Cascade and locally dissipative realizations of linear quantum systems for pure Gaussian state covariance assignment ⋆

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Abstract

This paper presents two realizations of linear quantum systems for covariance assignment corresponding to pure Gaussian states. The first one is called a cascade realization; given any covariance matrix corresponding to a pure Gaussian state, we can construct a cascaded quantum system generating that state. The second one is called a locally dissipative realization; given a covariance matrix corresponding to a pure Gaussian state, if it satisfies certain conditions, we can construct a linear quantum system that has only local interactions with its environment and achieves the assigned covariance matrix. Both realizations are illustrated by examples from quantum optics.

Key words: Linear quantum system, Cascade realization, Locally dissipative realization, Covariance assignment, Pure Gaussian state.

1 Introduction

For stochastic systems, many of the performance objectives are expressed in terms of the variances (or covariances) of the system states. In a large space structure, for example, the vibration at certain points on the structure must be reduced to an acceptable level. This objective in fact involves keeping the variances of some variables such as deflections within prescribed bounds. One way to achieve this is to assign an appropriate matrix value to the covariance of the state vector. This method, referred to as *covariance assignment*, has been extensively studied in a series of papers by Skelton and colleagues, e.g., in [1–3]. For linear stochastic systems with white noises, the covariance matrix can be computed by solving the Lyapunov equation for the system. In this case, the covariance assignment problem reduces to designing system matrices such that the corresponding Lyapunov equation has a prescribed solution.

Turning our attention to the quantum case, we find that a covariance matrix plays an essential role as well in the field of quantum information. In particular for a *linear quantum system*, the importance of a covariance matrix stands out, because it can fully characterize the *entanglement* property, which is indeed crucial for conducting quantum information processing [4,5]. Therefore it should be of great use to investigate the covariance assignment problem for linear quantum systems. In fact, there are several such proposals; [6] studies a quantum feedback control problem for covariance assignment, and [7–10] analyze systems that generate a *pure* Gaussian state. Note that, since a Gaussian state (with zero mean) is uniquely determined by its covariance matrix, the aforementioned covariance assignment problem is also known as the Gaussian state generation problem; thus, if a linear quantum system achieves a covariance matrix corresponding to a target Gaussian state, we call that the system generates this Gaussian state.

Let us especially focus on Refs. [7–10], which provide the basis of this paper. As mentioned before, in those papers pure Gaussian states are examined, which are a particularly important subclass of Gaussian states such that the highest performance of Gaussian quantum information processing can be realized [4, 5, 11, 12]. Then they provided several methods to construct a stable linear quantum system generating a given pure Gaussian state. Moreover, conditions for generating an arbitrary pure entangled Gaussian state are given there; surely these are important results, because such a state serves as an essential resource for Gaussian quan-

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tum information processing tasks. Of course in the literature several methods for generating various pure entangled Gaussian states have been proposed. For instance, [13] gives a systematic method to generate an arbitrary pure entangled Gaussian state; the idea is to construct a *coherent* process by applying a sequence of prescribed unitary operations (composed of beam splitters and squeezers in optics case) to an initial state. Thus this method is essentially a *closed*-system approach. In contrast, the approach we take here is an *open*-system one; that is, we aim to construct *dissipative* processes such that the system is stable and uniquely driven into a desired target pure Gaussian state. This strategy is categorized into the so-called *reservoir engineering* method [14–18]; in general, this approach has a clear advantage that the system has good robustness properties with respect to initial states and evolution time.

Now we describe the problem considered in this paper. The methods developed in [7–10] lead to infinitely many linear quantum systems that uniquely generate a target pure Gaussian state. Some of these systems are easy to implement, while others are not. Then a natural question is how to find a linear quantum system that is simple to implement, while still uniquely generates the desired pure Gaussian state.

In this paper, we provide two convenient realizations of a linear quantum system generating a target pure Gaussian state. The first one is a *cascade realization*, which is a typical system structure found in the literature [19–21]. We show that, given any covariance matrix corresponding to a pure Gaussian state, we can construct a cascaded quantum system uniquely generating that state. This cascaded system is a series connection of several subsystems in which the output of one is fed as the input to the next. A clear advantage of the cascade realization is that those subsystems can be placed at remote sites. Note that the cascade structure has also been widely studied in the classical control literature [22–24].

The second one is a *locally dissipative realization*, which is motivated by the specific system structure found in, e.g. [9,25–27]. Note that in these references the notion of *quasi-locality* has been studied, but in this paper we focus on a stronger notion, *locality*. Here "locally dissipative" means that all the system-environment interactions act only on one system component. Implementations of locally dissipative systems should be considerably easier than that of systems which have non-local interactions [28]. In this paper, we show that, given a covariance matrix corresponding to a pure Gaussian state, if it satisfies certain conditions, we can construct a locally dissipative quantum system generating that state.

Lastly we remark that the state generated by our method is an *internal* one confined in the system (e.g. an intra-cavity state in optics), rather than an external optical field state. This means that, if we aim to perform some quantum information processing with that Gaussian state, it must be extracted to outside by for instance the method developed in [29]. In particular by acting some non-Gaussian operations such as the cubic-phase gate or photon counting on that extracted Gaussian state, we can realize, e.g., entanglement distillation and universal quantum computation [5]. On the other hand, a generated internal Gaussian state is not necessarily extracted to outside for the purpose of precision measurement in the scenario of quantum metrology; for instance a spin squeezed state of an atomic ensemble can be directly used for ultra-precise magnetometry [30].

Notation. For a matrix $A = [A_{jk}]$ whose entries A_{jk} are complex numbers or operators, we define $A^\top = [A_{kj}]$, $A^\dagger = [A_{kj}^*]$, where the superscript * denotes either the complex conjugate of a complex number or the adjoint of an operator. $\mathrm{diag}[\tau_1,\cdots,\tau_n]$ denotes an $n\times n$ diagonal matrix with τ_j, $j=1,2,\cdots,n$, on its main diagonal. $\mathscr{P}_N$ is a $2N\times 2N$ permutation matrix defined by $\mathscr{P}_N[x_1\ x_2\ x_3\ x_4\ \cdots\ x_{2N}]^\top = [x_1\ x_3\ \cdots\ x_{2N-1}\ x_2\ x_4\ \cdots\ x_{2N}]^\top$ for any column vector $[x_1\ x_2\ x_3\ x_4\ \cdots\ x_{2N}]^\top$.

2 Preliminaries

We consider a linear quantum system G of N modes. Each mode is characterized by a pair of quadrature operators $\{\hat{q}_j,\hat{p}_j\}$, $j=1,2,\cdots,N$. Collecting them into an operator-valued vector $\hat{x}\triangleq[\hat{q}_1\ \cdots\ \hat{q}_N\ \ \hat{p}_1\ \cdots\ \hat{p}_N]^\top$, we write the canonical commutation relations as

$$\left[\hat{x},\hat{x}^\top\right]\triangleq\hat{x}\hat{x}^\top-\left(\hat{x}\hat{x}^\top\right)^\top=i\Sigma,\quad \Sigma\triangleq\begin{bmatrix}0 & I_N\\ -I_N & 0\end{bmatrix}. \tag{1}$$

Here we emphasize that the transpose operation $\top$, when applied to an operator-valued matrix (say, $\hat{x}\hat{x}^\top$), only exchanges the indices of the matrix and leaves the entries unchanged. Therefore $\left(\hat{x}\hat{x}^\top\right)^\top\neq\hat{x}\hat{x}^\top$. Let $\hat{H}$ be the Hamiltonian of the system, and let $\{\hat{c}_j\}$, $j=1,2,\cdots,K$, be Lindblad operators that represent the interactions between the system and its environment. For convenience, we collect all the Lindblad operators as an operator-valued vector $\hat{L}=\left[\hat{c}_1\ \hat{c}_2\ \cdots\ \hat{c}_K\right]^\top$ and call $\hat{L}$ the *coupling vector*. Suppose $\hat{H}$ is quadratic in $\hat{x}$, i.e., $\hat{H}=\frac{1}{2}\hat{x}^\top M\hat{x}$, with $M=M^\top\in\mathbb{R}^{2N\times 2N}$, and $\hat{L}$ is linear in $\hat{x}$, i.e., $\hat{L}=C\hat{x}$, with $C\in\mathbb{C}^{K\times 2N}$, then the quantum system G can be described by the following quantum stochastic differential equations (QSDEs)

$$\begin{cases} d\hat{x}(t)=\mathscr{A}\hat{x}(t)dt+\mathscr{B}\left[d\hat{A}^\top(t)\ d\hat{A}^\dagger(t)\right]^\top,\\ d\hat{Y}(t)=\mathscr{C}\hat{x}(t)dt+d\hat{A}(t),\end{cases} \tag{2}$$

where $\mathscr{A}=\Sigma(M+\mathrm{Im}(C^\dagger C))$, $\mathscr{B}=i\Sigma[-C^\dagger\ C^\top]$, $\mathscr{C}=C$ [8], [31, Chapter 6]. The input $d\hat{A}(t)=\left[d\hat{A}_1(t)\ \cdots\ d\hat{A}_K(t)\right]^\top$ represents K independent quantum stochastic processes, with $d\hat{A}_j(t)$, $j=1,2,\cdots,K$, satisfying the following quan-

tum Itō rules:

$$\begin{cases} d\hat{A}_j(t)d\hat{A}_k^*(t) = \delta_{jk}dt, \\ d\hat{A}_j(t)d\hat{A}_k(t) = d\hat{A}_j^*(t)d\hat{A}_k^*(t) = d\hat{A}_j^*(t)d\hat{A}_k(t) = 0, \end{cases} \tag{3}$$

where δ_{jk} is the Kronecker δ-function. The output $d\hat{Y}(t) = \begin{bmatrix} d\hat{Y}_1(t) & \cdots & d\hat{Y}_K(t) \end{bmatrix}^\top$ satisfies quantum Itō rules similar to (3) [8, 31–35]. The quantum expectation of the vector $\hat{x}$ is denoted by $\langle \hat{x} \rangle$ and the covariance matrix is given by $V = \frac{1}{2}\langle \triangle\hat{x}\triangle\hat{x}^\top + (\triangle\hat{x}\triangle\hat{x}^\top)^\top \rangle$, where $\triangle\hat{x} = \hat{x} - \langle \hat{x} \rangle$; see, e.g., [7, 8, 11]. The time evolutions of the mean vector $\langle \hat{x}(t) \rangle$ and the covariance matrix $V(t)$ can be derived from (2) by using the quantum Itō rule. They are given by

$$\begin{cases} \dfrac{d\langle \hat{x}(t) \rangle}{dt} = \mathscr{A}\langle \hat{x}(t) \rangle, & (4) \\ \dfrac{dV(t)}{dt} = \mathscr{A}V(t) + V(t)\mathscr{A}^\top + \dfrac{1}{2}\mathscr{B}\mathscr{B}^\dagger. & (5) \end{cases}$$

As in the classical case, a Gaussian state is completely characterized by the mean vector $\langle \hat{x} \rangle$ and the covariance matrix V. Since the mean vector $\langle \hat{x} \rangle$ contains no information about noise and entanglement, we will restrict our attention to zero-mean Gaussian states (i.e., $\langle \hat{x} \rangle = 0$). A Gaussian state is pure if and only if its covariance matrix V satisfies $\det(V) = 2^{-2N}$. In fact, when a Gaussian state is pure, its covariance matrix V can always be factored as

$$V = \frac{1}{2}SS^\top, \quad S = \begin{bmatrix} Y^{-\frac{1}{2}} & 0 \\ XY^{-\frac{1}{2}} & Y^{\frac{1}{2}} \end{bmatrix}, \tag{6}$$

where $X = X^\top \in \mathbb{R}^{N\times N}$, $Y = Y^\top \in \mathbb{R}^{N\times N}$ and $Y > 0$ [11, 36]. For example, the N-mode vacuum state is a special pure Gaussian state with $X = 0$ and $Y = I_N$. It can be seen from (6) that a pure Gaussian state is uniquely specified by a complex, symmetric matrix $Z \triangleq X + iY$, which is referred to as the *graph* corresponding to a pure Gaussian state [11]. Note also that the matrix S satisfies $S\Sigma S^\top = \Sigma$, which means that S is a symplectic matrix. The symplectic nature of S guarantees that the mapping $\hat{x} \longmapsto \hat{x}' \triangleq S\hat{x}$ preserves the canonical commutation relations (1), that is

$$\left[\hat{x}', \hat{x}'^\top\right] = \left[S\hat{x}, (S\hat{x})^\top\right] = S\left[\hat{x}, \hat{x}^\top\right]S^\top = S(i\Sigma)S^\top = i\Sigma.$$

Note that if the system G is initially in a Gaussian state, then the system G will always be Gaussian, with the mean vector $\langle \hat{x}(t) \rangle$ and the covariance matrix $V(t)$ obeying (4) and (5), respectively. We shall be particularly interested in the steady-state covariance matrix $V(\infty)$.

Assume that the system G is initially in a Gaussian state. The problem of pure Gaussian state covariance assignment is to find a Hamiltonian $\hat{H}$ and a coupling vector $\hat{L}$ such that the corresponding linear quantum system described by (2) is asymptotically stable and achieves the covariance matrix corresponding to a given pure Gaussian state. Since a pure Gaussian state (with zero mean) is uniquely specified by its covariance matrix, so if a linear quantum system achieves a covariance matrix corresponding to a pure Gaussian state, we can simply say that such a linear quantum system uniquely generates the pure Gaussian state. The problem can be expressed mathematically as:

$$\begin{array}{ll} \text{find} & M = M^\top \in \mathbb{R}^{2N\times 2N} \text{ and } C \in \mathbb{C}^{K\times 2N} \\ \text{subject to} & \mathscr{A} \text{ is Hurwitz,} \\ & \mathscr{A}V + V\mathscr{A}^\top + \dfrac{1}{2}\mathscr{B}\mathscr{B}^\dagger = 0, \end{array}$$

where V is the covariance matrix corresponding to the desired target pure Gaussian state. Here a matrix $\mathscr{A}$ is said to be *Hurwitz* if all its eigenvalues have strictly negative real parts. A system described by (2) is said to be *asymptotically stable* if the matrix $\mathscr{A}$ is a Hurwitz matrix. Recently, a necessary and sufficient condition has been developed in [7] for solving the pure Gaussian state covariance assignment problem. The result is summarized as follows.

Lemma 1 ([7, 8])**.** *Let V be the covariance matrix corresponding to a given N-mode pure Gaussian state. Assume that V is expressed in the factored form* (6)*. Then this pure Gaussian state is uniquely generated by the linear quantum system* (2) *if and only if*

$$M = \begin{bmatrix} XRX + YRY - \Gamma Y^{-1}X - XY^{-1}\Gamma^\top & -XR + \Gamma Y^{-1} \\ -RX + Y^{-1}\Gamma^\top & R \end{bmatrix}, \tag{7}$$

and

$$C = P^\top[-Z \;\; I_N], \tag{8}$$

where $R = R^\top \in \mathbb{R}^{N\times N}$, $\Gamma = -\Gamma^\top \in \mathbb{R}^{N\times N}$, and $P \in \mathbb{C}^{N\times K}$ are free matrices satisfying the following rank condition

$$\operatorname{rank}\left(\begin{bmatrix} P & QP & \cdots & Q^{N-1}P \end{bmatrix}\right) = N, \;\; Q \triangleq -iRY + Y^{-1}\Gamma. \tag{9}$$

Remark 2. From (8), we see that the resulting coupling vector $\hat{L}$ of the engineered system is $\hat{L} = C\hat{x} = P^\top[-Z \; I_N]\hat{x} = P^\top([\hat{p}_1 \; \cdots \; \hat{p}_N]^\top - Z[\hat{q}_1 \; \cdots \; \hat{q}_N]^\top)$. Therefore, all the components of $\hat{L}$ are *nullifiers* for the desired target pure Gaussian state [11]. As a special example, one can engineer a purely dissipative system (with $\hat{H} = 0$) to generate a pure Gaussian state. In this case, one could take $R = \Gamma = 0_{N\times N}$ and $P = I_N$ in Lemma 1. Then the resulting coupling vector $\hat{L}$ is the so-called *nullifier vector* for the desired target pure Gaussian state.

Remark 3. Lemma 1 has a simple interpretation in terms of symplectic transformations [8, 37]. As mentioned before, vacuum states are a special class of pure Gaussian states.

The covariance matrix corresponding to the N-mode vacuum state is $V = \frac{1}{2} I_N$. By using physical realizability conditions, it can be proved that the N-mode vacuum state can only be generated by an N-mode passive linear quantum system [37]. The converse is also true. That is, an N-mode passive linear quantum system, if it is asymptotically stable, must evolve toward the N-mode vacuum state [38]. Recall that for a passive linear quantum system, the Hamiltonian is always of the form $\hat{H} = \frac{1}{2}\hat{x}^\top \tilde{M}\hat{x}$, with $\tilde{M} = \begin{bmatrix} \tilde{R} & \tilde{\Gamma} \\ \tilde{\Gamma}^\top & \tilde{R} \end{bmatrix}$, $\tilde{R} = \tilde{R}^\top \in \mathbb{R}^{N\times N}$, and $\tilde{\Gamma} = -\tilde{\Gamma}^\top \in \mathbb{R}^{N\times N}$, and the coupling vector is always of the form $\hat{L} = \tilde{C}\hat{x}$, with $\tilde{C} = \tilde{P}^\top [-iI_N \ \ I_N]$, $\tilde{P} \in \mathbb{C}^{N\times K}$ [21,39–41]. Now we apply a symplectic transformation to $\hat{x}$, that is, we define $\hat{x}' \triangleq S\hat{x}$. Then, in terms of $\hat{x}'$, the Hamiltonian is rewritten as $\hat{H} = \frac{1}{2}\hat{x}'^\top S^{-\top}\tilde{M}S^{-1}\hat{x}'$ and the coupling vector is rewritten as $\hat{L} = \tilde{C}S^{-1}\hat{x}'$. We also observe that the relation between the covariance matrix V' of $\hat{x}'$ and the covariance matrix V of $\hat{x}$ is given as follows:

$$\begin{aligned} V' &= \frac{1}{2}\langle \triangle\hat{x}'\triangle\hat{x}'^\top + (\triangle\hat{x}'\triangle\hat{x}'^\top)^\top\rangle \\ &= \frac{1}{2}S\langle \triangle\hat{x}\triangle\hat{x}^\top + (\triangle\hat{x}\triangle\hat{x}^\top)^\top\rangle S^\top \\ &= SVS^\top. \end{aligned}$$

If the passive linear quantum system is asymptotically stable, then based on the result in [38], we have $V \to \frac{1}{2}I_N$, as $t \to +\infty$. As a result, $V' \to \frac{1}{2}SS^\top$, which gives the desired pure Gaussian state. Combining the results above, we conclude that for a given pure Gaussian state $V = \frac{1}{2}SS^\top$, a complete parametrization of the linear quantum system G that uniquely generates this pure Gaussian state is given by

$$\begin{cases} M = S^{-\top}\tilde{M}S^{-1}, & (10) \\ C = \tilde{C}S^{-1}, & (11) \end{cases}$$

where $(\tilde{M}, \tilde{C})$ form an asymptotically stable passive linear quantum system. Substituting $\tilde{M} = \begin{bmatrix} \tilde{R} & \tilde{\Gamma} \\ \tilde{\Gamma}^\top & \tilde{R} \end{bmatrix}$ and $\tilde{C} = \tilde{P}^\top [-iI_N, \ I_N]$ into (10), (11) and using some additional matrix transformations, we will obtain the formulas (7), (8), respectively. This is the idea behind Lemma 1. The rank constraint (9) indeed gives a sufficient and necessary stability condition for the original passive linear quantum system [40, 41]. As a result, it also guarantees the stability of the linear quantum system G based on the linear transformation theory in the control field [42].

3 The cascade realization

As we have seen in Lemma 1, the matrices R, Γ and P are free matrices, although they must satisfy the rank condition (9). By varying them we can obtain different linear quantum systems that uniquely generate a given pure Gaussian state. Based on this fact, we provide two feasible realizations of linear quantum systems for covariance assignment corresponding to pure Gaussian states, and this section is devoted to the first one, the cascade realization.

3.1 The cascade realization

For convenience, we denote a linear quantum system G with the Hamiltonian $\hat{H}$ and the coupling vector $\hat{L}$ as $G = (\hat{H}, \hat{L})$. Suppose we have two linear quantum systems $G_1 = (\hat{H}_1, \hat{L}_1)$ and $G_2 = (\hat{H}_2, \hat{L}_2)$. If we feed the output of the system G_1 into the input of the system G_2, we will obtain a cascaded quantum system $G = G_2 \lhd G_1$, as shown in Fig. 1. Based on the quantum theory of cascaded linear quantum

Fig. 1. The cascade connection of two linear quantum systems: $G = G_2 \lhd G_1$.

systems [43], the Hamiltonian $\hat{H}$ and the coupling vector $\hat{L}$ of the cascaded system G are, respectively, given by

$$\begin{cases} \hat{H} = \hat{H}_2 + \hat{H}_1 + \dfrac{1}{2i}\left(\hat{L}_2^\dagger \hat{L}_1 - \hat{L}_1^\dagger \hat{L}_2\right), \\ \hat{L} = \hat{L}_2 + \hat{L}_1. \end{cases} \quad (12)$$

This result can be extended to the cascade connection of N one-dimensional harmonic oscillators. Suppose we have N one-dimensional harmonic oscillators G_j with the Hamiltonian $\hat{H}_j = \frac{1}{2}\hat{\xi}_j^\top M_j \hat{\xi}_j$, $M_j = M_j^\top \in \mathbb{R}^{2\times 2}$, $\hat{\xi}_j \triangleq [\hat{q}_j \ \hat{p}_j]^\top$, and the coupling vector $\hat{L}_j = C_j\hat{\xi}_j$, $C_j \in \mathbb{C}^{K\times 2}$, $j = 1, 2, \cdots, N$. The system G is obtained by a cascade connection of these harmonic oscillators, that is, $G = G_N \lhd \cdots \lhd G_2 \lhd G_1$, as shown in Fig. 2. By repeatedly using (12), the Hamiltonian $\hat{H}$ and the coupling vector $\hat{L}$ of the cascaded system G are given by the following lemma.

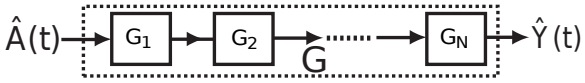

Fig. 2. The cascade connection of N one-dimensional harmonic oscillators: $G = G_N \lhd \cdots \lhd G_2 \lhd G_1$.

Lemma 4 ([20]). *Suppose that the system G is obtained via a cascade connection of the aforementioned N one-dimensional harmonic oscillators G_j, $j = 1, 2, \cdots, N$, that is, $G = G_N \lhd \cdots \lhd G_2 \lhd G_1$. Then the Hamiltonian $\hat{H}$ and the coupling vector $\hat{L}$ of the system G are, respectively, given by*

$$\begin{cases} \hat{H} = \dfrac{1}{2}\hat{x}^\top M\hat{x}, \quad M = \mathscr{P}_N \mathbb{M} \mathscr{P}_N^\top \\ \hat{L} = C\hat{x}, \quad C = [C_1 \ C_2 \ \cdots \ C_N]\, \mathscr{P}_N^\top, \end{cases}$$

where $\mathbb{M} = [\mathbb{M}_{jk}]_{j,k=1,\cdots,N}$ *is a symmetric block matrix with* $\mathbb{M}_{jj} = M_j$, $\mathbb{M}_{jk} = \mathrm{Im}(C_j^\dagger C_k)$ *whenever* $j > k$ *and* $\mathbb{M}_{jk} = \mathbb{M}_{kj}^\top$ *whenever* $j < k$.

It can be seen from Lemma 4 that due to the cascade feature, the Hamiltonian matrix M and the coupling matrix C of the cascaded system G depend on each other in a complicated way. Nevertheless, given any pure Gaussian state, we can always construct a cascade connection of several one-dimensional harmonic oscillators such that this cascaded quantum system is asymptotically stable and achieves the covariance matrix corresponding to the desired target pure Gaussian state. The result is stated as follows.

Theorem 5. *Any N-mode pure Gaussian state can be uniquely generated by constructing a cascade of N one-dimensional harmonic oscillators.*

Proof. We prove this result by construction. Recall that for an arbitrary N-mode pure Gaussian state, the corresponding covariance matrix V has the factorization shown in (6). Using the matrices X and Y obtained from (6), we construct a cascaded system $G = G_N \lhd \cdots \lhd G_2 \lhd G_1$ with the Hamiltonian $\hat{H}_j$ and the coupling vector $\hat{L}_j$, $j = 1, 2, \cdots, N$, given by

$$\begin{cases} \hat{H}_j = 0, \\ \hat{L}_j = C_j \hat{\xi}_j,\ C_j = iY^{-\frac{1}{2}} [-Z\ I_N]\, \mathscr{P}_N \begin{bmatrix} 0_{(2j-2)\times 2} \\ I_2 \\ 0_{(2N-2j)\times 2} \end{bmatrix}. \end{cases}$$

Using Lemma 4, we can calculate the Hamiltonian $\hat{H} = \frac{1}{2}\hat{x}^\top M \hat{x}$ and the coupling vector $\hat{L} = C\hat{x}$ for the cascaded system G. We find that $M = 0$ and $C = iY^{-1/2}[-Z\ \ I_N]$. Then it follows from the QSDE (2) that

$$\begin{aligned} \mathscr{A} &= \Sigma(M + \mathrm{Im}(C^\dagger C)) \\ &= \Sigma\, \mathrm{Im}\left(\begin{bmatrix} (X-iY)Y^{-1}(X+iY) & -(X-iY)Y^{-1} \\ -Y^{-1}(X+iY) & Y^{-1} \end{bmatrix} \right) \\ &= \Sigma\Sigma = -I_{2N}, \\ \mathscr{D} &\triangleq \frac{1}{2}\mathscr{B}\mathscr{B}^\dagger = \Sigma\, \mathrm{Re}(C^\dagger C)\Sigma^\top \\ &= \Sigma\, \mathrm{Re}\left(\begin{bmatrix} (X-iY)Y^{-1}(X+iY) & -(X-iY)Y^{-1} \\ -Y^{-1}(X+iY) & Y^{-1} \end{bmatrix} \right) \Sigma^\top \\ &= \begin{bmatrix} Y^{-1} & Y^{-1}X \\ XY^{-1} & XY^{-1}X + Y \end{bmatrix}. \end{aligned}$$

Clearly, $\mathscr{A}$ is Hurwitz. Furthermore, it can be verified that

$$\mathscr{A}V + V\mathscr{A}^\top + \mathscr{D} = 0. \tag{13}$$

The stability of $\mathscr{A}$ and the Lyapunov equation (13) guarantee that the cascaded system G constructed above is asymptotically stable and achieves the covariance matrix V. In other words, the cascaded system G uniquely generates the desired target pure Gaussian state. □

3.2 Example

Example 6. We consider the generation of *two-mode squeezed states* [11]. Two-mode squeezed states are highly symmetric entangled states, which are very useful in several quantum information protocols such as quantum teleportation [44]. The covariance matrix V corresponding to a two-mode squeezed state is

$$V = \frac{1}{2}\begin{bmatrix} \cosh(2\alpha) & \sinh(2\alpha) & 0 & 0 \\ \sinh(2\alpha) & \cosh(2\alpha) & 0 & 0 \\ 0 & 0 & \cosh(2\alpha) & -\sinh(2\alpha) \\ 0 & 0 & -\sinh(2\alpha) & \cosh(2\alpha) \end{bmatrix}, \tag{14}$$

where α is the squeezing parameter. Using the factorization (6), we have $X = 0$ and $Y = \begin{bmatrix} \cosh(2\alpha) & -\sinh(2\alpha) \\ -\sinh(2\alpha) & \cosh(2\alpha) \end{bmatrix}$. Therefore, the graph corresponding to a two-mode squeezed state is given by $Z = X + iY = \begin{bmatrix} i\cosh(2\alpha) & -i\sinh(2\alpha) \\ -i\sinh(2\alpha) & i\cosh(2\alpha) \end{bmatrix}$.

Next we provide two different cascade realizations. The first one, Realization 1, is constructed based on a heuristic derivation, while the second one, Realization 2, is constructed based on the proof of Theorem 5.

Realization 1. In this cascade realization, the subsystems $G_1 = (\hat{H}_1, \hat{L}_1)$ and $G_2 = (\hat{H}_2, \hat{L}_2)$ are, respectively, given by

$$\begin{cases} \hat{H}_1 = \frac{1}{2}\hat{\xi}_1^\top \begin{bmatrix} 2 & Q_1 \\ Q_1 & 2 \end{bmatrix} \hat{\xi}_1, \quad \hat{L}_1 = [iQ_2\ 1]\hat{\xi}_1, \\ \hat{H}_2 = -\frac{1}{2}\hat{\xi}_2^\top \begin{bmatrix} 2 & Q_1 \\ Q_1 & 2 \end{bmatrix} \hat{\xi}_2, \quad \hat{L}_2 = [iQ_2\ 1]\hat{\xi}_2, \end{cases}$$

where $Q_1 \triangleq \frac{\sinh^2(2\alpha)}{\cosh(2\alpha)} - \sinh(2\alpha)$ and $Q_2 \triangleq \sinh(2\alpha) - \cosh(2\alpha)$. It can be proved that the cascaded system $G = G_2 \lhd G_1$ is asymptotically stable and achieves the covariance matrix (14). The proof is similar to that of Theorem 5, and hence is omitted. Using the result in [45], a corresponding quantum optical realization is provided in Fig. 3. For each subsystem G_j, $j = 1, 2$, the Hamiltonian $\hat{H}_j$ is realized by a nonlinear crystal pumped by a classical field, and the coupling operator $\hat{L}_j$ is realized by implementing an auxiliary cavity. This auxiliary cavity interacts

with the subsystem via a cascade of a pumped crystal and a beam splitter. It has a fast mode that can be adiabatically eliminated.

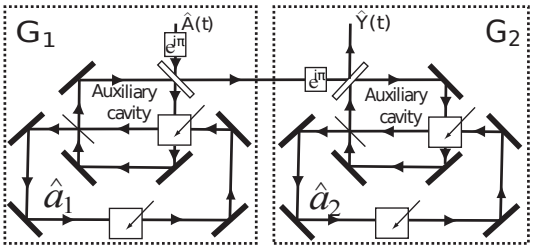

Fig. 3. An optical cascade realization of the two-mode linear quantum system that uniquely generates a two-mode squeezed state. The square with an arrow represents a pumped crystal. The symbol $e^{i\pi}$ with a square on it represents a phase shift π. Solid (dark) rectangles denote perfectly reflecting mirrors, while unfilled rectangles denote partially transmitting mirrors. The dark line "$\diagdown$" represents an optical beam splitter.

Realization 2. The second realization is constructed according to the method shown in the proof of Theorem 5. By direct calculation, the subsystems $G_1 = (\hat{H}_1,\ \hat{L}_1)$ and $G_2 = (\hat{H}_2,\ \hat{L}_2)$ are, respectively, given by

$$\begin{cases} \hat{H}_1 = 0, \quad \hat{L}_1 = \begin{bmatrix} \cosh(\alpha) & i\cosh(\alpha) \\ -\sinh(\alpha) & i\sinh(\alpha) \end{bmatrix} \hat{\xi}_1, \\ \hat{H}_2 = 0, \quad \hat{L}_2 = \begin{bmatrix} -\sinh(\alpha) & i\sinh(\alpha) \\ \cosh(\alpha) & i\cosh(\alpha) \end{bmatrix} \hat{\xi}_2. \end{cases}$$

Using the result in [45], a corresponding quantum optical realization of such a cascaded quantum system $G = G_2 \lhd G_1$ is provided in Fig. 4. This cascaded system G has two crucial features. First, because $\hat{H}_1 = \hat{H}_2 = 0$, implementations of the Hamiltonians involve no pumped crystals. Second, the first component of the coupling vector $\hat{L}_1 = [\hat{c}_{1,1}\ \hat{c}_{1,2}]^\top$ is $\hat{c}_{1,1} = \begin{bmatrix} \cosh(\alpha) & i\cosh(\alpha) \end{bmatrix} \begin{bmatrix} \hat{q}_1 \\ \hat{p}_1 \end{bmatrix} = \sqrt{2}\cosh(\alpha)\hat{a}_1$, where $\hat{a}_1 = (\hat{q}_1 + i\hat{p}_1)/\sqrt{2}$ denotes the annihilation operator of the first mode. This operator $\hat{c}_{1,1}$ represents the standard linear dissipation of a cavity mode into a continuum of field modes outside of the cavity. A similar case also occurs in the coupling vector $\hat{L}_2$. As can be seen in Fig. 4, Realization 2 requires two pumped crystals, in contrast to the case of Realization 1, where four pumped crystals are used. From this viewpoint, Realization 2, which is constructed based on our result, has a clear advantage over Realization 1.

4 The locally dissipative realization

In this section, we describe the second realization of linear quantum systems for covariance assignment corresponding to pure Gaussian states. Unlike the cascade realization, the locally dissipative realization cannot generate all pure Gaussian states, but as shown later the class of stabilizable states is fairly broad.

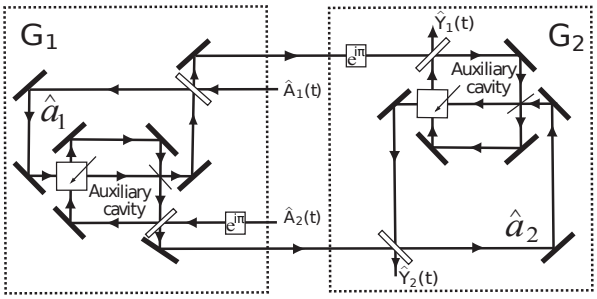

Fig. 4. Another optical cascade realization of the two-mode linear quantum system that uniquely generates a two-mode squeezed state.

4.1 The locally dissipative realization

As we have noted in Section 2, the coupling vector $\hat{L}$ is an operator-valued vector that consists of K elements, i.e., $\hat{L} = [\hat{c}_1\ \hat{c}_2\ \cdots\ \hat{c}_K]^\top$. Each element $\hat{c}_j,\ j = 1, 2, \cdots, K$, called a Lindblad operator, represents an interaction between the system and its environment. A Lindblad operator $\hat{c}_j$ is said to be local if it acts only on one system mode. As an example, consider the system depicted in Fig. 5. The Lindblad operator $\hat{c}_1 = \hat{q}_1 + \hat{p}_1$ acts only on the first system mode, so it is a local operator. On the other hand, the Lindblad operator $\hat{c}_2 = \hat{q}_1 + \hat{q}_2$ acts on two system modes, so by definition it is not a local operator. If all the Lindblad operators in $\hat{L}$ are local, then the system is called a locally dissipative quantum system. A locally dissipative quantum system could be relatively easy to implement in practice. Therefore, we would like to characterize the class of pure Gaussian states that can be generated using locally dissipative quantum systems. The result is given by the following theorem.

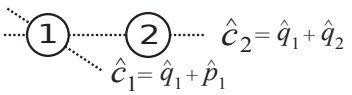

Fig. 5. An illustration of local Lindblad operators. $\hat{c}_1$ is a local Lindblad operator, while $\hat{c}_2$ is not a local one.

Theorem 7. *Let V be the covariance matrix corresponding to a given N-mode pure Gaussian state. Assume that it is expressed in the factored form* (6)*. Then this pure Gaussian state can be uniquely generated in an N-mode locally dissipative quantum system if and only if there exists an integer ℓ, $1 \le \ell \le N$, such that*

$$Z_{(\ell,j)} = Z_{(j,\ell)} = 0, \quad \forall j \neq \ell \ \text{and}\ 1 \le j \le N, \tag{15}$$

where $Z_{(\ell,j)}$ denotes the (ℓ, j) element of the graph matrix $Z = X + iY$ for the pure Gaussian state.

Proof. We prove the sufficiency part by construction. Equation (15) implies that there exists a row vector $\Upsilon = \begin{bmatrix} 0_{1\times(\ell-1)} & \tau_1 & 0_{1\times(N-\ell)} \end{bmatrix}$ with $\tau_1 \neq 0$, such that $\Upsilon Z = \begin{bmatrix} 0_{1\times(\ell-1)} & \tau_2 & 0_{1\times(N-\ell)} \end{bmatrix}$, where $\tau_2 = \tau_1 Z_{(\ell,\ell)}$. Using the Gram-Schmidt method, we can create three $N\times N$ matrices U_1, U_2 and Λ, where U_1 is a unitary matrix with the first column being $\frac{Y^{1/2}\Upsilon^\top}{\|Y^{1/2}\Upsilon^\top\|}$; U_2 is a unitary matrix with the first column being $\frac{1}{\sqrt{N}}\begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}^\top$ and Λ is a purely imaginary matrix $\Lambda = i\,\mathrm{diag}[\alpha_1,\cdots,\alpha_N]$, with $\alpha_j \in \mathbb{R}$, $j = 1,\cdots,N$, and $\alpha_j \neq \alpha_k$, $\forall j \neq k$.

Let $P = \Upsilon^\top$, $R = -Y^{-1/2}\,\mathrm{Im}(U_1U_2^\dagger\Lambda U_2U_1^\dagger)Y^{-1/2}$ and $\Gamma = Y^{1/2}\,\mathrm{Re}(U_1U_2^\dagger\Lambda U_2U_1^\dagger)Y^{1/2}$ in (7), (8). Then it can be verified that $R = R^\top$, $\Gamma = -\Gamma^\top$. Moreover, substituting $Q = -iRY + Y^{-1}\Gamma = Y^{-1/2}U_1U_2^\dagger\Lambda U_2U_1^\dagger Y^{1/2}$ into (9) yields

$$\begin{aligned}
&\mathrm{rank}\Big(\begin{bmatrix} P & QP & \cdots & Q^{N-1}P \end{bmatrix}\Big)\\
=&\mathrm{rank}\Big(\Big[U_2U_1^\dagger Y^{1/2}\Upsilon^\top \;\; \Lambda U_2U_1^\dagger Y^{1/2}\Upsilon^\top \cdots\\
&\qquad\qquad \Lambda^{N-1}U_2U_1^\dagger Y^{1/2}\Upsilon^\top\Big]\Big)\\
=&\mathrm{rank}\left(\frac{\|Y^{1/2}\Upsilon^\top\|}{\sqrt{N}}\begin{bmatrix} 1 & (i\alpha_1) & \cdots & (i\alpha_1)^{N-1}\\ 1 & (i\alpha_2) & \cdots & (i\alpha_2)^{N-1}\\ \vdots & \vdots & \cdots & \vdots\\ 1 & (i\alpha_N) & \cdots & (i\alpha_N)^{N-1}\end{bmatrix}\right)\\
=&N.
\end{aligned}$$

Here we have used the full rank property of a Vandermonde matrix. Hence the rank condition (9) is satisfied. Based on Lemma 1, we now obtain an N-mode locally dissipative quantum system that is asymptotically stable and achieves the given covariance matrix. The coupling vector $\hat{L}$ of the system, which consists of only one Lindblad operator, is given by

$$\begin{aligned}
C &= P^\top\begin{bmatrix} -Z & I_N\end{bmatrix} = \begin{bmatrix} -\Upsilon Z & \Upsilon\end{bmatrix}\\
&= \begin{bmatrix} 0_{1\times(\ell-1)} & -\tau_2 & 0_{1\times(N-\ell)} & 0_{1\times(\ell-1)} & \tau_1 & 0_{1\times(N-\ell)}\end{bmatrix},\\
\hat{L} &= C\hat{x} = -\tau_2\hat{q}_\ell + \tau_1\hat{p}_\ell.
\end{aligned}$$

We see that $\hat{L}$ acts only on the ℓth mode, and hence it is local. The Hamiltonian $\hat{H}$ of the system can also be obtained by directly substituting the matrices R and Γ above into (7). This completes the sufficiency part of the proof.

Next we prove the necessity part. Suppose an N-mode pure Gaussian state can be uniquely generated in an N-mode locally dissipative quantum system. Based on Lemma 1, there exists a $P \in \mathbb{C}^{N\times K}$, $P \neq 0$, such that the coupling vector $\hat{L} = P^\top\begin{bmatrix} -Z & I_N\end{bmatrix}\hat{x}$ is local. Let P_k, $1 \le k \le K$, be a nonzero column in the matrix P. Then the corresponding Lindblad operator $\hat{c}_k = P_k^\top\begin{bmatrix} -Z & I_N\end{bmatrix}\hat{x}$ is local. Suppose that $\hat{c}_k$ acts on the ℓth mode of the system. Then we have

$$\begin{aligned}
&P_k^\top\begin{bmatrix} -Z & I_N\end{bmatrix}\\
=&\begin{bmatrix} 0_{1\times(\ell-1)} & \tau_3 & 0_{1\times(N-\ell)} & 0_{1\times(\ell-1)} & \tau_4 & 0_{1\times(N-\ell)}\end{bmatrix},
\end{aligned}$$

where τ_3 and τ_4 are complex numbers. It follows that

$$P_k^\top Z = \begin{bmatrix} 0_{1\times(\ell-1)} & -\tau_3 & 0_{1\times(N-\ell)}\end{bmatrix}, \tag{16}$$

$$P_k^\top = \begin{bmatrix} 0_{1\times(\ell-1)} & \tau_4 & 0_{1\times(N-\ell)}\end{bmatrix}, \quad \tau_4 \neq 0. \tag{17}$$

Substituting (17) into (16) gives

$$\begin{aligned}
P_k^\top Z =&\tau_4\begin{bmatrix} Z_{(\ell,1)} & Z_{(\ell,2)} & \cdots & Z_{(\ell,N)}\end{bmatrix}\\
=&\begin{bmatrix} 0_{1\times(\ell-1)} & -\tau_3 & 0_{1\times(N-\ell)}\end{bmatrix}.
\end{aligned}$$

Since $\tau_4 \neq 0$, we have $Z_{(\ell,j)} = 0$, $\forall j \neq \ell$. Since $Z = Z^\top$, we have $Z_{(\ell,j)} = Z_{(j,\ell)} = 0$, $\forall j \neq \ell$. That is, Equation (15) holds. This completes the proof. □

Remark 8. The basic idea of Theorem 7 is that for any choice of $P \neq 0$, there always exist matrices $R = R^\top$ and $\Gamma = -\Gamma^\top$ such that the rank condition (9) is satisfied. So we can first specify a matrix P such that the coupling matrix C in (8) has a local structure. After obtaining P, we determine the other two matrices R and Γ to get a system Hamiltonian, under the rank constraint (9). Generally, for a given nonzero matrix P, we have infinite solutions $(R,\ \Gamma)$ that satisfy the rank condition (9). Different choices of $(R,\ \Gamma)$ lead to different system Hamiltonians. The optimization problem over these Hamiltonians is beyond the scope of this paper and is not considered, but in the next subsection we will show a specific recipe for determining those matrices $(R,\ \Gamma)$.

Remark 9. Suppose an N-mode pure Gaussian state is generated in an N-mode dissipative quantum system and the ℓth mode is locally coupled to the environment. Then from Equation (15), it is straightforward to see that the ℓth mode is not entangled with the rest of the system modes when the system achieves the steady state.

4.2 Examples

Example 10. We consider the generation of *canonical Gaussian cluster states*, which serve as an essential resource in quantum computation with continuous variables [4,11,12]. We mention that an interesting class of cluster states, called *bilayer square-lattice continuous–variable cluster states*, has been proposed recently in [46]. This class of cluster states has some practical advantages over canonical Gaussian cluster states for quantum computation [46]. For the sake of simplicity, we use canonical Gaussian cluster states to illustrate the developed theory. The covariance

matrix V corresponding to an N-mode canonical Gaussian cluster state is given by $V = \frac{1}{2}\begin{bmatrix} e^{2\alpha} I_N & e^{2\alpha} B \\ e^{2\alpha} B & e^{-2\alpha} I_N + e^{2\alpha} B^2 \end{bmatrix}$, where $B = B^\top \in \mathbb{R}^{N\times N}$ and α is the squeezing parameter. Note that in the limit $\alpha \to \infty$, the canonical Gaussian cluster state approximates the corresponding ideal cluster state. Using (6), we obtain $X = B$ and $Y = e^{-2\alpha} I_N$. The graph corresponding to a canonical Gaussian cluster state is given by $Z = X + iY = B + ie^{-2\alpha} I_N$.

Let us consider a simple case where

$$X = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2} \end{bmatrix}, \quad Y = e^{-2\alpha} I_4. \tag{18}$$

These matrices satisfy $X_{4j} = 0$ and $Y_{4j} = 0$ for all $j \neq 4$. Thus by Theorem 7, the corresponding canonical Gaussian cluster state can be generated in a *four*-mode locally dissipative system. To construct such a system, let us take $P = [0\,0\,0\,1]^\top$ in Lemma 1. The next step is to determine the other system parameters R and Γ. For a practical implementation, one of the basic requirements on the system is that, as mentioned before, the system has as few pumped crystals as possible. Motivated by the structure of the passive quantum systems described in Remark 3, we choose $R = 0_{4\times 4}$. As a result, the Hamiltonian matrix is $M = \begin{bmatrix} -e^{2\alpha}(\Gamma X + X\Gamma^\top) & e^{2\alpha}\Gamma \\ e^{2\alpha}\Gamma^\top & 0_{4\times 4} \end{bmatrix}$. The $(1,2)$ block in M is a skew matrix $e^{2\alpha}\Gamma$. So if we can additionally take the $(1,1)$ block to be a diagonal matrix, then the interaction Hamiltonian between the modes is passive and can be simply realized by beam splitters. According to this guideline, we now seek Γ such that $-e^{2\alpha}(\Gamma X + X\Gamma^\top)$ is a diagonal matrix. By direct calculation, we obtain

$$\Gamma = \begin{bmatrix} 0 & \gamma_1 & 0 & \gamma_2 \\ -\gamma_1 & 0 & \gamma_1 & \sqrt{2}\gamma_2 \\ 0 & -\gamma_1 & 0 & \gamma_2 \\ -\gamma_2 & -\sqrt{2}\gamma_2 & -\gamma_2 & 0 \end{bmatrix},$$

where $\gamma_1 \in \mathbb{R}$ and $\gamma_2 \in \mathbb{R}$. Substituting the matrices P, R and Γ above into the rank condition (9), we obtain that if $\gamma_1\gamma_2 \neq 0$, the resulting linear quantum system is asymptotically stable and achieves the covariance matrix corresponding to (18).

The Hamiltonian of this linear quantum system is now determined as

$$\hat{H} = -\gamma_1 e^{2\alpha}\hat{q}_1^2 + \gamma_1 e^{2\alpha}\hat{q}_3^2 + e^{2\alpha}\gamma_1(\hat{H}_{12}^{(\mathrm{BS})} + \hat{H}_{23}^{(\mathrm{BS})}) + e^{2\alpha}\gamma_2(\hat{H}_{14}^{(\mathrm{BS})} + \sqrt{2}\hat{H}_{24}^{(\mathrm{BS})} + \hat{H}_{34}^{(\mathrm{BS})}),$$

where $\hat{H}_{jk}^{(\mathrm{BS})} = (\hat{q}_j\hat{p}_k - \hat{p}_j\hat{q}_k) = i(\hat{a}_j\hat{a}_k^* - \hat{a}_j^*\hat{a}_k)$, where $\hat{a}_j = (\hat{q}_j + i\hat{p}_j)/\sqrt{2}$ and $\hat{a}_j^* = (\hat{q}_j - i\hat{p}_j)/\sqrt{2}$, is the Hamiltonian representing the coupling between the jth and kth optical modes at a beam splitter. Also the coupling vector is given by

$$\hat{L} = -(\sqrt{2} + e^{-2\alpha} i)\hat{q}_4 + \hat{p}_4,$$

which acts only on the fourth mode and hence it is local. Finally, using the result in [45], a corresponding optical realization of this linear quantum system is shown in Fig. 6. Note that three pumped crystals are used; we conjecture that this is the minimum number required for constructing a desired locally dissipative system.

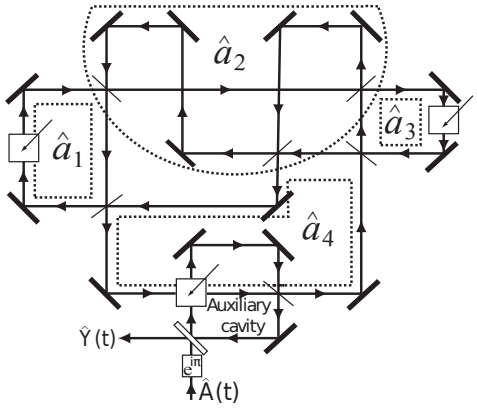

Fig. 6. The optical dissipative system that uniquely generates the canonical Gaussian cluster state (18). The coupling vector $\hat{L}$ acts only on the fourth mode, and hence it is local.

Example 11. We next consider a canonical Gaussian cluster state specified by the following matrices X and Y:

$$X = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad Y = e^{-2\alpha} I_4. \tag{19}$$

The strength of Theorem 7 is that it readily tells us that this canonical Gaussian cluster state cannot be generated in *any* four-mode locally dissipative system. Nonetheless let us take the same matrix P as before, i.e., $P = [0\,0\,0\,1]^\top$, and follow the same guideline as discussed in Example 10. That is, we set $R = 0_{4\times 4}$ and seek Γ such that $\Gamma X + X\Gamma^\top$ is a diagonal matrix. Then, again by direct calculation, we find

$$\Gamma = \begin{bmatrix} 0 & \gamma_1 & 0 & \gamma_2 \\ -\gamma_1 & 0 & \gamma_1+\gamma_2 & 0 \\ 0 & -(\gamma_1+\gamma_2) & 0 & \gamma_1 \\ -\gamma_2 & 0 & -\gamma_1 & 0 \end{bmatrix}.$$

Let us take $\gamma_1 = 1$ and $\gamma_2 = 0$. Then the corresponding system Hamiltonian is given by

$$\hat{H} = -e^{2\alpha}(\hat{q}_1^2 - \hat{q}_4^2) + e^{2\alpha}(\hat{H}_{12}^{(\mathrm{BS})} + \hat{H}_{23}^{(\mathrm{BS})} + \hat{H}_{34}^{(\mathrm{BS})}). \quad (20)$$

It can be verified that the rank condition (9) is satisfied, hence the system constructed here is asymptotically stable and achieves the desired covariance matrix corresponding to (19), though in this case the system needs to have the following non-local interaction with its environment:

$$\hat{L} = \left[\begin{array}{cccc|cccc} 0 & 0 & -1 & -e^{-2\alpha}i & 0 & 0 & 0 & 1 \end{array}\right] \hat{x} = -\hat{q}_3 - ie^{-2\alpha}\hat{q}_4 + \hat{p}_4.$$

An optical realization, which yet contains an abstract component corresponding to this non-local interaction, is depicted in Fig. 7. A practical implementation of the non-local interaction depicted in Fig. 7 could be experimentally difficult. Nonetheless this issue can be resolved by taking the following method: add an auxiliary system with a single mode $\hat{x}_A = [\hat{q}_A\ \hat{p}_A]^\top$, and specify the target canonical Gaussian cluster state as

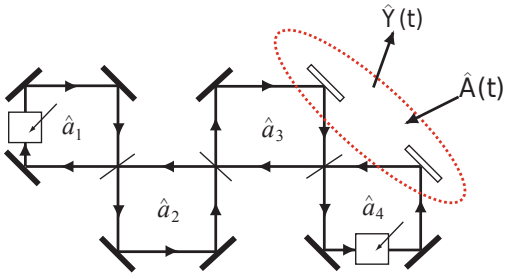

Fig. 7. The optical linear quantum system that uniquely generates the canonical Gaussian cluster state (19). The coupling vector $\hat{L}$ acts on the third and fourth modes, and hence it is not local.

$$\tilde{X} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda \end{bmatrix}, \quad \tilde{Y} = e^{-2\alpha} I_5. \quad (21)$$

Since $\tilde{X}_{5j} = 0$ and $\tilde{Y}_{5j} = 0$ for all $j \neq 5$, by Theorem 7, we can construct a five-mode locally dissipative system that uniquely generates the above canonical Gaussian cluster state (21). By choosing $P = [0\ 0\ 0\ 0\ 1]^\top$ and then taking a similar procedure as in the case of Example 10, we can obtain such a desired locally dissipative quantum system. Now we obtain an important observation: for an N-mode canonical Gaussian cluster state with the graph matrix $X = B$ and the squeezing matrix $Y = e^{-2\alpha} I_N$, it is *always* possible to generate this state in a locally dissipative quantum system by adding a single-mode auxiliary system and specifying the target state as $\tilde{X} = \mathrm{diag}[B, \lambda]$ and $\tilde{Y} = e^{-2\alpha} I_{N+1}$.

Remark 12. The method in [9] is based on essentially the same idea; given an N-mode pure Gaussian state with graph $Z = X + iY$, instead of generating it directly, we enlarge the system by adding a single-mode auxiliary system and then specify the target state as $\tilde{X} = \mathrm{diag}[X, \lambda]$ and $\tilde{Y} = \mathrm{diag}[Y, 1]$. By Theorem 7, this $(N+1)$-mode target state can be uniquely generated in an $(N+1)$-mode locally dissipative system. The original N-mode pure Gaussian state is then obtained as a reduced state of the target state.

5 Conclusion

In this paper, we have provided two feasible realizations of linear quantum systems for covariance assignment corresponding to pure Gaussian states: a cascade realization and a locally dissipative realization. First, we have shown that given any covariance matrix corresponding to a pure Gaussian state, we can construct a cascaded quantum system that achieves the assigned covariance matrix. This cascaded quantum system is constructed as a cascade connection of several one-dimensional harmonic oscillators, without any direct interaction Hamiltonians between these oscillators. Second, we have given a complete characterization of the class of pure Gaussian states that can be generated using locally dissipative quantum systems. In particular, we have shown a specific recipe for constructing a system having a relatively simple Hamiltonian coupling between the system modes. The results developed in this paper are potentially useful for the preparation of pure Gaussian states. In the examples, we have provided realizations of $(\hat{H}, \hat{L})$ in quantum optics using the result in [45]. The circuit figures shown in the examples are not necessarily the simplest realizations in quantum optics. Also, a system with $(\hat{H}, \hat{L})$ could be realized by other instances of linear quantum systems such as atomic ensembles and optomechanical systems [18, 47].

Appendix

Here we briefly review the synthesis theory of linear quantum systems in quantum optics developed in [45].

1. Realization of a quadratic Hamiltonian

Suppose a quadratic Hamiltonian is given by $\hat{H}_d = \frac{1}{2}\hat{\xi}^\top M_d \hat{\xi}$, where $\hat{\xi} = [\hat{q}\ \ \hat{p}]^\top$ and $M_d = M_d^\top \in \mathbb{R}^{2\times 2}$. This Hamiltonian can be realized by placing a crystal with a classical pump inside an optical cavity, as shown in Fig. 8. Working in the frame rotating at half the pump frequency, the Hamiltonian is written as

$$\begin{aligned} \hat{H}_r &= \triangle \hat{a}^* \hat{a} + \frac{i}{2}\left(\varepsilon(\hat{a}^*)^2 - \varepsilon^* \hat{a}^2\right) \\ &= \frac{1}{2}\hat{\xi}^\top \begin{bmatrix} \triangle - \mathrm{Im}(\varepsilon) & \mathrm{Re}(\varepsilon) \\ \mathrm{Re}(\varepsilon) & \triangle + \mathrm{Im}(\varepsilon) \end{bmatrix} \hat{\xi} - \frac{\triangle}{2}, \end{aligned} \quad (22)$$

where $\triangle = \omega_{\rm cav} - \omega_p/2$ is the detuning between the cavity mode frequency and the half pump frequency. ε is a measure of the effective pump intensity [34]. From (22), we see that by choosing the values of $\triangle$ and ε, one can make $\hat{H}_r = \hat{H}_d - \frac{\triangle}{2}$. Note that the constant term $-\frac{\triangle}{2}$ does not affect the dynamics of a linear quantum system, and hence can be ignored. Therefore, the desired Hamiltonian $\hat{H}_d$ can be realized in this scheme.

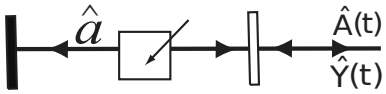

Fig. 8. A quadratic Hamiltonian can be realized by placing a crystal with a classical pump inside an optical cavity.

2. Realization of a beam-splitter-like interaction Hamiltonian

Suppose a Hamiltonian is given by $\hat{H}_d = h_d \hat{a}_1^* \hat{a}_2 + h_d^* \hat{a}_2^* \hat{a}_1$, where $h_d \in \mathbb{C}$. This Hamiltonian can be realized by implementing a beam splitter for the two incoming modes $\hat{a}_1$ and $\hat{a}_2$, as shown in Fig. 9. At the beam splitter, we have the following transformations

$$\begin{bmatrix} \hat{a}_3 \\ \hat{a}_4 \end{bmatrix} = \begin{bmatrix} t_2 & r_1 \\ r_2 & t_1 \end{bmatrix} \begin{bmatrix} \hat{a}_1 \\ \hat{a}_2 \end{bmatrix},$$

where $\hat{a}_3$ and $\hat{a}_4$ denote the outgoing modes, and $r_1, t_1 \in \mathbb{C}$ denote the (complex) reflectance and transmittance of the beam splitter, respectively. Note that r_1, t_1, r_2 and t_2 satisfy the following relations: $|r_2| = |r_1|$, $|t_2| = |t_1|$, $|r_1|^2 + |t_1|^2 = 1$, $r_1^* t_2 + r_2 t_1^* = 0$, and $r_1^* t_1 + r_2 t_2^* = 0$ [48]. Let us parametrize them as $r_1 = e^{-i\phi} sin\theta$, $r_2 = -e^{i\phi} sin\theta$, and $t_1 = t_2 = cos\theta$. Then the interaction Hamiltonian $\hat{H}^{(\rm BS)}$ for this beam splitter is given by

$$\hat{H}^{(\rm BS)} = i\theta e^{-i\phi} \hat{a}_1^* \hat{a}_2 - i\theta e^{i\phi} \hat{a}_2^* \hat{a}_1. \tag{23}$$

From (23), we see that by choosing the values of θ and ϕ, one can make $\hat{H}^{(\rm BS)} = \hat{H}_d$. That is, the desired beam-splitter-like interaction Hamiltonian $\hat{H}_d$ can be realized in this scheme.

3. Realization of a dissipative coupling $\hat{L}$

To realize a coupling operator $\hat{L}_d = c_1 \hat{q} + c_2 \hat{p} = \left(\frac{c_1 - ic_2}{\sqrt{2}}\right) \hat{a} + \left(\frac{c_1 + ic_2}{\sqrt{2}}\right) \hat{a}^*$, we consider the configuration shown in Fig. 10. The configuration consists of a ring cavity with mode $\hat{a}$ and an auxiliary ring cavity with mode $\hat{b}$. The cavity modes $\hat{a}$ and $\hat{b}$ interact through a crystal pumped by a classical beam, and a beam splitter. The frequency of the auxiliary cavity mode $\hat{b}$ is matched to half the pump frequency. Working in the frame rotating at half the pump frequency, the interaction Hamiltonian is written as

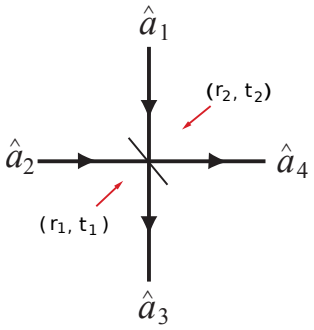

Fig. 9. A beam-splitter-like interaction Hamiltonian can be realized by placing a beam splitter for the two incoming modes $\hat{a}_1$ and $\hat{a}_2$.

$$\hat{H}_{ab} = \frac{i}{2}\left(\varepsilon_1 \hat{a}^* \hat{b}^* - \varepsilon_1^* \hat{a}\hat{b}\right) + \frac{i}{2}\left(\varepsilon_2 \hat{a}^* \hat{b} - \varepsilon_2^* \hat{a}\hat{b}^*\right),$$

where ε_1 determines the effective pump intensity and ε_2 determines the parameters of the beam splitter. Assume that the coupling coefficient γ of the partially transmitting mirror is large so that the mode $\hat{b}$ is heavily damped and can be adiabatically eliminated. Then after elimination of $\hat{b}$, the resulting coupling operator is given by

$$\hat{L}_r = \frac{1}{\sqrt{\gamma}}(-\varepsilon_2^* \hat{a} + \varepsilon_1 \hat{a}^*). \tag{24}$$

From (24), we see that by choosing the values of ε_1, ε_2, and γ with γ being large, we can make $\hat{L}_r = \hat{L}_d$. That is, the desired coupling operator $\hat{L}_d$ can be realized in this scheme. See [45] for details.

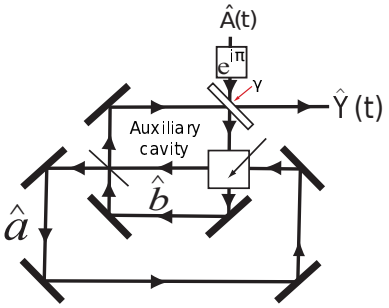

Fig. 10. Realization of a dissipative coupling operator $\hat{L}$.

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