho^2.$$

Combining this together with (A.4) and (A.5) yields

$$C_1 \geq J_i(u_i,\hat{u}_{-i}) \geq (\varepsilon_0 - \frac{C}{N})\|u_i\|_\rho^2 - 2C.$$

Let $N_0 = \inf\{m \in \mathbb{Z} | m > C/\varepsilon_0\}$. From this inequality, we obtain that there exists a constant C_2 such that for all $N \geq N_0$,

$$\|u_i\|_\rho^2 \leq \frac{N(C_1 + 2C)}{N\varepsilon_0 - C} \leq C_2.$$

$\qquad\square$

Lemma A.3 *Under **A1)-A4)**,*

$$|\bar{J}_i(\hat{u}_i) - J_i(\hat{u}_i,\hat{u}_{-i})| = O(\varepsilon_N + \frac{1}{\sqrt{N}}).$$

Proof. By Schwarz's inequality,

$$|\bar{J}_i(\hat{u}_i) - J_i(\hat{u}_i,\hat{u}_{-i})|$$
$$= \int_0^\infty e^{-\rho t} [\hat{p}(t)\hat{q}_i(t) - \bar{p}(t)\hat{q}_i(t)] dt$$
$$\leq \left[\int_0^\infty e^{-\rho t} |\hat{q}_i(t)|^2 dt \cdot \int_0^\infty e^{-\rho t} |\hat{p}(t) - \bar{p}(t))|^2 dt \right]^{\frac{1}{2}}$$
$$\leq O(\varepsilon_N + \frac{1}{\sqrt{N}}),$$

where the last inequality follows from Theorem 2. $\qquad\square$

Proof of Theorem 3. It suffices to show the first inequality in (27) under $J_i(u_i,\hat{u}_{-i}) \leq C_1$. By Lemma A.2, $J_i(u_i,\hat{u}_{-i}) \leq C_1$ implies

$$\|u_i\|_\rho^2 \leq C_2. \qquad (A.7)$$

It follows from (A.4) that

$$J_i(u_i, \hat{u}_{-i}) = \bar{J}_i(u_i) - I_1, \qquad (A.8)$$

where

$$I_1 = \left[E \int_0^\infty e^{-\rho t} |p(t) - \bar{p}(t)|^2 dt \cdot E \int_0^\infty e^{-\rho t} |q_i(t)|^2 dt \right]^{1/2}.$$

By (A.6), (A.7) and Theorem 2,

$$E \int_0^\infty e^{-\rho t} |p(t) - \bar{p}(t)|^2 dt \le O(\varepsilon_N^2 + \frac{1}{N}),$$

which with Lemma A.1 further implies $|I_1| \le O(\varepsilon_N + \frac{1}{\sqrt{N}})$. Thus, by (A.8) it follows that

$$J_i(u_i, \hat{u}_{-i}) = \bar{J}_i(u_i) - O(\varepsilon_N + \frac{1}{\sqrt{N}})$$
$$\ge \bar{J}_i(\hat{u}_i) - O(\varepsilon_N + \frac{1}{\sqrt{N}}). \qquad (A.9)$$

On the other hand, by Lemma A.3 we have

$$\bar{J}_i(\hat{u}_i) \ge J_i(\hat{u}_i, \hat{u}_{-i}) - O(\varepsilon_N + \frac{1}{\sqrt{N}}). \qquad (A.10)$$

Thus, the first inequality in (27) follows from (A.9) and (A.10). □

Appendix B: Proof of Theorem 4

Lemma B.1 *Assume that A1)-A3) and A5) hold. For the set of strategies $\{\check{u}_i, i = 1, \cdots, N\}$, we have*

$$\sup_{t \ge 0, N \ge 1} E \{ |\check{p}(t)|^2 + |\check{q}_i(t)|^2 + |\check{q}^{(N)}(t)|^2 + |\check{v}_i(t)|^2 \}$$
$$\le C_0, \qquad (B.1)$$
$$\sup_{t \ge 0} E \{ |\check{p}(t) - \bar{p}(t)|^2 + |\check{q}^{(N)}(t) - \bar{q}(t)|^2 + |\check{v}^{(N)}(t) - \bar{v}(t)|^2 \}$$
$$= O(\varepsilon_N^2 + 1/N), \qquad (B.2)$$

where $[\bar{q}, \bar{v}]^T = \int_\Theta y_\theta dF(\theta)$ is given by (30)-(33), and $\check{v}^{(N)} = \frac{1}{N} \sum_{i=1}^N \check{v}_i$.

Proof. It follows from (42) and (43) that

$$\begin{bmatrix} d\check{q}_i \\ d\check{v}_i \end{bmatrix} = A \begin{bmatrix} \check{q}_i \\ \check{v}_i \end{bmatrix} dt - \frac{1}{r} B_i B_i^T \check{s} dt + \begin{bmatrix} dw_i \\ 0 \end{bmatrix}.$$

Notice that A is Hurwitz and $\check{s} \in C_b([0, \infty), \mathbb{R}^2)$. By elementary linear estimates and **A3)** we can show that there exists a constant C_0 independent of (i, N) such that

$$\sup_{t \ge 0} E \{ |\check{q}_i(t)|^2 + |\check{v}_i(t)|^2 \} \le C_0,$$

which further gives $\sup_{t \ge 0} E \{ |\check{q}^{(N)}(t)|^2 \} \le C_0$. This together with (41) leads to (B.1). By (42) and (43), we have

$$\begin{bmatrix} d\check{q}^{(N)} \\ d\check{v}^{(N)} \end{bmatrix} = A \begin{bmatrix} \check{q}^{(N)} \\ \check{v}^{(N)} \end{bmatrix} dt - \frac{1}{Nr} \sum_{i=1}^N B_i B_i^T \check{s} dt$$
$$+ \begin{bmatrix} \frac{\sigma}{N} \sum_{i=1}^N dw_i \\ 0 \end{bmatrix}. \qquad (B.3)$$

Denote $\xi = [\check{q}^{(N)} - \bar{q}, \check{v}^{(N)} - \bar{v}]^T$. It follows from (30) and (B.3) that

$$d\xi = A\xi dt + \left[\frac{1}{r} \int_\Theta B_\theta B_\theta^T \check{s} dF(\theta) - \frac{1}{Nr} \sum_{i=1}^N B_i B_i^T \check{s} \right] dt$$
$$+ \begin{bmatrix} \frac{\sigma}{N} \sum_{i=1}^N dw_i \\ 0 \end{bmatrix},$$

where $\xi(0) = [\check{q}^{(N)}(0) - q_0, 0]^T$. Since A is Hurwitz,

$$E|\check{q}^{(N)}(t) - \bar{q}(t)|^2 + E|\check{v}^{(N)}(t) - \bar{v}(t)|^2$$
$$= O(\varepsilon_N^2 + 1/N). \qquad (B.4)$$

By (30) and (41) we have

$$\frac{d(\check{p} - \bar{p})}{dt} = -\alpha(\check{p} - \bar{p}) - \alpha(\check{q}^{(N)} - \bar{q}),$$
$$\check{p}(0) = \bar{p}(0),$$

which leads to

$$\check{p}(t) - \bar{p}(t) = -\alpha \int_0^t e^{-\alpha(t-\tau)} [\check{q}^{(N)}(\tau) - \bar{q}(\tau)] d\tau.$$

This together with (B.4) gives (B.2). □

Lemma B.2 *There exist constants $C_1 > 0, C_2 > 0$ and $C_3 > 0$ such that*

$$J_{\mathrm{soc}}^{(N)}(u) \ge C_1 \int_0^\infty e^{-\rho t} p^2 dt + \frac{C_2}{N} \int_0^\infty e^{-\rho t} \sum_{i=1}^N u_i^2 dt - C_3.$$

Proof. By **A3)**, assume $|b_i| \le \hat{b}$, $i = 1, \cdots, N$. From (2), we compute

$$\left| q^{(N)}(t) \right| \le \left| q^{(N)}(0) e^{-\mu t} \right| + \int_0^t e^{-\mu(t-\tau)} \left| \frac{1}{N} \sum_{i=1}^N b_i u_i(\tau) \right| d\tau$$
$$+ \left| \frac{\sigma}{N} \sum_{i=1}^N \int_0^t e^{-\mu(t-\tau)} dw_i(\tau) \right|.$$

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Thus, by Cauchy's inequality,

$$E \int_0^\infty e^{-\rho t} |q^{(N)}| dt$$

$$\leq C + E \int_0^\infty e^{-(\rho+\mu)t} \int_0^t e^{\mu\tau} \Big| \frac{1}{N} \sum_{i=1}^N b_i u_i(\tau) \Big| d\tau dt$$

$$= C + E \int_0^\infty e^{\mu\tau} \Big| \frac{1}{N} \sum_{i=1}^N b_i u_i(\tau) \Big| \int_\tau^\infty e^{-(\rho+\mu)t} dt d\tau$$

$$= C + \frac{1}{\rho+\mu} E \int_0^\infty e^{-\rho\tau} \Big| \frac{1}{N} \sum_{i=1}^N b_i u_i(\tau) \Big| d\tau$$

$$\leq C + \delta_1 E \int_0^\infty e^{-\rho\tau} \Big| \frac{1}{N} \sum_{i=1}^N b_i u_i(\tau) \Big|^2 d\tau$$

$$\leq C + \frac{\delta_1 \hat{b}^2}{N} E \int_0^\infty e^{-\rho\tau} \sum_{i=1}^N u_i^2(\tau) d\tau, \qquad (B.5)$$

where δ_1 is a sufficiently small positive number. Note

$$p(t) = e^{-\alpha t} p(0) + \int_0^t e^{-\alpha(t-\tau)} (-\alpha q^{(N)} + \alpha\beta) d\tau.$$

From (B.5) we have

$$E \int_0^\infty e^{-\rho t} |p(t)| dt$$

$$\leq E \int_0^\infty e^{-\rho t} \Big| e^{-\alpha t} p(0) + \int_0^t e^{-\alpha(t-\tau)} (-\alpha q^{(N)} + \alpha\beta) d\tau \Big| dt$$

$$\leq C + \alpha E \int_0^\infty e^{-(\rho+\alpha)t} \int_0^t e^{\alpha\tau} |q^{(N)}(\tau)| d\tau dt$$

$$= C + \frac{\alpha}{\rho+\alpha} E \int_0^\infty e^{-\rho\tau} |q^{(N)}(\tau)| d\tau$$

$$\leq C + \frac{\alpha\delta_1 \hat{b}^2}{N(\rho+\alpha)} E \int_0^\infty e^{-\rho\tau} \sum_{i=1}^N u_i^2(\tau) d\tau, \qquad (B.6)$$

where the fourth line is obtained by an exchange of order of the integration.

Consider the system

$$\frac{dp}{dt} = -\alpha p - \alpha \acute{u}, \ p(0) = p_0$$
$$y = -p$$
$$\acute{u} = q^{(N)} - \beta. \qquad (B.7)$$

By verifying the conditions in the positive real lemma (see e.g., [25, Lemma 6.2]), we obtain that the transfer function of the system (B.7) is positive real, which leads to the passivity of (B.7). This implies that there exists a constant $l > 0$ such that $\acute{u} y \geq \dot{V}(p)$ where $V(p) \overset{\Delta}{=} l p^2$, which further gives

$$E \int_0^\infty e^{-\rho t} (-p)(q^{(N)} - \beta) dt \geq E \int_0^\infty e^{-\rho t} d(l p^2)$$

$$= l E \Big[\rho \int_0^\infty p^2 e^{-\rho t} dt - p^2(0) \Big]. \qquad (B.8)$$

From this together with (B.5) and (B.6), we have

$$J_{\text{soc}}^{(N)}(u)$$

$$= E \int_0^\infty e^{-\rho t} \Big[(c-p) q^{(N)} + \frac{r}{N} \sum_{i=1}^N u_i^2(t) \Big] dt$$

$$= E \int_0^\infty e^{-\rho t} \Big[(-p)(q^{(N)} - \beta) - \beta p + c q^{(N)} + \frac{r}{N} \sum_{i=1}^N u_i^2(t) \Big] dt$$

$$\geq l E \Big[\rho \int_0^\infty p^2(t) e^{-\rho t} dt - p^2(0) \Big]$$

$$+ \frac{1}{N} (r - \frac{\beta\alpha\delta_1 \hat{b}^2}{\rho+\alpha} - c\delta_1 \hat{b}^2) E \int_0^\infty e^{-\rho t} \sum_{i=1}^N u_i^2(t) dt - C.$$

This completes the proof of the lemma. $\qquad\square$

Proof of Theorem 4. Notice $\inf_{u_i \in \mathscr{U}_c} J_{\text{soc}}^{(N)}(u) \leq J^{(N)}(\check{u}) \leq C$. It suffices to consider all $u_i \in \mathscr{U}_c$ satisfying

$$J_{\text{soc}}^{(N)}(u) \leq J^{(N)}(\check{u}) \leq C.$$

By Lemma B.2,

$$E \int_0^\infty e^{-\rho t} \Big(|p|^2 + \frac{1}{N} \sum_{i=1}^N |u_i|^2 \Big) dt < \infty. \qquad (B.9)$$

Let $\tilde{q}_i = q_i - \check{q}_i$, $\tilde{p} = p - \check{p}$, $\tilde{v}_i = v_i - \check{v}_i$ and

$$\tilde{u}_i = u_i - \check{u}_i = u_i + \frac{1}{r} [b_i, 0] \check{s}.$$

By (1), (2) and (41)-(43) we have

$$\frac{d\tilde{q}_i}{dt} = -\mu \tilde{q}_i + b_i \tilde{u}_i, \qquad (B.10)$$

$$\frac{d\tilde{p}}{dt} = -\alpha \tilde{p} - \alpha \tilde{q}^{(N)}, \qquad (B.11)$$

$$\frac{d\tilde{v}_i}{dt} = -\alpha \tilde{v}_i - \alpha \tilde{q}_i, \qquad (B.12)$$

where $\tilde{q}^{(N)} = \frac{1}{N} \sum_{i=1}^N \tilde{q}_i$, and $\tilde{q}_i(0) = \tilde{p}(0) = \tilde{v}_i(0) = 0$. Denote $\tilde{v}^{(N)} = \frac{1}{N} \sum_{i=1}^N \tilde{v}_i$. It follows from (B.12) that

$$\frac{d\tilde{v}^{(N)}}{dt} = -\alpha \tilde{v}^{(N)} - \alpha \tilde{q}^{(N)}, \quad \tilde{v}^{(N)}(0) = 0.$$

From this together with (B.11) we get $\tilde{v}^{(N)} = \tilde{p}$. We have

$$
\begin{aligned}
&J^{(N)}_{\text{soc}}(u)\\
&=E\int_0^\infty e^{-\rho t}\left[(c-p)q^{(N)}+\frac{1}{N}\sum_{i=1}^N ru_i^2\right]dt\\
&=J^{(N)}_{\text{soc}}(\check{u})+E\int_0^\infty e^{-\rho t}\left[\frac{1}{N}\sum_{i=1}^N r\tilde{u}_i^2-\tilde{p}\check{q}^{(N)}\right]dt\\
&\quad+E\int_0^\infty e^{-\rho t}\left[(c-\check{p})\tilde{q}^{(N)}-\tilde{p}\check{q}^{(N)}-\frac{1}{N}\sum_{i=1}^N 2\tilde{u}_i B_i^T s\right]dt\\
&\triangleq J^{(N)}_{\text{soc}}(\check{u})+\tilde{J}^{(N)}_{\text{soc}}(\tilde{u})+I^{(N)}.
\end{aligned}
\tag{B.13}
$$

To complete the proof of the theorem, we will show $J^{(N)}_{\text{soc}}(\tilde{u})\geq 0$ and $|I^{(N)}|=O(\frac{1}{\sqrt{N}}+\varepsilon_N)$. For the system

$$
\begin{aligned}
\frac{d\tilde{p}}{dt}&=-\alpha\tilde{p}-\alpha u,\quad \tilde{p}(0)=0\\
y&=-\tilde{p}\\
u&=\tilde{q}^{(N)},
\end{aligned}
$$

by verifying conditions in the positive real lemma (see e.g., [25]), we get that the system is passive. This implies there exists a constant $\tilde{l}>0$ such that

$$
E\int_0^\infty e^{-\rho t}(-\tilde{p})\tilde{q}^{(N)}dt\geq \tilde{l}\rho E\left[\int_0^\infty e^{-\rho t}\tilde{p}^2dt\right]\geq 0,\quad\text{(B.14)}
$$

which with (B.13) leads to $\tilde{J}^{(N)}_{\text{soc}}(\tilde{u})\geq 0$. We have

$$
\begin{aligned}
I^{(N)}&=\int_0^\infty e^{-\rho t}\left[(c-\bar{p})\tilde{q}^{(N)}-\tilde{p}\bar{q}-\frac{1}{N}\sum_{i=1}^N 2\tilde{u}_i B_i^T s\right]dt\\
&\quad+\int_0^\infty e^{-\rho t}\left[(\bar{p}-\check{p})\tilde{q}^{(N)}+\tilde{p}(\bar{q}-\check{q}^{(N)})\right]dt\\
&\triangleq \zeta_1+\zeta_2.
\end{aligned}
\tag{B.15}
$$

Applying Itô's formula to $e^{-\rho t}[\tilde{q}_i,\tilde{v}_i]\check{s}$ and using (31) and (32), we obtain

$$
\begin{aligned}
&-e^{-\rho T}E\left\{[\tilde{q}_i(T),\tilde{v}_i(T)]\check{s}(T)\right\}\\
&=E\int_0^T e^{-\rho t}\left[(c-\bar{p})\tilde{q}_i-\tilde{v}_i\bar{q}-2\tilde{u}_i B_i^T \check{s}\right]dt.
\end{aligned}
$$

Noting $\tilde{v}^{(N)}=\tilde{p}$, we get

$$
\zeta_1=-\lim_{T\to\infty}\sum_{i=1}^N e^{-\rho T}E\left\{[\tilde{q}_i(T),\tilde{v}_i(T)]\check{s}(T)\right\}.
$$

By (40) and (B.9), it follows that $E|\tilde{u}_i(t)|^2=O(e^{\rho t})$. This together with (B.10) and (B.12) leads to $E|\tilde{q}_i(t)|^2=O(e^{\rho t})$ and further gives $E|\tilde{v}_i(t)|^2=O(e^{\rho t})$. Noticing $\check{s}\in C_b([0,\infty),\mathbb{R}^2)$,

we have $\zeta_1=0$. As in the proof of Lemma A.1 in [17], we use Jenson's inequality to get that

$$
\begin{aligned}
&E\int_0^\infty e^{-\rho t}|q^{(N)}|^2dt\\
&\leq C+CE\int_1^\infty e^{-\rho t}\left[\int_0^t e^{-\mu(t-\tau)}|\frac{1}{N}\sum_{i=1}^N b_i u_i(\tau)|d\tau\right]^2dt\\
&\leq C+\frac{C}{\mu N(\rho+\mu)}E\int_1^\infty e^{-\rho t}\sum_{i=1}^N|u_i(t)|^2dt.
\end{aligned}
\tag{B.16}
$$

This together with (B.9) and (B.1) implies

$$
E\int_0^\infty e^{-\rho t}|\tilde{q}^{(N)}|^2dt<\infty.
$$

By Schwarz's inequality and Lemma B.1, we have

$$
\int_0^\infty e^{-\rho t}(\bar{p}-\check{p})\tilde{q}^{(N)}dt=O(\frac{1}{\sqrt{N}}+\varepsilon_N).
\tag{B.17}
$$

Furthermore, it follows from Schwarz's inequality, (B.9) and Lemma B.1 that

$$
\int_0^\infty e^{-\rho t}\tilde{p}(\bar{q}-\check{q}^{(N)})dt=O(\frac{1}{\sqrt{N}}+\varepsilon_N).
$$

Thus, by (B.15) and (B.17) we have

$$
|I^{(N)}|=|\zeta_2|=O(\frac{1}{\sqrt{N}}+\varepsilon_N).
$$

This completes the proof. □