A MATHEMATICAL MODEL FOR THE EVAPORATION OF A LIQUID FUEL DROPLET, SUBJECT TO NONLINEAR CONSTRAINTS

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We study the mathematical evolution of a liquid fuel droplet inside a vessel. In particular, we analyze the evolution of the droplet radius on a finite time interval. The model problem involves an hyperbolic system coupled with the pressure and velocity of the surrounding gas. Existence of bounded solutions for the mass fraction of the liquid, submitted to nonlinear constraints, is shown. Numerical simulations are given, in agreement with known physical experiments.

Keywords: Hyperbolic system, droplet radius, nonlinear parabolic equation.

AMS Subject Classification: 35Q35, 76T10.

1. Introduction

In this paper, we are mainly concerned with the mathematical analysis of the evaporation of a single droplet in a gas, in the continuation of our previous work[2].

Experimental studies around this subject are of course important for industrial purposes. Let us refer for instance to the works[3,4,5,6,7,8,11,13].

One of our main interest in this paper is to analyze the time evolution of the droplet radius, a study that we began in our previous work[2]. Let us just mention here that the experimental evolution of this radius is well known as the d^2 law[13,16], where d denotes the diameter of the droplet, see below for more details.

Let us recall the standard physical framework for this evolution.

The evaporation of a single droplet in a gas involves simultaneous heat and mass transfer processes. In particular, heat from evaporation is transferred to the droplet boundary by conduction and convection, while vapor is carried by convection and diffusion back into the gas stream. Evaporation rate depends on the pressure, temperature and physical properties of the gas, the temperature, volatility and diameter of the drop in the spray.

To fix the ideas, in the experimental study[13] of a single droplet evaporation performed by the LCSR (Combustion Laboratory of the University of campus from Orleans, France), the droplet is suspended from a silicate tube. The elliptic shape of the droplet is assimilated to a sphere of equal volume. Important quantities of interest for these experiments are time evolution of the droplet radius, as well as classical quantities such as mass fractions or temperatures of the liquid and gas. In the experimental studies performed above, the so-called d^2 law is used to simplify two-phase fluid models and then propose adequate numerical schemes. This law simply states that the time evolution of the radius behaves as $\frac{d^2}{d_0^2}$ in time flow, and is purely phenomenological.

Our purpose in this paper is exactly in the opposite sense. We start from phenomenological fluid (mixtures) PDE modeling the drop evaporation process, compute the time-evolution of the drop radius, and then deduce other quantities of interest such as mass fractions of the liquid and gas. In particular, our numerical experiments are in good agreement with this phenomenological d^2 law, at least for small time evolution.

Our framework is therefore as follows: we consider a droplet initially represented as a single component mixture (liquid chemical specie 1) while the surrounding gas at time $t = 0$ is made of only one (gas) chemical specie, say 2.
During the evaporation process, the liquid vapor is transferred into the gas, while by condensation at the droplet surface and then by diffusion, gas chemical specie 2 appears inside the droplet.
We make the important simplification that the moving interface between the droplet and the surrounding gas (i.e. between the two species) is spherical, with radius $R = R(t)$ evolving in time.
Let ρ_G (resp. ρ_L) denote gas density (resp. liquid density), and v_G (resp. v_L) denote the gas velocity (resp. liquid velocity). Then, one has the classical overall continuity and momentum conservation laws

$$\partial_t \rho_k + \operatorname{div}(\rho_k v_k) = 0 \tag{1.1}$$

$$\rho_k \partial_t v_k + \rho_k v_k . \nabla v_k = -\nabla p. \tag{1.2}$$

Above, subscript k refers to the gas G or to the liquid L, depending on whether one considers the gas or liquid.p the state equation of the gas.
Let Y_{L1}, Y_{L2} (resp. Y_{G1}, Y_{G2}) the mass fractions of the liquid (resp. gas) obtained after diffusion of species in the surrounding gas. Therefore for two species, one has

$$Y_{G1} + Y_{G2} = Y_{L1} + Y_{L2} = 1.$$

Along with equation (1.1), we have to add the equation giving species conservation. So for the liquid, we have

$$\rho_L \partial_t Y_{Lk} + \rho_L v_L . \nabla Y_{Lk} + \operatorname{div}(\rho_L Y_{Lk} v_{Lk}) = -\rho_L f(Y_{Lk}), \quad k = 1, 2, \tag{1.3}$$

Y_{Lk} denoting mass fraction of the liquid, and f being a continuous function modeling a friction or a resistance for the drop.
We assume that the liquid speed is so small that is can be settled to 0. Equations (1.1) and (1.3) can then be written under conservative form as

$$\partial_t(\rho \, \tilde{g}) + \operatorname{div}(\rho \, \tilde{g} v) = F(\, \tilde{g}) \tag{1.4}$$

or

$$\partial_t u + \frac{\partial}{\partial x}(f(u)) = F(u), \tag{1.5}$$

in a system of particular coordinates.

If Γ is a curve of discontinuity of u, then one has

$$[f(u)] = [u]\frac{dx}{dt}, \tag{1.6}$$

where [.] denotes the jump of the inner quantity, $s = \frac{dx}{dt}$ is the speed of discontinuity along Γ. The jump relation (1.6) is known as Rankine-Hugoniot condition. It merely means that discontinuities cannot be completely arbitrary. The above considerations are all classical facts[9,14,15,16].

In the case of our droplet, in order to find interface condition at the droplet surface, i.e. for $r = R(t)$, it is sufficient to use (1.5) and (1.4) in polar coordinates, getting

$$[\rho\, \tilde{g} v] = [\rho\, \tilde{g}]\frac{dR}{dt}. \tag{1.7}$$

Thus taking $\tilde{g} = 1$, one has

$$(\rho_G - \rho_L)\frac{dR}{dt} = \rho_G v_G - \rho_L v_L,$$

that is also with $v_L = 0$

$$\rho_G\left(v_G - \frac{dR}{dt}\right) = -\rho_L\frac{dR}{dt}. \tag{1.8}$$

Taking $\tilde{g} = Y$ in (1.7), Y denoting the mass fraction of the liquid or the gas after diffusion, we get

$$(\rho_G Y_{Gk} - \rho_L Y_{Lk})\frac{dR}{dt} = \rho_G Y_{Gk}(v_G + v_{Gk}) - \rho_L Y_{Lk} v_{Lk}$$

and this is equivalent to

$$\rho_G Y_{Gk}(v_G - R') + \rho_G Y_{Gk} v_{Gk} = -\rho_L Y_{Lk} R' + \rho_L Y_{Lk} v_{Lk}. \tag{1.9}$$

Above v_{Gk} (resp.v_{Lk}) is the speed of specie Gk (resp.Lk), $k = 1, 2$.

Combining relation (1.9) with Fick's law[7,9,16], that is

$$Y_{G1} v_{G1} =_{-} D_{12}\nabla Y_{G1},\ Y_{G2} v_{G2} = -D_{21}\nabla Y_{G2},$$

D_{12} and D_{21} being diffusion coefficients, and with equations relating the thermodynamic state at the interface $r = R(t)$

$$Y_{Gk} = K_k Y_{Lk},\ k = 1, 2,$$

we obtain, for the mass fraction of the liquid Y_{L1}, the boundary condition

$$\partial_r Y_{L1} + \frac{R'(t)(K_1 - 1)}{K_2 \rho_G(R(t), t) - K_3} Y_{L1} = 0 \text{ at } r = R(t), \tag{1.10}$$

using polar coordinates.
In our previous work[2], we have made huge mathematical and physical simplifications taking the state equation of the gas p as constant in (1.2), and considering gas velocity v_G as a given function of the time t. Thus in our previous work, system (1.1), (1.2) was reduced to equation (1.1) with a given $v_G(t)$.
In the present work, we consider the full hyperbolic system (1.1), (1.2) with an auxiliary state equation for the gas given by $p_1 = \rho^\gamma$. This of course extends our previous work, but considering such pressure laws has the advantage that we have been able to perform numerical comparisons. More general state laws will be studied in a future work.
Once $\rho_G(r,t)$ and $v_G(r,t)$ determined, radius $R(t)$ of the drop suspended in the gas will be computed through the ordinary differential equation (1.8). Then we shall determine the mass fraction Y_{L1} of liquid after evaporation process, through the PDE (1.3) along with boundary condition (1.10), for a given suitable function f.
For this last purpose, within the framework of weighted Sobolev spaces on initial data and for some continuous function f subject to increasing condition, we shall provide an unique local solution for the mass fraction Y_{L1} of the liquid. In addition, we shall show that if the initial condition is bounded, then so is our solution.
In the numerical applications (2nd example) we have chosen the experimental conditions made by the LCSR in the study of single drop evaporation and in this case the study of the radius of the drop shows us that the graphic associated to our mathematical model presents the same features as in the experimental curves.
Plan of the paper: In Section 2, by using Riemann invariants, we determine the droplet radius. This enables to get, in Section 3, the liquid mass fraction, using a variational method. Finally, we have presented some numerical simulations in the last Section, which shows that our model is is good agreement with experimental simulations, at least for short time.

2. Hyperbolic system and droplet radius

The gas velocity $v_G(r,t)$ and its density $\rho_G(r,t)$ satisfy the following system, using polar coordinates

$$\begin{aligned} \frac{\partial \rho_G}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \rho_G v_G\right) = 0 \\ \frac{\partial}{\partial t}(\rho_G v_G) + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \rho_G v_G^2\right) = -\frac{\partial p}{\partial r}. \end{aligned} \tag{2.1}$$

Setting $\rho(r,t) = r^2 \rho_G(r,t)$ and $v(r,t) = v_G(r,t)$, we have

$$\begin{aligned} \frac{\partial \rho}{\partial t} + v\frac{\partial \rho}{\partial r} + \rho\frac{\partial v}{\partial r} = 0 \\ \frac{\partial v}{\partial t} + v\frac{\partial v}{\partial r} = -\frac{1}{\rho}\frac{\partial p_1}{\partial r}, \end{aligned} \tag{2.2}$$

where $p_1(r,t)$ is an auxiliary function connected to the state equation of the gas $p(r,t)$, by

$$\frac{\partial p_1}{\partial r} = r^2 \frac{\partial p}{\partial r}.$$

In (2.2), according to the discussion in the Introduction, we choose the auxiliary function $p_1(r,t)$ as $p_1 {=} \rho^\gamma,\ \gamma > 1$.

With this choice, we get the following system

$$\begin{aligned} \frac{\partial\rho}{\partial t} + v\frac{\partial\rho}{\partial r} + \rho\frac{\partial v}{\partial r} = 0 \\ \frac{\partial v}{\partial t} + v\frac{\partial v}{\partial r} + \gamma\rho^{\gamma-2}\frac{\partial p}{\partial r} = 0. \end{aligned} \tag{2.3}$$

We note that (2.3) is equivalent to matrix form

$$\frac{\partial}{\partial t}\begin{pmatrix} \rho \\ v \end{pmatrix} + A.\frac{\partial}{\partial r}\begin{pmatrix} \rho \\ v \end{pmatrix} = 0, \tag{2.4}$$

A being the $(2,2)$ matrix

$$A = \begin{pmatrix} v & \rho \\ \gamma\rho^{\gamma-2} & v \end{pmatrix}.$$

Eigenvalues of A (characteristics speeds) are given $\lambda = v - c$ and $\mu = v + c,\ c = \sqrt{\gamma\rho^{\gamma-1}}$. Since $\lambda < \mu$, system (2.4) is therefore hyperbolic. Thus there exists two functions $W(\rho, v)$ and $Z(\rho, v)$ (Riemann invariants) such that

$$W(\rho, v) = \text{constant on } \frac{dX^1}{dt} = \lambda, \tag{2.5}$$

$$Z(\rho, v) = \text{constant on } \frac{dX^2}{dt} = \mu. \tag{2.6}$$

$W(\rho, v)$ is determined by the system $\dfrac{dv}{c/\rho} = \dfrac{d\rho}{1}$, vector $R_1 = (1, c/\rho)$ being and eigenvector associated to the eigenvalue μ. Thus

$$W(\rho, v) = v - \frac{2c}{\gamma - 1}. \tag{2.7}$$

Similarly, the Riemann invariant $Z(\rho, v)$ corresponding to λ is given by

$$Z(\rho, v) = v + \frac{2c}{\gamma - 1}. \tag{2.8}$$

Functions $W(\rho, v) = W(t, r)$ and $Z(\rho, v) = Z(t, r)$ satisfy the following system, equivalent to system (2.4)

$$\begin{aligned} \frac{\partial W}{\partial t} + \lambda(W, Z)\frac{\partial W}{\partial r} = 0 \\ \frac{\partial Z}{\partial t} + \mu(W, Z)\frac{\partial Z}{\partial r} = 0, \end{aligned} \tag{2.9}$$

where $\lambda(W, Z)$ and $\mu(W, Z)$ are given by

$$\begin{aligned} \lambda = -\left(\tfrac{\gamma-3}{4}\right)Z + \left(\tfrac{\gamma+1}{4}\right)W \\ \mu = \left(\tfrac{\gamma+1}{4}\right)Z - \left(\tfrac{\gamma-3}{4}\right)W, \end{aligned} \tag{2.10}$$

as follows from (2.7) and (2.8).

It is well known that a sufficient condition in order that (2.9) is authentically nonlinear is that $\dfrac{\partial\lambda}{\partial W} > 0$ and $\dfrac{\partial\mu}{\partial Z} > 0$, which is the case here according to (2.10).

Integration along the characteristics defined by

$$\frac{dX^1}{dt} = \lambda(W,Z),\ X^1(0) = \beta$$

gives

$$X^1_{(0,\beta)}(t) = \beta + \int_0^t \lambda(W(s,X^1(s)), Z(s,X^1(s)))ds. \tag{2.11}$$

Therefore, the solution of the initial value problem

$$W_t + \lambda(W,Z)W_r = 0,\ \ W(0,r) = W_0(r) \tag{2.12}$$

can be written as

$$W(t,r) = W_0\Big(X^1_{(0,\beta)}(0)\Big) = W_0(\beta), \tag{2.13}$$

where $\beta = r - \int_0^t \lambda(W(s,X^1(s)), Z(s,X^1(s)))ds$.
Similarly we have

$$Z(t,r) = Z_0\Big(X^2_{(0,\alpha)}(0)\Big) = Z_0(\alpha), \tag{2.14}$$

where

$$X^2_{(0,\alpha)}(t) = \alpha + \int_0^t \mu(W(s,X^2(s)), Z(s,X^2(s)))ds.$$

The above considerations lead to the following

Proposition 2.1. *Assume that $W_0'(\beta) < 0$ or $Z_0'(\alpha) < 0$. Then solution of system (2.3) is defined on a finite interval $[0,T[$.*

□

Proof: Differentiation of (2.5) and (2.11) with respect to β gives

$$\frac{dX^1_\beta}{dt} = \lambda_\beta(W,Z)\ \text{ with }\ X^1_\beta(t=0) = 1, \tag{2.15}$$

and in the same way

$$\frac{dX^2_\alpha}{dt} = \mu_\alpha(W,Z)\ \text{ with }\ X^2_\alpha(t=0) = 1. \tag{2.16}$$

Since $\lambda_\beta = \lambda_W W_\beta + \lambda_Z Z_\beta = \lambda_W W_0'(\beta)$ and $\mu_\alpha = \mu_Z Z_0'(\alpha)$, from (2.15) and (2.16), integrating w.r.t. t along the characteristics yields

$$X^1_\beta(t) = 1 + \int_0^t \lambda_W W_0'(\beta)dt = 1 + \left(\frac{\gamma+1}{4}\right)W_0'(\beta)t, \tag{2.17}$$

$$X^2_\alpha(t) = 1 + \int_0^t \mu_Z Z_0'(\alpha)dt = 1 + \left(\frac{\gamma+1}{4}\right)Z_0'(\alpha)t. \tag{2.18}$$

From (2.17), it follows that $X^1_\beta(t_1) = 0$ for $t_1 = \dfrac{-4}{(\gamma+1)W_0'(\beta)} > 0$. Similarly $X^2_\alpha(t_2) = 0$ for $t_2 = \dfrac{-4}{(\gamma+1)Z_0'(\alpha)} > 0$.

Hence $\dfrac{\partial W}{\partial r}(t,r)$ becomes infinite for $T = \inf\{t_1, t_2\}$, since $\dfrac{\partial W}{\partial r} = W_\beta . \dfrac{d\beta}{dX} = \dfrac{W_0'(\beta)}{X_\beta}$.

□

From Proposition 2.1, it follows that

Proposition 2.2. *System (2.9) admits an unique C^1 solution on $[0, T[$, for all $r \in \mathbb{R}^+$ and for initial data $\rho_G(0,r) = \rho_0(r)$ and $v_G(0,r) = v_0(r)$ belonging to $C^1(\mathbb{R}^+)$.*

□

Concerning the droplet radius, it follows from (1.8), that we have the following ode for this radius

$$\frac{dR(t)}{dt} = \frac{v_G(t, R(t))\rho_G(t, R(t))}{\rho_G(t, R(t)) - \rho_L}, \; R(0) = R_0. \tag{2.19}$$

We immediately deduce

Proposition 2.3. *The Cauchy problem (2.19) has an unique solution $R(t)$ on a maximal time interval $[0, T^*[$ with $T^* \leq T$, given initial data $\rho_0(r)$ and $v_0(r)$ such that $W_0'(r) < 0$ or $Z_0'(r) < 0$.*

□

3. Liquid Mass Fraction

The liquid mass fraction Y_{L1} satisfies the conservation equation of specie (1.3), which can be rewritten as

$$\partial_t Y_{L1} - \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r} Y_{L1}\right) + f(Y_{L1}) = 0. \tag{3.1}$$

We have used polar coordinates, and taken the diffusion constant D_{12} as being equal to 1. Of course, (3.1) is equivalent to

$$\partial_t Y_{L1} - \Delta Y_{L1} - \frac{2}{r}\frac{\partial}{\partial r} Y_{L1} + f(Y_{L1}) = 0, \text{ for } 0 < r < s(t), \tag{3.2}$$

where $s(t) = R(t)$ denotes the droplet radius determined in section 2.
The boundary condition at the surface $s(t)$ is given by the Rankine-Hugoniot condition connected to the thermodynamic equilibrium, i.e. formula (1.10).
Performing the change of variable $r = R(t)x$, function $Y_{L1}(t,r)$ turns to function $Y_{L1}(t, R(t)x) = u(t,x)$, which satisfies the following initial boundary value (i.b.v.) problem

$$\partial_t u - a(t)\left(\Delta u + \frac{2}{x}\partial_x u\right) - x\frac{R'(t)}{R(t)}\partial_x u + f(u) = 0, \;\; 0 < x < 1, \; t > 0, \tag{3.3}$$

$$|\lim_{x \to 0+} x u_x(t,x)| < \infty, \;\; u_x(t,1) + k(t)u(t,1) = 0, \tag{3.4}$$

$$u(0,x) = u_0(x), \tag{3.5}$$

where we used the following notations

$$a(t) = \frac{1}{R^2(t)}, \;\; k(t) = \frac{R(t)R'(t)(K_1 - 1)}{K_2\rho_G(t, R(t)) - K_3}. \tag{3.6}$$

Our purpose in this Section is to analyze the boundary value problem (3.3)-(3.5).

We shall do so by setting this problem in a variational framework, using weighted Sobolev spaces. Let $\Omega =]0,1[$ and define H as the Hilbert space given by

$$H = \{v : \Omega \to \mathbb{R},\text{ measurable and such that } \int_0^1 x^2 v^2(x)dx < +\infty\}.$$

Note that H is the closure of $C^0(\bar{\Omega})$ w.r.t. the norm $\|v\|_H = \Big(\int_0^1 x^2 v^2(x)dx\Big)^{1/2}$. We also introduce the real Hilbert space $V = \Big\{v \in H \ |v' \in H\Big\}$. In the following, we shall often use the fact that V is the closure of $C^1(\bar{\Omega})$ w.r.t. the norm $\|v\|_V = \Big(\|v\|_H^2 + \|v'\|_H^2\Big)^{1/2}$. V is continuously embedded in H. Identifying H with his dual H', one has $V \subset H \subset V'$with continuous injections.
Note also that the norms $\|.\|_H$ and $\|.\|_V$ can be defined, respectively, from the inner products $< u, v >= \int_0^1 x^2 u(x)v(x)dx$ and $< u, v > + < u', v' >$.
We then have the following results, the proofs of which can be found in the paper[12],

Lemma 3.1. *For every $v \in C^1([0,1])$, $\epsilon > 0$ and $x \in [0,1]$ we have*

$$\|v\|_0^2 \le \frac{1}{2}\ \|v'\|_0^2 + v^2(1),$$

$$v^2(1) \le \epsilon\|v'\|_0^2 + C_\epsilon\|v\|_0^2,$$

$$\Big|v(1)\Big| \le 2\ \|v\|_1,\ \Big|xv(x)\Big| \le \sqrt{5}\|v\|_1$$

where $C_\epsilon = 3 + \frac{1}{\epsilon}$ and $\|.\|_0 = \|.\|_H,\ \|.\|_1 = \|.\|_V$.

□

Lemma 3.2. *The embedding $V \subset H$ is compact.*

□

Remark 3.1. From Lemma 3.1, it follows that $\Big(\|v'\|_0^2 + v^2(1)\Big)^{1/2}$ and $\|v\|_1$ are two equivalent norms on V since $\frac{2}{3}\|v\|_1^2 \le v^2(1) + \|v'\|_0^2 \le 5\|v\|_1^2$, for all $v \in V$.

□

Remark 3.2. We have $xv(x) \in C^0([0,1])$, for all $v \in V$.
Indeed, on one hand, $\lim_{x\to 0+} xv(x) = 0,\ \forall v \in V$ (see the book[1], p.128), and on the other hand $v_{|[\epsilon,1]} \in C^0([\epsilon,1]),\ \forall\epsilon,\ 0 < \epsilon < 1$, since we have $H^1(\epsilon,1) \subset\ C^0([\epsilon,1])$ and $\epsilon\|v\|_{H1(\epsilon,1)} \le \|v\|_1\ \forall v \in V$ $\forall\epsilon,\ 0 < \epsilon < 1$.

□

If X is any Banach space, we denote by $\|.\|_X$ its norm, and by X' the dual space of X. We denote by $L^p(0,T;X),\ 1 \le p \le \infty$, the standard Banach space of real functions $u :\ (0,T) \to X$, measurable, such that

$$\|u\|_{L^p(0,T;X)} = \left(\int_0^T \|u(t)\|_X^p dt\right)^{1/p} < +\infty, \text{ for } 1 \le p < \infty$$

and

$$\|u\|_{L^\infty(0,T;X)} = \text{ess sup}_{0<t<T}\|u(t)\|_X, \text{ for } p = \infty.$$

Let $u(t),\ u'(t),\ u_x(t),\ u_{xx}(t)$ denote $u(t,x),\ \frac{\partial u}{\partial t}(t,x),\ \frac{\partial u}{\partial x}(t,x),\ \frac{\partial^2 u}{\partial x^2}(t,x)$ respectively.

We shall make the following set of <u>assumptions</u>:

- (H_1) $u_0 \in H$;
- (H_2) $a, k \in W^{1,\infty}(0,T)$, $a(t) \ge a_0 > 0$;
- (F_1) $f \in C(\mathbb{R},\mathbb{R})$;
- (F_2) There exists positive constants C_1, C_1', C_2 and $p,\ 1 < p < 3$, such that
  (i) $uf(u) \ge C_1|u|^p - C_1'$,
  (ii) $|f(u)| \le C_2(1 + |u|^{p-1})$.

Let $u \in C^2([0,T]\times[0,1])$ be a solution of problem (3.3)-(3.5).
Then, after multiplying equation (3.3) by $x^2 v,\ v \in V$ w.r.t. the scalar product of H, integrating by parts and taking into account boundary condition given by (3.4), we get

$$\frac{d}{dt} < u(t), v > + a(t)\int_0^1 x^2 u_x v_x dx + a(t)k(t)u(1)v(1) - \frac{R'(t)}{R(t)}\int_0^1 x^3 u_x v dx + < f(u), v >= 0$$

The <u>weak formulation</u> of the ibv problem (3.3)-(3.5) can then be given in the following way: Find $u(t)$, defined on the open set $(0,T)$, such that $u(t)$ satisfies the following variational problem

$$\frac{d}{dt} < u(t), v > + \tilde{a}(t; u(t), v) + < f(u(t)), v >= 0, \ \ \forall v \in V, \tag{3.7}$$

together with the initial condition

$$u(0) = u_0. \tag{3.8}$$

Above, we have used the following bilinear form

$$\tilde{a}(t; u, v) = a(t)\int_0^1 x^2 u_x v_x dx + a(t)k(t)u(1)v(1) - \frac{R'(t)}{R(t)}\int_0^1 x^3 u_x v dx,\ u, v \in V. \tag{3.9}$$

We first note the following lemma, the proof of which can be found in our previous paper[2]

Lemma 3.3. *There exists constants $K_T,\ \alpha_T$ and β_T depending on T, such that*

$$|\ \tilde{a}(t; u, v)| \le K_T\|u\|_1\|v\|_1, \text{ for all } u, v \in V, \tag{3.10}$$

$$\tilde{a}(t; u, u) \ge \alpha_T\|u\|_1^2 - \beta_T\|u\|_0^2,\ u, v \in V. \tag{3.11}$$

$\square$

We then have the following existence theorem

Theorem 3.1. *Let $T > 0$ and assumptions (H_1),(H_2),(F_1),(F_2) hold true. Then, there exists a solution u of the variational problem (3.7),(3.8) such that*

$$u \in L^2(0,T;V) \cap L^\infty(0,T;H),\ x^{2/p}u \in L^p(Q_T),$$

$$tu \in L^\infty(0,T;V),\ tu_t \in L^2(0,T;H).$$

Furthermore, if f satisfies the additional condition

$$(f(u) - f(v))(u - v) \geq -\delta|u - v|^2,$$

for all $u, v \in \mathbb{R}$, for some $\delta \in \mathbb{R}$, then the above solution u is unique.

$\square$

Proof of Theorem 3.1. We divide it in several steps.

• Step 1, Galerkin method.

Denote by $\{w_j\},\ j = 1, 2, ...,$ an orthonormal basis of the separable Hilbert space V. We wish to find $u_m(t)$ of the form

$$u_m(t) = \sum_{j=1}^{m} c_{mj}(t) w_j, \tag{3.12}$$

where $c_{mj}(t)$ satisfy the following system of nonlinear differential equations

$$< u'_m(t), w_j > + \tilde{a}(t; u_m(t), w_j) + < f(u_m(t)), w_j >= 0,\ \ 1 \leq j \leq m, \tag{3.13}$$

together with the initial condition

$$u_m(0) = u_{0m}, \tag{3.14}$$

and

$$u_{0m} \to u_0 \ \text{ strongly in } H. \tag{3.15}$$

Clearly, for each m, there exists an unique local solution $u_m(t)$ of the form (3.12), which satisfies (3.13) and (3.14) almost everywhere on $0 \leq t \leq T_m$, for some $T_m,\ 0 < T_m \leq T$. The following estimates allow us to take $T_m = T$ for all m.

• Step 2, A priori estimates.

(a) First estimate.

Multiplying j^{th} equation of system (3.13) by $c_{mj}(t)$ and summing up w.r.t. j, we have

$$\frac{1}{2}\frac{d}{dt}\|u_m(t)\|_0^2 + \ \ \tilde{a}(t; u_m(t), u_m(t)) + < f(u_m(t)), u_m(t) >= 0. \tag{3.16}$$

Using assumption (H_2), (F_2,i), Lemma 3.1 and Remark 3.1, it follows from (3.16) that

$$\frac{d}{dt}\|u_m(t)\|_0^2 + 2\alpha_T\|u_m(t)\|_1^2 + 2C_1\int_0^1 x^2|u_m(t,x)|^p dx \leq \frac{2C'_1}{3} + 2\beta_T\|u_m(t)\|_0^2. \tag{3.17}$$

Integrating (3.17), using (3.15), it follows that

$$S_m(t) \le C_0 + \frac{2}{3}TC_1' + 2\beta_T \int_0^t S_m(s)ds, \tag{3.18}$$

where

$$S_m(t) = \|u_m(t)\|_0^2 + 2\alpha_T \int_0^t \|u_m(s)\|_1^2 ds + 2C_1 \int_0^t ds \int_0^1 x^2 |u_m(s,x)|^p dx, \tag{3.19}$$

and C_0 is a constant depending only on u_0 with $\|u_{0m}\|_0^2 \le C_0 \ \forall m$.
Applying Gronwall's lemma, we obtain from (3.18)

$$S_m(t) \le \left(C_0 + \frac{2}{3}TC_1'\right)\exp(2\beta_T t) \le M_T, \ \ \forall m, \ \forall t, \ 0 \le t \le T_m \le T, \tag{3.20}$$

that is $T_m = T$.
In the following, we denote by M_T any generic constant depending only on T.

(b) Second estimate.
Replacing w_j by $t^2 u_m$ in (3.8) gives

$$\begin{aligned} &\|tu_m'\|_0^2 + \frac{1}{2}\frac{d}{dt}\left[a(t)\|tu_m\|_0^2 + a(t)k(t)t^2u_m^2(1)\right] + \frac{1}{2}\frac{d}{dt}\left[t^2\int_0^1 x^2 \hat{f}(u_m)dx\right] \\ &= \|u_{mx}\|_0^2 \frac{d}{dt}\left[t^2 a(t)\right] + \frac{1}{2}u_m^2(1)\frac{d}{dt}(t^2 a(t)k(t)) + 2t\int_0^1 x^2 \hat{f}(u_m)dx \\ &+ \frac{R'(t)t^2}{R(t)}\int_0^1 x^3 u_{mx} u_m' dx \end{aligned} \tag{3.21}$$

where

$$\hat{f}(z) = \int_0^z f(y)dy. \tag{3.22}$$

Integrating (3.21) w.r.t. time variable from 0 to t, we have, after some rearrangements

$$\begin{aligned} &2\int_0^t \|su_m'(s)\|_0^2 ds + a(t)\|tu_{mx}(t)\|_0^2 + a(t)t^2u_m^2(t,1) \\ &= a(t)(1-k(t))t^2u_m^2(t,1) + \int_0^t \left[s^2 a(s)\right]' \|u_{mx}(s)\|_0^2 ds \\ &+ \int_0^t \left[s^2 a(s)k(s)\right]' u_m^2(s,1)ds + 2\int_0^t \frac{R'(s)}{R(s)} s^2 < xu_{mx}(s), u_m'(s) > ds \\ &+4\int_0^t sds\int_0^1 x^2 \hat{f}(u_m(s,x))dx - 2t^2\int_0^1 x^2 \hat{f}(u_m)dx. \end{aligned} \tag{3.23}$$

By means of assumption (H_2) and Remark 3.1, we get

$$a(t)\|tu_m(t)\|_0^2 + a(t)t^2u_m^2(t,1) \ge \frac{2}{3}a_0\|tu_m(t)\|_1^2 \quad \forall t \in [0,T], \ \forall m. \tag{3.24}$$

We fix $\epsilon > 0$ such that

$$\|a\|_\infty \|1-k\|_\infty \epsilon < \frac{a_0}{3}, \tag{3.25}$$

where $\|.\|_\infty = \|.\|_{L\infty(0,T)}$.

Using again Lemma 3.1, Remark 3.1 with $\epsilon > 0$ as in (3.25) and first estimate (3.20), the terms on the r.h.s. of (3.23) can be estimated as follows

$$\begin{aligned} a(t)(1-k(t))t^2u_m^2(t,1) &\leq \|a\|_\infty \|1-k\|_\infty \Big(\epsilon \|tu_m(t)\|_1^2 + C_\epsilon \|tu_m(t)\|_0^2\Big) \\ &\leq \frac{a_0}{3}\|tu_m(t)\|_1^2 + M_T, \end{aligned} \tag{3.26}$$

$$\begin{aligned} &\int_0^t \big(s^2a(s)\big)'\|u_{mx}(s)\|_0^2 ds + \int_0^t \big(s^2a(s)k(s)\big)' u_m^2(s,1)ds \\ &\leq \big[\big\|(t^2a)'\big\|_\infty + \big\|(t^2ak)'\big\|_\infty\big] \int_0^t \Big[\|u_{mx}(s)\|_0^2 + u_m^2(s,1)\Big] ds \\ &\leq 5\big[\big\|(t^2a)'\big\|_\infty + \big\|(t^2ak)'\big\|_\infty\big] \int_0^t \|u_m(s)\|_1^2 ds \leq M_T, \end{aligned} \tag{3.27}$$

$$2\left|\int_0^t s^2 \frac{R'(t)}{R(t)} < xu_{mx}(s), u_m'(s) > ds\right| \leq \int_0^t \|su_m'(s)\|_0^2 ds + \left\|\frac{R'}{R}\right\|_\infty^2 \int_0^t \|su_m(s)\|_1^2 ds. \tag{3.28}$$

From assumptions (F_1) and (F_2), we note also that

$$-\hat{m}_0 = -\int_{-z_0}^{z_0} |f(y)|dy \leq \hat{f}(z) = \int_0^z f(y)dy \leq C_2\Big(|z| + \frac{|z|^p}{p}\Big), \quad \forall z \in \mathbb{R}, \tag{3.29}$$

where $z_0 = (C_1'/C_1)^{1/p}$.
Using first estimate (3.20), (3.29) and Lemma 3.1, we obtain

$$\begin{aligned} &\left|4\int_0^t sds\int_0^1 x^2\hat{f}(u_m(s,x))dx - 2t^2\int_0^1 x^2\hat{f}(u_m(t,x)dx\right| \\ &\leq 4C_2\int_0^t sds\int_0^1 x^2\Big(|u_m(s,x)| + \frac{1}{p}|u_m(s,x)|^p\Big)dx + 2t^2\int_0^1 x^2\hat{m}_0 dx \\ &\leq 4C_2\int_0^t s\|u_m(s)\|_0 ds + \frac{4}{p}C_2 t\int_0^t ds\int_0^1 x^2|u_m(s,x)|^p dx + \frac{2}{3}T^2\hat{m}_0 \\ &\qquad \leq 2C_2T\sqrt{M_T} + \frac{2C_2}{pC_1}TM_T + \frac{2}{3}T^2\hat{m}_0 \leq M_T. \end{aligned} \tag{3.30}$$

Hence, we deduce from (3.23), (3.24), (3.26)-(3.28) and (3.30) that

$$\int_0^t \|su_m'(s)\|_0^2 ds + \frac{a_0}{3}\|tu_m(t)\|_1^2 \leq M_T + \left\|\frac{R'}{R}\right\|_\infty^2 \int_0^t \|su_m(s)\|_1^2 ds. \tag{3.31}$$

By Gronwall's lemma, we get

$$\int_0^t \|su_m'(s)\|_0^2 ds + \frac{a_0}{3}\|tu_m(t)\|_1^2 \leq M_T \exp\left(\left\|\frac{R'}{R}\right\|_\infty^2 . \frac{3T}{a_0}\right) \leq M_T, \quad \forall t \in [0,T]. \tag{3.32}$$

Finally, using (3.20) and assumption (F_2,ii) we have also

$$\int_0^t ds\int_0^1 \Big|x^{2/p'}f(u_m(s,x))\Big|^{p'} dx \leq (2C_2)^{p'}\int_0^t ds\int_0^1 x^2|u_m(s,x)|^p dx \leq M_T, \tag{3.33}$$

with $p' = \dfrac{p}{p-1}$.

- Step 3, the limiting process.

From (3.20), (3.32) and (3.33), we deduce that there exists a subsequence of $\{u_m\}$, still denoted $\{u_m\}$ such that

$$\begin{aligned} &u_m \to u \text{ weakly } * \text{ in } L^\infty(0,T;H)\ , \\ &u_m \to u \text{ weakly in } L^2(0,T;V)\ , \\ &x^{2/p}u_m \to x^{2/p}u \text{ weakly in } L^p(Q_T)\ \ , \\ &tu_m \to tu \text{ weakly } * \text{ in } L^\infty(0,T;V)\ , \\ &(tu_m)' \to (tu)' \text{weakly in } L^2(0,T;H). \end{aligned} \tag{3.34}$$

Using a standard compactness lemma[10] (p.57) together with (3.34), we can extract from the sequence $\{u_m\}$, a subsequence still denoted by $\{u_m\}$ such that

$$tu_m \to tu \ \text{ strongly in } L^2(0,T;H). \tag{3.35}$$

Continuity of f also implies (up to a sub-sequence)

$$f(u_m(t,x)) \to f(u(t,x)) \quad \text{a.e. } (t,x) \in Q_T = (0,T)\times(0,1). \tag{3.36}$$

Applying a standard weak convergence lemma[10], we have also

$$x^{2/p'}f(u_m) \to x^{2/p'}f(u) \text{ weakly in } L^{p'}(Q_T).$$

Passing to the limit in (3.13) and (3.14), it follows from (3.15), (3.34) and (3.36), that function $u(t)$ satisfies the i.b.v. problem (3.7), (3.8).

• Step 4. Uniqueness of the solutions.
First of all, we note the following slight extension of a lemma used in our previous paper[2] (see also the book[10])

Lemma 3.4. *Let w be the weak solution of the following i.b.v. problem*

$$\begin{aligned} &w_t - a(t)(w_{xx} + \frac{2}{x}w_x) = \ \tilde{f}(t,x),\ 0<t<T,\ 0<x<1, \\ &\left|\lim_{x\to 0+} xw_x(t,x)\right| < +\infty, \quad w_x(t,1)+k(t)w(t,1)=0,\ w(0,x)=0, \\ &w \in L^2(0,T;V)\cap L^\infty(0,T;H),\ x^{2/p}w \in L^p(Q_T), \\ &tw \in L^\infty(0,T;V),\ tw_t \in L^2(0,T;H). \end{aligned}$$

Then

$$\frac{1}{2}\|w(t)\|_0^2 + \int_0^t a(s)\Big[\|w_x(s)\|_0^2 + k(s)w^2(s,1)\Big]ds - \int_0^t <\tilde{f}(s), w(s)> ds = 0,$$
$$\text{a.e. } t\in(0,T)$$

□

Uniqueness of solutions for our initial i.b.v problem will then be deduced as follows. Let u and v be two weak solutions of (3.3)-(3.5). Then $w = u - v$ is a weak solution of problem mentioned in Lemma 3.4, with r.h.s. given by $\tilde{f}(t,x) = \dfrac{xR'(t)}{R(t)}w_x - f(u) + f(v)$. Therefore, Lemma 3.4 implies

$$\frac{1}{2}\|w(t\|_0^2+\int_0^t \tilde{a}(s;w(s),w(s))ds+2\int_0^t < f(u(s))-f(v(s)),w(s)>ds=0.$$

Using Lemma 3.3 and assumption (F_3) we obtain

$$\|w(t)\|_0^2+2\alpha_T\int_0^t\|w(s)\|_1^2\le 2(\delta+\beta_T)\int_0^t\|w(s)\|_0^2 ds. \tag{3.37}$$

If $\delta+\beta_T\ge 0$ we have $\|w(t)\|_0=0$ by applying Gronwall's lemma. In the case where $\delta+\beta_T<0$ the result is clearly still true.
This ends the proof of Theorem 3.1.

□

We now turn to the boundness of the above solutions.
For this purpose, we shall make use of the following assumptions

- (H'$_1$) $u_0\in L^\infty(0,1)$, $|u_0(x)|\le M,\ a.e.\ x\in(0,1)$
- (H'$_2$) $a,k\in W^{1,\infty}(0,T)$, $a(t)\ge a_0>0$, $k(t)\ge k_0>0$
- (F'$_1$) $uf(u)\ge 0\ \ \forall u\in\mathbb{R}$ such that $|u|\ge\|u_0\|_\infty$, for a.e. $x\in(0,1)$.

We then have the following result

Theorem 3.2. *Let (H'$_1$), (H'$_2$), (F$_1$)-(F$_3$) and (F'$_1$) hold. Then the unique weak solution of the ibv problem (3.7)-(3.9), as given by theorem 1, belongs to $L^\infty(Q_T)$.*

□

Proof of Theorem 3.2. Firstly, we note that $Z=u-M$ satisfies the i.b.v. problem

$$\partial_t Z-a(t)\left(\Delta Z+\frac{2}{x}\partial_x Z\right)-x\frac{R'(t)}{R(t)}\partial_x Z+f(Z+M)=0,\ 0<x<1,\ t\in(0,T), \tag{3.38}$$

$$|\lim_{x\to 0+}xZ_x(t,x)|<\infty,\ \ Z_x(t,1)+k(t)[Z(t,1)+M]=0, \tag{3.39}$$

$$Z(0,x)=u_0(x)-M \tag{3.40}$$

Multiplying equation (3.38) by x^2v, for $v\in V$, integrating by parts w.r.t. variable x and taking into account boundary condition (3.39), one has

$$\begin{gathered}\int_0^1 x^2Z_tv dx+a(t)\int_0^1 x^2Z_xv_xdx+a(t)k(t)Z(t,1)v(1)-\frac{R'(t)}{R(t)}\int_0^1 x^3Z_xvdx\\ +\int_0^1 x^2f(Z+M)vdx=-Ma(t)k(t)v(1),\ \ \forall v\in V,\end{gathered} \tag{3.41}$$

hence for $v=Z^+=\frac{1}{2}\Big(Z+|Z|\Big)$, since $u_0\in L^\infty(0,1)$. It follows that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_0^1 x^2|Z^+|^2dx+a(t)\int_0^1 x^2|(Z^+)_x|^2dx+a(t)k(t)|Z^+(t,1)|^2\\ &-\frac{R'(t)}{R(t)}\int_0^1 x^3Z_x^+Z^+dx+\int_0^1 x^2f(Z^++M)Z^+dx=-Ma(t)k(t)Z^+(t,1)\le 0,\end{aligned}$$

since

$$\int_0^1 x^2Z_tZ^+dx=\int_{0,Z>0}^1 x^2(Z^+)_tZ^+dx=\frac{1}{2}\frac{d}{dt}\int_0^1 x^2|Z^+|^2dx.$$

On the other hand, by assumption (H'$_2$) and Remark 3.1, one has

$$a(t)\int_0^1 x^2\big|Z_x^+\big|^2dx + a(t)k(t)\big|Z^+(t,1)\big|^2 \geq \tilde{C}_0\big\|Z^+(t)\big\|_1^2, \tag{3.42}$$

where $\tilde{C}_0 = \frac{2}{3}a_0 \min\{1, k_0\}$.
Using the monotonicity of $f(u) + \delta u$ and (F'$_1$), we have

$$\begin{aligned}\int_0^1 x^2 f(Z^+ + M)Z^+dx &= \int_0^1 x^2\big[f(Z^+ + M) - f(M)\big]Z^+dx + \int_0^1 f(M)x^2Z^+dx \\ &\geq -\delta\int_0^1 x^2\big|Z^+\big|^2dx + \int_0^1 f(M)x^2Z^+dx \ \geq -\delta\big\|Z^+\big\|_0^2.\end{aligned} \tag{3.43}$$

(3.41)-(3.43) together with Cauchy's inequality applied to the term $-\dfrac{R'(t)}{R(t)}\displaystyle\int_0^1 x^3Z_x^+Z^+dx$ yields

$$\frac{d}{dt}\big\|Z^+(t)\big\|_0^2 + \tilde{C}_0\big\|Z^+(t)\big\|_1^2 \leq \left(\frac{1}{\tilde{C}_0}\left\|\frac{R'}{R}\right\|_\infty^2 + 2|\delta|\right)\big\|Z^+(t)\big\|_0^2. \tag{3.44}$$

Integrating (3.44), we get

$$\big\|Z^+(t)\big\|_0^2 \leq \big\|Z^+(0)\big\|_0^2 + \left(\frac{1}{\tilde{C}_0}\left\|\frac{R'}{R}\right\|_\infty^2 + 2|\delta|\right)\int_0^t \big\|Z^+(s)\big\|_0^2ds. \tag{3.45}$$

Since $Z^+(0) = \Big(u(0,x) - M\Big)^+ = \Big(u_0(x) - M\Big)^+ = 0$, Gronwall's lemma yields $\|Z^+(t)\|_0 = 0$. Thus $u(t,x) \leq M$ a.e. $(t,x) \in Q_T$.
The case $u_0(x) \geq -M$ is similar, by considering $Z = u + M$ and $Z^- = \frac{1}{2}\Big(|Z| - Z\Big)$. Thus we get $Z^- = 0$ and hence $u(t,x) \geq -M$ a.e. $(t,x) \in Q_T$.
All in all, one obtains $|u(t,x)| \leq M$ a.e. $(t,x) \in Q_T$ and this ends the proof of Theorem 3.2.

□

4. Numerical applications

For the numerical applications, we have taken in (2.3) $\gamma = 3$, so that equation (2.9) reduces to Burger's equation

$$\begin{cases} W_t + WW_r = 0, \ \ W(0,r) = W_0(r) \\ Z_t + ZZ_r = 0, \ \ Z(0,r) = Z_0(r). \end{cases} \tag{4.1}$$

It is well known that classical Burger's equation

$$u_t + uu_r = 0, \ \ u(0,r) = u_0(r)$$

admits the solution $u(t,r) = u_0(\xi(t,r))$, $\xi(t,r)$ being defined by the parametrization $r = u_0(\xi)t + \xi$.
Having in mind (4.1), we have considered two examples.

- First example.

For the first example, we have chosen the initial conditions $W_0(r) = 1,\ r > 0;\ W_0(0) = 0$ and $Z_0(r) = 2$, $r > 0$; $Z_0(0) = 0$. The continuous solutions of (4.1) are then given by

$$W(t,r) = \begin{cases} 1 \text{ if } 0 \le t \le r, \\ \frac{r}{t} \text{ if } 0 \le r \le t, \end{cases}$$

$$Z(t,r) = \begin{cases} 2 \text{ if } 0 \le 2t \le r, \\ \frac{r}{t} \text{ if } 0 \le r \le 2t. \end{cases}$$

According to Section 2, the droplet radius is given by formula (2.19) which we consider here with an initial condition taken equal to be 1

$$\frac{dR(t)}{dt} = \frac{v_G(t,R(t))\rho_G(t,R(t))}{\rho_G(t,R(t)) - \rho_L},\ R(0) = 1, \tag{4.2}$$

and where

$$\begin{cases} v_G(t,r) = \dfrac{1}{2}(W(t,r) + Z(t,r)) \\ \rho_G(t,r) = \dfrac{1}{2\sqrt{3}r^2}(Z(t,r) - W(t,r)). \end{cases} \tag{4.3}$$

In figure 1 below, we have drawn the curve $t \longrightarrow R(t)$ on the time interval $[0,1]$ with a step $h = 0.05$ and $\rho_L = 0.9$.

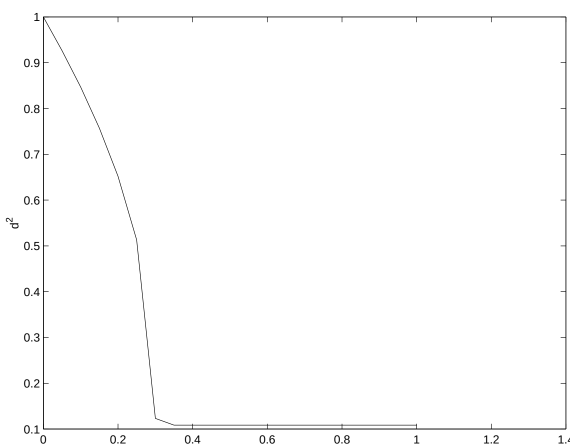

Fig. 1

• Second example.

For the second example, we have chosen the truly experimental conditions made by the LCSR in the study of single drop evaporation, the drop being suspended from a silicate tube. Drops are made up of n-heptane fuel ($\rho_L = 683\ kg/mm^3$)) in air at normalized atmospheric pressure and with an initial speed $v_G(0,r) = C_1 = 35mm/s$. The initial density $\rho_G(0,r)$ of the gas is taken as $C_2 = \dfrac{348}{T_0},\ T_0 = 373K$.

In this case, the solution of (4.1) are given by

$$W(t,r) = C_1 - \sqrt{3}C_2\xi^2,\ \xi = \frac{1 + \sqrt{1 - 4(r - C_1t)\sqrt{3}C_2t}}{2\sqrt{3}C_2t},$$

$$Z(t,r) = C_1 + \sqrt{3}C_2\eta^2,\ \eta = \frac{-1+\sqrt{1+4(r-C_1t)\sqrt{3}C_2t}}{2\sqrt{3}C_2t}.$$

We then compute v_G and ρ_G by formula (4.3), and then solve the ode for R given by (4.2).
We note that $R'(0) = \frac{C_1C_2}{C_2-\rho_L} < 0$ where $C_2 = \frac{348}{372} < 1 < \rho_L = 683$.
On the other hand, one has

$$v_G(t,r) = \frac{1}{4\sqrt{3}C_2t}[C_1 + \frac{\sqrt{3}}{12t^2C_2}(\alpha + \sqrt{1-\alpha} - \sqrt{1+\alpha})],$$

where

$$\alpha = \alpha(t,r) = 4(r-C_1t)\sqrt{3}C_2t,\ |\alpha| \leq 1.$$

One has $\operatorname{sgn} v_G(t,r) = \operatorname{sgn}[C_1 12t^2C_2 + \sqrt{3}(\alpha + \sqrt{1-\alpha} - \sqrt{1+\alpha})]$. Since $|\alpha| \leq 1$, it follows that $\sqrt{3}(\alpha + \sqrt{1-\alpha} - \sqrt{1+\alpha}) \geq -\sqrt{6}$.
Thus $v_G(t,r) > 0$ if $C_1 12t^2C_2 \geq \sqrt{6}$ i.e. $t \geq t_2 = \frac{1}{C_1 2^{\frac{1}{4}}} \simeq \frac{1}{35}$ and $v_G(t,r) < 0$ on the interval $[t_1, t_2[$.
Similarly $0 \leq \rho_G(t,r) \leq \frac{1}{C_2^2t^2 6.2.r^2}[2 + \sqrt{1-\alpha} - \sqrt{1+\alpha}] \leq \frac{1}{3t^2r^2}$.
Thus, if $t \geq t_m$, $r \geq r_m$, $\rho_G(t,r) \leq \frac{1}{3t_m^2r_m^2} \leq \rho_L \simeq 683$, that is for $t_mr_m \geq \frac{1}{3.683} \simeq \frac{1}{45}$.
Since $R'(0) < 0$ it follows that $R(t)$ is decreasing on $[0, t_1[$, increasing on $[t_1, t_2[$ and then from the starting point t_2 always non increasing.
Figures 2 and 3 represent resp. the velocity $v_G(t,r)$ and the pressure $\rho_G(t,r)$ given by (4.3) for $(t,r) \in (0,1) \times (0,1)$.

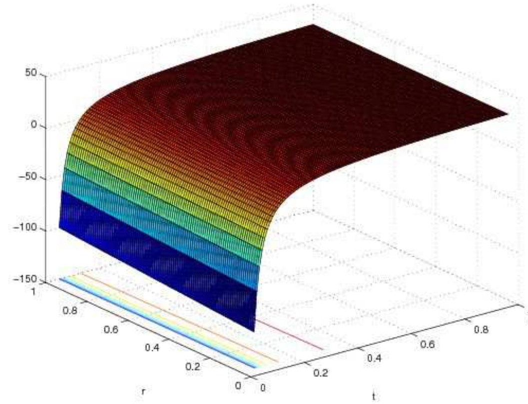

Fig. 2

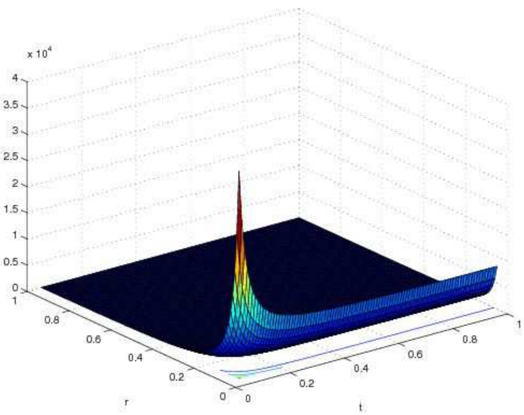

Fig. 3

The curve of the radius $t \longrightarrow R(t)$ for this case is drawn in figure 4.

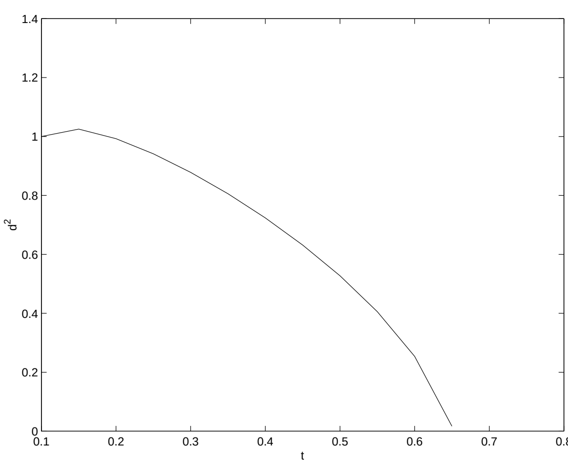

Fig. 4

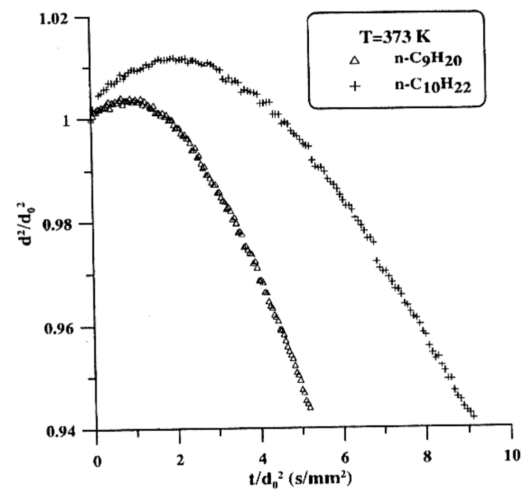

Fig. 5

Since $W_0'(\xi) < 0$ the maximal existence interval is finite (Proposition 2.1) as can be seen in our graphic. Let us remark that looking on the experimental curves [11] made by the LCSR (figure 5) at the beginning, the function $t \longrightarrow R(t)$ is increasing around the vicinity of $t = 0$. This fact is confirmed by our model which represents a good improvement of our previous model[2] in which the velocity $v_G(t)$ was a given function of t.

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