# CONVERGENCE OF A FLUX-SPLITTING FINITE VOLUME SCHEME FOR CONSERVATION LAWS DRIVEN BY LÉVY NOISE.

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Abstract. We explore numerical approximation of multidimensional stochastic balance laws driven by multiplicative Lévy noise via flux- splitting finite volume method. The convergence of the approximations is proved towards the unique entropy solution of the underlying problem.

## 1. Introduction

Let $\big(\Omega, \mathbb{P}, \mathcal{F}, \{\mathcal{F}_t\}_{t\ge 0}\big)$ be a filtered probability space satisfying the usual hypothesis i.e $\{\mathcal{F}_t\}_{t\ge 0}$ is a right-continuous filtration such that $\mathcal{F}_0$ contains all the $\mathbb{P}$-null subsets of $(\Omega, \mathcal{F})$. In this paper, we are interested in the study of numerical scheme and numerical approximation for multi-dimensional nonlinear stochastic balance laws of type

$$du(t,x) + \mathrm{div}_x\big(\vec{v}(t,x) f(u(t,x))\big)\,dt = \int_{\mathbf{E}} \eta(u(t,x);z)\,\tilde{N}(dz,dt), \quad (t,x) \in \Pi_T, \tag{1.1}$$
$$u(0,x) = u_0(x), \quad x \in \mathbb{R}^d,$$

where $\Pi_T = [0,T) \times \mathbb{R}^d$ with $T > 0$ fixed. Here, $f : \mathbb{R} \to \mathbb{R}$ is a given real valued flux function, $\vec{v}$ is a given vector valued function, $u_0(x)$ is a given initial function and $\tilde{N}(dz,dt) = N(dz,dt) - m(dz)\,dt$, where $N$ is a Poisson random measure on $(\mathbf{E}, \mathcal{E})$ with intensity measure $m(dz)$, where $(\mathbf{E}, \mathcal{E}, m)$ is a $\sigma$-finite measure space. Furthermore, $(u,z) \mapsto \eta(u,z)$ is a given real valued functions signifying the multiplicative nature of the noise.

This type of equation arises in many different fields where non-Gaussianity plays an important role. As for example, it has been used in models of neuronal activity accounting for synaptic transmissions occurring randomly in time as well as at different locations on a spatially extended neuron, chemicals reaction-diffusion systems, market fluctuations both for risk management and option pricing purpose, stochastic turbulence, etc. The study of well-posedness theory for this kind of equation is of great importance in the light of current applications in continuum physics.

**Remark 1.1.** We will carry out our analysis under the structural assumption $\mathbf{E} = \mathcal{O} \times \mathbb{R}^*$ where $\mathcal{O}$ is a subset of the Euclidean space. The measure $m$ on $\mathbf{E}$ is defined as $\gamma \times \mu^*$ where $\gamma$ is a Radon measure on $\mathcal{O}$ and $\mu^*$ is so-called Lévy measure on $\mathbb{R}^*$. Such a noise would be called an impulsive white noise with jump position intensity $\gamma$ and jump size intensity $\mu^*$. We refer to [30] for more on Lévy sheet and related impulsive white noise.

In the case $\eta = 0$, the equation (1.1) becomes a standard conservation law in $\mathbb{R}^d$ and there exists a satisfactory well-posedness theory based on Kruzkov's pioneering idea to pick up the physically relevant solution in an unique way, called *entropy solution*. We refer to [19, 27, 28, 33] and references therein for more on entropy solution theory for deterministic conservation laws.

The study of stochastic balance laws driven by noise is comparatively new area of pursuit. Only recently balance laws with stochastic forcing have attracted the attention of many authors [2, 4, 5, 8, 9, 10, 11, 12, 14, 15, 16, 18, 22, 23] and resulted a significant momentum in the theoretical development of such problems. Due to nonlinear nature of the underlying problem, explicit solution formula is hard to obtain and hence robust numerical schemes for approximating such equation are very important. In the

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last decade, there has been a growing interest in numerical approximation and numerical experiments for entropy solution to the related Cauchy problem driven by stochastic forcing. The first documented development in this direction is [21], where the authors established existence of weak solution (possibly non-unique) of one dimensional balance law driven by Brownian noise via splitting method. In a recent paper [26], Kröker and Rodhe established the convergence of monotone semi-discrete finite volume scheme by using stochastic compensated compactness method. Bauzet [3] revisited the paper of Holden and Risebro [21], and generalized the operator-splitting method for the same Cauchy problem but in a bounded domain of $\mathbb{R}^d$. Using Young measure theory, the author established the convergence of approximate solutions to an entropy solution. We also refer to see [25], where the time splitting method was analyzed for more general noise coefficient in the spirit of Malliavin calculus and Young measure theory. In a recent papers [6, 7], Bauzet et. al. have studied fully discrete scheme via flux-splitting and monotone finite volume schemes for stochastic conservation laws driven by multiplicative Brownian noise and established its convergence by using Young measure technique.

The study of numerical schemes for stochastic balance laws driven by Lévy noise is more sparse than the previous case. A semi-discrete finite difference scheme for conservation laws driven by a homogeneous multiplicative Lévy noise has been studied by Koley et al.[24]. Using BV estimates, the authors showed the convergence of approximate solutions, generated by the finite difference scheme, to the unique entropy solution as the spatial mesh size $\Delta x \to 0$ and established rate of convergence which is of order $\frac{1}{2}$.

The above discussions clearly highlight the lack of the study of fully discrete scheme and its convergence for stochastic balance laws driven by Lévy noise. In this paper, drawing primary motivation from [6], we propose a fully discrete flux-splitting finite volume scheme for (1.1), and address the convergence of the scheme. First we establish few essential *a priori* estimates for approximate solutions and then using these estimates, we deduce entropy inequality for approximate solutions. Using Young measure theory, we conclude that the finite volume approximate solutions tend to a generalized entropy solution of (1.1).

The rest of the paper is organized as follows. In Sections 2 and 3, we collect all the assumptions for the subsequent analysis, then we define the numerical scheme and finally state the main result of this article. Section 4 deals with few a priori estimates on the finite volume approximate solutions and using these *a priori* estimates, in Section 5, we establish discrete and continuous version of entropy inequalities on approximate solutions. The Section 6 is devoted to the proof of the main theorem along with short discussion of Young measure theory and its compactness, is presented in Appendix 7.

## 2. Preliminaries and technical framework

It is well-known that due to nonlinear flux term in (1.1), solutions to (1.1) are not necessarily smooth even if initial data is smooth, and hence must be interpreted via weak sense. Before introducing the concept of weak solutions, we first assume that $\big(\Omega, \mathbb{P}, \mathcal{F}, \{\mathcal{F}_t\}_{t\ge 0}\big)$ be a filtered probability space satisfying the usual hypothesis, i.e., $\{\mathcal{F}_t\}_{t\ge 0}$ is a right-continuous filtration such that $\mathcal{F}_0$ contains all the $\mathbb{P}$-null subsets of $(\Omega, \mathcal{F})$. Moreover, by a predictable $\sigma$-field on $[0,T]\times\Omega$, denoted by $\mathcal{P}_T$, we mean that the $\sigma$-field is generated by the sets of the form: $\{0\}\times A$ and $(s,t]\times B$ for any $A\in\mathcal{F}_0; B\in\mathcal{F}_s,\ 0<s,t\le T$. The notion of stochastic weak solution is defined as follows:

**Definition 2.1** (Weak solution)**.** A square integrable $L^2(\mathbb{R}^d)$-valued $\{\mathcal{F}_t : t\ge 0\}$-predictable stochastic process $u(t) = u(t,x)$ is called a stochastic weak solution of (1.1) if for all test functions $\psi \in C_c^\infty([0,T)\times\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} \psi(0,x)u_0(x)\,dx + \int_{\Pi_T}\Big\{\partial_t\psi(t,x)u(t,x) + \vec{v}(t,x)f(u(t,x))\cdot\nabla_x\psi(t,x)\Big\}\,dt\,dx$$
$$+ \int_{\Pi_T}\int_{\mathbf{E}} \eta(u(t,x);z)\psi(t,x)\,\tilde{N}(dz,dt)\,dx = 0, \quad \mathbb{P}\text{-a.s.}$$

However, since there are infinitely many weak solutions, one needs to define an extra admissibility criteria to select physically relevant solution in a unique way, and one such condition is called entropy condition. Let us begin with the notion of entropy flux pair.

**Definition 2.2** (Entropy flux pair)**.** $(\beta,\phi)$ is called an entropy flux pair if $\beta\in C^2(\mathbb{R})$ and $\zeta:\mathbb{R}\mapsto\mathbb{R}$ is such that

$$\zeta'(r) = \beta'(r)f'(r).$$

An entropy flux pair $(\beta, \zeta)$ is called convex if $\beta''(s) \ge 0$.

Let $\mathcal{A} = \big\{\beta \in C^2(\mathbb{R}), \text{convex such that support of } \beta'' \text{ is compact}\big\}$. In the sequel, we will use specific entropy flux pairs. For any $a \in \mathbb{R}$ and $\beta \in \mathcal{A}$, define $F^\beta(a) = \displaystyle\int_0^a \beta'(s) f'(s)\,ds$ . Note that, $F^\beta(\cdot)$ is a Lipschitz continuous function on $\mathbb{R}$ and $(\beta, F^\beta)$ is an entropy flux-pair. To this end, we define the notion of stochastic entropy solution of (1.1).

**Definition 2.3** (Stochastic entropy solution)**.** An $L^2(\mathbb{R}^d)$-valued $\{\mathcal{F}_t : t \ge 0\}$-predictable stochastic process $u(t) = u(t,x)$ is called a stochastic entropy solution of (1.1) if the following hold:

i) For each $T > 0$

$$\sup_{0\le t\le T} \mathbb{E}\Big[||u(\cdot,t)||_2^2\Big] < +\infty.$$

ii) For each $0 \le \psi \in C_c^\infty([0,\infty)\times\mathbb{R}^d)$ and $\beta \in \mathcal{A}$, there holds

$$\int_{\mathbb{R}^d} \psi(x,0)\beta(u_0(x))\,dx + \int_{\Pi_T} \Big\{\partial_t\psi(t,x)\beta(u(t,x)) + F^\beta(u(t,x))\vec{v}(t,x)\cdot\nabla\psi(t,x)\Big\}\,dx\,dt$$
$$+ \int_{\Pi_T}\int_{\mathbf{E}}\int_0^1 \eta(u(t,x);z)\beta'\big(u(t,x) + \lambda\eta(u(t,x);z)\big)\psi(t,x)\,d\lambda\,\tilde{N}(dz,dt)\,dx$$
$$+ \int_{\Pi_T}\int_{\mathbf{E}}\int_0^1 (1-\lambda)\eta^2(u(t,x);z)\beta''\big(u(t,x) + \lambda\eta(u(t,x);z)\big)\psi(t,x)\,d\lambda\,m(dz)\,dt\,dx \ge 0, \quad \mathbb{P}\text{-a.s.}$$

Due to nonlocal nature of the Itô-Lévy formula and the missing noise-noise interaction, the Definition 2.3 does not alone give the $L^1$-contraction principle in the sense of average when one tries to compare two entropy solutions directly, and hence fails to give uniqueness. For the details, we refer to see [12, 18]. However, in view of [2, 9], we can look for so called *generalized entropy solution* which are $L^2\big(\mathbb{R}^d\times(0,1)\big)$-valued $\{\mathcal{F}_t : t \ge 0\}$-predictable stochastic process.

**Definition 2.4** (Generalized entropy solution)**.** An $L^2\big(\mathbb{R}^d\times(0,1)\big)$-valued $\{\mathcal{F}_t : t \ge 0\}$-predictable stochastic process $u(t) = u(t,x,\alpha)$ is called a generalized stochastic entropy solution of (1.1) provided

(1) For each $T > 0$

$$\sup_{0\le t\le T} \mathbb{E}\Big[||u(t,\cdot,\cdot)||_2^2\Big] < +\infty.$$

(2) For all test functions $0 \le \psi \in C_c^{1,2}([0,\infty)\times\mathbb{R}^d)$, and any $\beta \in \mathcal{A}$, the following inequality holds

$$\int_{\mathbb{R}^d} \beta(u_0(x))\psi(x,0)\,dx + \int_{\Pi_T}\int_0^1 \Big\{\beta(u(t,x,\alpha))\partial_t\psi(t,x) + F^\beta(u(t,x,\alpha))\vec{v}(t,x)\cdot\nabla_x\psi(t,x)\Big\}\,d\alpha\,dt\,dx$$
$$+ \int_{\Pi_T}\int_{\mathbf{E}}\int_0^1\int_0^1 \eta(u(t,x,\alpha);z)\beta'\big(u(t,x,\alpha) + \lambda\eta(u(t,x,\alpha);z)\big)\psi(t,x)\,d\alpha\,d\lambda\,\tilde{N}(dz,dt)\,dx$$
$$+ \int_{\Pi_T}\int_{\mathbf{E}}\int_0^1\int_0^1 (1-\lambda)\eta^2(u(t,x,\alpha);z)\beta''\big(u(t,x,\alpha) + \lambda\eta(u(t,x,\alpha);z)\big)$$
$$\times\,\psi(t,x)\,d\alpha\,d\lambda\,m(dz)\,dt\,dx \ge 0, \quad \mathbb{P}\text{-a.s.} \tag{2.1}$$

The aim of this paper is to establish convergence of approximate solutions, constructed via flux-splitting finite volume scheme (cf. Section 3), to the unique entropy solution of (1.1), and we will do so under the following assumptions:

**A**.1 $f : \mathbb{R} \mapsto \mathbb{R}$ is $C^2$ and Lipschitz continuous with $f(0) = 0$.

**A**.2 $\vec{v} : [0,T]\times\mathbb{R}^d \mapsto \mathbb{R}^d$ is a $C^1$ function with $\mathrm{div}_x\vec{v}(t,x) = 0$ for all $(t,x) \in \Pi_T$. Furthermore, there exists $V < +\infty$ such that $|\vec{v}(t,x)| \le V$ for all $(t,x) \in \Pi_T$.

**A**.3 There exist positive constants $0 < \lambda^* < 1$ and $C^* > 0$, and $h_1 \in L^2(\mathbf{E}, m)$ with $0 \le h_1(z) \le 1$ such that for all $u, v \in \mathbb{R};\ z \in \mathbf{E}$

$$|\eta(u;z) - \eta(v;z)| \le \lambda^*|u-v|h_1(z); \quad |\eta(u,z)| \le C^* h_1(z).$$

Moreover, $\eta(0;z) = 0$ for all $z \in \mathbf{E}$.

**A**.4 The initial function $u_0(x)$ is a $L^2(\mathbb{R}^d)$-valued $\mathcal{F}_0$ measurable random variable satisfying

$$\mathbb{E}\Big[||u_0||_2^2\Big] < +\infty.$$

We have the following existence and uniqueness theorems whose proofs are postponed to the Appendix.

**Theorem 2.1.** *Let the assumptions* **A**.1-**A**.4 *hold. Then there exists a generalized entropy solution of* (1.1) *in the sense of Definition* 2.4.

**Theorem 2.2.** *Under the assumptions* **A**.1-**A**.4*, the generalized entropy solution of* (1.1) *is unique. Moreover, it is the unique stochastic entropy solution.*

**Remark 2.1.** Note that we need the assumption **A**.1 to get entropy solution for the initial data in $L^2(\mathbb{R}^d)$ to control the multi-linear integrals terms. The assumption **A**.3 is needed to handle the nonlocal nature of the entropy inequalities. Boundedness of $\eta$ is needed to validate Proposition 5.2.

Throughout this paper, we use the letter $C$ to denote various generic constant which may change line to line. We denote by $c_f$ the Lipschitz constant of $f$ and $c_\eta$, the finite constant (which exists thanks to **A**.3) as $c_\eta = \displaystyle\int_{\mathbf{E}} h_1^2(z)\, m(dz)$. The Euclidean norm on $\mathbb{R}^d$ is denoted by $|\cdot|$.

## 3. Flux-splitting finite volume scheme

Our main point of interest is numerical approximation for the problem (1.1). Let us first introduce the space discretization by finite volumes (control volumes). For that we need to recall the definition of so called admissible meshes for finite volume scheme (cf. [17]).

**Definition 3.1** (Admissible mesh)**.** An admissible mesh $\mathcal{T}$ of $\mathbb{R}^d$ is a family of disjoint polygonal connected subset of $\mathbb{R}^d$ satisfying the following:

i) $\mathbb{R}^d$ is the union of the closure of the elements (called control volume) of $\mathcal{T}$.
ii) The common interface of any two elements of $\mathcal{T}$ is included in a hyperplane of $\mathbb{R}^d$.
iii) There exists a nonnegative constant $\alpha$ such that

$$\begin{cases} \alpha h^d \le |K|, \\ |\partial K| \le \dfrac{1}{\alpha} h^{d-1}, \quad \forall K \in \mathcal{T}, \end{cases} \tag{3.1}$$

where $h = \sup\Big\{\text{diam}(K) : K \in \mathcal{T}\Big\} < +\infty$, $|K|$ denotes the $d$-dimensional Lebesgue measure of $K$, and $|\partial K|$ represents the $(d-1)$-dimensional Lebesgue measure of $\partial K$.

In the sequel, we denote the followings:

- $\mathcal{E}_K$: the set of interfaces of the control volume $K$.
- $\mathcal{N}(K)$: the set of control volumes neighbors of the control volume $K$.
- $K|L$: the common interface between $K$ and $L$ for any $L \in \mathcal{N}(K)$.
- $\mathcal{E}$: the set of all the interfaces of the mesh.
- $n_{K,\sigma}$: the unit normal to the interface $\sigma$, outward to the control volume $K$, for any $\sigma \in \mathcal{E}_K$.

Consider an admissible mesh $\mathcal{T}$ in the sense of Definition 3.1. In order to discretize the time variable, we split the time interval $[0,T]$ as follows: Let $N$ be a positive integer and we set $\Delta t = \frac{T}{N}$. Define $t_n = n\Delta t$, $n = 0, 1, \cdots, N$. Then $\{t_n : n = 0, 1, \cdots, N\}$ splits the time interval $[0,T]$ into equal step with a length equal to $\Delta t$.

It is well known that, the main idea behind flux-splitting finite volume method is to express a flux function $f$ as the sum of a nondecreasing function $f_1$ and a non increasing function $f_2$. Since the flux function $f$ is Lipschitz continuous such a decomposition is always possible.

We propose the following flux-splitting finite volume scheme to approximate the solution of (1.1): for any $K \in \mathcal{T}$, and $n \in \{0, 1, 2, \cdots, N-1\}$, we define the discrete unknowns $u_K^n$ as follows

$$u_K^0 = \frac{1}{|K|} \int_K u_0(x)\, dx,$$

$$\frac{|K|}{\Delta t}\big(u_K^{n+1}-u_K^n\big)+\sum_{\substack{\sigma\in\mathcal{E}_K\\ \sigma=K|L}}|\sigma|\Big\{(\vec{v}\cdot n_{K,\sigma})^+\big(f_1(u_K^n)+f_2(u_L^n)\big)-(\vec{v}\cdot n_{K,\sigma})^-\big(f_1(u_L^n)+f_2(u_K^n)\big)\Big\}$$
$$=\frac{|K|}{\Delta t}\int_{t_n}^{t_{n+1}}\int_{\mathbf{E}}\eta(u_K^n;z)\tilde{N}(dz,dt), \tag{3.2}$$

where, by denoting $d\nu$ the $d-1$ dimensional Lebesgue measure

$$(\vec{v}\cdot n_{K,\sigma})^+=\frac{1}{\Delta t|\sigma|}\int_{t_n}^{t_{n+1}}\int_\sigma\big(\vec{v}(t,x)\cdot n_{K,\sigma}\big)^+\,d\nu(x)\,dt,$$
$$(\vec{v}\cdot n_{K,\sigma})^-=\frac{1}{\Delta t|\sigma|}\int_{t_n}^{t_{n+1}}\int_\sigma\big(\vec{v}(t,x)\cdot n_{K,\sigma}\big)^-\,d\nu(x)\,dt.$$

Since $\mathrm{div}_x\vec{v}(t,x)=0$ for any $(t,x)\in\Pi_T$, an elementary estimate yields

$$\sum_{\sigma\in\mathcal{E}_K}|\sigma|\vec{v}\cdot n_{K,\sigma}=0. \tag{3.3}$$

We define approximate finite volume solution on $\Pi_T$ as a piecewise constant given by

$$u^h_{\mathcal{T},\Delta t}(t,x)=u_K^n \text{ for } x\in K \text{ and } t\in[t_n,t_{n+1}). \tag{3.4}$$

**Remark 3.1.** In view of the properties of stochastic integral with respect to compensated Poisson random measure, each $u_K^n$ is $\mathcal{F}_{n\Delta t}$ - measurable for $K\in\mathcal{T}$ and $n\in\{0,1,\cdots,N\}$. Thus, $u^h_{\mathcal{T},\Delta t}(t,\cdot)$ is an $L^2(\mathbb{R}^d)$-valued $\mathcal{F}_t$- predictable stochastic process as $u_0$ satisfies **A**.4.

Finally, we state the main theorem of this paper.

**Main Theorem.** Let the assumptions **A**.1-**A**.4 be true and $\mathcal{T}$ be an admissible mesh on $\mathbb{R}^d$ with size $h$ in the sense of Definition 3.1. Let $\Delta t$ be the time step as discuss above and assume that

$$\frac{\Delta t}{h}\to 0 \text{ as } h\to 0.$$

Let $u^h_{\mathcal{T},\Delta t}(t,x)$ be the finite volume approximation as prescribed by (3.4). Then, there exists a $L^2(\mathbb{R}^d\times(0,1))$-valued $\{\mathcal{F}_t: t\ge 0\}$-predictable process $u=u(t,x,\alpha)$ such that

i) $u(t,x,\alpha)$ is a generalized entropy solution of (1.1) and $u^h_{\mathcal{T},\Delta t}(t,x)\to u(t,x,\alpha)$ in the sense of Young measure.

(ii) $u^h_{\mathcal{T},\Delta t}(t,x)\to\bar{u}(t,x)$ in $L^p_{\mathrm{loc}}(\mathbb{R}^d;L^p(\Omega\times(0,T)))$ for $1\le p<2$, where $\bar{u}(t,x)=\int_0^1 u(t,x,\alpha)\,d\alpha$ is the unique stochastic entropy solution of (1.1).

**Remark 3.2.** Under the CFL condition

$$\Delta t\le\frac{(1-\xi)\alpha^2 h}{c_f V},\ \text{ for some } \xi\in(0,1), \tag{3.5}$$

we have uniform moment estimate and weak $BV$ estimate on $u^h_{\mathcal{T},\Delta t}$ for $\xi=0$ and $\xi\in(0,1)$ respectively (see Lemmas 4.1 and 4.2). In the deterministic case, condition (3.5) is sufficient to establish the convergence of approximate solutions to the unique entropy solution of the problem. But in the stochastic case, only this condition is not enough and hence, we assume the stronger condition, namely $\frac{\Delta t}{h}\to 0$ as $h\to 0$.

**Remark 3.3.** Since every Lipschitz continuous function can be expressed as the sum of nondecreasing function and a non increasing one, it suffices to prove the main theorem (cf. Theorem 3) for a nondecreasing Lipschitz continuous flux function $f$.

For a nondecreasing Lipschitz continuous function $f$, the finite volume scheme (3.2) reduces to an upwind finite volume scheme

$$\begin{cases}\dfrac{|K|}{\Delta t}\big(u_K^{n+1}-u_K^n\big)+\displaystyle\sum_{\sigma\in\mathcal{E}_K}|\sigma|\vec{v}\cdot n_{K,\sigma}f(u_\sigma^n)=\frac{|K|}{\Delta t}\int_{t_n}^{t_{n+1}}\int_{\mathbf{E}}\eta(u_K^n;z)\tilde{N}(dz,dt),\\ u_K^0=\dfrac{1}{|K|}\displaystyle\int_K u_0(x)\,dx,\end{cases} \tag{3.6}$$

where $u_\sigma^n$ represents the upstream value at time $t_n$ with respect to $\sigma$. More precisely, if $\sigma$ is the interface between two control volumes $K$ and $L$, then

$$u_\sigma^n = \begin{cases} u_K^n \text{ if } \vec{v}\cdot n_{K,\sigma} \ge 0, \\ u_L^n \text{ if } \vec{v}\cdot n_{K,\sigma} < 0. \end{cases}$$

The upwind finite volume approximate solution $u^h_{\mathcal{T},\Delta t}(t,x)$ on $\Pi_T$ is defined as

$$u^h_{\mathcal{T},\,\Delta t}(t,x) = u_K^n \text{ for } x\in K \text{ and } t\in[t_n,t_{n+1}), \tag{3.7}$$

where the discrete unknown $u_K^n, K\in\mathcal{T}, n\in\{0,1,\cdots,N-1\}$ is computed from (3.6).

## 4. A priori estimates

This section is devoted to *a priori* estimates for the upwind finite volume approximate solution $u^h_{\mathcal{T},\Delta t}$ which will be very useful to prove its convergence. We start with the following lemma which is essentially a uniform moment estimate.

**Lemma 4.1.** *Let $T>0$ and the assumptions* **A**.1-**A**.4 *hold. Consider an admissible mesh $\mathcal{T}$ on $\mathbb{R}^d$ with size $h$ in the sense of Definition* 3.1. *Let $\Delta t = \frac{T}{N}$ be the time step for some $N\in\mathbb{N}^*$, satisfying the Courant-Friedrichs-Levy (CFL) condition*

$$\Delta t \le \frac{\alpha^2 h}{c_f V}.$$

*Then the upwind finite volume approximate solution $u^h_{\mathcal{T},\,\Delta t}$ satisfies the following bound*

$$\|u^h_{\mathcal{T},\,\Delta t}\|^2_{L^\infty(0,T;L^2(\Omega\times\mathbb{R}^d))} \le e^{c_\eta T}\mathbb{E}\big[||u_0||_2^2\big]. \tag{4.1}$$

*As a consequence, we see that $u^h_{\mathcal{T},\,\Delta t}$ satisfies the following bound*

$$\|u^h_{\mathcal{T},\,\Delta t}\|^2_{L^2(\Omega\times\Pi_T)} \le T\,e^{c_\eta T}\mathbb{E}\big[||u_0||_2^2\big].$$

*Proof.* To prove (4.1), it is enough to prove: for $n\in\{0,1,\cdots,N-1\}$, the following property holds

$$\sum_{K\in\mathcal{T}}|K|\mathbb{E}\big[(u_K^n)^2\big] \le (1+\Delta t c_\eta)^n\mathbb{E}\big[||u_0||_2^2\big]. \tag{4.2}$$

Observe that

$$\begin{aligned}\sum_{K\in\mathcal{T}}|K|\mathbb{E}\big[(u_K^0)^2\big] &= \sum_{K\in\mathcal{T}}|K|\mathbb{E}\Big[\big(\frac{1}{|K|}\int_K u_0(x)\,dx\big)^2\Big] \\ &\le \mathbb{E}\big[||u_0||_2^2\big] = (1+\Delta t c_\eta)^0\mathbb{E}\big[||u_0||_2^2\big].\end{aligned}$$

Set $n\in\{0,1,\cdots,N-1\}$ and suppose that (4.2) holds for $n$. We will show that (4.2) holds for $n+1$. In view of (3.3), one has $\sum_{\sigma\in\mathcal{E}_K}|\sigma|\vec{v}\cdot n_{K,\sigma}f(u_K^n)=0$ and hence the scheme (3.6) reduces to

$$\frac{|K|}{\Delta t}\big(u_K^{n+1}-u_K^n\big) + \sum_{\sigma\in\mathcal{E}_K}|\sigma|\vec{v}\cdot n_{K,\sigma}\big(f(u_\sigma^n)-f(u_K^n)\big) = \frac{|K|}{\Delta t}\int_{t_n}^{t_{n+1}}\int_{\mathbf{E}}\eta(u_K^n;z)\tilde{N}(dz,dt).$$

Again, in view of the definition of $u_\sigma^n$, the above finite volume scheme is equivalent to

$$\frac{|K|}{\Delta t}\big(u_K^{n+1}-u_K^n\big) + \sum_{\substack{\sigma\in\mathcal{E}_K\\ \sigma=K|L}}|\sigma|(\vec{v}\cdot n_{K,\sigma})^-\big(f(u_K^n)-f(u_L^n)\big) = \frac{|K|}{\Delta t}\int_{t_n}^{t_{n+1}}\int_{\mathbf{E}}\eta(u_K^n;z)\tilde{N}(dz,dt). \tag{4.3}$$

Multiplying (4.3) by $u_K^n$ and using the fact that $ab=\frac12[(a+b)^2-a^2-b^2]$ for any $a,b\in\mathbb{R}$, we obtain

$$\begin{aligned}\frac{|K|}{2}\Big[(u_K^{n+1})^2-(u_K^n)^2\Big] &= \frac{|K|}{2}\big(u_K^{n+1}-u_K^n\big)^2 - \Delta t\sum_{\substack{\sigma\in\mathcal{E}_K\\ \sigma=K|L}}|\sigma|(\vec{v}\cdot n_{K,\sigma})^-\big(f(u_K^n)-f(u_L^n)\big)u_K^n \\ &\quad + |K|\int_{t_n}^{t_{n+1}}\int_{\mathbf{E}}\eta(u_K^n;z)u_K^n\tilde{N}(dz,dt).\end{aligned}$$

Taking expectation, and using the fact that for any two constants $T_1, T_2 \ge 0$ with $T_1 < T_2$,

$$\mathbb{E}\Big[X_{T_1} \int_{T_1}^{T_2} \int_{\mathbf{E}} \zeta(t,z)\, \tilde{N}(dz,dt)\Big] = 0,$$

where $\zeta$ is a predictable integrand with $\mathbb{E}\Big[\int_0^T \int_{\mathbf{E}} \zeta^2(t,z)\, m(dz)\, dt\Big] < +\infty$ and $X$ is an adapted process, we obtain, thanks to Itô isometry

$$\begin{aligned}\frac{|K|}{2}\mathbb{E}\Big[(u_K^{n+1})^2 - (u_K^n)^2\Big] &= \frac{|K|}{2}\mathbb{E}\Big[\int_{t_n}^{t_{n+1}} \int_{\mathbf{E}} \eta^2(u_K^n;z)\, m(dz)\, dt\Big] \\ &\quad + \frac{(\Delta t)^2}{2|K|}\mathbb{E}\Big[\Big(\sum_{\substack{\sigma\in\mathcal{E}_K\\ \sigma=K|L}} |\sigma|(\vec{v}\cdot n_{K,\sigma})^-\big(f(u_K^n) - f(u_L^n)\big)\Big)^2\Big] \\ &\quad - \Delta t\,\mathbb{E}\Big[\sum_{\substack{\sigma\in\mathcal{E}_K\\ \sigma=K|L}} |\sigma|(\vec{v}\cdot n_{K,\sigma})^-\big(f(u_K^n) - f(u_L^n)\big)u_K^n\Big],\end{aligned}$$

where we have used (4.3) to replace $u_K^{n+1} - u_K^n$. Note that, thanks to (3.1), the following inequality holds

$$\frac{|\partial K|}{|K|} \le \frac{1}{\alpha^2 h}.$$

Therefore, $\sum_{\sigma\in\mathcal{E}_K} |\sigma||\vec{v}\cdot n_{K,\sigma}| \le V|\partial K| \le \frac{V}{\alpha^2 h}|K|$, and hence

$$\sum_{\sigma\in\mathcal{E}_K} |\sigma|(\vec{v}\cdot n_{K,\sigma})^- \le \frac{V}{\alpha^2 h}|K|. \tag{4.4}$$

We use Cauchy-Schwartz inequality, the assumption **A**.3 on $\eta$, and (4.4) to have

$$\begin{aligned}\frac{|K|}{2}\mathbb{E}\Big[(u_K^{n+1})^2 - (u_K^n)^2\Big] &\le \Delta t E\Bigg[\sum_{\substack{\sigma\in\mathcal{E}_K\\ \sigma=K|L}} |\sigma|(\vec{v}\cdot n_{K,\sigma})^-\Big\{\frac{\Delta t V}{2\alpha^2 h}\big(f(u_K^n) - f(u_L^n)\big)^2 - \big(f(u_K^n) - f(u_L^n)\big)u_K^n\Big\}\Bigg] \\ &\quad + \Delta t\, c_\eta \frac{|K|}{2}\mathbb{E}\big[(u_K^n)^2\big] \\ &\equiv \mathbf{A} + \Delta t\, c_\eta \frac{|K|}{2}\mathbb{E}\big[(u_K^n)^2\big].\end{aligned} \tag{4.5}$$

To estimate **A**, we use [17, Lemma 4.5] and have: for any $a, b \in \mathbb{R}$

$$b\big(f(b) - f(a)\big) \ge \phi(b) - \phi(a) + \frac{1}{2c_f}\big(f(b) - f(a)\big)^2,$$

where $\phi(a) = \int_0^a s f'(s)\, ds$. Note that $0 \le \phi(a) \le c_f a^2$. Thus using the CFL condition $\frac{\Delta t V}{2\alpha^2 h} \le \frac{1}{2c_f}$, we get

$$\mathbf{A} \le \mathbb{E}\Bigg[\sum_{\substack{\sigma\in\mathcal{E}_K\\ \sigma=K|L}} |\sigma|(\vec{v}\cdot n_{K,\sigma})^-\big(\phi(u_L^n) - \phi(u_K^n)\big)\Bigg]. \tag{4.6}$$

Combining (4.5) and (4.6), we obtain the following inequality after summing over all $K \in \mathcal{T}$

$$\begin{aligned}&\frac{1}{2}\sum_{K\in\mathcal{T}} |K|\mathbb{E}\Big[(u_K^{n+1})^2 - (u_K^n)^2\Big] \\ &\le \frac{\Delta t\, c_\eta}{2}\sum_{K\in\mathcal{T}} |K|\mathbb{E}\big[(u_K^n)^2\big] + \sum_{K\in\mathcal{T}} \Delta t \sum_{\substack{\sigma\in\mathcal{E}_K\\ \sigma=K|L}} |\sigma|(\vec{v}\cdot n_{K,\sigma})^-\mathbb{E}\big[\phi(u_L^n) - \phi(u_K^n)\big] \\ &\equiv \frac{\Delta t\, c_\eta}{2}\sum_{K\in\mathcal{T}} |K|\mathbb{E}\big[(u_K^n)^2\big] + \mathbf{B}.\end{aligned}$$

Since $\operatorname{div}_x \vec{v}(t,x) = 0$ for any $(t,x) \in \Pi_T$, one can show that $\mathbf{B} = 0$, yielding

$$\sum_{K\in\mathcal{T}} |K|\mathbb{E}\big[(u_K^{n+1})^2\big] \le \big(1+\Delta t\, c_\eta\big) \sum_{K\in\mathcal{T}} |K|\mathbb{E}\big[(u_K^n)^2\big] \le \big(1+\Delta t\, c_\eta\big)^{n+1}\mathbb{E}\big[||u_0||_2^2\big].$$

Thus (4.2) holds by mathematical induction. In other words, (4.1) holds as well. As a consequence, we have

$$\|u^h_{\mathcal{T},\Delta t}\|^2_{L^2(\Omega\times\Pi_T)} = \sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}} \Delta t|K|\mathbb{E}\big[(u_K^n)^2\big] \le T\, e^{c_\eta T}\mathbb{E}\big[||u_0||_2^2\big].$$

This completes the proof. □

**Lemma 4.2** (Weak BV estimate)**.** *Suppose $T > 0$, and the assumptions* **A**.1-**A**.4 *be true. Let $\mathcal{T}$ be an admissible mesh with size $h$ in the sense of Definition* 3.1*. Let $\Delta t = \frac{T}{N}$ be the time step for some $N \in \mathbb{N}^*$, satisfying the CFL condition*

$$\Delta t \le \frac{(1-\xi)\alpha^2 h}{c_f V}, \text{ for some } \xi \in (0,1). \tag{4.7}$$

*Let $u_K^n : K \in \mathcal{T},\ n \in \{0,1,\cdots,N-1\}$ be the discrete unknowns as in* (3.6)*. Then the followings hold:*

a) *There exists a positive constant $C$, only depending on $T, u_0, \xi, c_f, c_\eta$ such that*

$$\sum_{K\in\mathcal{T}}\sum_{n=0}^{N-1}\Delta t \sum_{\sigma\in\mathcal{E}_K} |\sigma||\vec{v}.n_{K,\sigma}|\mathbb{E}\Big[\big(f(u_\sigma^n) - f(u_K^n)\big)^2\Big] \le C. \tag{4.8}$$

b) *Let $R > 0$ be such that $h < R$. Define $\mathcal{T}_R = \{K \in \mathcal{T} : K \subset B(0,R)\}$ and $\mathcal{E}^R$ be the set of all interfaces of the mesh $\mathcal{T}_R$. Then there exists a positive constant $C_1$, only depending on $R, d, T, u_0, \xi, c_f, c_\eta$ such that*

$$\sum_{n=0}^{N-1}\Delta t \sum_{\sigma\in\mathcal{E}^R} |\sigma||\vec{v}\cdot n_{K,\sigma}|\mathbb{E}\Big[\big|f(u_\sigma^n) - f(u_K^n)\big|\Big] \le C_1 h^{-\frac{1}{2}}. \tag{4.9}$$

*Proof.* Multiplying (4.3) by $\Delta t\, u_K^n$, taking expectation and summing over $n = 0,1,\cdots,N-1$ and $K \in \mathcal{T}$, we obtain

$$\sum_{K\in\mathcal{T}}\sum_{n=0}^{N-1}|K|\mathbb{E}\Big[\big(u_K^{n+1} - u_K^n\big)u_K^n\Big] + \sum_{K\in\mathcal{T}}\sum_{n=0}^{N-1}\Delta t|\sigma|(\vec{v}\cdot n_{K,\sigma})^- \sum_{\substack{\sigma\in\mathcal{E}_K\\ \sigma=K|L}} \mathbb{E}\Big[\big(f(u_K^n) - f(u_L^n)\big)u_K^n\Big] = 0$$

i.e., $\quad \bar{\mathbf{A}} + \bar{\mathbf{B}} = 0.$

Let us first consider $\bar{\mathbf{A}}$. Using the formula $ab = \frac{1}{2}\big[(a+b)^2 - a^2 - b^2\big]$ and (4.3), we rewrite $\bar{\mathbf{A}}$ as

$$\begin{aligned}\bar{\mathbf{A}} &= \frac{1}{2}\sum_{K\in\mathcal{T}}|K|\mathbb{E}\Big[(u_K^N)^2 - (u_K^0)^2\Big] - \frac{1}{2}\sum_{K\in\mathcal{T}}\sum_{n=0}^{N-1}|K|\mathbb{E}\Big[\int_{t_n}^{t_{n+1}}\int_{\mathbf{E}}\eta^2(u_K^n;z)\,m(dz)\,dt\Big] \\ &\qquad - \sum_{K\in\mathcal{T}}\sum_{n=0}^{N-1}\frac{(\Delta t)^2}{2|K|}\mathbb{E}\Bigg[\Big(\sum_{\substack{\sigma\in\mathcal{E}_K\\ \sigma=K|L}}|\sigma|(\vec{v}\cdot n_{K,\sigma})^-\big(f(u_K^n) - f(u_L^n)\big)\Big)^2\Bigg] \\ &\equiv \bar{\mathbf{A}}_1 + \bar{\mathbf{A}}_2 + \bar{\mathbf{A}}_3.\end{aligned}$$

Thanks to Cauchy-Schwartz inequality, the CFL condition (4.7), the inequality (4.4), and the assumption **A**.3

$$\bar{\mathbf{A}}_3 \ge -\frac{1}{2}\sum_{K\in\mathcal{T}}\sum_{n=0}^{N-1}\Delta t\frac{(1-\xi)}{c_f}\mathbb{E}\Bigg[\sum_{\substack{\sigma\in\mathcal{E}_K\\ \sigma=K|L}}|\sigma|(\vec{v}\cdot n_{K,\sigma})^-\big(f(u_K^n) - f(u_L^n)\big)^2\Bigg],$$

$$\bar{\mathbf{A}}_2 \ge -\frac{1}{2}\sum_{K\in\mathcal{T}}\sum_{n=0}^{N-1}\Delta t|K|c_\eta\mathbb{E}\big[(u_K^n)^2\big].$$

Therefore, by using Lemma 4.1, we arrive at

$$\begin{aligned}\bar{\mathbf{A}} &\ge -\frac{1}{2}T\, c_\eta e^{T\,c_\eta}\mathbb{E}\big[||u_0||_2^2\big] - \frac{1}{2}\mathbb{E}\big[||u_0||_2^2\big] - \frac{(1-\xi)}{2\,c_f}\sum_{K\in\mathcal{T}}\sum_{n=0}^{N-1}\Delta t \sum_{\substack{\sigma\in\mathcal{E}_K\\ \sigma=K|L}}|\sigma|(\vec{v}\cdot n_{K,\sigma})^-\mathbb{E}\Big[\big(f(u_K^n)-f(u_L^n)\big)^2\Big]\\
&\ge -\frac{(1-\xi)}{2\,c_f}\sum_{K\in\mathcal{T}}\sum_{n=0}^{N-1}\Delta t \sum_{\substack{\sigma\in\mathcal{E}_K\\ \sigma=K|L}}|\sigma|(\vec{v}\cdot n_{K,\sigma})^-\mathbb{E}\Big[\big(f(u_K^n)-f(u_L^n)\big)^2\Big] - \frac{1}{2}\tilde{C},\end{aligned}$$

for some constant $\tilde{C}>0$, depending only on $T, c_\eta, u_0$. A similar argumentations (cf. estimation of $\mathbf{A}$) reveal that

$$\bar{\mathbf{B}} \ge \frac{1}{2\,c_f}\sum_{K\in\mathcal{T}}\sum_{n=0}^{N-1}\Delta t \sum_{\substack{\sigma\in\mathcal{E}_K\\ \sigma=K|L}}|\sigma|(\vec{v}\cdot n_{K,\sigma})^-\mathbb{E}\Big[\big(f(u_K^n)-f(u_L^n)\big)^2\Big].$$

Since $\bar{\mathbf{A}}+\bar{\mathbf{B}}=0$, there exists positive constant $C=C(T,u_0,\xi,c_f,c_\eta)>0$ such that

$$\sum_{K\in\mathcal{T}}\sum_{n=0}^{N-1}\Delta t \sum_{\substack{\sigma\in\mathcal{E}_K\\ \sigma=K|L}}|\sigma|(\vec{v}\cdot n_{K,\sigma})^-\mathbb{E}\Big[\big(f(u_K^n)-f(u_L^n)\big)^2\Big] \le C, \tag{4.10}$$

or equivalently (4.8) holds.

Let $\mathcal{T}_R=\{K\in\mathcal{T}: K\subset B(0,R)\}$. Following [6], there exists $C_1=C_1(R,d,T,u_0,\xi,c_f,c_\eta)>0$ such that

$$\sum_{K\in\mathcal{T}_R}\sum_{n=0}^{N-1}\Delta t \sum_{\substack{\sigma\in\mathcal{E}_K\\ \sigma=K|L}}|\sigma|(\vec{v}\cdot n_{K,\sigma})^-\mathbb{E}\Big[\big|f(u_K^n)-f(u_L^n)\big|\Big] \le C_1 h^{-\frac{1}{2}}, \tag{4.11}$$

holds as well. Let $\mathcal{E}^R$ denotes the set of all interfaces of $\mathcal{T}_R$. Then (4.11) is equivalent to (4.9). This completes the proof of the lemma. □

## 5. On entropy inequality for approximate solution

In this section, we establish entropy inequality for finite volume approximate solution. Since we are in stochastic set up, one needs to encounter the Itô calculus, and therefore it is natural to consider a time-continuous approximate solution constructed from $u^h_{\mathcal{T},\Delta t}$.

5.1. **Time-continuous approximate solution.** Since $\mathrm{div}_x\vec{v}(t,x)=0$ for any $(t,x)\in\Pi_T$, the upwind finite volume scheme (3.6) can be rewritten as: for any $K\in\mathcal{T}$, and $n\in\{0,1,\cdots,N-1\}$

$$\begin{cases} u_K^{n+1} = u_K^n + \dfrac{\Delta t}{|K|}\displaystyle\sum_{\sigma\in\mathcal{E}_K}|\sigma|(\vec{v}\cdot n_{K,\sigma})^-\big(f(u_\sigma^n)-f(u_K^n)\big) + \int_{t_n}^{t_{n+1}}\int_{\mathbf{E}}\eta(u_K^n;z)\tilde{N}(dz,dt),\\ u_K^0 = \dfrac{1}{|K|}\displaystyle\int_K u_0(x)\,dx.\end{cases}$$

We define a time-continuous discrete approximation, denoted by $v_K^n(\omega,s)$ on $\Omega\times[t_n,t_{n+1}]$, $n\in\{0,1,\cdots,N-1\}$ and $K\in\mathcal{T}$ from the discrete unknowns $u_K^n$ as

$$v_K^n(\omega,s) = u_K^n + \int_{t_n}^s \sum_{\sigma\in\mathcal{E}_K}|\sigma|(\vec{v}\cdot n_{K,\sigma})^-\frac{f(u_\sigma^n)-f(u_K^n)}{|K|} + \int_{t_n}^s\int_{\mathbf{E}}\eta(u_K^n;z)\tilde{N}(dz,dt). \tag{5.1}$$

Note that,

$$\begin{cases} v_K^n(\omega,t_n) = u_K^n\\ v_K^n(\omega,t_{n+1}) = u_K^{n+1}.\end{cases}$$

We drop $\omega$ and write $v_K^n(\cdot)$ instead of $v_K^n(\omega,\cdot)$. Define a time-continuous approximate solution $v^h_{\mathcal{T},\Delta t}(s,x)$ on $[0,T]\times\mathbb{R}^d$ by

$$v^h_{\mathcal{T},\Delta t}(t,x) = v_K^n(t),\; x\in K,\; t\in[0,T). \tag{5.2}$$

Next, we estimate the $L^2$-error between $u^h_{\mathcal{T},\Delta t}$ and $v^h_{\mathcal{T},\Delta t}$. We have the following proposition.

**Proposition 5.1.** *Let the assumptions of Lemma 4.2 hold and $u^h_{\mathcal{T},\Delta t}$ be the finite volume approximate solution defined by (3.6) and (3.7), and $v^h_{\mathcal{T},\Delta t}$ be the corresponding time-continuous approximate solution prescribed by (5.1)-(5.2). Then there exit two constants $C, C_1 \in \mathbb{R}^*_+$, independent of $h$ and $\Delta t$ such that*

$$\|v^h_{\mathcal{T},\Delta t} - u^h_{\mathcal{T},\Delta t}\|^2_{L^2(\Omega\times\Pi_T)} \le Ch + C_1\Delta t.$$

*Proof.* In view of Lemmas 4.1 - 4.2, and the estimate (4.4) along with (4.7), we have

$$\begin{aligned}
&\|v^h_{\mathcal{T},\Delta t} - u^h_{\mathcal{T},\Delta t}\|^2_{L^2(\Omega\times\Pi_T)} \\
&= \sum_{K\in\mathcal{T}}\sum_{n=0}^{N-1}\int_{t_n}^{t_{n+1}}\int_K \mathbb{E}\Big[\int_{t_n}^s\int_{\mathbf{E}} \eta^2(u_K^n;z)m(dz)\,dt\Big]\,dx\,ds \\
&\quad + \sum_{K\in\mathcal{T}}\sum_{n=0}^{N-1}\int_{t_n}^{t_{n+1}}\int_K \mathbb{E}\Big[\Big(\frac{s-\Delta t}{|K|}\sum_{\sigma\in\mathcal{E}_K}|\sigma|(\vec{v}\cdot n_{K,\sigma})^-\big(f(u^n_\sigma)-f(u^n_K)\big)\Big)^2\Big]\,dx\,ds \\
&\le \sum_{K\in\mathcal{T}}\sum_{n=0}^{N-1}\frac{(\Delta t)^3}{|K|}\frac{V|K|}{\alpha^2 h}\sum_{\sigma\in\mathcal{E}_K}|\sigma|(\vec{v}\cdot n_{K,\sigma})^-\mathbb{E}\Big[\big(f(u^n_\sigma)-f(u^n_K)\big)^2\Big] + c_\eta\Delta t\|u^h_{\mathcal{T},\Delta t}\|^2_{L^2(\Omega\times\Pi_T)} \\
&\le \frac{(\Delta t)^2 V}{\alpha^2 h}\sum_{K\in\mathcal{T}}\sum_{n=0}^{N-1}\Delta t\sum_{\sigma\in\mathcal{E}_K}|\sigma|(\vec{v}\cdot n_{K,\sigma})^-\mathbb{E}\Big[\big(f(u^n_\sigma)-f(u^n_K)\big)^2\Big] + c_\eta\Delta t\|u^h_{\mathcal{T},\Delta t}\|^2_{L^2(\Omega\times\Pi_T)} \\
&\le h\frac{(1-\xi)^2\alpha^2}{c_f^2 V}C + c_\eta\Delta t\|u^h_{\mathcal{T},\Delta t}\|^2_{L^2(\Omega\times\Pi_T)} \le Ch + C_1\Delta t,
\end{aligned}$$

where $C$, $C_1 \in \mathbb{R}^*_+$ are two constants, independent of $h$ and $\Delta t$. This finishes the proof. □

5.2. **Entropy inequalities for the approximate solution.** This subsection is devoted to derive the entropy inequalities for the finite volume approximate solution which will be used to prove the convergence of the numerical scheme and hence the existence of entropy solution of the underlying problem (1.1). To do so, we start with the following proposition related to the entropy inequalities for the discrete unknowns $u^n_K$, $K \in \mathcal{T}$, $n \in \{0, 1, 2, \dots, N-1\}$.

**Proposition 5.2** (Discrete entropy inequalities)**.** *Let the assumptions* **A**.1-**A**.4 *hold, and $T > 0$ be fixed. Consider an admissible mesh $\mathcal{T}$ on $\mathbb{R}^d$ with size $h$ in the sense of Definition 3.1. Let $\Delta t = \frac{T}{N}$ be the time step for some $N \in \mathbb{N}^*$, satisfying*

$$\frac{\Delta t}{h} \to 0 \text{ as } h \to 0.$$

*Then, $\mathbb{P}$-a.s. in $\Omega$, for any $\beta \in \mathcal{A}$ and for any nonnegative test function $\psi \in C_c^\infty([0,T)\times\mathbb{R}^d)$, the following inequality holds:*

$$\begin{aligned}
&-\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\int_K\big(\beta(u^{n+1}_K)-\beta(u^n_K)\big)\psi(t_n,x)\,dx \\
&\quad+\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\int_{t_n}^{t_{n+1}}\int_K F^\beta(u^n_K)\vec{v}(t,x)\cdot\nabla_x\psi(t_n,x)\,dx\,dt \\
&\quad+\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\int_{t_n}^{t_{n+1}}\int_{\mathbf{E}}\int_K\int_0^1 \eta(u^n_K;z)\beta'\big(u^n_K+\lambda\eta(u^n_K;z)\big)\psi(t_n,x)d\lambda\,dx\,\tilde{N}(dz,dt) \\
&\quad+\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\int_{t_n}^{t_{n+1}}\int_{\mathbf{E}}\int_K\int_0^1 (1-\lambda)\eta^2(u^n_K;z)\beta''\big(u^n_K+\lambda\eta(u^n_K;z)\big)\psi(t_n,x)d\lambda\,dx\,m(dz)\,dt \ge R^{h,\Delta t},
\end{aligned}$$

*where $R^{h,\Delta t}$ satisfies the following condition: for any $\mathbb{P}$-measurable set $B$, $\mathbb{E}\Big[\mathbf{1}_B R^{h,\Delta t}\Big] \to 0$ as $h \to 0$.*

*Proof.* Let $T>0$ be fixed and $\mathcal{T}$ be an admissible mesh on $\mathbb{R}^d$ with size $h$ in the sense of Definition 3.1. Let $\Delta t=\frac{T}{N}$ be the time step for some $N\in\mathbb{N}^*$ and $t_n=n\Delta t, n\in\{0,1,\cdots,N\}$. Let $\beta\in\mathcal{A}$. Applying Itô-Lévy formula to $\beta(v_K^n)$, where $v_K^n$ is prescribed by the equation (5.1), we have

$$\begin{aligned}\beta(v_K^n(t_{n+1}))=\beta(v_K^n(t_n))&+\int_{t_n}^{t_{n+1}}\beta'(v_K^n(t))\sum_{\sigma\in\mathcal{E}_K}|\sigma|(\vec{v}\cdot n_{K,\sigma})^-\frac{f(u_\sigma^n)-f(u_K^n)}{|K|}\,dt\\&+\int_{t_n}^{t_{n+1}}\int_{\mathbf{E}}\int_0^1\eta(u_K^n;z)\beta'\big(v_K^n(t)+\lambda\eta(u_K^n;z)\big)d\lambda\,\tilde{N}(dz,dt)\\&+\int_{t_n}^{t_{n+1}}\int_{\mathbf{E}}\int_0^1(1-\lambda)\eta^2(u_K^n;z)\beta''\big(v_K^n(t)+\lambda\eta(u_K^n;z)\big)d\lambda\,m(dz)\,dt.\end{aligned}\tag{5.3}$$

Let $\psi\in C_c^\infty([0,T)\times\mathbb{R}^d)$ be a nonnegative test function. Then there exists $R>h$ such that supp$\psi\subset[0,T)\times B(0,R-h)$. Also define $\mathcal{T}_R=\{K\in\mathcal{T}:K\subset B(0,R)\}$. We multiply the equation (5.3) by $|K|\psi_K^n$ where $\psi_K^n=\frac{1}{|K|}\int_K\psi(t_n,x)\,dx$ and then we sum over all $K\in\mathcal{T}_R$ and $n\in\{0,1,\cdots,N-1\}$. The resulting expression reads to

$$\begin{aligned}&\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\big[\beta(u_K^{n+1})-\beta(u_K^n)\big]|K|\psi_K^n\\&=\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\int_{t_n}^{t_{n+1}}\beta'(v_K^n(t))\sum_{\sigma\in\mathcal{E}_K}|\sigma|(\vec{v}\cdot n_{K,\sigma})^-\big(f(u_\sigma^n)-f(u_K^n)\big)\psi_K^n\,dt\\&+\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\int_{t_n}^{t_{n+1}}\int_{\mathbf{E}}\int_0^1\eta(u_K^n;z)\beta'\big(v_K^n(t)+\lambda\eta(u_K^n;z)\big)|K|\psi_K^n d\lambda\,\tilde{N}(dz,dt)\\&+\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\int_{t_n}^{t_{n+1}}\int_{\mathbf{E}}\int_0^1(1-\lambda)\eta^2(u_K^n;z)\beta''\big(v_K^n(t)+\lambda\eta(u_K^n;z)\big)|K|\psi_K^n d\lambda\,m(dz)\,dt\\&\text{i.e. }\;A^{h,\Delta t}=B^{h,\Delta t}+M^{h,\Delta t}+D^{h,\Delta t}.\end{aligned}\tag{5.4}$$

Following [6], we express $B^{h,\Delta t}$ as follows.

$$B^{h,\Delta t}=B^{h,\Delta t}-B_1^{h,\Delta t}+B_1^{h,\Delta t}-B_2^{h,\Delta t}+B_2^{h,\Delta t},$$

where

$$\begin{aligned}B_1^{h,\Delta t}&=\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\int_{t_n}^{t_{n+1}}\beta'(u_K^n)\sum_{\sigma\in\mathcal{E}_K}|\sigma|(\vec{v}\cdot n_{K,\sigma})^-\big(f(u_\sigma^n)-f(u_K^n)\big)\psi_K^n\,dt,\\B_2^{h,\Delta t}&=\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\int_{t_n}^{t_{n+1}}\sum_{\sigma\in\mathcal{E}_K}|\sigma|(\vec{v}\cdot n_{K,\sigma})^-\big(F^\beta(u_\sigma^n)-F^\beta(u_K^n)\big)\psi_K^n\,dt.\end{aligned}$$

Observe that

$$B_1^{h,\Delta t}-B_2^{h,\Delta t}=\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\Delta t\sum_{\sigma\in\mathcal{E}_K}|\sigma|(\vec{v}\cdot n_{K,\sigma})^-\Big\{\beta'(u_K^n)\big(f(u_\sigma^n)-f(u_K^n)\big)-\big(F^\beta(u_\sigma^n)-F^\beta(u_K^n)\big)\Big\}\psi_K^n.$$

Thanks to nondecreasingness of the functions $f$ and $\beta'$, one has

$$\beta'(u_K^n)\big(f(u_\sigma^n)-f(u_K^n)\big)-\big(F^\beta(u_\sigma^n)-F^\beta(u_K^n)\big)=\int_{u_K^n}^{u_\sigma^n}\big(\beta'(u_K^n)-\beta'(s)\big)f'(s)ds\le 0,$$

and hence $B_1^{h,\Delta t}-B_2^{h,\Delta t}\le 0$.

By the assumption **A**.2, we have $\sum_{\sigma\in\mathcal{E}_K}|\sigma|\vec{v}\cdot n_{K,\sigma}F^\beta(u_K^n)\psi_K^n=0$, and therefore

$$B_2^{h,\Delta t}=-\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\Delta t\sum_{\sigma\in\mathcal{E}_K}|\sigma|(\vec{v}\cdot n_{K,\sigma})F^\beta(u_\sigma^n)\psi_K^n.$$

Let $x_\sigma$ be the center of the edge $\sigma$ and $\psi_\sigma^n$ be the value of $\psi(t_n, x_\sigma)$. Then,

$$\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\Delta t\sum_{\sigma\in\mathcal{E}_K}|\sigma|(\vec{v}\cdot n_{K,\sigma})F^\beta(u_\sigma^n)\psi_\sigma^n = 0.$$

A similar argument (as described in Bauzet et al.[6, Proposition 4]) reveals that

$$B_2^{h,\Delta t} = \sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\int_{t_n}^{t_n+1}\int_K F^\beta(u_K^n)\vec{v}(t,x)\cdot\nabla_x\psi(t_n,x)\,dx\,dt + R_1^{h,\Delta t} + R_2^{h,\Delta t},$$

where

$$R_1^{h,\Delta t} = \sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\Delta t\sum_{\sigma\in\mathcal{E}_K}|\sigma|(\vec{v}\cdot n_{K,\sigma})\big[F^\beta(u_K^n) - F^\beta(u_\sigma^n)\big]\big(\psi_K^n - \psi_\sigma^n\big),$$
$$R_2^{h,\Delta t} = \sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\Delta t\sum_{\sigma\in\mathcal{E}_K}\Big\{|\sigma|(\vec{v}\cdot n_{K,\sigma})\psi_\sigma^n - \int_\sigma(\vec{v}\cdot n_{K,\sigma})\psi(t_n,x)d\nu(x)\Big\}F^\beta(u_K^n).$$

Combining all these, we obtain that

$$B^{h,\,\Delta t} \le \sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\int_{t_n}^{t_n+1}\int_K F^\beta(u_K^n)\vec{v}(t,x)\cdot\nabla_x\psi(t_n,x)\,dx\,dt$$
$$+ B^{h,\Delta t} - B_1^{h,\Delta t} + R_1^{h,\Delta t} + R_2^{h,\Delta t}. \tag{5.5}$$

Next we consider the term $M^{h,\Delta t}$. It can be decompose as follows:

$$M^{h,\Delta t} = M^{h,\Delta t} - M_1^{h,\Delta t} + M_1^{h,\Delta t}, \tag{5.6}$$

where

$$M_1^{h,\Delta t} = \sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\int_{t_n}^{t_{n+1}}\int_{\mathbf{E}}\int_K\int_0^1 \eta(u_K^n;z)\beta'\big(u_K^n + \lambda\eta(u_K^n;z)\big)\psi(t_n,x)\,d\lambda\,dx\,\tilde{N}(dz,dt).$$

Similarly, we rewrite $D^{h,\Delta t}$ as

$$D^{h,\Delta t} = D^{h,\Delta t} - D_1^{h,\Delta t} + D_1^{h,\Delta t}, \tag{5.7}$$

where

$$D_1^{h,\Delta t}$$
$$= \sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\int_{t_n}^{t_{n+1}}\int_{\mathbf{E}}\int_K\int_0^1(1-\lambda)\eta^2(u_K^n;z)\beta''\big(u_K^n + \lambda\eta(u_K^n;z)\big)\psi(t_n,x)\,d\lambda\,dx\,m(dz)\,dt.$$

In view of (5.5), (5.6), (5.7), and (5.4) we have

$$-\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\int_K\big(\beta(u_K^{n+1}) - \beta(u_K^n)\big)\psi(t_n,x)\,dx$$
$$+\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\int_{t_n}^{t_{n+1}}\int_K F^\beta(u_K^n)\vec{v}(t,x)\cdot\nabla_x\psi(t_n,x)\,dx\,dt$$
$$+\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\int_{t_n}^{t_{n+1}}\int_{\mathbf{E}}\int_K\int_0^1\eta(u_K^n;z)\beta'\big(u_K^n + \lambda\eta(u_K^n;z)\big)\psi(t_n,x)\,d\lambda\,dx\,\tilde{N}(dz,dt)$$
$$+\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\int_{t_n}^{t_{n+1}}\int_{\mathbf{E}}\int_K\int_0^1(1-\lambda)\eta^2(u_K^n;z)\beta''\big(u_K^n + \lambda\eta(u_K^n;z)\big)\psi(t_n,x)d\lambda\,dx\,m(dz)\,dt$$
$$\ge \big(B_1^{h,\Delta t} - B^{h,\Delta t}\big) - R_1^{h,\Delta t} - R_2^{h,\Delta t} + \big(M_1^{h,\Delta t} - M^{h,\Delta t}\big) + \big(D_1^{h,\Delta t} - D^{h,\Delta t}\big) \equiv R^{h,\Delta t}.$$

To complete the proof of the proposition, it is only required to show: for any $\mathbb{P}$-measurable set $B$, $\mathbb{E}\Big[\mathbf{1}_B R^{h,\Delta t}\Big] \to 0$ as $h \to 0$. Now we assume that $\dfrac{\Delta t}{h} \to 0$ as $h \to 0$. In this manner, with out loss of generality, we may assume that the CFL condition

$$\Delta t \le \frac{(1-\xi)h}{c_f V}\alpha^2$$

holds for some $\xi \in (0,1)$ and hence the estimates given in Lemmas 4.1 and 4.2 hold as well. To proceed further, we will separately show the convergence of $\mathbb{E}\Big[\mathbf{1}_B\big(B_1^{h,\Delta t} - B^{h,\Delta t}\big)\Big]$, $\mathbb{E}\Big[\mathbf{1}_B\big(M_1^{h,\Delta t} - M^{h,\Delta t}\big)\Big]$, $\mathbb{E}\Big[\mathbf{1}_B\big(D_1^{h,\Delta t} - D^{h,\Delta t}\big)\Big]$, $\mathbb{E}\big[\mathbf{1}_B\, R_1^{h,\Delta t}\big]$, and $\mathbb{E}\big[\mathbf{1}_B\, R_2^{h,\Delta t}\big]$.

1. **Study of $\mathbb{E}\big[\mathbf{1}_B\big(B_1^{h,\Delta t} - B^{h,\Delta t}\big)\big]$:** Let $B$ be any $\mathbb{P}$-measurable set. Then, by using (5.1) we get

$$\begin{aligned}
&\Big|\mathbb{E}\big[\mathbf{1}_B\big(B_1^{h,\Delta t} - B^{h,\Delta t}\big)\big]\Big| \\
&= \Big|\mathbb{E}\Big[\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\int_{t_n}^{t_{n+1}} \mathbf{1}_B\beta''(\xi_K^n)\big(v_K^n(s)) - u_K^n)\big)\sum_{\sigma\in\mathcal{E}_K}|\sigma|(\vec{v}\cdot n_{K,\sigma})^-\big(f(u_\sigma^n) - f(u_K^n)\big)\psi_K^n\,ds\Big]\Big| \\
&\le \Big|\mathbb{E}\Big[\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\mathbf{1}_B\int_{t_n}^{t_{n+1}} \beta''(\xi_K^n)\frac{s-t_n}{|K|}\Big(\sum_{\sigma\in\mathcal{E}_K}|\sigma|(\vec{v}\cdot n_{K,\sigma})^-\big(f(u_\sigma^n) - f(u_K^n)\big)\Big)^2\psi_K^n\,ds\Big]\Big| \\
&+ \Big|\mathbb{E}\Big[\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\mathbf{1}_B\int_{t_n}^{t_{n+1}} \beta''(\xi_K^n)\int_{t_n}^{s}\int_{\mathbf{E}}\eta(u_K^n;z)\tilde{N}(dz,dr)\sum_{\sigma\in\mathcal{E}_K}|\sigma|(\vec{v}\cdot n_{K,\sigma})^-\big(f(u_\sigma^n) - f(u_K^n)\big)\psi_K^n\,ds\Big]\Big| \\
&\equiv T_1^{h,\Delta t} + T_2^{h,\Delta t}.
\end{aligned}$$

Following computations as in [6, estimation for $\tilde{T}_1^{h,k}$], it can be shown that

$$T_1^{h,\Delta t} \le \frac{\Delta t}{h}||\beta''||_{L^\infty}||\psi||_{L^\infty}\frac{V}{\alpha^2}C.$$

Next, we move on to estimate $T_2^{h,\Delta t}$. Note that

$$\begin{aligned}
\big|T_2^{h,\Delta t}\big|^2 &\le \mathbb{E}\Big[\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\int_{t_n}^{t_{n+1}}\Big(\mathbf{1}_B\beta''(\xi_K^n)\psi_K^n\sum_{\sigma\in\mathcal{E}_K}|\sigma|(\vec{v}\cdot n_{K,\sigma})^-\big(f(u_\sigma^n) - f(u_K^n)\big)\Big)^2\,ds\Big] \\
&\qquad \times \mathbb{E}\Big[\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\int_{t_n}^{t_{n+1}}\Big(\int_{t_n}^{s}\int_{\mathbf{E}}\eta(u_K^n;z)\tilde{N}(dz,dr)\Big)^2\,ds\Big] \\
&\le ||\beta''||_{L^\infty}^2||\psi||_{L^\infty}^2\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\Delta t\Big(\sum_{\sigma\in\mathcal{E}_K}|\sigma|(\vec{v}\cdot n_{K,\sigma})^-\Big)\sum_{\sigma\in\mathcal{E}_K}|\sigma|(v\cdot n_{K,\sigma})^-\mathbb{E}\Big[\big(f(u_\sigma^n) - f(u_K^n)\big)^2\Big] \\
&\qquad \times \mathbb{E}\Big[\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\int_{t_n}^{t_{n+1}}\int_{t_n}^{s}\int_{\mathbf{E}}\eta^2(u_K^n;z)\,m(dz)\,dr\,ds\Big] \\
&\le \frac{V}{\alpha^2 h}||\beta''||_{L^\infty}^2||\psi||_{L^\infty}^2\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\Delta t\sum_{\sigma\in\mathcal{E}_K}|\sigma|(\vec{v}\cdot n_{K,\sigma})^-\mathbb{E}\Big[\big(f(u_\sigma^n) - f(u_K^n)\big)^2\Big] \\
&\qquad \times c_\eta\Delta t\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}|K|\Delta t\,\mathbb{E}\big[(u_K^n)^2\big] \\
&\le \frac{\Delta t}{h}\frac{c_\eta V}{\alpha^2}||\beta''||_{L^\infty}^2||\psi||_{L^\infty}^2\|u_{\mathcal{T},\Delta}\|_{L^2(\Omega\times\Pi_T)}^2 C.
\end{aligned}$$

In the above, the first inequality follows from Cauchy-Schwartz inequality, second inequality follows from Cauchy-Schwartz inequality and Itô-Lévy isometry. In view of (4.4) and the assumption **A**.3 on $\eta$, the third inequality holds true. In the last inequality, we have used the constant $C$ given by Lemma 4.2. Here we note that the assumption $\dfrac{\Delta t}{h} \to 0$ as $h \to 0$ is crucial. Passing to the limit as $h \to 0$, we conclude that $\mathbb{E}\Big[\mathbf{1}_B\big(B_1^{h,\Delta t} - B^{h,\Delta t}\big)\Big] \to 0$.

2. **Study of $\mathbb{E}\big[\mathbf{1}_B\big(M_1^{h,\Delta t} - M^{h,\Delta t}\big)\big]$:** In view of triangle inequality, one has

$$\Big|\mathbb{E}\Big[\mathbf{1}_B\big(M_1^{h,\Delta t} - M^{h,\Delta t}\big)\Big]\Big| \le \mathcal{M}_1^{h,\Delta t} + \mathcal{M}_2^{h,\Delta t},$$

where

$$\mathcal{M}_1^{h,\Delta t} := \Bigg|\mathbb{E}\Big[\mathbf{1}_B \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}_R} \int_{t_n}^{t_{n+1}} \int_{\mathbf{E}} \int_K \int_0^1 \eta(u_K^n; z)\Big\{\beta'\big(u_K^n + \lambda\eta(u_K^n; z)\big) - \beta'\big(v_K^n(t) + \lambda\eta(u_K^n; z)\big)\Big\} \times \big(\psi(t_n, x) - \psi(t, x)\big) d\lambda\, dx\, \tilde{N}(dz, dt)\Big]\Bigg|$$

$$\mathcal{M}_2^{h,\Delta t} := \Bigg|\mathbb{E}\Big[\mathbf{1}_B \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}_R} \int_{t_n}^{t_{n+1}} \int_{\mathbf{E}} \int_K \int_0^1 \eta(u_K^n; z)\Big\{\beta'\big(u_K^n + \lambda\eta(u_K^n; z)\big) - \beta'\big(v_K^n(t) + \lambda\eta(u_K^n; z)\big)\Big\} \times \psi(t, x) d\lambda\, dx\, \tilde{N}(dz, dt)\Big]\Bigg|.$$

Let us turn our focus on the term $\mathcal{M}_1^{h,\Delta t}$. Note that $\operatorname{supp}\psi \subset B(0, R-h) \times [0, T)$ for some $R > h$. Using Cauchy-Schwartz inequality along with the assumptions **A**.1-**A**.4, and Itô-Lévy isometry, we get

$$\begin{aligned}
\Big|\mathcal{M}_1^{h,\Delta t}\Big|^2 &\le \Bigg(\sum_{n=0}^{N-1} \Bigg\{\sum_{K \in \mathcal{T}_R} \int_K \mathbb{E}\Big[\Big(\int_{t_n}^{t_{n+1}} \int_{\mathbf{E}} \int_0^1 \big(\beta'\big(u_K^n + \lambda\eta(u_K^n; z)\big) - \beta'\big(v_K^n(t) + \lambda\eta(u_K^n; z)\big)\big) \\
&\qquad \times \eta(u_K^n; z)\big(\psi(t_n, x) - \psi(t, x)\big) d\lambda \tilde{N}(dz, dt)\Big)^2\Big] dx\Bigg\}^{\frac{1}{2}}\Bigg)^2 |B(0, R)| \\
&\le 2||\beta'||_{L^\infty}^2 |B(0, R)| \Bigg(\sum_{n=0}^{N-1} \Big\{\sum_{K \in \mathcal{T}_R} \int_K \mathbb{E}\Big[\int_{t_n}^{t_{n+1}} \int_{\mathbf{E}} \eta^2(u_K^n; z)\big(\psi(t_n, x) - \psi(t, x)\big)^2 m(dz)\, dt\Big] dx\Big\}^{\frac{1}{2}}\Bigg)^2 \\
&\le 2\Delta t ||\beta'||_{L^\infty}^2 |B(0, R)|\, ||\partial_t \psi||_{L^\infty}^2 c_\eta \Big(\sum_{n=0}^{N-1} \Delta t \Big\{\sum_{K \in \mathcal{T}_R} |K| \mathbb{E}\big[(u_K^n)^2\big]\Big\}^{\frac{1}{2}}\Big)^2 \\
&\le (\Delta t) C\big(\beta', \partial_t\psi, c_\eta, R\big) T \|u_{\mathcal{T},\Delta t}^h\|_{L^\infty(0,T;L^2(\Omega\times\mathbb{R}^d))}^2 \\
&\le h C\big(\xi, \alpha, c_f, V, \beta', \partial_t\psi, c_\eta, R\big) T \|u_{\mathcal{T},\Delta t}^h\|_{L^\infty(0,T;L^2(\Omega\times\mathbb{R}^d))}^2 \text{ (by (4.7))}.
\end{aligned}$$

Thus, we see that

$$\mathcal{M}_1^{h,\Delta t} \longrightarrow 0 \text{ as } h \to 0.$$

Now, we estimate $\mathcal{M}_2^{h,\Delta t}$. Here we note that the boundedness of $\eta$ i.e. $|\eta(u, z)| \le Ch_1(z)$ for any $u \in \mathbb{R}$ and $z \in \mathbf{E}$ is crucial. In view of the Cauchy-Schwartz inequality and Itô-Lévy isometry, we obtain

$$\begin{aligned}
\Big|\mathcal{M}_2^{h,\Delta t}\Big|^2 &\le \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}_R} \int_K \mathbb{E}\Big[\int_{t_n}^{t_{n+1}} \int_{\mathbf{E}} \int_0^1 \Big\{\beta'\big(u_K^n + \lambda\eta(u_K^n; z)\big) - \beta'\big(v_K^n(t) + \lambda\eta(u_K^n; z)\big)\Big\}^2 \\
&\qquad \times \eta^2(u_K^n; z)\psi^2(t, x)\, d\lambda\, m(dz)\, dt\Big] dx |B(0, R)| \\
&\le C(R)||\beta''||_\infty^2 ||\psi||_\infty^2 \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}_R} \int_K \mathbb{E}\Big[\int_{t_n}^{t_{n+1}} \int_{\mathbf{E}} |u_K^n - v_K^n(t)|^2 \eta^2(u_K^n; z)\, m(dz)\, dt\Big] dx \\
&\quad \text{(by the boundedness of } \eta) \\
&\le C\big(R, \beta'', \psi, \big) \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}_R} \int_K \int_{t_n}^{t_{n+1}} \mathbb{E}\big[(u_K^n - v_K^n(t))^2\big]\, dt\, dx \Big(\int_{\mathbf{E}} h_1^2(z) m(dz)\Big)
\end{aligned}$$

$$= C\big(R,\beta'',\psi,\big)c_\eta \|u^h_{\mathcal{T},\Delta t} - v^h_{\mathcal{T},\Delta t}\|^2_{L^2(\Omega\times\Pi_T)} \longrightarrow 0 \,\text{as}\, h\to 0,$$

where in the last line, we have invoked Proposition 5.1 and the CFL condition (4.7). Hence

$$\mathbb{E}\Big[\mathbf{1}_B\big(M_1^{h,\Delta t} - M^{h,\Delta t}\big)\Big] \to 0 \quad \text{as} \quad h\to 0.$$

3. **Study of $\mathbb{E}\big[\mathbf{1}_B\big(D_1^{h,\Delta t} - D^{h,\Delta t}\big)\big]$:** Observe that

$$\begin{aligned}
&\Big|\mathbb{E}\big[\mathbf{1}_B\big(D_1^{h,\Delta t} - D^{h,\Delta t}\big)\big]\Big| \\
&\le ||\beta'''||_\infty\, ||\psi||_\infty \mathbb{E}\Big[\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\int_{t_n}^{t_{n+1}}\int_{\mathbf{E}}\int_{B(0,R)} \eta^2(u_K^n;z)\big|u_K^n - v_K^n(t)\big|dx\, m(dz)\, dt\Big] \\
&\le C(\beta''',\psi)\mathbb{E}\Big[\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\int_{t_n}^{t_{n+1}}\int_{B(0,R)}\big|u_K^n - v_K^n(t)\big|dx\,dt\Big]\Big(\int_{\mathbf{E}} h_1^2(z)\, m(dz)\Big) \\
&= C(\beta'',\psi,c_\eta)\|u^h_{\mathcal{T},\Delta t} - v^h_{\mathcal{T},\Delta t}\|_{L^1\left(\Omega\times B(0,R)\times[0,T)\right)} \longrightarrow 0 \text{ as } h\to 0.
\end{aligned}$$

In the above, the second inequality follows from the boundedness condition on $\eta$, and the last line holds because of Proposition 5.1 and the CFL condition (4.7).

4. **Study of $\mathbb{E}\big[\mathbf{1}_B\, R_1^{h,\Delta t}\big]$ and $\mathbb{E}\big[\mathbf{1}_B\, R_2^{h,\Delta t}\big]$:** Following computations as in Bauzet et al. [6, Proposition 4] we infer that

$$\Big|\mathbb{E}\big[\mathbf{1}_B\, R_1^{h,\Delta t}\big]\Big| \le C||\psi_x||_\infty ||\beta'||_\infty\sqrt{h};\; \Big|\mathbb{E}\big[\mathbf{1}_B\, R_2^{h,\Delta t}\big]\Big| \le ||\beta'||_\infty \frac{V c_f}{\alpha^2}\bar{\varepsilon}(h)\|u^h_{\mathcal{T},\Delta t}\|_{L^1\left(\Omega\times B(0,R)\times[0,T)\right)},$$

where $\bar{\varepsilon}(r)\to 0$ as $r\to 0$.

We now combine all the above estimates to conclude: for any $\mathbb{P}$-measurable set $B$,

$$\mathbb{E}\Big[\mathbf{1}_B R^{h,\Delta t}\Big] \to 0 \text{ as } h\to 0.$$

This completes the proof of the proposition. □

To prove convergence of the proposed scheme and hence existence of entropy solution for (1.1), one also needs a continuous entropy inequality on the discrete solutions. Regarding this, we have the following proposition which essentially gives the entropy inequality for the finite volume approximate solution $u^h_{\mathcal{T},\Delta t}$.

**Proposition 5.3** (Entropy inequality for approximate solution)**.** *Let the assumptions* **A**.1-**A**.4 *hold, and $T>0$ be fixed. Let $\mathcal{T}$ be an admissible mesh on $\mathbb{R}^d$ with size $h$ in the sense of Definition* 3.1*. Let $\Delta t = \frac{T}{N}$ be the time step for some $N\in\mathbb{N}^*$ satisfying*

$$\frac{\Delta t}{h} \to 0 \text{ as } h\to 0.$$

*Then, $\mathbb{P}$-a.s. in $\Omega$, for any $\beta\in\mathcal{A}$ and for any nonnegative test function $\psi\in C_c^\infty([0,T)\times\mathbb{R}^d)$, the following inequality holds:*

$$\begin{aligned}
&\int_{\mathbb{R}^d}\beta(u_0(x))\psi(0,x)\,dx + \int_{\Pi_T}\Big\{\beta(u^h_{\mathcal{T},\,\Delta t})\partial_t\psi(t,x) + F^\beta(u^h_{\mathcal{T},\,\Delta t})\vec{v}(t,x)\cdot\nabla_x\psi(t,x)\Big\}\,dt\,dx \\
&+ \int_{\mathbb{R}^d}\int_0^T\int_{\mathbf{E}}\int_0^1 \eta(u^h_{\mathcal{T},\,\Delta t};z)\beta'\big(u^h_{\mathcal{T},\,\Delta t} + \lambda\eta(u^h_{\mathcal{T},\,\Delta t};z)\big)\psi(t,x)\,d\lambda\,\tilde{N}(dz,dt)\,dx \\
&+ \int_{\mathbb{R}^d}\int_0^T\int_{\mathbf{E}}\int_0^1 (1-\lambda)\eta^2(u^h_{\mathcal{T},\,\Delta t};z)\beta''\big(u^h_{\mathcal{T},\,\Delta t} + \lambda\eta(u^h_{\mathcal{T},\,\Delta t};z)\big)\psi(t,x)d\lambda\, m(dz)\,dt\,dx \\
&\ge \mathcal{R}^{h,\,\Delta t},
\end{aligned} \tag{5.8}$$

*where for any $\mathbb{P}$-measurable set $B$, $\mathbb{E}\Big[\mathbf{1}_B\mathcal{R}^{h,\,\Delta t}\Big] \to 0$ as $h\to 0$.*

*Proof.* Let the assumptions of the proposition hold true. Since $\dfrac{\Delta t}{h} \to 0$ as $h \to 0$, we may assume that the CFL condition

$$\Delta t \le \frac{(1-\xi)h}{c_f V}\alpha^2$$

holds for some $\xi \in (0,1)$ and hence the estimates given in Lemmas 4.1 and 4.2 and Proposition 5.2 hold as well. Let $\psi \in C_c^\infty([0,T) \times \mathbb{R}^d)$ be a nonnegative test function. Then there exists $R > h$ such that supp $\psi \subset [0,T) \times B(0, R-h)$. Also define $\mathcal{T}_R = \{K \in \mathcal{T} : K \subset B(0,R)\}$.

Note that $\psi(t_N, x) = 0$ for any $x \in \mathbb{R}^d$. Using the summation by parts formula,

$$\sum_{n=1}^{N} a_n\big(b_n - b_{n-1}\big) = a_N b_N - a_0 b_0 - \sum_{n=0}^{N-1} b_n\big(a_{n+1} - a_n\big)$$

one has

$$\begin{aligned} &-\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\int_K \big(\beta(u_K^{n+1}) - \beta(u_K^n)\big)\psi(t_n,x)\,dx \\ &\quad= \sum_{K\in\mathcal{T}_R}\int_K \beta(u_K^0)\psi(0,x)\,dx + \int_{\Delta t}^T\int_{\mathbb{R}^d}\beta(u_{\mathcal{T},\Delta t}^h)\partial_t\psi(t-\Delta t,x)\,dx\,dt. \end{aligned} \tag{5.9}$$

Let $R^{h,\Delta t}$ be the quantity as in Proposition 5.2. Define

$$\begin{aligned} \mathcal{R}^{h,\Delta t} =& R^{h,\Delta t} + \bigg\{\int_{\mathbb{R}^d}\beta(u_0(x))\psi(0,x)\,dx - \sum_{K\in\mathcal{T}_R}\int_K \beta(u_K^0)\psi(0,x)\,dx\bigg\} \\ &+\bigg\{\int_{\Pi_T}\beta(u_{\mathcal{T},\Delta t})\partial_t\psi(t,x)\,dt\,dx - \int_{\Delta t}^T\int_{\mathbb{R}^d}\beta(u_{\mathcal{T},\Delta t}^h)\partial_t\psi(t-\Delta t,x)\,dx\,dt\bigg\} \\ &+\bigg\{\int_{\Pi_T}F^\beta(u_{\mathcal{T},\Delta t}^h)\vec{v}(t,x)\cdot\nabla_x\psi(t,x)\,dt\,dx - \sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\int_{t_n}^{t_{n+1}}\int_K F^\beta(u_K^n)\vec{v}(t,x)\cdot\nabla_x\psi(t_n,x)\,dx\,dt\bigg\} \\ &+\bigg\{\int_{\mathbb{R}^d}\int_0^T\int_{\mathbf{E}}\int_0^1 \eta(u_{\mathcal{T},\Delta t}^h;z)\beta'\big(u_{\mathcal{T},\Delta t}^h + \lambda\eta(u_{\mathcal{T},\Delta t}^h;z)\big)\psi(t,x)\,d\lambda\,\tilde{N}(dz,dt)\,dx \\ &\quad-\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\int_{t_n}^{t_{n+1}}\int_{\mathbf{E}}\int_K\int_0^1 \eta(u_K^n;z)\beta'\big(u_K^n + \lambda\eta(u_K^n;z)\big)\psi(t_n,x)\,d\lambda\,dx\,\tilde{N}(dz,dt)\bigg\} \\ &+\bigg\{\int_{\mathbb{R}^d}\int_0^T\int_{\mathbf{E}}\int_0^1 (1-\lambda)\eta^2(u_{\mathcal{T},\Delta t}^h;z)\beta''\big(u_{\mathcal{T},\Delta t}^h + \lambda\eta(u_{\mathcal{T},\Delta t}^h;z)\big)\psi(t,x)\,d\lambda\,m(dz)\,dt\,dx \\ &\quad-\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\int_{t_n}^{t_{n+1}}\int_{\mathbf{E}}\int_K\int_0^1 (1-\lambda)\eta^2(u_K^n;z)\beta''\big(u_K^n + \lambda\eta(u_K^n;z)\big)\psi(t_n,x)\,d\lambda\,dx\,m(dz)\,dt\bigg\} \\ &\equiv R^{h,\Delta t} + \mathcal{I}^{h,\Delta t} + \mathcal{T}^{h,\Delta t} + \mathcal{D}^{h,\Delta t} + \mathcal{M}^{h,\Delta t} + \mathcal{A}^{h,\Delta t}. \end{aligned}$$

In view of Proposition 5.2 and the definition of $\mathcal{R}^{h,\Delta t}$ along with (5.9), we note that (5.8) holds.

In order to prove the proposition, it remains to prove the convergence of the following quantities: $\mathbb{E}\Big[\mathbf{1}_B R^{h,\Delta t}\Big]$, $\mathbb{E}\Big[\mathbf{1}_B\mathcal{I}^{h,\Delta t}\Big]$, $\mathbb{E}\Big[\mathbf{1}_B\mathcal{T}^{h,\Delta t}\Big]$, $\mathbb{E}\Big[\mathbf{1}_B\mathcal{D}^{h,\Delta t}\Big]$, $\mathbb{E}\Big[\mathbf{1}_B\mathcal{M}^{h,\Delta t}\Big]$ and $\mathbb{E}\Big[\mathbf{1}_B\mathcal{A}^{h,\Delta t}\Big]$, where $B$ is any $\mathbb{P}$-measurable subset of $\Omega$.

1. **Convergence of $\mathbb{E}\big[\mathbf{1}_B\,\mathcal{I}^{h,\Delta t}\big]$:** Note that, due to Lebesgue differentiation theorem, for almost all $x \in K$, $\big|u_0(x) - u_K^0\big| \longrightarrow 0$ as diameter of $K$ tends to zero (i.e., $h \to 0$). Now

$$\Big|\mathbb{E}\big[\mathbf{1}_B\,\mathcal{I}^{h,\Delta t}\big]\Big| = \Big|\mathbb{E}\Big[\mathbf{1}_B\sum_{K\in\mathcal{T}_R}\int_K \big(\beta(u_0(x)) - \beta(u_K^0)\big)\psi(x,0)\,dx\Big]\Big|$$

$$\leq ||\beta'||_\infty \mathbb{E}\Big[\sum_{K\in\mathcal{T}_R}\int_K \big|u_0(x)-u_K^0\big|\psi(x,0)\,dx\Big],$$

and hence $\mathbb{E}\big[\mathbf{1}_B\,\mathcal{I}^{h,\Delta t}\big] \longrightarrow 0$ as $h\to 0$.

2. **Convergence of $\mathbb{E}\big[\mathbf{1}_B\,\mathcal{T}^{h,\Delta t}\big]$:** In view of Lemma 4.2 and the CFL condition (4.7), we obtain

$$\begin{aligned}
&\Big|\mathbb{E}\big[\mathbf{1}_B\,\mathcal{T}^{h,\Delta t}\big]\Big| \\
=&\Big|\mathbb{E}\Big[\mathbf{1}_B\int_0^{\Delta t}\int_{\mathbb{R}^d}\beta(u^h_{\mathcal{T},\Delta t})\partial_t\psi(x,t)\,dx\,dt\Big]+\mathbb{E}\Big[\mathbf{1}_B\int_{\Delta t}^T\int_{\mathbb{R}^d}\beta(u^h_{\mathcal{T},\Delta t})\big(\partial_t\psi(t,x)-\partial_t\psi(t-\Delta t,x)\big)\,dx\,dt\Big]\Big| \\
\leq& ||\beta'||_\infty\,\Delta t\Big(||\partial_t\psi||_\infty\|u^h_{\mathcal{T},\Delta t}\|_{L^\infty\left(0,T;L^1(\Omega\times B(0,R))\right)}+||\partial_{tt}\psi||_\infty\,\|u^h_{\mathcal{T},\Delta t}\|_{L^1\left(\Omega\times B(0,R)\times[0,T)\right)}\Big),
\end{aligned}$$

and hence $\mathbb{E}\big[\mathbf{1}_B\,\mathcal{T}^{h,\Delta t}\big] \longrightarrow 0$ as $h\to 0$.

3. **Convergence of $\mathbb{E}\big[\mathbf{1}_B\,\mathcal{D}^{h,\Delta t}\big]$:** In view of the assumption **A**.3, one has

$$\begin{aligned}
\Big|\mathbb{E}\big[\mathbf{1}_B\,\mathcal{D}^{h,\Delta t}\big]\Big| &\leq V||\nabla_x\partial_t\psi||_\infty\Delta t\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\int_K\int_{t_n}^{t_{n+1}}\mathbb{E}\Big[\big|F^\beta(u_K^n)\big|\Big]\,dx\,dt \\
&\leq V||\nabla_x\partial_t\psi||_\infty\Delta t||\beta'||_\infty c_f\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\int_K\int_{t_n}^{t_{n+1}}\mathbb{E}\big[|u_K^n|\big]\,dx\,dt \\
&\leq V||\nabla_x\partial_t\psi||_\infty\Delta t||\beta'||_\infty c_f\|u^h_{\mathcal{T},\Delta t}\|_{L^1\left(\Omega\times B(0,R)\times[0,T)\right)}\longrightarrow 0 \text{ as } h\to 0.
\end{aligned}$$

Thanks to Lemma 4.1 and the CFL condition (4.7), one can pass to the limit in the last line as well.

4. **Convergence of $\mathbb{E}\big[\mathbf{1}_B\,\mathcal{M}^{h,\Delta t}\big]$:** By using Cauchy-Schwartz inequality, Itô-Lévy isometry, the CFL condition (4.7) and Lemma 4.1, we obtain

$$\begin{aligned}
\Big|E\big[\mathbf{1}_B\,\mathcal{M}^{h,\Delta t}\big]\Big|^2 \leq& |B(0,R)|\Bigg(\sum_{n=0}^{N-1}\Bigg\{\mathbb{E}\Big[\sum_{K\in\mathcal{T}_R}\int_{t_n}^{t_{n+1}}\int_{\mathbf{E}}\int_K\int_0^1\eta^2(u_K^n;z)\beta'^2\big(u_K^n+\lambda\eta(u_K^n;z)\big) \\
&\times\big(\psi(t,x)-\psi(t_n,x)\big)^2\,d\lambda\,dx\,m(dz)\,dt\Big]\Bigg\}^{\frac12}\Bigg)^2 \\
\leq& C(R,\psi,c_\eta)\Delta t\Bigg(\sum_{n=0}^{N-1}\Delta t\Big(\sum_{K\in\mathcal{T}_R}|K|\mathbb{E}\big[(u_K^n)^2\big]\Big)^{\frac12}\Bigg)^2 \\
\leq& C(R,\psi,c_\eta,T)\Delta t\|u^h_{\mathcal{T},\Delta t}\|^2_{L^\infty(0,T;L^2(\Omega\times\mathbb{R}^d))}\longrightarrow 0 \text{ as } h\to 0.
\end{aligned}$$

5. **Convergence of $\mathbb{E}\big[\mathbf{1}_B\,\mathcal{A}^{h,\Delta t}\big]$:** Note that

$$\begin{aligned}
\mathcal{A}^{h,\Delta t}=&\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\int_{t_n}^{t_{n+1}}\int_{\mathbf{E}}\int_K\int_0^1(1-\lambda)\eta^2(u_K^n;z)\beta''\big(u_K^n+\lambda\eta(u_K^n;z)\big) \\
&\times\Big\{\psi(t,x)-\psi(t_n,x)\Big\}\,d\lambda\,dx\,m(dz)\,dt.
\end{aligned}$$

Therefore, by (4.7) and Lemma 4.1, we obtain

$$\begin{aligned}
\Big|\mathbb{E}\big[\mathbf{1}_B\,\mathcal{A}^{h,\Delta t}\big]\Big| &\leq ||\beta''||_\infty\Delta t||\partial_t\psi||_\infty\mathbb{E}\Big[\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\int_{t_n}^{t_{n+1}}\int_{\mathbf{E}}\int_K\eta^2(u_K^n;z)\,dx\,m(dz)\,dt\Big] \\
&\leq C(\beta,\psi,c_\eta)\Delta t\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\int_{t_n}^{t_{n+1}}\int_K\mathbb{E}\big[(u_K^n)^2\big]\,dx\,dt\longrightarrow 0 \text{ as } h\to 0.
\end{aligned}$$

6. **Convergence of $\mathbb{E}\big[\mathbf{1}_B\, R^{h,\Delta t}\big]$:** Thanks to Proposition 5.2, we have seen that

$$\mathbb{E}\big[\mathbf{1}_B\, R^{h,\Delta t}\big] \longrightarrow 0 \text{ as } h \to 0,$$

for any $\mathbb{P}$-measurable set $B$. □

## 6. Proof of the main theorem

In this section, we establish the convergence of the scheme and hence existence of entropy solution to the underlying problem (1.1). Note that *a-priori* estimates on $u^h_{\mathcal{T},\Delta t}(x,t)$ given by Lemma 4.1 only guarantee weak compactness of the family $\{u^h_{\mathcal{T},\Delta t}\}_{h>0}$, which is inadequate in view of the nonlinearities in the equation. The concept of Young measure theory is appropriate in this case. We now recapitulate the results we shall use from Young measure theory due to Dafermos [13] and Panov [29] for the deterministic setting, and Balder [1] for the stochastic version of the theory.

6.1. **Young measure and convergence of approximate solutions.** Roughly speaking a Young measure is a parametrized family of probability measures where the parameters are drawn from a measure space. Let $(\Theta, \Sigma, \mu)$ be a $\sigma$-finite measure space and $\mathcal{P}(\mathbb{R})$ be the space of probability measures on $\mathbb{R}$.

**Definition 6.1** (Young Measure)**.** A Young measure from $\Theta$ into $\mathbb{R}$ is a map $\tau \mapsto \mathcal{P}(\mathbb{R})$ such that for any $\phi \in C_b(\mathbb{R})$, $\theta \mapsto \langle \tau(\theta), \phi\rangle := \int_{\mathbb{R}} \phi(\xi)\tau(\theta)(d\xi)$ is measurable from $\Theta$ to $\mathbb{R}$. The set of all Young measures from $\Theta$ into $\mathbb{R}$ is denoted by $\mathcal{R}(\Theta, \Sigma, \mu)$.

In this context, we mention that with an appropriate choice of $(\Theta, \Sigma, \mu)$, the family $\{u^h_{\mathcal{T},\Delta t}\}_{h>0}$ can be thought of as a family of Young measures. We are interested in finding a subsequences out of this family that "converges" to a Young measure in a suitable sense. To this end, we consider the predictable $\sigma$-field of $\Omega \times (0,T)$ with respect to $\{\mathcal{F}_t\}$, denoted by $\mathcal{P}_T$, and set

$$\Theta = \Omega \times (0,T) \times \mathbb{R}^d, \quad \Sigma = \mathcal{P}_T \times \mathcal{L}(\mathbb{R}^d) \quad \text{and} \quad \mu = P \otimes \lambda_t \otimes \lambda_x,$$

where $\lambda_t$ and $\lambda_x$ are respectively the Lebesgue measures on $(0,T)$ and $\mathbb{R}^d$. Moreover, for $M \in \mathbb{N}$, set $\Theta_M = \Omega \times (0,T) \times B_M$, where $B_M$ be the ball of radius $M$ around zero in $\mathbb{R}^d$. We sum up the necessary results in the following lemma to carry over the subsequent analysis. For a proof of this lemma, consult [2, 9].

**Proposition 6.1.** *Let $\{u^h_{\mathcal{T},\Delta t}(t,x)\}_{h>0}$ be a sequence of $L^2(\mathbb{R}^d)$-valued predictable processes such that (4.1) holds. Then there exists a subsequence $\{h_n\}$ with $h_n \to 0$ and a Young measure $\tau \in \mathcal{R}(\Theta, \Sigma, \mu)$ such that the following hold:*

(A) *If $g(\theta, \xi)$ is a Carathéodory function on $\Theta \times \mathbb{R}$ such that supp$(g) \subset \Theta_M \times \mathbb{R}$ for some $M \in \mathbb{N}$ and $\{g(\theta, u^{h_n}_{\mathcal{T},\Delta t}(\theta))\}_n$ (where $\theta \equiv (\omega; t, x)$) is uniformly integrable, then*

$$\lim_{h_n \to 0} \int_\Theta g(\theta, u^{h_n}_{\mathcal{T},\Delta t}(\theta))\, \mu(d\theta) = \int_\Theta \Big[\int_{\mathbb{R}} g(\theta,\xi)\tau(\theta)(\,d\xi)\Big]\, \mu(d\theta).$$

(B) *Denoting a triplet $(\omega, x, t) \in \Theta$ by $\theta$, we define*

$$u(\theta, \alpha) = \inf\Big\{c \in \mathbb{R} : \tau(\theta)\big((-\infty, c)\big) > \alpha\Big\} \quad \text{for} \quad \alpha \in (0,1) \text{ and } \theta \in \Theta.$$

*Then, $u(\theta,\alpha)$ is non-decreasing, right continuous on $(0,1)$ and $\mathcal{P}_T \times \mathcal{L}(\mathbb{R}^d \times (0,1))$- measurable. Moreover, if $g(\theta,\xi)$ is a nonnegative Carathéodory function on $\Theta \times \mathbb{R}$, then*

$$\int_\Theta \Big[\int_{\mathbb{R}} g(\theta,\xi)\tau(\theta)(\,d\xi)\Big]\, \mu\,(d\theta) = \int_\Theta \int_{\alpha=0}^1 g(\theta, u(\theta,\alpha))\, d\alpha\, \mu(d\theta).$$

6.2. **Proof of the main theorem.** Having all the necessary *a priori* bounds and entropy inequality on $u^h_{\mathcal{T},\Delta t}$, we are now ready to prove the main theorem (cf. Main Theorem 3). Here we mentioned that $u(\theta,\alpha)$ given by Proposition 6.1 will serve as a possible generalized entropy solution to (1.1) for the above choice of the measure space $(\Theta, \Sigma, \mu)$. In view of (5.8), we have for any $B \in \mathcal{F}_T$

$$\mathbb{E}\Big[\mathbf{1}_B \int_{\Pi_T} \Big\{\beta(u^h_{\mathcal{T},\,\Delta t})\partial_t\psi(t,x) + F^\beta(u^h_{\mathcal{T},\,\Delta t})\vec{v}(t,x)\cdot\nabla_x\psi(t,x)\Big\}\, dt\, dx\Big]$$

$$+\mathbb{E}\Big[\mathbf{1}_B\int_{\mathbb{R}^d}\int_0^T\int_{\mathbf{E}}\int_0^1 \eta(u^h_{\mathcal{T},\Delta t};z)\beta'\big(u^h_{\mathcal{T},\Delta t}+\lambda\eta(u^h_{\mathcal{T},\Delta t};z)\big)\psi(t,x)\,d\lambda\,\tilde{N}(dz,dt)\,dx\Big]$$
$$+\mathbb{E}\Big[\mathbf{1}_B\int_{\mathbb{R}^d}\int_0^T\int_{\mathbf{E}}\int_0^1 (1-\lambda)\eta^2(u^h_{\mathcal{T},\Delta t};z)\beta''\big(u^h_{\mathcal{T},\Delta t}+\lambda\eta(u^h_{\mathcal{T},\Delta t};z)\big)\psi(t,x)\,d\lambda\,m(dz)\,dt\,dx\Big]$$
$$+\mathbb{E}\Big[\mathbf{1}_B\int_{\mathbb{R}^d}\beta(u_0(x))\psi(0,x)\,dx\Big]\ge \mathbb{E}\Big[\mathbf{1}_B\mathcal{R}^{h,\Delta t}\Big]$$
$$\text{i.e.,}\qquad \mathcal{T}_1+\mathcal{T}_2+\mathcal{T}_3+\mathbb{E}\Big[\mathbf{1}_B\int_{\mathbb{R}^d}\beta(u_0(x))\psi(0,x)\,dx\Big]\ge \mathbb{E}\Big[\mathbf{1}_B\mathcal{R}^{h,\Delta t}\Big]. \tag{6.1}$$

We would like to pass the limit in (6.1) as $h$ approaches to zero. To do this, here we use the technique of Young measure theory in stochastic setting. Let $(\Theta,\Sigma,\mu)$ be a $\sigma$- finite measure space as mentioned previously. Note that $L^2(\Theta,\Sigma,\mu)$ is a closed subspace of the larger space $L^2\Big(0,T;L^2((\Omega,\mathcal{F}_T),L^2(\mathbb{R}^d))\Big)$ and hence the weak convergence in $L^2\big(\Theta,\Sigma,\mu\big)$ would imply weak convergence in $L^2\Big(0,T;L^2((\Omega,\mathcal{F}_T),L^2(\mathbb{R}^d))\Big)$. Now, for any $B\in\mathcal{F}_T$, the functions $\mathbf{1}_B\partial_t\psi(t,x)$, $\mathbf{1}_B\partial_{x_i}\psi(t,x)$ and $\mathbf{1}_B\psi(t,x)$ are all members of $L^2\Big(0,T;L^2((\Omega,\mathcal{F}_T),L^2(\mathbb{R}^d))\Big)$. Therefore, in view of Proposition 6.1 and the above discussion, one has

$$\begin{aligned}\lim_{h\to 0}\mathcal{T}_1 &=\lim_{h\to 0}\mathbb{E}\Big[\mathbf{1}_B\int_{\Pi_T}\Big\{\beta(u^h_{\mathcal{T},\Delta t})\partial_t\psi(t,x)+F^\beta(u^h_{\mathcal{T},\Delta t})\vec{v}(t,x)\cdot\nabla_x\psi(t,x)\Big\}\,dt\,dx\Big]\\ &=\mathbb{E}\Big[\mathbf{1}_B\int_{\Pi_T}\int_0^1\Big\{\beta(u(t,x,\alpha))\partial_t\psi(t,x)+F^\beta(u(t,x,\alpha))\vec{v}(t,x)\cdot\nabla_x\psi(t,x)\Big\}\,d\alpha\,dt\,dx\Big].\end{aligned} \tag{6.2}$$

Next we want to pass to the limit in $\mathcal{T}_3$. For this, we fix $(\lambda,z)$, and define a Carathéodory function

$$G_{\lambda,z}(r,x,\omega,\xi)=\mathbf{1}_B(\omega)(1-\lambda)\eta^2(\xi,z)\beta''\big(\xi+\lambda\eta(\xi,z)\big)\psi(r,x).$$

Note that $\{G_{\lambda,z}(r,x,\omega,u^{h_n}_{\mathcal{T},\Delta t}(r,x,\omega))\}_n$ is uniformly integrable in $L^1((\Theta,\Sigma,\mu);\mathbb{R})$. Thus, in view of Proposition 6.1 we have, for fixed $(\lambda,z)\in(0,1)\times\mathbf{E}$

$$\begin{aligned}&\lim_{h\to 0}\mathbb{E}\Big[\int_{\Pi_T}\mathbf{1}_B(1-\lambda)\eta^2(u^h_{\mathcal{T},\Delta t};z)\beta''\big(u^h_{\mathcal{T},\Delta t}+\lambda\eta(u^h_{\mathcal{T},\Delta t};z)\big)\psi(t,x)\,dt\,dx\Big]\\ &=\mathbb{E}\Big[\int_{\Pi_T}\int_0^1\mathbf{1}_B(1-\lambda)\eta^2(u(t,x,\alpha);z)\beta''\big(u(t,x,\alpha)+\lambda\eta(u(t,x,\alpha);z)\big)\psi(t,x)\,d\alpha\,dt\,dx\Big].\end{aligned}$$

Thanks to the assumption **A**.3, and Lemma 4.1, we invoke dominated convergence theorem and have

$$\begin{aligned}\lim_{h\to 0}\mathcal{T}_3 &=\lim_{h\to 0}\mathbb{E}\Big[\mathbf{1}_B\int_{\Pi_T}\int_{\mathbf{E}}\int_0^1(1-\lambda)\eta^2(u^h_{\mathcal{T},\Delta t};z)\beta''\big(u^h_{\mathcal{T},\Delta t}+\lambda\eta(u^h_{\mathcal{T},\Delta t};z)\big)\psi(t,x)\,d\lambda\,m(dz)\,dt\,dx\Big]\\ &=\mathbb{E}\Big[\mathbf{1}_B\int_{\Pi_T}\int_{\mathbf{E}}\int_0^1\int_0^1(1-\lambda)\eta^2(u(t,x,\alpha);z)\beta''\big(u(t,x,\alpha)+\lambda\eta(u(t,x,\alpha);z)\big)\\ &\qquad\qquad\times\psi(t,x)\,d\alpha\,d\lambda\,m(dz)\,dt\,dx\Big].\end{aligned} \tag{6.3}$$

Now passage to the limit in the martingale term requires some additional reasoning. Let $\Gamma=\Omega\times[0,T]\times\mathbf{E}$, $\mathcal{G}=\mathcal{P}_T\times\mathcal{L}(\mathbf{E})$ and $\varsigma=\mathbb{P}\otimes\lambda_t\otimes m(dz)$, where $\mathcal{L}(\mathbf{E})$ represents a Lebesgue $\sigma$- algebra on $\mathbf{E}$. The space $L^2\big((\Gamma,\mathcal{G},\varsigma);\mathbb{R}\big)$ represents the space of square integrable predictable integrands for Itô-Lévy integrals with respect to the compensated Poisson random measure $\tilde{N}(dz,dt)$. Moreover, by Itô-Lévy isometry and martingale representation theorem, it follows that Itô-Lévy integral defines isometry between two Hilbert spaces $L^2\big((\Gamma,\mathcal{G},\varsigma);\mathbb{R}\big)$ and $L^2\big((\Omega,\mathcal{F}_T);\mathbb{R}\big)$. In other words, if $\mathcal{I}$ denotes the Itô-Lévy integral operator, i.e., the application

$$\begin{aligned}\mathcal{I}:L^2\big((\Gamma,\mathcal{G},\varsigma);\mathbb{R}\big)&\to L^2\big((\Omega,\mathcal{F}_T);\mathbb{R}\big)\\ v&\mapsto\int_0^T\int_{\mathbf{E}}v(\omega,z,r)\tilde{N}(dz,dr)\end{aligned}$$

and $\{X_n\}_n$ be sequence in $L^2\big((\Gamma,\mathcal{G},\varsigma);\mathbb{R}\big)$ weakly converging to $X$; then $\mathcal{I}(X_n)$ will converge weakly to $\mathcal{I}(X)$ in $L^2\big((\Omega,\mathcal{F}_T);\mathbb{R}\big)$. Note that, for fixed $z\in\mathbf{E}$, $G(t,x,\omega,\xi)=\Big(\beta\big(\xi+\eta(\xi;z)\big)-\beta(\xi)\Big)\psi(t,x)$

is a Carathéodory function and $\{G(t,x,\omega,u^{h_n}_{\mathcal{T},\Delta t}(t,x,\omega))\}_n$ is uniformly integrable in $L^1((\Theta,\Sigma,\mu);\mathbb{R})$. Therefore, one can apply Proposition 6.1 and conclude that for $m(dz)$-almost every $z\in\mathbf{E}$ and $g(t,z)\in L^2\big((\Gamma,\mathcal{G},\varsigma);\mathbb{R}\big)$,

$$\begin{aligned}&\lim_{h\to 0}\mathbb{E}\Big[\int_0^T\int_{\mathbb{R}^d}\Big(\beta\big(u^h_{\mathcal{T},\Delta t}+\eta(u^h_{\mathcal{T},\Delta t};z)\big)-\beta\big(u^h_{\mathcal{T},\Delta t}\big)\Big)\psi(r,x)g(r,z)\,dx\,dr\Big]\\&\quad=\mathbb{E}\Big[\int_0^T\int_{\mathbb{R}^d}\int_0^1\Big(\beta\big(u(r,x,\alpha)+\eta(u(r,x,\alpha);z)\big)-\beta\big(u(r,x,\alpha)\big)\Big)\psi(r,x)g(r,z)\,d\alpha\,dx\,dr\Big].\end{aligned}$$

We apply dominated convergence theorem along with Lemma 4.1 and the assumption $\mathbf{A}.3$ to have

$$\begin{aligned}&\lim_{h\to 0}\mathbb{E}\Big[\int_0^T\int_{\mathbf{E}}\int_{\mathbb{R}^d}\Big(\beta\big(u^h_{\mathcal{T},\Delta t}+\eta(u^h_{\mathcal{T},\Delta t};z)\big)-\beta\big(u^h_{\mathcal{T},\Delta t}\big)\Big)\psi(r,x)h(r,z)\,dx\,m(dz)\,dr\Big]\\&\quad=\mathbb{E}\Big[\int_0^T\int_{\mathbf{E}}\Big\{\int_{\mathbb{R}^d}\int_0^1\Big(\beta\big(u(r,x,\alpha)+\eta(u(r,x,\alpha);z)\big)-\beta\big(u(r,x,\alpha)\big)\Big)\\&\qquad\qquad\times\psi(r,x)h(r,z)\,d\alpha\,dx\Big\}\,m(dz)\,dr\Big].\end{aligned}$$

Hence, if we denote

$$X_n(t,z)=\int_{\mathbb{R}^d}\Big(\beta\big(u^h_{\mathcal{T},\Delta t}+\eta(u^h_{\mathcal{T},\Delta t};z)\big)-\beta\big(u^h_{\mathcal{T},\Delta t}\big)\Big)\psi(t,x)\,dx$$

and

$$X(t,z)=\int_{\mathbb{R}^d}\int_0^1\Big(\beta\big(u(t,x,\alpha)+\eta(u(t,x,\alpha);z)\big)-\beta\big(u(t,x,\alpha)\big)\Big)\psi(t,x)\,d\alpha\,dx$$

then, $X_n$ converges to $X$ in $L^2\big((\Gamma,\mathcal{G},\varsigma);\mathbb{R}\big)$ which implies, in view of the above discussion

$$\int_0^T\int_{\mathbf{E}}X_n(t,z)\tilde{N}(dz,dt)\rightharpoonup\int_0^T\int_{\mathbf{E}}X(t,z)\tilde{N}(dz,dt)\quad\text{in}\quad L^2\big((\Omega,\mathcal{F}_T);\mathbb{R}\big).$$

In other words, since $B\in\mathcal{F}_T$, we obtain

$$\begin{aligned}\lim_{h\to 0}\mathcal{T}_2&=\lim_{h\to 0}\mathbb{E}\Big[\mathbf{1}_B\int_{\Pi_T}\int_{\mathbf{E}}\int_0^1\eta(u^h_{\mathcal{T},\Delta t};z)\beta'\big(u^h_{\mathcal{T},\Delta t}+\lambda\eta(u^h_{\mathcal{T},\Delta t};z)\big)\psi(t,x)\,d\lambda\,\tilde{N}(dz,dt)\,dx\Big]\\&=\mathbb{E}\Big[\mathbf{1}_B\int_{\Pi_T}\int_{\mathbf{E}}\int_0^1\int_0^1\eta(u(t,x,\alpha);z)\beta'\big(u(t,x,\alpha)+\lambda\eta(u(t,x,\alpha);z)\big)\\&\qquad\qquad\times\psi(t,x)\,d\alpha\,d\lambda\,\tilde{N}(dz,dt)\,dx\Big].\end{aligned}\tag{6.4}$$

By (6.2), (6.3) and (6.4) and the fact that $\mathbb{E}\Big[\mathbf{1}_B\mathcal{R}^{h,\Delta t}\Big]\longrightarrow 0$ as $h\to 0$ (cf. Proposition 5.3), one can pass to the limit in (6.1) yielding (2.1). Also, in view of Proposition 6.1 and the uniform moment estimate (4.1) along with Fatou's lemma, we have

$$\sup_{0\le t\le T}\mathbb{E}\Big[||u(t,\cdot,\cdot)||_2^2\Big]<+\infty.$$

This implies that $u(t,x,\alpha)$ is a generalized entropy solution of (1.1). Again, thanks to Theorem 2.2, we conclude that $u(t,x,\alpha)$ is an independent function of variable $\alpha$ and $\bar{u}(t,x)=\int_0^1 u(t,x,\tau)d\tau=u(t,x,\alpha)$ (for almost all $\alpha$) is the unique stochastic entropy solution. Moreover, since $u^h_{\mathcal{T},\Delta t}$ is bounded in $L^2(\Omega\times\Pi_T)$, we conclude that $u^h_{\mathcal{T},\Delta t}$ converges to $\bar{u}$ in $L^p_{\rm loc}(\mathbb{R}^d;L^p(\Omega\times(0,T))$, for $1\le p<2$. This completes the proof.

## 7. Appendix

In this section, we study existence and uniqueness of entropy solution for the underlying problem (1.1).

7.1. **Existence of weak solution for viscous problem.** Just as the deterministic problem, here also we study the corresponding regularized problem by adding a small diffusion operator and derive some *a priori* bounds. Due to the nonlinearity in equation, one cannot expect classical solution and instead seeks a weak solution.

For a small parameter $\varepsilon > 0$, we consider the following viscous approximation of (1.1)

$$\begin{aligned} du(t,x) + \mathrm{div}_x(\vec{v}(t,x)f(u(t,x)))\,dt &= \int_{\mathbf{E}} \eta(u(t,x);z)\tilde{N}(dz,dt) + \varepsilon\Delta u(t,x)\,dt,\ (t,x)\in\Pi_T \\ u(0,x) &= u_0^\varepsilon(x),\ x\in\mathbb{R}^d, \end{aligned} \tag{7.1}$$

where $u_0^\varepsilon \in L^2(\mathbb{R}^d)$. To establish existence of a weak solution for (7.1), we follow [11, 32] and use an implicit time discretization scheme. Let $\Delta t = \frac{T}{N}$ for some fixed positive integer $N \geq 1$. Set $t_n = n\,\Delta t$ for $n = 0, 1, 2 \cdots, N$. Define

$$\begin{aligned} \mathcal{N} &= L^2(\Omega; H^1(\mathbb{R}^d)), \quad \mathcal{N}_n = \{\text{the } \mathcal{F}_{n\Delta t} \text{ measurable elements of } \mathcal{N}\}, \\ \mathcal{H} &= L^2(\Omega; L^2(\mathbb{R}^d)), \quad \mathcal{H}_n = \{\text{the } \mathcal{F}_{n\Delta t} \text{ measurable elements of } \mathcal{H}\}. \end{aligned}$$

The following proposition holds.

**Proposition 7.1.** *Assume that $\Delta t$ is small with $\Delta t < \dfrac{2\varepsilon}{V^2 c_f^2}$. Then, for any given $u_n \in \mathcal{H}_n$, there exists a unique $u_{n+1} \in \mathcal{N}_{n+1}$ such that $\mathbb{P}$-a.s. for any $v \in H^1(\mathbb{R}^d)$, the following variational formula holds:*

$$\begin{aligned} &\int_{\mathbb{R}^d} \Big( (u_{n+1} - u_n)v + \Delta t\big\{\varepsilon \nabla u_{n+1}\cdot\nabla v - \vec{v}(t_n,x)f(u_{n+1})\cdot\nabla v\big\}\Big)\,dx \\ &= \int_{\mathbb{R}^d}\int_{t_n}^{t_{n+1}}\int_{\mathbf{E}} \eta(u_n;z)\,v\,\tilde{N}(dz,ds)\,dx. \end{aligned} \tag{7.2}$$

*Proof.* Let us define a map

$$\begin{aligned} T :& \mathcal{H}_{n+1} \mapsto \mathcal{H}_{n+1} \\ & S \mapsto u = T(S) \end{aligned}$$

via the variational problem in $\mathcal{N}_{n+1}$ : for all $v \in \mathcal{N}_{n+1}$

$$\begin{aligned} &\mathbb{E}\Big[\int_{\mathbb{R}^d}\Big((u-u_n)v + \Delta t\big\{\varepsilon\nabla u\cdot\nabla v - \vec{v}(t_n,x)f(S)\cdot\nabla v\big\}\Big)\,dx\Big] \\ &= \mathbb{E}\Big[\int_{\mathbb{R}^d}\int_{t_n}^{t_{n+1}}\int_{\mathbf{E}}\eta(u_n;z)\,v\,\tilde{N}(dz,ds)\,dx\Big]. \end{aligned}$$

Thanks to Lax-Milgram theorem, $T$ is a well-defined function. Moreover, for any $S_1, S_2 \in \mathcal{H}_{n+1}$, we see that

$$\begin{aligned} &\mathbb{E}\Big[\int_{\mathbb{R}^d}|T(S_1)-T(S_2)|^2\,dx + \Delta t\varepsilon\int_{\mathbb{R}^d}\big|\nabla(T(S_1)-T(S_2))\big|^2\,dx\Big] \\ &= \Delta t\mathbb{E}\Big[\int_{\mathbb{R}^d}\vec{v}(t_n,x)(f(S_1)-f(S_2))\cdot\nabla(T(S_1)-T(S_2))\,dx\Big] \end{aligned}$$

and hence, by Young's inequality and the assumptions **A**.1 and **A**.2

$$\begin{aligned} &\mathbb{E}\Big[\int_{\mathbb{R}^d}|T(S_1)-T(S_2)|^2\,dx + \frac{\Delta t}{2}\varepsilon\int_{\mathbb{R}^d}\big|\nabla(T(S_1)-T(S_2))\big|^2\,dx\Big] \\ &\le \frac{\Delta t}{2\,\varepsilon}\mathbb{E}\Big[\int_{\mathbb{R}^d}|\vec{v}(t_n,x)|^2|f(S_1)-f(S_2)|^2\,dx\Big] \le \frac{\Delta t\,V^2 c_f^2}{2\,\varepsilon}\mathbb{E}\Big[\int_{\mathbb{R}^d}|S_1-S_2|^2\,dx\Big]. \end{aligned}$$

Thus, if $\Delta t < \dfrac{2\varepsilon}{V^2 c_f^2}$, then $T$ is a contractive mapping in $\mathcal{H}_{n+1}$ which completes the proof. □

7.1.1. ***A priori estimate***. Note that, since $\mathrm{div}_x \vec{v}(t,x) = 0$ for all $(t,x) \in \Pi_T$, for any $\theta \in \mathcal{D}(\mathbb{R}^d)$, $\int_{\mathbb{R}^d} \vec{v}(t,x) f(\theta) \cdot \nabla\theta \, dx = 0$ and hence true for any $\theta \in H^1(\mathbb{R}^d)$ by density argument. We choose a test function $v = u_{n+1}$ in (7.2) and have

$$\begin{aligned}
&\int_{\mathbb{R}^d} (u_{n+1} - u_n) u_{n+1} \, dx + \varepsilon \, \Delta t \int_{\mathbb{R}^d} |\nabla u_{n+1}|^2 \, dx = \int_{\mathbb{R}^d} \int_{t_n}^{t_{n+1}} \int_{\mathbf{E}} \eta(u_n; z) \, \tilde{N}(dz, ds) u_{n+1} \, dx \\
&\le \int_{\mathbb{R}^d} \int_{t_n}^{t_{n+1}} \int_{\mathbf{E}} \eta(u_n; z) u_n \, \tilde{N}(dz, ds) \, dx + \frac{\alpha}{2} ||u_{n+1} - u_n||^2_{L^2(\mathbb{R}^d)} \\
&\quad + \frac{1}{2\alpha} \int_{\mathbb{R}^d} \Big( \int_{t_n}^{t_{n+1}} \int_{\mathbf{E}} \eta(u_n; z) \, \tilde{N}(dz, ds) \Big)^2 dx, \quad \text{for some } \alpha > 0. 
\end{aligned} \tag{7.3}$$

Therefore, thanks to the assumption **A**.3, and Itô-Lévy isometry

$$\frac{1}{2}\Big[ ||u_{n+1}||^2_{\mathcal{H}} + ||u_{n+1} - u_n||^2_{\mathcal{H}} - ||u_n||^2_{\mathcal{H}} \Big] + \varepsilon \, \Delta t ||\nabla u_{n+1}||^2_{\mathcal{H}} \le \frac{\alpha}{2} ||u_{n+1} - u_n||^2_{\mathcal{H}} + \frac{C \, \Delta t}{2\alpha} \big( 1 + ||u_n||^2_{\mathcal{H}} \big).$$

Since $\alpha > 0$ is arbitrary, one can choose $\alpha > 0$ so that

$$||u_n||^2_{\mathcal{H}} + \sum_{k=0}^{n-1} ||u_{k+1} - u_k||^2_{\mathcal{H}} + \varepsilon \Delta t \sum_{k=0}^{n-1} ||\nabla u_{k+1}||^2_{\mathcal{H}} \le C_1 + C_2 \Delta t \sum_{k=0}^{n-1} ||u_k||^2_{\mathcal{H}},$$

for some constants $C_1, C_2 > 0$. Hence an application of discrete Gronwall's lemma implies

$$||u_n||^2_{\mathcal{H}} + \sum_{k=0}^{n-1} ||u_{k+1} - u_k||^2_{\mathcal{H}} + \varepsilon \Delta t \sum_{k=0}^{n-1} ||\nabla u_{k+1}||^2_{\mathcal{H}} \le C. \tag{7.4}$$

For fixed $\Delta t = \frac{T}{N}$, we define

$$u^{\Delta t}(t) = \sum_{k=1}^{N} u_k \mathbf{1}_{[t_{k-1}, t_k)}(t); \quad \tilde{u}^{\Delta t}(t) = \sum_{k=1}^{N} \Big[ \frac{u_k - u_{k-1}}{\Delta t} (t - t_{k-1}) + u_{k-1} \Big] \mathbf{1}_{[t_{k-1}, t_k)}(t)$$

with $u^{\Delta t}(t) = u_0$ for $t < 0$. Similarly, we define

$$\tilde{B}^{\Delta t}(t) = \sum_{k=1}^{N} \Big[ \frac{B_k - B_{k-1}}{\Delta t} (t - t_{k-1}) + B_{k-1} \Big] \mathbf{1}_{[t_{k-1}, t_k)}(t),$$

where

$$B_n = \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \int_{\mathbf{E}} \eta(u_k; z) \tilde{N}(dz, ds) = \int_0^{t_n} \int_{\mathbf{E}} \eta(u^{\Delta t}(s - \Delta t); z) \tilde{N}(dz, ds).$$

A straightforward calculation shows that

$$\begin{cases}
\|u^{\Delta t}\|_{L^\infty(0,T;\mathcal{H})} = \max\limits_{k=1,2,\cdots,N} \|u_k\|_{\mathcal{H}}; \quad \|\tilde{u}^{\Delta t}\|_{L^\infty(0,T;\mathcal{H})} = \max\limits_{k=0,1,\cdots,N} \|u_k\|_{\mathcal{H}}, \\
\|u^{\Delta t} - \tilde{u}^{\Delta t}\|^2_{L^2(0,T;\mathcal{H})} \le \Delta t \sum\limits_{k=0}^{N-1} \|u_{k+1} - u_k\|^2_{\mathcal{H}}.
\end{cases}$$

In view of the above definitions and *a priori* estimate (7.4), we have the following lemma.

**Lemma 7.2.** *Assume that $\Delta t$ is small. Then $u^{\Delta t}$, $\tilde{u}^{\Delta t}$ are bounded sequences in $L^\infty(0,T;\mathcal{H})$; $\sqrt{\epsilon} u^{\Delta t}$ is a bounded sequence in $L^2(0,T;\mathcal{N})$ and $||u^{\Delta t} - \tilde{u}^{\Delta t}||^2_{L^2(0,T;\mathcal{H})} \le C\Delta t$. Moreover, $u^{\Delta t} - u^{\Delta t}(\cdot - \Delta t) \to 0$ in $L^2(\Omega \times \Pi_T)$.*

Next, we want to find some upper bound for $\tilde{B}^{\Delta t}(t)$. Regarding this, we have the following lemma.

**Lemma 7.3.** *$\tilde{B}^{\Delta t}$ is a bounded sequence in $L^2(\Omega \times \Pi_T)$ and*

$$\Big\| \tilde{B}^{\Delta t}(\cdot) - \int_0^{\cdot} \int_{\mathbf{E}} \eta(u^{\Delta t}(s - \Delta t); z) \tilde{N}(dz, ds) \Big\|^2_{L^2(\Omega \times \mathbb{R}^d)} \le C\Delta t.$$

*Proof.* First we prove the boundedness of $\tilde{B}^{\Delta t}(t)$. By using the definition of $\tilde{B}^{\Delta t}(t)$, the assumption **A**.3, and the boundedness of $u^{\Delta t}$ in $L^\infty(0,T;\mathcal{H})$ along with Itô-Lévy isometry, we obtain

$$\begin{aligned}
\|\tilde{B}^{\Delta t}\|^2_{L^2\left(0,T;L^2(\Omega,L^2(\mathbb{R}^d))\right)} &\le \Delta t \sum_{k=0}^{N} ||B_k||^2_{L^2(\Omega\times\mathbb{R}^d)} \\
&\le \Delta t \sum_{k=0}^{N} \mathbb{E}\Big[\int_{\mathbb{R}^d} \Big|\int_0^{t_k}\int_{\mathbf{E}} \eta(u^{\Delta t}(s-\Delta t);z)\tilde{N}(dz,ds)\Big|^2 dx\Big] \\
&\le C\Delta t \sum_{k=0}^{N} E\Big[\int_{\mathbb{R}^d}\int_0^{t_k} |u^{\Delta t}(s-\Delta t)|^2\,dx\,ds\Big] \le C\|u^{\Delta t}\|_{L^\infty(0,T;L^2(\Omega\times\mathbb{R}^d))} \le C.
\end{aligned}$$

Thus, $\tilde{B}^{\Delta t}$ is a bounded sequence in $L^2(\Omega\times\Pi_T)$.

To prove second part of the lemma, we see that for any $t\in\big[t_n,t_{n+1}\big)$,

$$\begin{aligned}
&\tilde{B}^{\Delta t}(t) - \int_0^t\int_{\mathbf{E}} \eta(u^{\Delta t}(s-\Delta t);z)\tilde{N}(dz,ds) \\
=&\frac{t-t_n}{\Delta t}\int_{t_n}^{t_{n+1}}\int_{\mathbf{E}}\eta(u_n;z)\tilde{N}(dz,ds) - \int_{t_n}^t\int_{\mathbf{E}}\eta(u_n;z)\tilde{N}(dz,ds).
\end{aligned}$$

Therefore, in view of (7.4) and the assumption **A**.3, we have

$$\begin{aligned}
&\Big\|\tilde{B}^{\Delta t}(t) - \int_0^t\int_{\mathbf{E}}\eta(u^{\Delta t}(s-\Delta t);z)\tilde{N}(dz,ds)\Big\|^2_{L^2(\Omega\times\mathbb{R}^d)} \\
\le&2\int_{\mathbb{R}^d}\mathbb{E}\Bigg[\Big(\frac{t-t_n}{\Delta t}\Big)^2\int_{t_n}^{t_{n+1}}\int_{\mathbf{E}}\eta^2(u_n;z)\,m(dz)\,ds + \int_{t_n}^t\int_{\mathbf{E}}\eta^2(u_n;z)\,m(dz)\,ds\Bigg]dx \\
\le&C||u_n||^2_{\mathcal{H}}\Big[\frac{(t-t_n)^2}{\Delta t} + (t-t_n)\Big] \le C\,\Delta t.
\end{aligned}$$

This completes the proof. □

7.1.2. **Convergence of $u^{\Delta t}(t,x)$.** Thanks to Lemma 7.2 and Lipschitz property of $f$ and $\eta$, there exist $u, f_u$ and $\eta_u$ such that (up to a subsequence)

$$\begin{cases}
u^{\Delta t} \rightharpoonup^* u & \text{in } L^\infty\big(0,T;L^2(\Omega\times\mathbb{R}^d)\big) \\
u^{\Delta t} \rightharpoonup u & \text{in } L^2\big((0,T)\times\Omega;H^1(\mathbb{R}^d)\big) \quad \text{(for fixed } \varepsilon>0) \\
f(u^{\Delta t}) \rightharpoonup f_u & \text{in } L^2\big((0,T)\times\Omega;H^1(\mathbb{R}^d)\big) \\
\eta(u^{\Delta t}(\cdot-\Delta t);\cdot) \rightharpoonup \eta_u & \text{in } L^2\big(\Omega\times\Pi_T\times\mathbf{E}\big).
\end{cases} \tag{7.5}$$

Let $v^{\Delta t}(t) = \sum_{k=1}^{N}\vec{v}(t_k,\cdot)\mathbf{1}_{[t_{k-1},t_k)}(t)$. Then, for any $\theta\in H^1(\mathbb{R}^d)$, we can rewrite (7.2), in terms of $u^{\Delta t}, \tilde{u}^{\Delta t}, \tilde{B}^{\Delta t}$ and $v^{\Delta t}$ as

$$\Big\langle\frac{\partial}{\partial t}\big(\tilde{u}^{\Delta t}-\tilde{B}^{\Delta t}\big)(t),\theta\Big\rangle + \int_{\mathbb{R}^d}\big\{\varepsilon\nabla u^{\Delta t}(t) - v^{\Delta t}(t)f(u^{\Delta t}(t))\big\}\cdot\nabla\theta\,dx = 0. \tag{7.6}$$

In view of (7.6), one needs to show the boundedness of $\frac{\partial}{\partial t}(\tilde{u}^{\Delta t}-\tilde{B}^{\Delta t})$ in $L^2(\Omega\times(0,T);H^{-1}(\mathbb{R}^d))$ and then identify the weak limit. Regarding this, we have the following lemma.

**Lemma 7.4.** *The sequence* $\Big\{\frac{\partial}{\partial t}(\tilde{u}^{\Delta t}-\tilde{B}^{\Delta t})(t)\Big\}$ *is bounded in* $L^2\big(\Omega\times(0,T);H^{-1}(\mathbb{R}^d)\big)$*, and*

$$\frac{\partial}{\partial t}(\tilde{u}^{\Delta t}-\tilde{B}^{\Delta t}) \rightharpoonup \frac{\partial}{\partial t}\Big(u - \int_0^{\cdot}\int_{\mathbf{E}}\eta_u\tilde{N}(dz,ds)\Big) \quad \text{in } L^2\big(\Omega\times(0,T);H^{-1}(\mathbb{R}^d)\big)$$

*where $u$ is given by* (7.5).

*Proof.* To prove the lemma, we use similar argumentation (cf. passage to the limit in $\mathcal{T}_2$) as in Section 6. Note that Itô-Lévy integral defines a linear operator from $L^2\big((\Gamma,\mathcal{G},\varsigma);\mathbb{R}\big)$ to $L^2\big((\Omega,\mathcal{F}_T);\mathbb{R}\big)$ and it preserves the norm (cf. for example [30]). Therefore, in view of (7.5) and Lemma 7.3, we have

$$\tilde{B}^{\Delta t} \rightharpoonup \int_0^{\cdot}\int_{\mathbf{E}} \eta_u \tilde{N}(dz,ds) \quad \text{in } L^2(\Omega\times\Pi_T).$$

Again, note that

$$\frac{\partial}{\partial t}\big(\tilde{u}^{\Delta t}-\tilde{B}^{\Delta t}\big)(t)=\sum_{k=1}^{N}\frac{(u_k-u_{k-1})-(B_k-B_{k-1})}{\Delta t}\mathbf{1}_{\left[t_{k-1},t_k\right)}.$$

From (7.2), we see that for any $\theta\in H^1(\mathbb{R}^d)$,

$$\begin{aligned}
&\int_{\mathbb{R}^d}\Big(\frac{u_{n+1}-u_n}{\Delta t}-\frac{1}{\Delta t}\int_{t_n}^{t_{n+1}}\int_{\mathbf{E}}\eta(u_n;z)\tilde{N}(dz,ds)\Big)\theta\,dx\\
&=-\varepsilon\int_{\mathbb{R}^d}\nabla u_{n+1}\cdot\nabla\theta\,dx-\int_{\mathbb{R}^d}\vec{v}(t_n,\cdot)f(u_{n+1})\cdot\nabla\theta\,dx\\
&\le\Big\{\varepsilon\big\|\nabla u_{n+1}\big\|_{L^2(\mathbb{R}^d)}+c_fV\,\big\|u_{n+1}\big\|_{L^2(\mathbb{R}^d)}\Big\}||\theta||_{H^1(\mathbb{R}^d)},
\end{aligned}$$

and hence

$$\begin{aligned}
&\sup_{\theta\in H^1(\mathbb{R}^d)\setminus\{0\}}\frac{\int_{\mathbb{R}^d}\Big(\frac{u_{n+1}-u_n}{\Delta t}-\frac{1}{\Delta t}\int_{t_n}^{t_{n+1}}\int_{\mathbf{E}}\eta(u_n;z)\tilde{N}(dz,ds)\Big)\theta\,dx}{||\theta||_{H^1(\mathbb{R}^d)}}\\
&\qquad\le\varepsilon\big\|\nabla u_{n+1}\big\|_{L^2(\mathbb{R}^d)}+c_fV\big\|u_{n+1}\big\|_{L^2(\mathbb{R}^d)}.
\end{aligned}$$

This implies that $\frac{\partial}{\partial t}(\tilde{u}^{\Delta t}-\tilde{B}^{\Delta t})(t)$ is a bounded sequence in $L^2(\Omega\times(0,T);H^{-1}(\mathbb{R}^d))$.

To prove the second part of the lemma, we recall that $\tilde{B}^{\Delta t}\rightharpoonup\int_0^{\cdot}\int_{\mathbf{E}}\eta_u\tilde{N}(dz,ds)$ and $\tilde{u}^{\Delta t}\rightharpoonup u$ in $L^2(\Omega\times\Pi_T)$. In view of the first part of this lemma, one can conclude that, up to a subsequence

$$\frac{\partial}{\partial t}(\tilde{u}^{\Delta t}-\tilde{B}^{\Delta t})\rightharpoonup\frac{\partial}{\partial t}\Big(u-\int_0^{\cdot}\int_{\mathbf{E}}\eta_u\tilde{N}(dz,ds)\Big)\quad\text{in } L^2(\Omega\times(0,T);H^{-1}(\mathbb{R}^d)).$$

This completes the proof. □

In view of (7.5) and Lemma 7.4, one can pass to the limit in (7.6) and has, for $\theta\in H^1(\mathbb{R}^d)$

$$\Big\langle\frac{\partial}{\partial t}\Big(u-\int_0^{\cdot}\int_{\mathbf{E}}\eta_u\tilde{N}(dz,ds)\Big),\theta\Big\rangle+\int_{\mathbb{R}^d}\big\{\varepsilon\nabla u(t)-\vec{v}(t,\cdot)f_u\big\}\cdot\nabla\theta\,dx=0.$$

We denote by $\|\cdot\|_2$ the norm in $L^2(\mathbb{R}^d)$. An application of Itô-Lévy formula [20, similar to Theorem 3.4] to the functional $e^{-ct}\|u(t)\|_2^2$ yields

$$\begin{aligned}
&e^{-ct}\mathbb{E}\Big[\|u(t)\|_2^2\Big]+2\varepsilon\int_0^t e^{-cs}\mathbb{E}\big[\|\nabla u(s)\|_2^2\big]\,ds-2\int_0^t\mathbb{E}\Big[\int_{\mathbb{R}^d}e^{-cs}\vec{v}(s,x)f_u\cdot\nabla u\,dx\Big]\,ds\\
&=\mathbb{E}\big[\|u_0\|_2^2\big]-c\int_0^t e^{-cs}\mathbb{E}\big[\|u(s)\|_2^2\big]\,ds+\mathbb{E}\Big[\int_{\mathbf{E}}\int_0^t e^{-cs}\|\eta_u\|_2^2\,ds\,m(dz)\Big].
\end{aligned}\tag{7.7}$$

By choosing $\alpha>0$ suitably in (7.3) and multiplying by $e^{-ct_n}$ for positive $c>0$, we have

$$\begin{aligned}
&\mathbb{E}\Big[\int_{\mathbb{R}^d}\Big(e^{-ct_n}|u_{n+1}|^2-e^{-ct_{n-1}}|u_n|^2\Big)\,dx\Big]+2\varepsilon\,\Delta te^{-ct_n}\mathbb{E}\Big[\int_{\mathbb{R}^d}|\nabla u_{n+1}|^2\,dx\Big]\\
&\le\Delta te^{-ct_n}\mathbb{E}\Big[\int_{\mathbf{E}}\int_{\mathbb{R}^d}\eta^2(u_n;z)\,dx\,m(dz)\Big]+\Big(e^{-ct_n}-e^{-ct_{n-1}}\Big)\mathbb{E}\Big[\int_{\mathbb{R}^d}|u_n|^2\,dx\Big].
\end{aligned}\tag{7.8}$$

Therefore, by summing over $n$ from 0 to $k$ in (7.8) we get

$$
\begin{aligned}
&e^{-ct_k}\mathbb{E}\big[\|u_{k+1}\|_2^2\big] + 2\varepsilon\sum_{n=0}^{k}\Delta t e^{-ct_n}\mathbb{E}\Big[\|\nabla u_{n+1}\|_2^2\Big] \\
&\le e^{c\Delta t}\mathbb{E}\big[\|u_0\|_2^2\big] + \Delta t\sum_{n=0}^{k} e^{-ct_n}\mathbb{E}\Big[\int_{\mathbf{E}}\int_{\mathbb{R}^d}\eta^2(u_n;z)\,dx\,m(dz)\Big] + \sum_{n=0}^{k}\Big(e^{-ct_n}-e^{-ct_{n-1}}\Big)\mathbb{E}\Big[\|u_n\|_2^2\Big].
\end{aligned}
$$

Note that

$$
\begin{aligned}
&\sum_{n=0}^{k}\Big(e^{-ct_n}-e^{-ct_{n-1}}\Big)\mathbb{E}\Big[\|u_n\|_2^2\Big] \\
&= \big(1-e^{c\Delta t}\big)\mathbb{E}\big[\|u_0\|_2^2\big] + \sum_{n=1}^{k}\Big(e^{-ct_n}-e^{-ct_{n-1}}\Big)\mathbb{E}\Big[\|u_n\|_2^2\Big] \\
&= \big(1-e^{c\Delta t}\big)\mathbb{E}\big[\|u_0\|_2^2\big] - c\sum_{n=1}^{k}\int_{t_{n-1}}^{t_n} e^{-cs}\,ds\mathbb{E}\Big[\|u_n\|_2^2\Big] \\
&\le \big(1-e^{c\Delta t}\big)\mathbb{E}\big[\|u_0\|_2^2\big] - ce^{-c\Delta t}\int_0^{t_k} e^{-cs}\mathbb{E}\big[\|u^{\Delta t}(s)\|_2^2\big]\,ds,
\end{aligned}
$$

and

$$
\Delta t\sum_{n=0}^{k} e^{-ct_n}\mathbb{E}\Big[\int_{\mathbf{E}}\int_{\mathbb{R}^d}\eta^2(u_n;z)\,dx\,m(dz)\Big] \le \int_0^{t_k} e^{-cs}\mathbb{E}\Big[\int_{\mathbf{E}}\int_{\mathbb{R}^d}\eta^2(u^{\Delta t};z)\,dx\,m(dz)\Big]\,ds.
$$

Thus, we obtain, for $t\in[t_k,t_{k+1})$

$$
\begin{aligned}
&e^{-ct}\mathbb{E}\big[\|u^{\Delta t}(t)\|_2^2\big] + 2\varepsilon\int_0^t e^{-cs}\mathbb{E}\big[\|\nabla u^{\Delta t}\|_2^2\big]\,ds \\
&\le \mathbb{E}\big[\|u_0\|_2^2\big] + \int_0^t e^{-cs}\mathbb{E}\Big[\int_{\mathbf{E}}\int_{\mathbb{R}^d}\eta^2(u^{\Delta t};z)\,dx\,m(dz)\Big]\,ds \\
&\qquad - ce^{-c\Delta t}\int_0^t e^{-cs}\mathbb{E}\big[\|u^{\Delta t}\|_2^2\big]\,ds. 
\end{aligned}
\tag{7.9}
$$

Note that, for any $\theta\in H^1(\mathbb{R}^d)$ and any $s\in[0,T]$, there holds $\int_{\mathbb{R}^d}\vec{v}(s,x)f(\theta)\nabla\theta\,dx = 0$. Thus, using (7.9) we obtain

$$
\begin{aligned}
&e^{-ct}\mathbb{E}\big[\|u^{\Delta t}(t)\|_2^2\big] + 2\varepsilon\int_0^t e^{-cs}\mathbb{E}\big[\|\nabla(u^{\Delta t}-u)\|_2^2\big]\,ds \\
&\qquad - 2\int_0^t e^{-cs}\mathbb{E}\big[\int_{\mathbb{R}^d}\vec{v}(s,x)[f(u^{\Delta t})-f(u)]\nabla(u^{\Delta t}-u)\,dx\big]\,ds \\
&\le \mathbb{E}\big[\|u_0\|_2^2\big] + \int_0^t e^{-cs}\mathbb{E}\Big[\int_{\mathbf{E}}\|\eta(u^{\Delta t};z)-\eta(u;z)\|^2\,m(dz)\Big]\,ds \\
&- \int_0^t e^{-cs}\mathbb{E}\Big[\int_{\mathbf{E}}\|\eta(u;z)\|^2\,m(dz)\Big]\,ds + 2\int_0^t e^{-cs}\mathbb{E}\Big[\int_{\mathbf{E}}\int_{\mathbb{R}^d}\eta(u^{\Delta t};z)\eta(u;z)\,dx\,m(dz)\Big]\,ds \\
&- ce^{-c\Delta t}\int_0^t e^{-cs}\mathbb{E}\big[\|u^{\Delta t}-u\|_2^2\big]\,ds + ce^{-c\Delta t}\int_0^t e^{-cs}\mathbb{E}\big[\|u\|_2^2\big]\,ds + 2\varepsilon\int_0^t e^{-cs}\mathbb{E}\big[\|\nabla u\|_2^2\big]\,ds \\
&- 2ce^{-c\Delta t}\int_0^t e^{-cs}\mathbb{E}\big[\int_{\mathbb{R}^d}u^{\Delta t}u\,dx\big]\,ds + 2\int_0^t e^{-cs}\mathbb{E}\big[\int_{\mathbb{R}^d}\vec{v}(s,x)f(u^{\Delta t})\nabla u\,dx\big]\,ds \\
&+ 2\int_0^t e^{-cs}\mathbb{E}\big[\int_{\mathbb{R}^d}\vec{v}(s,x)f(u)\nabla u^{\Delta t}\,dx\big]\,ds - 4\varepsilon\int_0^t e^{-cs}\mathbb{E}\big[\int_{\mathbb{R}^d}\nabla u^{\Delta t}\nabla u\,dx\big]\,ds.
\end{aligned}
\tag{7.10}
$$

In view of Young's inequality, one has

$$\begin{aligned}
&-2\varepsilon\int_0^t e^{-cs}\mathbb{E}\big[\|\nabla(u^{\Delta t}-u)\|_2^2\big]\,ds+2\int_0^t e^{-cs}\mathbb{E}\big[\int_{\mathbb{R}^d}\vec{v}(s,x)[f(u^{\Delta t})-f(u)]\nabla(u^{\Delta t}-u)\,dx\big]\,ds\\
&\le -\varepsilon\int_0^t e^{-cs}\mathbb{E}\big[\|\nabla(u^{\Delta t}-u)\|_2^2\big]\,ds+\frac{1}{\varepsilon}\int_0^t e^{-cs}\mathbb{E}\big[\|\vec{v}(s,\cdot)[f(u^{\Delta t})-f(u)]\|_2^2\big]\,ds, \qquad (7.11)
\end{aligned}$$

and by choosing $c>0$ with $\frac{1}{\varepsilon}V^2C_f^2+c_\eta\le ce^{-c\Delta t}$, one arrive at

$$\begin{aligned}
&\frac{1}{\varepsilon}\int_0^t e^{-cs}\mathbb{E}\big[\|\vec{v}(s,\cdot)[f(u^{\Delta t})-f(u)]\|_2^2\big]\,ds+\int_0^t e^{-cs}\mathbb{E}\Big[\int_{\mathbf{E}}\|\eta(u^{\Delta t};z)-\eta(u;z)\|_2^2\,m(dz)\Big]\,ds\\
&\qquad -ce^{-c\Delta t}\int_0^t e^{-cs}\mathbb{E}\big[\|u^{\Delta t}-u\|_2^2\big]\,ds\le 0. \qquad (7.12)
\end{aligned}$$

We use (7.11)-(7.12) in (7.10) for the above choice of $c>0$ along with (7.5) and (7.7) to have

$$\begin{aligned}
&\limsup_{\Delta t}\int_0^T e^{-ct}\mathbb{E}\big[\|u^{\Delta t}(t)\|_2^2\big]\,dt+\int_0^T\int_0^t e^{-cs}\mathbb{E}\Big[\int_{\mathbf{E}}\|\eta_u-\eta(u;z)\|_2^2\,m(dz)\Big]\,ds\,dt\\
&\le\int_0^T e^{-ct}\mathbb{E}\big[\|u(t)\|_2^2\big]\,dt.
\end{aligned}$$

Thus, we obtain $\eta_u=\eta(u;z)$ and $u^{\Delta t}\to u$ in $L^2(\Omega\times\Pi_T)$. Moreover, one can show that $f_u=f(u)$. Thus $u$ is a weak solution to the viscous problem (7.1). Since it depends on $\varepsilon>0$, we denote it by $u_\varepsilon$.

7.1.3. ***A priori* bounds for viscous solutions.** Note that for fixed $\varepsilon>0$, there exists a weak solution $u_\varepsilon\in H^1(\mathbb{R}^d)$ satisfying: $\mathbb{P}$-a.s., and for a.e. $t\in(0,T)$

$$\Big\langle\frac{\partial}{\partial t}[u_\varepsilon-\int_0^t\int_{\mathbf{E}}\eta(u_\varepsilon(s,\cdot);z)\tilde{N}(dz,ds)],v\Big\rangle+\int_{\mathbb{R}^d}\Big\{\vec{v}(t,x)f(u_\varepsilon(t,x))+\varepsilon\nabla u_\varepsilon(t,x)\Big\}\cdot\nabla v(x)\,dx=0, \qquad (7.13)$$

for any $v\in H^1(\mathbb{R}^d)$. We apply Itô-Lévy formula to $\beta(u)=\|u\|_2^2$, and then take expectation. The result is

$$\mathbb{E}\Big[\big\|u_\varepsilon(t)\big\|_2^2\Big]+2\varepsilon\int_0^t\mathbb{E}\Big[\big\|\nabla u_\varepsilon\big\|_2^2\Big]\,ds\le\mathbb{E}\Big[\big\|u_\varepsilon(0)\big\|_2^2+C\int_0^t\mathbb{E}\Big[\big\|u_\varepsilon(s)\big\|_2^2\Big]\,ds.$$

An application of Gronwall's inequality yields

$$\sup_{0\le t\le T}\mathbb{E}\Big[\big\|u_\varepsilon(t)\big\|_2^2\Big]+\varepsilon\int_0^T\mathbb{E}\Big[\big\|\nabla u_\varepsilon(s)\big\|_2^2\Big]\,ds\le C.$$

The following lemma states that $\frac{\partial}{\partial t}\Big[u_\varepsilon-\int_0^t\int_{\mathbf{E}}\eta(u_\varepsilon;z)\tilde{N}(dz,ds)\Big]\in L^2(\Omega\times\Pi_T)$ if the initial data $u_0^\varepsilon\in H^1(\mathbb{R}^d)$.

**Lemma 7.5.** *Suppose that $u_0^\varepsilon\in H^1(\mathbb{R}^d)$. Then, a weak solution $u_\varepsilon$ of* (7.1) *satisfies the following regularity properties:* $\frac{\partial}{\partial t}\big[u_\varepsilon-\int_0^t\int_{\mathbf{E}}\eta(u_\varepsilon;z)\tilde{N}(dz,ds)\big]$, $\Delta u_\varepsilon\in L^2(\Omega\times\Pi_T)$.

*Proof.* Let $u_0^\varepsilon\in H^1(\mathbb{R}^d)$. By choosing $v=u_{n+1}-u_n-\int_{t_n}^{t_{n+1}}\int_{\mathbf{E}}\eta(u_n;z)\tilde{N}(dz,ds)$ in (7.2), we obtain

$$\begin{aligned}
&\big\|u_{n+1}-u_n-\int_{t_n}^{t_{n+1}}\int_{\mathbf{E}}\eta(u_n;z)\tilde{N}(dz,ds)\big\|_{L^2(\mathbb{R}^d)}^2\\
&\quad+\Delta t\varepsilon\int_{\mathbb{R}^d}\nabla u_{n+1}\cdot\nabla\big[u_{n+1}-u_n-\int_{t_n}^{t_{n+1}}\int_{\mathbf{E}}\eta(u_n;z)\tilde{N}(dz,ds)\big]\,dx\\
&=-\Delta t\int_{\mathbb{R}^d}\vec{v}(t_n,x)f'(u_{n+1})\cdot\nabla u_{n+1}\Big(u_{n+1}-u_n-\int_{t_n}^{t_{n+1}}\int_{\mathbf{E}}\eta(u_n;z)\tilde{N}(dz,ds)\Big)\,dx\\
&\le\frac{1}{2}\big\|u_{n+1}-u_n-\int_{t_n}^{t_{n+1}}\int_{\mathbf{E}}\eta(u_n;z)\tilde{N}(dz,ds)\big\|_{L^2(\mathbb{R}^d)}^2+\frac{1}{2}C(V,f')(\Delta t)^2\|\nabla u_{n+1}\|_{L^2(\mathbb{R}^d)^d}^2.
\end{aligned}$$

Note that, $\mathbb{E}\Big[\nabla u_n \int_{t_n}^{t_{n+1}}\int_{\mathbf{E}} \nabla\eta(u_n;z)\tilde{N}(dz,ds)\Big]=0$. Since $\tilde{N}$ is a compensated Poisson random measure, an application of differentiation under integral sign, the assumption **A**.3 along with Young's inequality and Itô-Lévy isometry reveals that

$$\begin{aligned}
&\mathbb{E}\Big[\int_{\mathbb{R}^d} \nabla u_{n+1}\cdot\nabla\big[u_{n+1}-u_n-\int_{t_n}^{t_{n+1}}\int_{\mathbf{E}}\eta(u_n;z)\tilde{N}(dz,ds)\big]\,dx\Big]\\
&=\frac{1}{2}\mathbb{E}\Big[\|\nabla u_{n+1}\|^2_{L^2(\mathbb{R}^d)^d}-\|\nabla u_n\|^2_{L^2(\mathbb{R}^d)^d}+\|\nabla[u_{n+1}-u_n]\|^2_{L^2(\mathbb{R}^d)^d}\Big]\\
&\qquad-\mathbb{E}\Big[\int_{\mathbb{R}^d}\nabla[u_{n+1}-u_n]\int_{t_n}^{t_{n+1}}\int_{\mathbf{E}}\nabla\eta(u_n;z)\tilde{N}(dz,ds)\,dx\Big]\\
&\ge\frac{1}{2}\mathbb{E}\Big[\|\nabla u_{n+1}\|^2_{L^2(\mathbb{R}^d)^d}-\|\nabla u_n\|^2_{L^2(\mathbb{R}^d)^d}+\frac{1}{2}\|\nabla[u_{n+1}-u_n]\|^2_{L^2(\mathbb{R}^d)^d}-2\lambda^*\Delta t\|\nabla u_n\|^2_{L^2(\mathbb{R}^d)^d}\int_{\mathbf{E}}h_1^2(z)\,m(dz)\Big],
\end{aligned}$$

and hence

$$\begin{aligned}
&\mathbb{E}\Big[\big\|u_{n+1}-u_n-\int_{t_n}^{t_{n+1}}\int_{\mathbf{E}}\eta(u_n;z)\tilde{N}(dz,ds)\big\|^2_{L^2(\mathbb{R}^d)}\Big]\\
&+\Delta t\varepsilon\,\mathbb{E}\Big[\|\nabla u_{n+1}\|^2_{L^2(\mathbb{R}^d)^d}-\|\nabla u_n\|^2_{L^2(\mathbb{R}^d)^d}+\frac{1}{2}\|\nabla[u_{n+1}-u_n]\|^2_{L^2(\mathbb{R}^d)^d}\Big]\\
&\le 2\lambda^*(\Delta t)^2c_\eta\mathbb{E}\Big[\|\nabla u_n\|^2_{L^2(\mathbb{R}^d)^d}\Big]+C(V,f')(\Delta t)^2\mathbb{E}\Big[\|\nabla u_{n+1}\|^2_{L^2(\mathbb{R}^d)^d}\Big].
\end{aligned}$$

Thus, for any $k\in\{0,1,\cdots,N-1\}$,

$$\begin{aligned}
&\sum_{n=0}^{k}\Delta t\mathbb{E}\Big[\|\frac{u_{n+1}-u_n-\int_{t_n}^{t_{n+1}}\int_{\mathbf{E}}\eta(u_n;z)\tilde{N}(dz,ds)}{\Delta t}\|^2_{L^2(\mathbb{R}^d)}\Big]\\
&+\varepsilon\mathbb{E}\Big[\|\nabla u_{k+1}\|^2_{L^2(\mathbb{R}^d)^d}\Big]+\frac{\varepsilon}{2}\sum_{n=0}^{k}\mathbb{E}\Big[\|\nabla[u_{n+1}-u_n]\|^2_{L^2(\mathbb{R}^d)^d}\Big]\\
&\le\varepsilon\mathbb{E}\Big[\|\nabla u_0^\varepsilon\|^2_{L^2(\mathbb{R}^d)^d}\Big]+C(V,f',c_\eta,\lambda^*)\Delta t\sum_{n=0}^{k+1}\mathbb{E}\Big[\|\nabla u_n\|^2_{L^2(\mathbb{R}^d)^d}\Big]\le C.
\end{aligned}$$

Therefore, in view of the definitions of $u^{\Delta t}$, $\tilde{u}^{\Delta t}$, $\tilde{B}^{\Delta t}$, we see that $u^{\Delta t}$, $\tilde{u}^{\Delta t}$ are bounded in $L^\infty(0,T;L^2(\Omega;H^1(\mathbb{R}^d)))$, and the sequence $\Big\{\frac{\partial}{\partial t}(\tilde{u}^{\Delta t}-\tilde{B}^{\Delta t})(t)\Big\}$ is bounded in $L^2\big(\Omega\times(0,T);L^2(\mathbb{R}^d)\big)$. Moreover, second part of Lemma 7.4 reveals that $\frac{\partial}{\partial t}\Big[u_\varepsilon-\int_0^t\int_{\mathbf{E}}\eta(u_\varepsilon;z)\tilde{N}(dz,ds)\Big]\in L^2(\Omega\times\Pi_T)$ and hence by using the equation (7.1) we arrive at the conclusion that $\Delta u_\varepsilon\in L^2(\Omega\times\Pi_T)$. Furthermore, (7.13) holds with an integral over $\mathbb{R}^d$ instead of the duality bracket if the initial data $u_0^\varepsilon\in H^1(\mathbb{R}^d)$. □

In addition, if $u_0^\varepsilon\in L^{2p}(\mathbb{R}^d)$, $p\ge1$, then a straightforward argumentation as in the proof of [2, Proposition $A.5$] gives $u_\varepsilon\in L^\infty\big(0,T;L^{2p}(\Omega\times\mathbb{R}^d)\big)$.

The achieved results can be summarized into the following theorem.

**Theorem 7.6.** *Let $\varepsilon>0$ is fixed and $u_0^\varepsilon\in H^1(\mathbb{R}^d)$. Then there exists a weak solution $u_\varepsilon$ of (7.1) such that $\frac{\partial}{\partial t}\Big[u_\varepsilon-\int_0^t\int_{\mathbf{E}}\eta(u_\varepsilon;z)\tilde{N}(dz,ds)\Big]$, $\Delta u_\varepsilon\in L^2(\Omega\times\Pi_T)$. Moreover the following estimate holds:*

$$\sup_{0\le t\le T}\mathbb{E}\Big[\big\|u_\varepsilon(t)\big\|_2^2\Big]+\varepsilon\int_0^T\mathbb{E}\Big[\big\|\nabla u_\varepsilon(s)\big\|_2^2\Big]\,ds\le C.$$

*Furthermore, if $u_0^\varepsilon\in L^{2p}(\mathbb{R}^d)$, $p\ge1$, then $u_\varepsilon\in L^\infty\big(0,T;L^{2p}(\Omega\times\mathbb{R}^d)\big)$.*

7.2. **Proof of Theorem 2.1.** In this subsection, we prove existence of generalized entropy solution in the sense of Definition 2.4. For this, fix a nonnegative test function $\psi\in C_c^\infty([0,\infty)\times\mathbb{R}^d)$, $B\in\mathcal{F}_T$ and convex entropy flux pair $(\beta,\zeta)$. For any $\varepsilon>0$, we consider the viscous problem (7.1) with initial data

$u_0^\varepsilon \in \mathcal{D}(\mathbb{R}^d)$. We apply Itô-Lévy formula to the functional $F(t,u_\varepsilon) = \int_{\mathbb{R}^d} \beta(u_\varepsilon)\psi(t,x)\,dx$ and conclude

$$\begin{aligned}
0 \le & \mathbb{E}\Big[\mathbf{1}_B \int_{\mathbb{R}^d} \beta(u_0^\varepsilon(x))\psi(0,x)\,dx\Big] - \varepsilon\mathbb{E}\Big[\mathbf{1}_B \int_{\Pi_T} \beta'(u_\varepsilon(t,x))\nabla u_\varepsilon(t,x)\cdot\nabla\psi(t,x)\,dx\,dt\Big] \\
&+ \mathbb{E}\Big[\mathbf{1}_B \int_{\Pi_T} \Big(\beta(u_\varepsilon(t,x))\partial_t\psi(t,x) + \nabla\psi(t,x)\cdot\vec{v}(t,x)\zeta(u_\varepsilon(t,x))\Big)\,dx\,dt\Big] \\
&+ \mathbb{E}\Big[\mathbf{1}_B \int_{\Pi_T}\int_{\mathbf{E}}\int_0^1 \eta(u_\varepsilon(t,x);z)\beta'(u_\varepsilon(t,x) + \theta\,\eta(u_\varepsilon(t,x);z))\psi(t,x)\,d\theta\,\tilde{N}(dz,dt)\,dx\Big] \\
&+ \mathbb{E}\Big[\mathbf{1}_B \int_{\Pi_T}\int_{\mathbf{E}}\int_0^1 (1-\theta)\eta^2(u_\varepsilon(t,x);z)\beta''(u_\varepsilon(t,x) + \theta\,\eta(u_\varepsilon(t,x);z))\psi(t,x)\,d\theta\,m(dz)\,dt\,dx\Big] \qquad (7.14)
\end{aligned}$$

We use Young measure technique (cf. Subsection 6.2) to pass to the limit in (7.14) as $\varepsilon \to 0$. Moreover, there exists a $L^2(\mathbb{R}^d\times(0,1))$-valued predictable limit process $u \in L^\infty\big(0,T;L^2(\Omega\times\mathbb{R}^d\times(0,1))\big)$ such that

$$\begin{aligned}
& \mathbb{E}\Big[\mathbf{1}_B \int_{\Pi_T}\int_0^1 \Big(\beta(u(t,x,\alpha))\partial_t\psi(t,x) + \nabla\psi(t,x)\cdot\vec{v}(t,x)F^\beta(u(t,x,\alpha))\Big)\,d\alpha\,dx\,dt\Big] \\
&+ \mathbb{E}\Big[\mathbf{1}_B \int_{\Pi_T}\int_{\mathbf{E}}\int_0^1\int_0^1 \eta(u(t,x,\alpha);z)\beta'(u(t,x,\alpha) + \theta\,\eta(u(t,x,\alpha);z))\psi(t,x)\,d\alpha\,d\theta\,\tilde{N}(dz,dt)\,dx\Big] \\
&+ \mathbb{E}\Big[\mathbf{1}_B \int_{\Pi_T}\int_{\mathbf{E}}\int_0^1\int_0^1 (1-\theta)\eta^2(u(t,x,\alpha);z)\beta''(u(t,x,\alpha) + \theta\,\eta(u(t,x,\alpha);z))\psi(t,x)\,d\alpha\,d\theta\,m(dz)\,dt\,dx\Big] \\
&\qquad + \mathbb{E}\Big[\mathbf{1}_B \int_{\mathbb{R}^d} \beta(u_0(x))\psi(0,x)\,dx\Big] \ge 0. \qquad (7.15)
\end{aligned}$$

Since (7.15) holds for every $B \in \mathcal{F}_T$, we conclude that $\mathbb{P}$-a.s., inequality (2.1) holds true as well. In other words, $u(t,x,\alpha)$ is a generalized entropy solution to the problem (1.1).

7.3. **Proof of Theorem 2.2.** To prove uniqueness of generalized entropy solutions, we follow the same argumentations as in [9]. Let $\rho$ and $\varrho$ be the standard nonnegative mollifiers on $\mathbb{R}$ and $\mathbb{R}^d$ respectively such that $\operatorname{supp}(\rho) \subset [-1,0]$ and $\operatorname{supp}(\varrho) = B_1(0)$, where $B_1(0)$ denotes the bounded ball of radius 1 around 0 in $\mathbb{R}^d$. We define $\rho_{\delta_0}(r) = \frac{1}{\delta_0}\rho(\frac{r}{\delta_0})$ and $\varrho_\delta(x) = \frac{1}{\delta^d}\varrho(\frac{x}{\delta})$, where $\delta$ and $\delta_0$ are two positive constants. Given a nonnegative test function $\psi \in C_c^{1,2}([0,\infty)\times\mathbb{R}^d)$ and two positive constants $\delta$ and $\delta_0$, we define

$$\phi_{\delta,\delta_0}(t,x,s,y) = \rho_{\delta_0}(t-s)\varrho_\delta(x-y)\psi(s,y).$$

Let $\beta:\mathbb{R}\to\mathbb{R}$ be a $C^\infty$ function satisfying

$$\beta(0) = 0, \quad \beta(-r) = \beta(r), \quad \beta'(-r) = -\beta'(r), \quad \beta'' \ge 0,$$

and

$$\beta'(r) = \begin{cases} -1 & \text{when } r \le -1, \\ \in [-1,1] & \text{when } |r| < 1, \\ +1 & \text{when } r \ge 1. \end{cases}$$

For any $\vartheta > 0$, define $\beta_\vartheta:\mathbb{R}\to\mathbb{R}$ by $\beta_\vartheta(r) = \vartheta\beta(\frac{r}{\vartheta})$. Then

$$|r| - M_1\vartheta \le \beta_\vartheta(r) \le |r| \quad \text{and} \quad |\beta_\vartheta''(r)| \le \frac{M_2}{\vartheta}\mathbf{1}_{|r|\le\vartheta},$$

where $M_1 = \sup_{|r|\le 1}\big||r| - \beta(r)\big|$ and $M_2 = \sup_{|r|\le 1}|\beta''(r)|$. For $\beta = \beta_\vartheta$ we define

$$\mathcal{F}^{\beta_\vartheta}(a,b) = \int_b^a \beta_\vartheta'(\sigma - b)f'(\sigma)\,d(\sigma).$$

Let $v(t,x,\alpha)$ be a generalized entropy solution of (1.1). Moreover, let $\varsigma$ be the standard symmetric nonnegative mollifier on $\mathbb{R}$ with support in $[-1,1]$ and $\varsigma_l(r) = \frac{1}{l}\varsigma(\frac{r}{l})$ for $l > 0$. Given $k \in \mathbb{R}$, the function $\beta_\vartheta(\cdot - k)$ is a smooth convex function and $(\beta_\vartheta(\cdot - k), \mathcal{F}^{\beta_\vartheta}(\cdot,k))$ is a convex entropy pair. Consider the entropy inequality for $v(t,x,\alpha)$, based on the entropy pair $(\beta_\vartheta(\cdot - k), \mathcal{F}^{\beta_\vartheta}(\cdot,k))$, and then multiply by $\varsigma_l(u_\varepsilon(s,y)-k)$, integrate with respect to $s,y,k$ and take the expectation. The result is

$$\begin{aligned}
0 \le & \mathbb{E}\Big[\int_{\Pi_T}\int_{\mathbb{R}^d}\int_{\mathbb{R}} \beta_\vartheta(v_0(x)-k)\phi_{\delta,\delta_0}(0,x,s,y)\varsigma_l(u_\varepsilon(s,y)-k)\,dk\,dx\,dy\,ds\Big] \\
&+ \mathbb{E}\Big[\int_{\Pi_T^2}\int_0^1\int_{\mathbb{R}} \beta_\vartheta(v(t,x,\alpha)-k)\partial_t\phi_{\delta,\delta_0}(t,x,s,y)\varsigma_l(u_\varepsilon(s,y)-k)\,dk\,d\alpha\,dx\,dt\,dy\,ds\Big] \\
&+ \mathbb{E}\Big[\int_{\Pi_T^2}\int_{\mathbb{R}}\int_{\mathbf{E}}\int_0^1 \Big(\beta_\vartheta\big(v(t,x,\alpha)+\eta(v(t,x,\alpha);z)-k\big) - \beta_\vartheta(v(t,x,\alpha)-k)\Big) \\
&\qquad\qquad \times \phi_{\delta,\delta_0}\,dx\,d\alpha\,\tilde{N}(dz,dt)\varsigma_l(u_\varepsilon(s,y)-k)\,dk\,dy\,ds\Big] \\
&+ \mathbb{E}\Big[\int_{\Pi_T^2}\int_{\mathbf{E}}\int_{\mathbb{R}}\int_0^1 \Big(\beta_\vartheta\big(v(t,x,\alpha)+\eta(v(t,x,\alpha);z)-k\big) - \beta_\vartheta(v(t,x,\alpha)-k) \\
&\qquad\qquad - \eta(v(t,x,\alpha);z)\beta_\vartheta'(v(t,x,\alpha)-k)\Big)\phi_{\delta,\delta_0}\varsigma_l(u_\varepsilon(s,y)-k)\,d\alpha\,dk\,dx\,m(dz)\,dt\,dy\,ds\Big] \\
&+ \mathbb{E}\Big[\int_{\Pi_T^2}\int_0^1\int_{\mathbb{R}} \mathcal{F}^{\beta_\vartheta}(v(t,x,\alpha),k)\vec{v}(t,x)\cdot\nabla_x\phi_{\delta,\delta_0}(t,x,s,y)\varsigma_l(u_\varepsilon(s,y)-k)\,dk\,d\alpha\,dx\,dt\,dy\,ds\Big] \\
=: & I_1 + I_2 + I_3 + I_4 + I_5. \qquad (7.16)
\end{aligned}$$

Since $u_\varepsilon(s,y)$ is a viscous solution to the problem (7.1), one has

$$\begin{aligned}
0 \le & \mathbb{E}\Big[\int_{\Pi_T}\int_{\mathbb{R}^d}\int_0^1\int_{\mathbb{R}} \beta_\vartheta(u_\varepsilon(0,y)-k)\phi_{\delta,\delta_0}(t,x,0,y)\varsigma_l(v(t,x,\alpha)-k)dk\,d\alpha dx\,dydt\Big] \\
&+ \mathbb{E}\Big[\int_{\Pi_T^2}\int_0^1\int_{\mathbb{R}} \beta_\vartheta(u_\varepsilon(s,y)-k)\partial_s\phi_{\delta,\delta_0}(t,x,s,y)\varsigma_l(v(t,x,\alpha)-k)\,dk\,d\alpha\,dy\,ds\,dx\,dt\Big] \\
&+ \mathbb{E}\Big[\int_{\Pi_T^2}\int_{\mathbf{E}}\int_{\mathbb{R}}\int_0^1 \Big(\beta_\vartheta\big(u_\varepsilon(s,y)+\eta_\varepsilon(y,u_\varepsilon(s,y);z)-k\big) - \beta_\vartheta(u_\varepsilon(s,y)-k)\Big) \\
&\qquad\qquad \times \phi_{\delta,\delta_0}(t,x,s,y)\varsigma_l(v(t,x,\alpha)-k)\,dy\,d\alpha\,dk\,\tilde{N}(dz,ds)\,dx\,dt\Big] \\
&+ \mathbb{E}\Big[\int_{\Pi_T^2}\int_{\mathbf{E}}\int_{\mathbb{R}}\int_0^1 \Big(\beta_\vartheta\big(u_\varepsilon(s,y)+\eta_\varepsilon(y,(u_\varepsilon(s,y);z)-k\big) - \eta_\varepsilon(y,u_\varepsilon(s,y);z)\beta_\vartheta'(u_\varepsilon(s,y)-k) \\
&\qquad\qquad - \beta_\vartheta(u_\varepsilon(s,y)-k)\Big)\phi_{\delta,\delta_0}(t,x;s,y)\varsigma_l(v(t,x,\alpha)-k)\,d\alpha\,dk\,dy\,m(dz)\,ds\,dx\,dt\Big] \\
&+ \mathbb{E}\Big[\int_{\Pi_T^2}\int_0^1\int_{\mathbb{R}} \mathcal{F}^{\beta_\vartheta}(u_\varepsilon(s,y),k)\vec{v}(s,y)\cdot\nabla_y\varrho_\delta(x-y)\psi(s,y)\rho_{\delta_0}(t-s)\varsigma_l(v(t,x,\alpha)-k)\,dk\,d\alpha\,dx\,dt\,dy\,ds\Big] \\
&+ \mathbb{E}\Big[\int_{\Pi_T^2}\int_0^1\int_{\mathbb{R}} \mathcal{F}^{\beta_\vartheta}(u_\varepsilon(s,y),k)\vec{v}(s,y)\cdot\nabla_y\psi(s,y)\varrho_\delta(x-y)\rho_{\delta_0}(t-s)\varsigma_l(v(t,x,\alpha)-k)\,dk\,d\alpha\,dx\,dt\,dy\,ds\Big] \\
&- \varepsilon\mathbb{E}\Big[\int_{\Pi_T^2}\int_0^1\int_{\mathbb{R}} \beta_\vartheta'(u_\varepsilon(s,y)-k)\nabla_y u_\varepsilon(s,y)\cdot\nabla_y\phi_{\delta,\delta_0}(t,x,s,y)\varsigma_l(v(t,x,\alpha)-k)\,dk\,d\alpha\,dy\,ds\,dx\,dt\Big] \\
=: & J_1 + J_2 + J_3 + J_4 + J_5 + J_6 + J_7. \qquad (7.17)
\end{aligned}$$

We now add (7.16) and (7.17), and compute limits with respect to the various parameters involved. In [9], convergence of the terms $I_i (i = 1,2,3,4)$ and $J_j (j = 1,2,3,4,7)$ has been studied in details. Therefore, we only study the terms involving flux function namely the terms $I_5$, $J_5$ and $J_6$ in details.

We first consider the term $I_5 + J_5$ and prove the following lemma.

**Lemma 7.7.** *There holds*

$$\limsup_{\delta\downarrow 0,\ \vartheta\downarrow 0,\ \varepsilon\downarrow 0\ l\downarrow 0\ \delta_0\downarrow 0} \big|I_5 + J_5\big| = 0.$$

*Proof.* Note that

$$
\begin{aligned}
&\Big|I_5 - \mathbb{E}\Big[\int_{\Pi_T}\int_{\mathbb{R}^d}\int_0^1\int_{\mathbb{R}} \mathcal{F}^{\beta_\vartheta}(v(s,x,\alpha),k)\vec{v}(s,x)\cdot\nabla_x\varrho_\delta(x-y)\psi(s,y)\varsigma_l(u_\varepsilon(s,y)-k)\,dk\,d\alpha\,dx\,dy\,ds\Big]\Big| \\
&\le \mathbb{E}\Big[\int_{\Pi_T^2}\int_0^1\int_{\mathbb{R}} \big|\mathcal{F}^{\beta_\vartheta}(v(t,x,\alpha),k)-\mathcal{F}^{\beta_\vartheta}(v(s,x,\alpha),k)\big||\vec{v}(t,x)||\nabla_x\varrho_\delta(x-y)|\psi(s,y) \\
&\qquad\qquad \times \rho_{\delta_0}(t-s)\varsigma_l(u_\varepsilon(s,y)-k)\,dk\,d\alpha\,dx\,dy\,dt\,ds\Big] \\
&\quad + \Big|\mathbb{E}\Big[\int_{\Pi_T}\int_{\mathbb{R}^d}\int_0^1\int_{\mathbb{R}} \mathcal{F}^{\beta_\vartheta}(v(s,x,\alpha),k)\vec{v}(s,x)\cdot\nabla_x\varrho_\delta(x-y)\,\psi(s,y)\Big(1-\int_{t=0}^T\rho_{\delta_0}(t-s)\,dt\Big) \\
&\qquad\qquad \times \varsigma_l(u_\varepsilon(s,y)-k)\,dk\,d\alpha\,dx\,dy\,ds\Big]\Big| \\
&\quad + \mathbb{E}\Big[\int_{\Pi_T^2}\int_0^1\int_{\mathbb{R}} |\mathcal{F}^{\beta_\vartheta}(v(s,x,\alpha),k)||\vec{v}(s,x)-\vec{v}(t,x)||\nabla_x\varrho_\delta(x-y)|\,\psi(s,y) \\
&\qquad\qquad \times \rho_{\delta_0}(t-s)\varsigma_l(u_\varepsilon(s,y)-k)\,dk\,d\alpha\,dx\,dy\,ds\,dt\Big]\Big| \\
&\le \mathbb{E}\Big[\int_{s=\delta_0}^T\int_{\Pi_T}\int_{\mathbb{R}^d}\int_0^1\int_{\mathbb{R}} \big|\mathcal{F}^{\beta_\vartheta}(v(t,x,\alpha),k)-\mathcal{F}^{\beta_\vartheta}(v(s,x,\alpha),k)\big||\vec{v}(t,x)||\nabla_x\varrho_\delta(x-y)|\psi(s,y) \\
&\qquad\qquad \times \rho_{\delta_0}(t-s)\varsigma_l(u_\varepsilon(s,y)-k)\,dk\,d\alpha\,dx\,dy\,dt\,ds\Big] + \mathcal{O}(\delta_0) \\
&+ C\mathbb{E}\Big[\int_{s=0}^{\delta_0}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_0^1\int_{\mathbb{R}} \big|\mathcal{F}^{\beta_\vartheta}(v(s,x,\alpha),k)\vec{v}(s,x)\cdot\nabla_x\varrho_\delta(x-y)\big|\psi(s,y)\varsigma_l(u_\varepsilon(s,y)-k)\,dk\,d\alpha\,dx\,dy\,ds\Big] \\
&\qquad \text{(we have used the fact that } \int_0^T\rho_{\delta_0}(t-s)\,dt\le 1, \text{ equality holds if } s\ge\delta_0). \\
&\le C\mathbb{E}\Big[\int_{s=\delta_0}^T\int_{\Pi_T}\int_{\mathbb{R}^d}\int_0^1 |\nabla_x\varrho_\delta(x-y)|\big|v(t,x,\alpha)-v(s,x,\alpha)\big|\psi(s,y)\rho_{\delta_0}(t-s)\,d\alpha\,dx\,dy\,dt\,ds\Big] + \mathcal{O}(\delta_0) \\
&\qquad \text{(we have used the Lipschitz continuity of } \mathcal{F}^{\beta_\vartheta}(\cdot,k) \text{ in above)} \\
&\le C\Big(\mathbb{E}\Big[\int_{s=\delta_0}^T\int_{\Pi_T}\int_0^1 |v(t,x,\alpha)-v(s,x,\alpha)|^2\rho_{\delta_0}(t-s)d\alpha dx\,dt\,ds\Big]\Big)^{\frac12} + \mathcal{O}(\delta_0) \\
&\le C\Big(\mathbb{E}\Big[\int_{r=0}^1\int_{\Pi_T}\int_0^1 |v(t+\delta_0\,r,x,\alpha)-v(t,x,\alpha)|^2\rho(-r)\,d\alpha\,dt\,dx\,dr\Big]\Big)^{\frac12} + \mathcal{O}(\delta_0).
\end{aligned}
$$

In the above, we have used the notation $\mathcal{O}(\delta_0)$ to denote quantities that depend on $\delta_0$ and are bounded above by $C\delta_0$. Note that, $\lim_{\delta_0\downarrow 0}\int_{\Pi_T}\int_0^1 |v(t+\delta_0 r,x,\alpha)-v(t,x,\alpha)|^2\,d\alpha\,dx\,dt \to 0$ almost surely for all $r\in[0,1]$. Therefore, by the bounded convergence theorem,

$$
\lim_{\delta_0\downarrow 0}\mathbb{E}\Big[\int_{r=0}^1\int_{\Pi_T}\int_0^1 |v(t+\delta_0 r,x,\alpha)-v(t,x,\alpha)|^2\rho(-r)\,d\alpha\,dx\,dt\,dr\Big] = 0.
$$

Since $\nabla_y\varrho_\delta(x-y) = -\nabla_x\varrho_\delta(x-y)$, we see that

$$
\begin{aligned}
&\Big|I_5 + \mathbb{E}\Big[\int_{\Pi_T}\int_{\mathbb{R}^d}\int_0^1\int_{\mathbb{R}} \mathcal{F}^{\beta_\vartheta}(v(s,x,\alpha),u_\varepsilon(s,y)-k)\vec{v}(s,x)\cdot\nabla_y\varrho_\delta(x-y)\,\psi(s,y)\varsigma_l(k)\,dk\,d\alpha\,dx\,dy\,ds\Big]\Big| \\
&\le A(\delta_0) + \mathcal{O}(\delta_0),
\end{aligned}
$$

for some $A(\delta_0)$, with the property that $A(\delta_0)\to 0$ as $\delta_0\to 0$. In a similar manner, one has

$$
\begin{aligned}
&\Big|J_5 - \mathbb{E}\Big[\int_{\Pi_T}\int_{\mathbb{R}^d}\int_{\mathbb{R}}\int_0^1 \mathcal{F}^{\beta_\vartheta}(u_\varepsilon(s,y),v(s,x,\alpha)-k)\vec{v}(s,y)\cdot\nabla_y\varrho_\delta(x-y)\,\psi(s,y)\varsigma_l(k)\,d\alpha\,dk\,dx\,dy\,ds\Big]\Big| \\
&\le B(\delta_0) + \mathcal{O}(\delta_0),
\end{aligned}
$$

where $B(\delta_0)$ is a quantity satisfying $B(\delta_0) \to 0$ as $\delta_0 \to 0$. Note that, since $\operatorname{div}_x \vec{v}(t,x) = 0$ for all $(t,x) \in \Pi_T$, and $\nabla_y \varrho_\delta(x-y) = -\nabla_x \varrho_\delta(x-y)$, integration by parts formula yields

$$\mathbb{E}\Big[\int_{\Pi_T}\int_{\mathbb{R}^d}\int_{\mathbb{R}}\int_0^1 \mathcal{F}^{\beta_\vartheta}(u_\varepsilon(s,y), v(s,y,\alpha)-k)[\vec{v}(s,x)-\vec{v}(s,y)]\cdot\nabla_y\varrho_\delta(x-y) \times \psi(s,y)\varsigma_l(k)\,dk\,d\alpha\,dx\,dy\,ds\Big] = 0.$$

Hence, we have

$$\begin{aligned}\big|I_5+J_5\big| \le \Big|&\mathbb{E}\Big[\int_{\Pi_T}\int_{\mathbb{R}^d}\int_{\mathbb{R}}\int_0^1 \Big(\mathcal{F}^{\beta_\vartheta}(v(s,x,\alpha), u_\varepsilon(s,y)-k) - \mathcal{F}^{\beta_\vartheta}(u_\varepsilon(s,y), v(s,x,\alpha)-k)\Big) \\ &\qquad \times \vec{v}(s,x)\cdot\nabla_x\varrho_\delta(x-y)\psi(s,y)\varsigma_l(k)\,d\alpha\,dk\,dx\,ds\,dy\Big] \\ &+\mathbb{E}\Big[\int_{\Pi_T}\int_{\mathbb{R}^d}\int_{\mathbb{R}}\int_0^1 \Big(\mathcal{F}^{\beta_\vartheta}(u_\varepsilon(s,y), v(s,x,\alpha)-k) - \mathcal{F}^{\beta_\vartheta}(u_\varepsilon(s,y), v(s,y,\alpha)-k)\Big) \\ &\qquad \times \big(\vec{v}(s,x)-\vec{v}(s,y)\big)\cdot\nabla_x\varrho_\delta(x-y)\psi(s,y)\varsigma_l(k)\,d\alpha\,dk\,dx\,ds\,dy\Big]\Big| \\ &+ A(\delta_0) + B(\delta_0) + \mathcal{O}(\delta_0).\end{aligned}$$

Define $\mathcal{F}(a,b) = \operatorname{sign}(a-b)\big(f(a)-f(b)\big)$. Then, $\mathcal{F}$ is symmetric (*i.e.*, $\mathcal{F}(a,b) = \mathcal{F}(b,a)$) and Lipschitz continuous in both of its variables. Moreover,

$$\big|\mathcal{F}^{\beta_\vartheta}(a,b) - \mathcal{F}(a,b)\big| \le \vartheta c_f. \tag{7.18}$$

Therefore, one has

$$\Big|I_5+J_5\Big| \le C(c_f,v,\psi)\delta + C(c_f,V,\psi)\frac{\vartheta}{\delta} + C(c_f,V,\psi)\frac{l}{\delta} + A(\delta_0) + B(\delta_0) + \mathcal{O}(\delta_0)$$

and hence

$$\limsup_{\delta\downarrow 0,\ \vartheta\downarrow 0,\ \varepsilon\downarrow 0\ l\downarrow 0\ \delta_0\downarrow 0} \big|I_5+J_5\big| = 0.$$

□

**Lemma 7.8.** *It holds that*

$$\begin{aligned} J_6 &\underset{\delta_0\to 0}{\to} \mathbb{E}\Big[\int_{\Pi_T}\int_{\mathbb{R}^d}\int_0^1\int_{\mathbb{R}} \mathcal{F}^{\beta_\vartheta}(u_\varepsilon(s,y),k)\vec{v}(s,y)\cdot\nabla_y\psi(s,y)\varrho_\delta(x-y)\varsigma_l(v(s,x,\alpha)-k)\,dk\,d\alpha\,dx\,dy\,ds\Big] \\ &\underset{l\to 0}{\to} \mathbb{E}\Big[\int_{\Pi_T}\int_{\mathbb{R}^d}\int_0^1 \mathcal{F}^{\beta_\vartheta}(u_\varepsilon(s,y), v(s,x,\alpha))\vec{v}(s,y)\cdot\nabla_y\psi(s,y)\,\varrho_\delta(x-y)\,d\alpha\,dx\,dy\,ds\Big] \\ &\underset{\varepsilon\to 0}{\to} \mathbb{E}\Big[\int_{\Pi_T}\int_{\mathbb{R}^d}\int_0^1\int_0^1 \mathcal{F}^{\beta_\vartheta}(u(s,y,\gamma), v(s,x,\alpha))\vec{v}(s,y)\cdot\nabla_y\psi(s,y)\varrho_\delta(x-y)\,d\gamma\,d\alpha\,dx\,dy\,ds\Big] \\ &\underset{\vartheta\to 0}{\to} \mathbb{E}\Big[\int_{\Pi_T}\int_{\mathbb{R}^d}\int_0^1\int_0^1 \mathcal{F}(u(s,y,\gamma), v(s,x,\alpha))\vec{v}(s,y)\cdot\nabla_y\psi(s,y)\varrho_\delta(x-y)\,d\gamma\,d\alpha\,dx\,dy\,ds\Big] \\ &\underset{\delta\to 0}{\to} \mathbb{E}\Big[\int_{\Pi_T}\int_0^1\int_0^1 \mathcal{F}(u(s,y,\gamma), v(s,y,\alpha))\cdot\nabla_y\psi(s,y)\,d\gamma\,d\alpha\,dy\,ds\Big]. \end{aligned}$$

*Proof.* The proof is divided into five steps.

**Step 1:** We will justify the $\delta_0 \to 0$ limit. Define

$$\begin{aligned}\mathcal{B}_1 &:= \Big|J_6 - \mathbb{E}\Big[\int_{\Pi_T}\int_{\mathbb{R}^d}\int_{\mathbb{R}}\int_0^1 \mathcal{F}^{\beta_\vartheta}(u_\varepsilon(s,y),k)\vec{v}(s,y)\cdot\nabla_y\psi(s,y)\varrho_\delta(x-y)\varsigma_l(v(s,x,\alpha)-k)d\alpha\,dk\,dy\,dx\,ds\Big]\Big| \\ &= \Big|\mathbb{E}\Big[\int_{\Pi_T^2}\int_{\mathbb{R}}\int_0^1 \Big(\mathcal{F}^{\beta_\vartheta}(u_\varepsilon(s,y), v(t,x,\alpha)-k) - \mathcal{F}^{\beta_\vartheta}(u_\varepsilon(s,y), v(s,x,\alpha)-k)\Big)\vec{v}(s,y)\cdot\nabla_y\psi(s,y) \\ &\qquad \times \rho_{\delta_0}(t-s)\varrho_\delta(x-y)\varsigma_l(k)\,d\alpha\,dk\,dy\,ds\,dx\,dt\Big]\end{aligned}$$

$$
\begin{aligned}
&-\mathbb{E}\Big[\int_{\Pi_T}\int_{\mathbb{R}^d}\int_{\mathbb{R}}\int_0^1 \mathcal{F}^{\beta_\vartheta}(u_\varepsilon(s,y),v(s,x,\alpha)-k)\vec{v}(s,y)\cdot\nabla_y\psi(s,y)\,\varrho_\delta(x-y)\\
&\qquad\qquad\times\Big(1-\int_0^T\rho_{\delta_0}(t-s)\,dt\Big)\varsigma_l(k)d\alpha\,dk\,dy\,dx\,ds\Big]\Big|\\
&\le C\mathbb{E}\Big[\int_{\Pi_T^2}\int_0^1|\nabla_y\psi(s,y)|\,\rho_{\delta_0}(t-s)\varrho_\delta(x-y)|v(s,x,\alpha)-v(t,x,\alpha)|\,d\alpha\,dy\,ds\,dx\,dt\Big]\\
&\quad+C\mathbb{E}\Big[\int_0^{\delta_0}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}}\int_0^1\big|\mathcal{F}^{\beta_\vartheta}(u_\varepsilon(s,y),k)||\nabla_y\psi(s,y)\big|\,\varrho_\delta(x-y)\varsigma_l(v(s,x,\alpha)-k)d\alpha\,dk\,dy\,dx\,ds\Big]\\
&\le C\mathbb{E}\Big[\int_{\delta_0}^T\int_{\mathbb{R}^d}\int_0^T\int_0^1|v(s,x,\alpha)-v(t,x,\alpha)|\rho_{\delta_0}(t-s)\,d\alpha\,dt\,dx\,ds\Big]+\mathcal{O}(\delta_0)\\
&\le C\Big(\mathbb{E}\Big[\int_{\delta_0}^T\int_0^T\int_{\mathbb{R}^d}\int_0^1|v(s,x,\alpha)-v(t,x,\alpha)|^2\rho_{\delta_0}(t-s)d\alpha\,dx\,dt\,ds\Big]\Big)^{\frac12}+\mathcal{O}(\delta_0)\longrightarrow 0 \text{ as } \delta_0\to 0,
\end{aligned}
$$

and therefore the first step follows.

**Step 2:** We will justify the $l\to 0$ limit. Let

$$
\begin{aligned}
\mathcal{B}_2 :=&\mathbb{E}\Big[\int_{\Pi_T}\int_{\mathbb{R}^d}\int_0^1\int_{\mathbb{R}}\mathcal{F}^{\beta_\vartheta}(u_\varepsilon(s,y),k)\vec{v}(s,y)\cdot\nabla_y\psi(s,y)\varrho_\delta(x-y)\varsigma_l(v(s,x,\alpha)-k)\,dk\,d\alpha\,dx\,dy\,ds\Big]\\
&-\mathbb{E}\Big[\int_{\Pi_T}\int_{\mathbb{R}^d}\int_0^1\mathcal{F}^{\beta_\vartheta}(u_\varepsilon(s,y),v(s,x,\alpha))\vec{v}(s,y)\cdot\nabla_y\psi(s,y)\,\varrho_\delta(x-y)\,d\alpha\,dx\,dy\,ds\Big]\\
=&\,\mathbb{E}\Big[\int_{\Pi_T}\int_{\mathbb{R}^d}\int_0^1\int_{\mathbb{R}}\Big(\mathcal{F}^{\beta_\vartheta}(u_\varepsilon(s,y),k)-\mathcal{F}^{\beta_\vartheta}(u_\varepsilon(s,y),v(s,x,\alpha))\Big)\vec{v}(s,y)\cdot\nabla_y\psi(s,y)\\
&\qquad\qquad\times\varrho_\delta(x-y)\varsigma_l(v(s,x,\alpha)-k)\,dk\,d\alpha\,dx\,dy\,ds\Big].
\end{aligned}
$$

By using the boundedness of $\vec{v}$ and Lipschitz property of $\mathcal{F}^{\beta_\vartheta}$, we arrive at

$$
|\mathcal{B}_2|\le Cl\int_{\Pi_T}|\nabla_y\psi(s,y)|\,dy\,ds\to 0\quad\text{as } l\to 0.
$$

**Step 3:** We now justify the passage to the limit $\varepsilon\to 0$. Let

$$
G_x(s,y,\omega,\xi)=\int_{\mathbb{R}^d}\int_0^1\mathcal{F}^{\beta_\vartheta}(\xi,v(s,x,\alpha))\vec{v}(s,y)\cdot\nabla_y\psi(s,y)\,\varrho_\delta(x-y)\,d\alpha\,dx.
$$

Then $G_x(s,y,\omega,\xi)$ is a Carathéodory function for every $x\in\mathbb{R}^d$ and $\{G_x(s,y,\omega,u_{\varepsilon_n}(s,y))\}_n$ is bounded in $L^2((\Theta,\Sigma,\mu);\mathbb{R})$ and uniformly integrable. Thus, by Proposition 6.1 we conclude that

$$
\begin{aligned}
&\lim_{\varepsilon\to 0}\mathbb{E}\Big[\int_{\mathbb{R}^d}\int_{\Pi_T}\int_0^1\mathcal{F}^{\beta_\vartheta}(u_\varepsilon(s,y),v(s,x,\alpha))\vec{v}(s,y)\cdot\nabla_y\psi(s,y)\,\varrho_\delta(x-y)\,d\alpha\,dx\,ds\,dy\Big]\\
&=\mathbb{E}\Big[\int_{\mathbb{R}^d}\int_{\Pi_T}\int_0^1\int_0^1\mathcal{F}^{\beta_\vartheta}(u(s,y,\gamma),v(s,x,\alpha))\vec{v}(s,y)\cdot\nabla_y\psi(s,y)\,\varrho_\delta(x-y)\,d\gamma d\alpha dx ds dy\Big].
\end{aligned}
$$

**Step 4:** Justification of the limit $\vartheta\to 0$. Let

$$
\begin{aligned}
\mathcal{B}_3 :=\mathbb{E}\Big[\int_{\mathbb{R}^d}\int_{\Pi_T}\int_0^1\int_0^1&\Big(\mathcal{F}^{\beta_\vartheta}(u(s,y,\gamma),v(s,x,\alpha))-\mathcal{F}(u(s,y,\gamma),v(s,x,\alpha))\Big)\vec{v}(s,y)\cdot\nabla_y\psi(s,y)\\
&\times\varrho_\delta(x-y)\,d\gamma d\alpha dx ds dy\Big].
\end{aligned}
$$

In view of (7.18) and the assumption **A**.2, we see that

$$
|\mathcal{B}_3|\le C(V,c_f)\vartheta\int_{\Pi_T}|\nabla_y\psi(s,y)|\,dy\,ds\to 0\quad\text{as}\quad\vartheta\to 0.
$$

**Step 5:** Justification of the limit $\delta\to 0$.

$$
\mathcal{B}_4 :=\Big|\mathbb{E}\Big[\int_{\Pi_T}\int_{\mathbb{R}^d}\int_0^1\int_0^1\mathcal{F}(u(s,y,\gamma),v(s,x,\alpha))\vec{v}(s,y)\cdot\nabla_y\psi(s,y)\,\varrho_\delta(x-y)\,d\gamma\,d\alpha\,dx\,dy\,ds\Big]
$$

$$- \mathbb{E}\Big[\int_{\Pi_T}\int_0^1\int_0^1 \mathcal{F}(u(s,y,\gamma),v(s,y,\alpha))\vec{v}(s,y)\cdot\nabla_y\psi(s,y)\,d\gamma\,d\alpha\,dy\,ds\Big]\Big|$$

$$\le C\mathbb{E}\Big[\int_{\Pi_T}\int_{\mathbb{R}^d}\int_0^1\int_0^1 \big|\mathcal{F}(u(s,y,\gamma),v(s,x,\alpha))-\mathcal{F}(u(s,y,\gamma),v(s,y,\alpha))\big| \times|\nabla_y\psi(s,y)|\,\varrho_\delta(x-y)\,d\gamma\,d\alpha\,dx\,dy\,ds\Big].$$

Since $\mathcal{F}$ is Lipschitz continuous in both of its variables, by Cauchy-Schwartz's inequality, we have

$$|\mathcal{B}_4|\le C\Big(E\Big[\int_{\Pi_T}\int_{\mathbb{R}^d}\int_0^1 |v(s,y,\gamma)-v(s,y+\delta z,\gamma)|^2\varrho(z)\,d\gamma\,dy\,dz\,ds\Big]\Big)^{\frac12}\to 0 \quad \text{as } \delta\to 0.$$

This completes the proof. □

Following [9], we arrive at

**Lemma 7.9.** *The following hold:*

$$\lim_{(\delta,\vartheta,\varepsilon,l,\delta_0)\to 0}(I_1+J_1)=\mathbb{E}\Big[\int_{\mathbb{R}^d}|v_0(x)-u_0(x)|\psi(0,x)\,dx\Big];\qquad \limsup_{(\varepsilon,l,\delta_0)\to(0)}|J_7|=0.$$

$$\lim_{(\delta,\vartheta,\varepsilon,l,\delta_0)\to 0}(I_2+J_2)=\mathbb{E}\Big[\int_{\Pi_T}\int_0^1\int_0^1|u(s,y,\gamma)-v(s,y,\alpha)|\partial_s\psi(s,y)\,d\gamma\,d\alpha\,dy\,ds\Big].$$

$$\begin{aligned}\lim_{(l,\delta_0)\to 0}(I_4+J_4)=\mathbb{E}\Big[\int_{\Pi_T}\int_{\mathbb{R}^d}\int_{\mathbf{E}}\int_0^1\int_0^1 &\Big\{\beta_\vartheta''\big(u_\varepsilon(s,y)-v(s,x,\alpha)+\lambda\eta(u_\varepsilon(s,y);z)\big)|\eta(u_\varepsilon(s,y);z)|^2\\ &+\beta_\vartheta''\big(v(s,x,\alpha)-u_\varepsilon(s,y)+\lambda\eta(v(s,x,\alpha);z)\big)|\eta(v(s,x,\alpha);z)|^2\Big\}\\ &\times(1-\lambda)\psi(s,y)\varrho_\delta(x-y)\,d\alpha\,d\lambda\,m(dz)\,dy\,dy\,ds\Big].\end{aligned}$$

Let us consider the stochastic integrals. Note that $J_3=0$. In view of Itô-Lévy formula, we see that

$$\begin{aligned}I_3=&\,\mathbb{E}\Big[\int_{\mathbb{R}}\int_{\Pi_T}J[\beta_\vartheta',\phi_{\delta,\delta_0}](s;y,k)\Big(\int_{s-\delta_0}^s\varsigma_l(u_\varepsilon(\sigma,y)-k)\operatorname{div}(f(u_\varepsilon(\sigma,y))\vec{v}(\sigma,y))\,d\sigma\Big)\,ds\,dy\,dk\Big]\\ &-\mathbb{E}\Big[\int_{\mathbb{R}}\int_{\Pi_T}J[\beta_\vartheta',\phi_{\delta,\delta_0}](s;y,k)\Big(\int_{s-\delta_0}^s\varsigma_l(u_\varepsilon(\sigma,y)-k)\,\varepsilon\Delta u_\varepsilon(\sigma,y))\,d\sigma\Big)\,ds\,dy\,dk\Big]\\ &+\mathbb{E}\Big[\int_{\Pi_T}\int_{\mathbb{R}}\int_{r=s-\delta_0}^s\int_{\mathbb{R}^d}\int_{\mathbf{E}}\int_0^1\Big(\beta_\vartheta\big(v(r,x,\alpha)+\eta(v(r,x,\alpha);z)-k\big)-\beta_\vartheta\big(v(r,x,\alpha)-k\big)\Big)\\ &\qquad\times\Big(\varsigma_l(u_\varepsilon(r,y)+\eta(u_\varepsilon(r,y);z)-k)-\varsigma_l(u_\varepsilon(r,y)-k)\Big)\\ &\qquad\times\rho_{\delta_0}(r-s)\,\psi(s,y)\,\varrho_\delta(x-y)\,d\alpha\,m(dz)\,dx\,dr\,dk\,dy\,ds\Big]\\ &+\mathbb{E}\Big[\int_{\mathbb{R}}\int_{\Pi_T}J[\beta_\vartheta,\phi_{\delta,\delta_0}](s;y,k)\Big\{\int_{s-\delta_0}^s\int_{\mathbf{E}}\int_0^1(1-\lambda)|\eta(u_\varepsilon(\sigma,y);z)|^2\\ &\qquad\times\varsigma_l''(u_\varepsilon(\sigma,y)-k+\lambda\eta(u_\varepsilon(\sigma,y);z))\,d\lambda\,m(dz)\,d\sigma\Big\}\,dy\,ds\,dk\Big]\\ =:&\,A_1^{l,\varepsilon}(\delta,\delta_0)+A_2^{l,\varepsilon}(\delta,\delta_0)+B^{\varepsilon,l}+A_3^{l,\varepsilon}(\delta,\delta_0),\end{aligned}$$

where

$$\begin{aligned}J[\beta_\vartheta,\phi_{\delta,\delta_0}](s;y,k):=\int_{\Pi_T}\int_{\mathbf{E}}\int_0^1&\Big(\beta_\vartheta\big(v(r,x,\alpha)+\eta(v(r,x,\alpha);z)-k\big)-\beta_\vartheta\big(v(r,x,\alpha)-k\big)\Big)\\ &\times\phi_{\delta,\delta_0}(r,x,s,y)\,d\alpha\,\tilde{N}(dz,dr)\,dx.\end{aligned}$$

Thanks to the assumption **A**.2, by following the arguments as in the proof of [9, Lemma 5.6], we infer that $A_1^{l,\varepsilon}(\delta,\delta_0)$, $A_2^{l,\varepsilon}(\delta,\delta_0)$, $A_3^{l,\varepsilon}(\delta,\delta_0)\to 0$ as $\delta_0\to 0$, and

$$\lim_{(l,\delta_0)\to 0}B^{l,\varepsilon}(\delta,\delta_0)=\mathbb{E}\Big[\int_{\Pi_T}\int_{\mathbb{R}^d}\int_{\mathbf{E}}\int_0^1\Big\{\beta_\vartheta\big(v(r,x,\alpha)+\eta(v(r,x,\alpha);z)-u_\varepsilon(r,y)-\eta(u_\varepsilon;z)\big)$$

$$- \beta_\vartheta\big(v(r,x,\alpha)+\eta(v(r,x,\alpha);z)-u_\varepsilon(r,y)\big)+\beta_\vartheta\big(v(r,x,\alpha)-u_\varepsilon(r,y)\big)$$
$$- \beta_\vartheta\big(v(r,x,\alpha)-u_\varepsilon(r,y)-\eta(u_\varepsilon(r,y);z)\big)\Big\}\psi(r,y)\,\varrho_\delta(x-y)\,d\alpha\,m(dz)\,dx\,dy\,dr\Big],$$

which yields

$$\lim_{l\to 0}\lim_{\delta_0\to 0}\Big((I_3+J_3)+(I_4+J_4)\Big)$$
$$=\mathbb{E}\Big[\int_{\Pi_T}\int_{\mathbb{R}^d}\Big(\int_{\mathbf{E}}\int_0^1\int_0^1 b^2(1-\theta)\beta_\vartheta''(a+\theta\, b)\,d\theta\,d\alpha\,m(dz)\Big)\,\psi(t,y)\varrho_\delta(x-y)\,dx\,dy\,dt\Big],$$

where $a=v(t,x,\alpha)-u_\varepsilon(t,y)$ and $b=\eta(v(t,x,\alpha);z)-\eta(u_\varepsilon(t,y);z)$. In view of **A**.3, one has (cf. proof of [9, Lemma 5.11]) $b^2\beta_\vartheta''(a+\theta\, b)\le 2(1-\lambda^*)^{-2}\vartheta h_1^2(z)$, and thus

$$\limsup_{\delta\to 0,\,\vartheta\to 0}\;\limsup_{\varepsilon\to 0}\Big[\lim_{l\to 0}\lim_{\delta_0\to 0}\Big((I_3+J_3)+(I_4+J_4)\Big)\Big]=0.$$

Finally, we add (7.16) and (7.17), and pass to the limits $\delta_0\to 0$, $l\to 0$, $\varepsilon\to 0$, $\vartheta\to 0$ and $\delta\to 0$ to arrive at the following Kato inequality

$$\mathbb{E}\Big[\int_{\mathbb{R}^d}|v_0(x)-u_0(x)|\psi(0,x)\,dx\Big]+\mathbb{E}\Big[\int_{\Pi_T}\int_0^1\int_0^1|v(t,x,\alpha)-u(t,x,\gamma)|\partial_t\psi(t,x)\,d\alpha\,d\gamma\,dx\,dt\Big]$$
$$+\mathbb{E}\Big[\int_{\Pi_T}\int_0^1\int_0^1\mathcal{F}\big(v(t,x,\alpha),u(t,x,\gamma)\big)\vec{v}(t,x)\cdot\nabla_x\psi(t,x)\,d\alpha\,d\gamma\,dx\,dt\Big]\ge 0,\quad (7.19)$$

where $0\le\psi\in H^1([0,\infty)\times\mathbb{R}^d)$ with compact support. One can choose special test function $\psi$ and $u_0=v_0$ in (7.19) to conclude that $u(t,x,\gamma)=v(t,x,\alpha)$ for a.e. $(t,x)\in\Pi_T$ and a.e. $(\alpha,\gamma)\in(0,1)^2$ (cf. proof of [9, Theorem 2.2]). This finishes the proof.

7.4. **On Poisson random measure.** For the convenience of the reader, we recapitulate the basics of Poisson random measure. Let $\{\tau_n\}_{n\ge 1}$ be a sequence of independent exponential random variables with parameter $\iota$ and $T_n=\sum_{i=1}^n\tau_i$. Then the process

$$N_t=\sum_{n\ge 1}\mathbf{1}_{t\ge T_n}$$

counts the number of random times $T_n$ which arise between 0 and $t$. The jump times $T_1,T_2,\cdots$ form a random configuration of points on $[0,\infty)$. This counting procedure defines a measure on $[0,\infty)$ as follows: for any measurable set $A\subset(0,\infty)$, set $N(\omega,A)=\#\{i\ge 1: T_i(\omega)\in A\}$. Clearly, $N(\omega,\cdot)$ is a positive integer-valued measure and for fixed $A$, $N(\cdot,A)$ is a Poisson random variable with parameter $\iota|A|$, where $|A|$ denotes the Lebesgue measure of $A$. Also, if $A$ and $B$ are two disjoint sets then $N(\cdot,A)$ and $N(\cdot,B)$ are two independent random variables. This can be extended to a general setting. Let $\bar{\mathbb{Z}_+}=\mathbb{Z}_+\cup\{+\infty\}$.

**Definition 7.1.** Let $(\Theta,\mathcal{B},\rho)$ be a $\sigma$-finite measure space. A family of $\bar{\mathbb{Z}_+}$-valued random variables $\{N(B): B\in\mathcal{B}\}$ is called a Poisson random measure on $\Theta$ with intensity measure $\rho$, if
i) For each $B$, $N(B)$ has a Poisson distribution with mean $\rho(B)$.
ii) If $B_1,B_2,\cdots,B_m$ are disjoint, then $N(B_1),N(B_2),....,N(B_m)$ are independent.
iii) For every $\omega\in\Omega$, $N(.,\omega)$ is a measure on $\Theta$.

**Construction of a Poisson random measure:** Let $(\Theta,\mathcal{B},\rho)$ be a $\sigma$-finite measure space. We want to construct a Poisson random measure $\{N(B): B\in\mathcal{B}\}$ on $\Theta$ with intensity measure $\rho$ on some probability space $(\Omega,\mathcal{F},\mathbb{P})$. Assume that $\rho(\Theta)<\infty$. If $\rho=0$, then we choose $N(B)=0$. Assume that $\rho(\Theta)>0$. Then on some probability space $(\Omega,\mathcal{F},\mathbb{P})$, one can construct a sequence $\{Z_n: n=1,2,3......\}$ of i.i.d random variables on $\Theta$ each having distribution $(\rho(\Theta))^{-1}\rho$ and a Poisson random variable $Y$ with mean $\rho(\Theta)$ such that $Y$ and $\{Z_n\}$ are independent. Define

$$N(B)=\begin{cases}0, & \text{if } Y=0\\ \sum_{j=1}^Y\chi_B(Z_j), & \text{if } Y\ge 1.\end{cases}$$

Then, $\{N(B): B\in\mathcal{B}\}$ is called a Poisson random measure on $\Theta$ with intensity measure $\rho$, see [31, Proposition 19.4]. Next we consider the case $\rho(\Theta)=\infty$. By $\sigma$-finiteness, there exist disjoint sets $\Theta_1,\Theta_2,....\in\mathcal{B}$ such that $\cup_{k=1}^\infty\Theta_k=\Theta$ and $\rho(\Theta_k)<\infty$, for each $k$. Define $\rho_k(B)=\rho(B\cap\Theta_k)$. Then, one can construct independent Poisson random measures $\{N_k(B) : B \in \mathcal{B}\}$, $k = 1,2,3,\cdots$, with intensity measure $\rho_k$, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Then

$$N(B) := \sum_{k=1}^{\infty} N_k(B), \quad B \in \mathcal{B}$$

is a Poisson random measure on $\Theta$ with intensity measure $\rho$ (cf. [31, Proposition 19.4]).

The construction of a Poisson random measure shows that it is a counting measure associated to a random sequence of points $X_n(\omega)$ in $\Theta$ such that

$$N(\omega, B) = \sum_{n\geq 1} \mathbf{1}_B(X_n(\omega)).$$

Let us give an example of a Poisson random measure on $[0,\infty)\times\mathbb{R}^d$. Let $L = \{L_t\}_{t\geq 0}$ be a Lévy process taking values in $\mathbb{R}^d$ on a given filtered probability space $\big(\Omega, \mathbb{P}, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}\big)$. For $t \geq 0$ and $A \in \mathcal{B}(\mathbb{R}_0^d)$ , where $\mathbb{R}_0^d = \mathbb{R}^d \setminus \{0\}$, we define

$$N\big([0,t], A\big) = \#\Big\{0 \leq s \leq t : L_s - L_{s-} \in A\Big\} = \sum_{s\leq t} \mathbf{1}_A(\Delta L(s))$$

where $\Delta L(s) = L_s - L_{s-}$. It counts the jumps $\Delta L(s)$ of the process $L$ of size in $A$ up to time $t$. Let $\nu(A) = \mathbb{E}\big[N\big([0,1], A\big)\big]$, the expected number of jumps of $L_t$ per unit time, whose size belongs to $A$. The Lévy measure $\nu(dz)$ may be infinite but satisfies $\int_{\mathbb{R}_0^d}(|z|^2\wedge 1)\nu(dz) < +\infty$. It is a Radon measure with a possible singularity at $z = 0$ i.e., $\nu(dz)$ restricted to each $\mathbb{R}^d \setminus B(0,r), r > 0$ is a finite measure. One can show the following properties:

i). $N\big([0,t], A\big)$ is a random variable on $\big(\Omega, P, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}\big)$.
ii). $t \mapsto N\big([0,t], A\big)$ is a Poisson process with intensity $t\nu(A)$.
iii). $N\big([0,t], \emptyset\big) = 0$ and for any disjoint sets $A_1, A_2, \cdots, A_m$, the random variables $N\big([0,t], A_1\big)$, $N\big([0,t], A_2\big), \cdots, N\big([0,t], A_m\big)$ are independent.

The compensated Poisson random measure is defined by

$$\tilde{N}\big([0,t], A\big) = N\big([0,t], A\big) - t\nu(A).$$

Next we define stochastic integral with respect to compensated Poisson random measure $\tilde{N}(dz, dt) = N(dz, dt) - \nu(dz)dt$, where $N(dz, dt)$ is a Poisson random measure and $\nu(dz)$ is a Lévy measure. To do so, let us first define it so called for simple predictable functions. A simple predictable function $f(s,z) : \Omega \times [0,T] \times \mathbb{R}_0^d \to \mathbb{R}$ is of the form

$$f(s,z) = \sum_{i=1}^{n}\sum_{j=1}^{m} \xi_{ij}\mathbf{1}_{(\tau_i,\tau_{i+1}]}(t)\mathbf{1}_{A_j}(z),$$

where $0 = \tau_0 < \tau_1 < \cdots < \tau_{n+1} = T$ are stopping times, $\xi_{i1}, \cdots, \xi_{im} \in \mathcal{F}_{\tau_i}$ for $i = 1, \cdots, n$ with $\xi_{ij}$ are bounded for all $i, j$, and $A_1, \cdots A_m \in \mathcal{B}(\mathbb{R}_0^d)$ are disjoint sets with $\nu(A_1), \cdots \nu(A_m) < \infty$. For simple predictable function $f(s,z)$ of the above form, we define

$$I_t(f) =: \int_0^t \int_{\mathbb{R}_0^d} f(s,z)\tilde{N}(dz, ds) = \sum_{i=1}^{n}\sum_{j=1}^{m} \xi_{ij}\tilde{N}\big((\tau_i \wedge t, \tau_{i+1}\wedge t], A_j\big).$$

**Lemma 7.10.** *Let $f : \Omega\times[0,T]\times\mathbb{R}_0^d \to \mathbb{R}$ be a simple predictable function. Then $I_t(f)$ is a $L^2$- martingale and satisfies the isometry property*

$$\mathbb{E}\Big[\big|I_t(f)\big|^2\Big] = \mathbb{E}\Big[\int_0^t \int_{\mathbb{R}_0^d} \big|f(s,z)\big|^2 \nu(dz)\, ds\Big]. \tag{7.20}$$

In view of the isometry property, one can extend the integral to the closure of the space of simple predictable functions in $L^2\big(\Omega\times[0,T]\times\mathbb{R}_0^d\big)$ with respect to $\sigma$-algebra $\mathcal{F}_T\otimes\mathcal{B}([0,T])\otimes\mathcal{B}(\mathbb{R}_0^d)$ and product measure $\mathbb{P}\otimes dt\otimes \nu(dz)$. Note that above mentioned closure contains all $\mathcal{P}_T\otimes\mathcal{B}(\mathbb{R}_0^d)$-measurable functions $f:\Omega\times[0,T]\times\mathbb{R}_0^d\to\mathbb{R}$ such that

$$\mathbb{E}\Big[\int_0^T\int_{\mathbb{R}_0^d}\big|f(s,z)\big|^2\nu(dz)\,ds\Big]<+\infty. \tag{7.21}$$

Thus, for any predictable functions satisfying (7.21), one can define the integral via limiting argument. One can also show that $t\mapsto\int_0^t\int_{\mathbb{R}_0^d}f(s,z)\tilde{N}(dz,ds)$ is a martingale. Moreover (7.20) holds.

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