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Numerical Solution of Stochastic Nash Games with State-Dependent Noise for Weakly Coupled Large-Scale Systems

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Abstract. This paper discusses the infinite horizon stochastic Nash games with state-dependent noise. After establishing the asymptotic structure along with the positive semidefiniteness for the solutions of the cross-coupled stochastic algebraic Riccati equations (CSAREs), a new algorithm that combines Newton's method with two fixed point algorithms for solving the CSAREs is derived. As a result, it is shown that the proposed algorithm attains quadratic convergence and the reduced-order computations for sufficiently small parameter ε. As another important feature, the high-order approximate strategy that is based on the iterative solutions is proposed. Using such strategy, the degradation of the cost functional is investigated. Finally, in order to demonstrate the efficiency of the proposed algorithms, computational examples are provided.

keywords: stochastic Nash games, cross-coupled stochastic algebraic Riccati equations (CSAREs), Newton's method, Newton-Kantorovich theorem, fixed point algorithm.

1 Introduction

The stochastic control problems governed by Itô's differential equation have become a popular research topic in a past decade. Recently, stochastic H_∞ control problem with state- and control-dependent noise was considered [1, 2]. It has attracted much attention and has been widely applied to various fields. Particularly, the stochastic H_2/H_∞ control with state-dependent noise has been addressed [3].

Recently, linear quadratic Nash games and their applications have been widely investigated in many literatures. Particularly, the linear quadratic Nash games and related topics for weakly coupled large-scale systems have been discussed in [6, 7, 8, 9]. These results are based on the deterministic systems. However, to the best of our knowledge, no results have been obtained for stochastic Nash games with state-dependent noise.

In this paper, the stochastic Nash games for weakly coupled large-scale systems governed by Itô differential equations with state-dependent noise are addressed as an extension of the existing result of [6, 7]. Specifically, this paper focuses on the development of the numerical algorithm for solving the cross-coupled stochastic algebraic Riccati equations (CSAREs). First, 2-player stochastic Nash games are formulated by applying the results of stochastic linear quadratic control problems [3, 4] for the first time. It should be noted that although the stochastic games for weakly coupled large-scale systems have been studied in [5], the state-dependent noise has not been considered. Moreover, it may be noted that the considered CSAREs is quite different from the existing results in [6, 7] in the sense that the CSAREs have the additional linear equations. Thus, these terms would result in the complication for the analysis of the existence of the solutions. Second, in order to choose the appropriate initial conditions, the uniqueness and boundedness of the solution to the CSAREs and their asymptotic structure are investigated. After establishing these properties of the solutions, the numerical algorithm that is based on Newton's method is considered. The quadratic convergence and the local uniqueness of the solutions are proved for sufficiently small parameter ε via the Newton-Kantorovich theorem [10]. Additionally, in order to overcome the computation of large dimensional matrix that arises in Newton's method, two fixed point algorithms are combined. As another important feature, the high-order approximate strategy set that is based on the iterative solutions is proposed. As a result, the better performance is attained. Finally, in order to demonstrate the efficiency of the algorithm, computational examples are included.

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Notation: The notations used in this paper are fairly standard. The superscript T denotes the matrix transpose. I_n denotes the $n \times n$ identity matrix. **block diag** denotes the block diagonal matrix. $\|\cdot\|$ denotes its Euclidean norm for a matrix. E denotes the expectation. $\otimes$ denotes the Kronecker product. vec$M$ denotes the column vector of the matrix M.

2 Stochastic Nash Games

Consider stochastic linear time-invariant weakly coupled large-scale systems.

$$dx(t) = [A_\varepsilon x(t) + B_{1\varepsilon} u_1(t) + B_{2\varepsilon} u_2(t)]dt + \sum_{p=1}^{N} A_{p\varepsilon} x(t) dw_p(t),\ x(0) = x^0, \tag{1}$$

where

$$x(t) := \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix},\ A_\varepsilon := \begin{bmatrix} A_{11} & \varepsilon A_{12} \\ \varepsilon A_{21} & A_{22} \end{bmatrix},\ A_{p\varepsilon} := \begin{bmatrix} A_{p11} & \varepsilon A_{p12} \\ \varepsilon A_{p21} & A_{p22} \end{bmatrix},\ B_{1\varepsilon} := \begin{bmatrix} B_{11} \\ \varepsilon B_{21} \end{bmatrix},\ B_{2\varepsilon} := \begin{bmatrix} \varepsilon B_{12} \\ B_{22} \end{bmatrix}.$$

$x_i(t) \in \mathbf{R}^{n_i}$ are the state vectors, $u_i(t) \in \mathbf{R}^{m_i},\ i = 1,\ 2$ are the control inputs. $w_p(t) \in \mathbf{R},\ p = 1,\ \ldots, N$ to be one-dimensional standard Wiener process is defined on the filtered probability space [1, 2, 3, 4]. Without loss of generality, it is assumed that $w_i(t),\ w_j(t)$ are mutually independent for all $i,\ j = 1,\ \ldots, N$ and $E[w(t)w^T(t)] = I_N$, where $w(t) := \begin{bmatrix} w_1(t) & \cdots & w_N(t) \end{bmatrix}^T$. ε denotes a relatively small positive coupling parameter that connects the linear system with other subsystems.

The cost functional for each strategy subset is defined by

$$J_i(u_1,\ u_2,\ x(0)) = E\int_0^\infty \Big[x^T(t) Q_{i\varepsilon} x(t) + u_i^T(t) R_{ii} u_i(t) + \varepsilon u_j^T(t) R_{ij} u_j(t)\Big]dt, \tag{2}$$

where $i,\ j = 1,\ 2,\ i \neq j$,

$$Q_{1\varepsilon} = Q_{1\varepsilon}^T = \begin{bmatrix} Q_{111} & \varepsilon Q_{112} \\ \varepsilon Q_{112}^T & \varepsilon Q_{122} \end{bmatrix} \geq 0,\ Q_{2\varepsilon} = Q_{2\varepsilon}^T = \begin{bmatrix} \varepsilon Q_{211} & \varepsilon Q_{212} \\ \varepsilon Q_{212}^T & Q_{222} \end{bmatrix} \geq 0,$$
$$R_{ii} = R_{ii}^T > 0 \in \mathbf{R}^{m_i \times m_i},\ R_{ij} = R_{ij}^T \geq 0 \in \mathbf{R}^{m_j \times m_j}.$$

The stabilizability, which is an essential assumption in this paper is introduced [3, 4].

Definition 1 *The stochastic controlled system governed by Itô equation is called stabilizable, if there exist the feedback laws such that the closed-loop system is asymptotically mean square stable, i.e.,* $\lim_{t\to\infty} Ex^T(t)x(t) = 0$.

For the matrices $A_\varepsilon,\ B_{i\varepsilon},\ i = 1,\ \ldots, M,\ A_{p\varepsilon},\ p = 1,\ \ldots, N$, the set $\mathcal{F}_M$ is defined by $\mathcal{F}_M := \Big\{(F_{1\varepsilon},\ \ldots, F_{M\varepsilon})\ |$ The closed-loop system $dx(t) = [A_\varepsilon + \sum_{i=1}^{M} B_{i\varepsilon} F_{i\varepsilon}]x(t)dt + \sum_{p=1}^{N} A_{p\varepsilon} x(t) dw_p(t)$ is asymptotically mean square stable. $\Big\}$.

In the sequel, the following assumption is introduced [1].

Assumption 1 *There exists a matrix* $F_{i\varepsilon} \in \Re^{m_i \times \bar{n}}$, $\bar{n} := n_1 + n_2$ *such that* $A_\varepsilon + B_{1\varepsilon} F_{1\varepsilon} + B_{2\varepsilon} F_{2\varepsilon}$ *is a stable matrix with* $\|\exp[(A_\varepsilon + B_{1\varepsilon} F_{1\varepsilon} + B_{2\varepsilon} F_{2\varepsilon})t]\| \leq \alpha e^{-\beta t}$, $\exists\ \alpha,\ \beta > 0$ *and* $\alpha^2/\beta \sum_{p=1}^{N} \|A_{p\varepsilon}\|^2 \leq v < 2/(N+1)$.

Assumption 1 implies that the system

$$dx(t) = [A_\varepsilon + B_{1\varepsilon} F_{1\varepsilon} + B_{2\varepsilon} F_{2\varepsilon}]x(t)dt + \sum_{p=1}^{N} A_{p\varepsilon} x(t) dw_p(t) \tag{3}$$

is exponentially mean square stable. Indeed, using the representation of the solution of equation (3) in the form

$$x(t) = \exp[(A_\varepsilon + B_{1\varepsilon}F_{1\varepsilon} + B_{2\varepsilon}F_{2\varepsilon})(t-s)]x(0) + \sum_{p=1}^{N}\int_s^t \exp[(A_\varepsilon + B_{1\varepsilon}F_{1\varepsilon} + B_{2\varepsilon}F_{2\varepsilon})(t-\tau)]A_{p\varepsilon}x(\tau)dw_p(\tau) \tag{4}$$

and the independence of the Wiener processes $w_i(t)$ results in

$$E\|x(t)\|^2 \le (N+1)\|\exp[(A_\varepsilon + B_{1\varepsilon}F_{1\varepsilon} + B_{2\varepsilon}F_{2\varepsilon})(t-s)]\|^2 E\|x(0)\|^2 + (N+1)\sum_{p=1}^{N}\int_s^t \|\exp[(A_\varepsilon + B_{1\varepsilon}F_{1\varepsilon} + B_{2\varepsilon}F_{2\varepsilon})(t-\tau)]\|^2\|A_{p\varepsilon}\|^2 E\|x(\tau)\|^2 d\tau. \tag{5}$$

Thus, the conditions $\|\exp[(A_\varepsilon + B_{1\varepsilon}F_{1\varepsilon} + B_{2\varepsilon}F_{2\varepsilon})t]\| \le \alpha e^{-\beta t}$, $\exists\ \alpha,\ \beta > 0$ and $\alpha^2/\beta\sum_{p=1}^{N}\|A_{p\varepsilon}\|^2 \le v$ imply that

$$e^{2\beta(t-s)}E\|x(t)\|^2 \le (N+1)\alpha^2 E\|x(0)\|^2 + (N+1)\beta v\int_s^t e^{2\beta(\tau-s)}E\|x(\tau)\|^2 d\tau. \tag{6}$$

From the Bellman-Gronwall inequality [11], it follows that

$$E\|x(t)\|^2 \le 2\alpha^2 E\|x(0)\|^2 e^{\beta[(N+1)v-2](t-s)}. \tag{7}$$

Since v has been chosen such that $v < 2/(N+1)$, then equation (3) is exponentially mean square stable.

The stochastic Nash equilibrium strategy pair $(F_{1\varepsilon}^*,\ F_{2\varepsilon}^*)$ is defined as satisfying the following conditions.

$$J_1(F_{1\varepsilon}^*x(t),\ F_{2\varepsilon}^*x(t),\ x(0)) \le J_1(F_{1\varepsilon}x(t),\ F_{2\varepsilon}^*x(t),\ x(0)), \tag{8a}$$
$$J_2(F_{1\varepsilon}^*x(t),\ F_{2\varepsilon}^*x(t),\ x(0)) \le J_2(F_{1\varepsilon}^*x(t),\ F_{2\varepsilon}x(t),\ x(0)), \tag{8b}$$

where

$$u_i(t) := F_{i\varepsilon}x(t),\ i = 1,\ 2, \tag{9}$$

for all $x(0)$ and for all $(F_{1\varepsilon},\ F_{2\varepsilon})$ that satisfy $(F_{1\varepsilon}^*,\ F_{2\varepsilon}) \in \mathcal{F}_2$, $(F_{1\varepsilon},\ F_{2\varepsilon}^*) \in \mathcal{F}_2$, and $(F_{1\varepsilon}^*,\ F_{2\varepsilon}^*) \in \mathcal{F}_2$.

It should be noted that the systems governed by Itô differential equations are disturbed by deterministic noise and the strategy spaces are of the static linear feedback form.

2.1 One-Player Case

First, one-player case is discussed. The result obtained for that particular case will be the basis for the derivation of results for 2-player case.

Consider a linear time-invariant stochastic stabilizable system

$$dx(t) = [A_\varepsilon x(t) + B_{1\varepsilon}u_1(t)]dt + \sum_{p=1}^{N}A_{p\varepsilon}x(t)dw_p(t),\ x(0) = x^0, \tag{10}$$

where $u_1(t) := F_{1\varepsilon}x(t)$, $F_{1\varepsilon} \in \mathcal{F}_1$. The cost functional is given below.

$$J(u_1,\ x(0)) = E\int_0^\infty \left[x^T(t)Q_{1\varepsilon}x(t) + u_1^T(t)R_{11}u_1(t)\right]dt. \tag{11}$$

Theorem 1 *Assume that for any* $u_1(t)$*, the closed-loop system is asymptotically mean square stable. Suppose that the following stochastic algebraic Riccati equation* (SARE) *has a solution* $P_\varepsilon > 0$.

$$P_\varepsilon A_\varepsilon + A_\varepsilon^T P_\varepsilon + \sum_{p=1}^{N} A_{p\varepsilon}^T P_\varepsilon A_{p\varepsilon} - P_\varepsilon S_{1\varepsilon} P_\varepsilon + Q_{1\varepsilon} = 0, \tag{12}$$

where $S_{1\varepsilon} := B_{1\varepsilon} R_{11}^{-1} B_{1\varepsilon}^T$.

The strategy that minimizes the cost functional (11) *is given below.*

$$u_1^*(t) = F_{1\varepsilon}^* x(t) = -R_{11}^{-1} B_{1\varepsilon}^T P_\varepsilon x(t). \tag{13}$$

Proof : Since the assumption that for any $u_1(t)$, the closed-loop system is asymptotically mean square stable, $\lim_{t\to\infty} Ex^T(t)x(t) = 0$. Thus, applying Itô's formula to (10) and considering (12) results in

$$J(u_1,\ x(0)) = x^T(0) P_\varepsilon x(0) + E \int_0^\infty \|u_1(t) - u_1^*(t)\|_{R_{11}}^2 dt \geq x^T(0) P_\varepsilon x(0). \tag{14}$$

Hence,

$$J(u_1,\ x(0)) \geq J(u_1^*,\ x(0)) = x^T(0) P_\varepsilon x(0). \tag{15}$$

This is the desired result. ■

2.2 Stochastic Nash Equilibrium Strategies

The solution of the stochastic Nash games is given below.

Theorem 2 *Suppose that there exist the real symmetric matrices* $P_{i\varepsilon}$ *such that*

$$\begin{aligned}\mathcal{G}_i(\varepsilon,\ P_{1\varepsilon},\ P_{2\varepsilon}) = P_{i\varepsilon}\left(A_\varepsilon - S_{j\varepsilon} P_{j\varepsilon}\right) + \left(A_\varepsilon - S_{j\varepsilon} P_{j\varepsilon}\right)^T P_{i\varepsilon} + \sum_{p=1}^{N} A_{p\varepsilon}^T P_{i\varepsilon} A_{p\varepsilon} \\ - P_{i\varepsilon} S_{i\varepsilon} P_{i\varepsilon} + \varepsilon P_{j\varepsilon} S_{ij\varepsilon} P_{j\varepsilon} + Q_{i\varepsilon} = 0,\end{aligned} \tag{16}$$

where $i,\ j = 1,\ 2, i \neq j$, $S_{i\varepsilon} := B_{i\varepsilon} R_{ii}^{-1} B_{i\varepsilon}^T$, $S_{ij\varepsilon} := B_{j\varepsilon} R_{jj}^{-1} R_{ij} R_{jj}^{-1} B_{j\varepsilon}^T$.

Define the strategy pair $(F_{1\varepsilon}^*,\ F_{2\varepsilon}^*)$ *by*

$$u_i^*(t) := F_{i\varepsilon}^* x(t) = -R_{ii}^{-1} B_{i\varepsilon}^T P_{i\varepsilon} x(t),\ i = 1,\ 2. \tag{17}$$

Then, $(F_{1\varepsilon}^*,\ F_{2\varepsilon}^*) \in \mathcal{F}_2$ *and this strategy set is a stochastic Nash equilibrium. Furthermore, the minimal value of cost functional satisfies* $J_i(F_{1\varepsilon}^*,\ F_{2\varepsilon}^*,\ x(0)) = x^T(0) P_{i\varepsilon} x(0)$.

Proof : Now let us consider the following problem that the cost functional (18) is minimal at $F_{i\varepsilon} = F_{i\varepsilon}^*$.

$$\phi(F_\varepsilon) := E \int_0^\infty x^T(t)(Q_{i\varepsilon} + F_{i\varepsilon}^T R_{ii} F_{i\varepsilon} + \varepsilon P_{j\varepsilon}^T S_{ij} P_{j\varepsilon}) x(t) dt, \tag{18}$$

where $x(t)$ follows from

$$dx(t) = (A_\varepsilon - S_{j\varepsilon} P_{j\varepsilon} + B_{i\varepsilon} F_{i\varepsilon}) x(t) dt + \sum_{p=1}^{N} A_{p\varepsilon} x(t) dw_p(t),\ x(0) = x^0,\ i,\ j = 1,\ 2,\ i \neq j. \tag{19}$$

Note that the function ϕ coincides with the cost functional $J(u_1,\ x(0))$ in Theorem 1. Applying Theorem 1 to this minimization problem as

$$A_\varepsilon - S_{j\varepsilon} P_{j\varepsilon} \Rightarrow A_\varepsilon,\ B_{i\varepsilon} \Rightarrow B_{1\varepsilon},\ Q_{i\varepsilon} + \varepsilon P_{j\varepsilon}^T S_{ij} P_{j\varepsilon} \Rightarrow Q_{1\varepsilon},\ R_{ii} \Rightarrow R_{11}$$

yields the fact that the function ϕ is minimal at

$$F_{1\varepsilon}^* = -R_{11}^{-1} B_{1\varepsilon}^T P_\varepsilon = -R_{ii}^{-1} B_{i\varepsilon}^T P_{i\varepsilon} = F_{i\varepsilon}^*. \tag{20}$$

Moreover, the minimal value is equal to $x^T(0) P_{i\varepsilon} x(0)$. ■

3 Asymptotic Structure of the CSAREs

Firstly, in order to obtain the strategy set that is based on the numerical solutions, the asymptotic structure of the CSAREs (16) is established. Since $A_\varepsilon,\ A_{p\varepsilon},\ S_{i\varepsilon}$ and $S_{ij\varepsilon}$ include the term of the parameter ε, the solution $P_{i\varepsilon}$ of the CSAREs (16), if it exists, must contain the parameter ε. Taking this fact into account, the solution $P_{i\varepsilon}$ of the CSAREs (16) with the following structure is considered.

$$P_{1\varepsilon} = \begin{bmatrix} P_{111} & \varepsilon P_{112} \\ \varepsilon P_{112}^T & \varepsilon P_{122} \end{bmatrix},\ P_{2\varepsilon} = \begin{bmatrix} \varepsilon P_{211} & \varepsilon P_{212} \\ \varepsilon P_{212}^T & P_{222} \end{bmatrix}. \tag{21}$$

Substituting the matrices A_ε, $A_{1\varepsilon}$, $S_{i\varepsilon}$, $S_{ij\varepsilon}$, $Q_{i\varepsilon}$ and $P_{i\varepsilon}$ into the CSAREs (16), letting $\varepsilon = 0$, and partitioning the CSAREs (16), the following reduced-order stochastic algebraic Riccati equation (SARE) are obtained, where $\bar{P}_{iii},\ i = 1,\ 2$ be the 0-order solutions of the CSAREs (16) as $\varepsilon = 0$.

$$\bar{P}_{iii}A_{ii} + A_{ii}^T\bar{P}_{iii} + \sum_{p=1}^{N} A_{pii}^T\bar{P}_{iii}A_{pii} - \bar{P}_{iii}S_{ii}\bar{P}_{iii} + Q_{iii} = 0,\ i = 1,\ 2, \tag{22}$$

where $S_{ii} := B_{ii}R_{ii}^{-1}B_{ii}^T$.

The following condition is assumed.

Assumption 2 *$(A_{ii},\ B_{ii})$ is stabilizable, $(\sqrt{Q}_{iii},\ A_{ii})$ is observable, and*

$$\inf_{K_{ii}} \left\| \int_0^\infty \exp[(A_{ii} - B_{ii}K_{ii})^T t] \left(\sum_{p=1}^{N} A_{pii}^T A_{pii} \right) \exp[(A_{ii} - B_{ii}K_{ii})t]dt \right\| < 1.$$

If the above assumption holds, there exists the unique positive definite stabilizing solution $\bar{P}_{iii} > 0$ of the SARE (22) such that $D_{ii} := A_{ii} - S_{ii}\bar{P}_{iii}$ is stable.

The asymptotic expansion of the CSAREs (16) at $\varepsilon = 0$ is described by the following theorem.

Theorem 3 *Under Assumption 2, there exists the small constant σ^* such that for all $\varepsilon \in (0,\ \sigma^*)$ the CSAREs (16) admits a positive semidefinite solution $P_{i\varepsilon}^*$ that can be written as*

$$P_{i\varepsilon} := P_{i\varepsilon}^* = \bar{P}_i + O(\varepsilon), \tag{23}$$

where

$$\bar{P}_1 = \textbf{block diag} \begin{pmatrix} \bar{P}_{111} & 0 \end{pmatrix},\ \bar{P}_2 = \textbf{block diag} \begin{pmatrix} 0 & \bar{P}_{222} \end{pmatrix}.$$

Proof : The proof can be done by using the implicit function theorem to the CSAREs (16). To do so, it is enough to show that the corresponding Jacobian is nonsingular at $\varepsilon = 0$. The derivative of the function $\mathcal{G}_i(\varepsilon,\ P_{1\varepsilon},\ P_{2\varepsilon})$ at the matrix $P_{i\varepsilon}$ is given by

$$\begin{aligned} \mathbf{J}_{ii} &:= \frac{\partial}{\partial \mathrm{vec} P_{i\varepsilon}} \mathrm{vec}\mathcal{G}_i(\varepsilon,\ P_{1\varepsilon},\ P_{2\varepsilon})^T \\ &= (A_\varepsilon - S_{1\varepsilon}P_{1\varepsilon} - S_{2\varepsilon}P_{2\varepsilon})^T \otimes I_{\bar{n}} + I_{\bar{n}} \otimes (A_\varepsilon - S_{1\varepsilon}P_{1\varepsilon} - S_{2\varepsilon}P_{2\varepsilon})^T + \sum_{p=1}^{N} A_{p\varepsilon}^T \otimes A_{p\varepsilon}^T, \qquad (24a) \\ \mathbf{J}_{ij} &:= \frac{\partial}{\partial \mathrm{vec} P_{j\varepsilon}} \mathrm{vec}\mathcal{G}_i(\varepsilon,\ P_{1\varepsilon},\ P_{2\varepsilon})^T = -(S_{j\varepsilon}P_{i\varepsilon} - \varepsilon S_{ij\varepsilon}P_{j\varepsilon})^T \otimes I_{\bar{n}} - I_{\bar{n}} \otimes (S_{j\varepsilon}P_{i\varepsilon} - \varepsilon S_{ij\varepsilon}P_{j\varepsilon})^T. \qquad (24b) \end{aligned}$$

Using the fact that $S_{j\varepsilon}P_{i\varepsilon} = O(\varepsilon)$, after some algebra, the Jacobian of the CSAREs (16) in the limit as $\varepsilon \to +0$ is given by

$$\mathbf{J} = \begin{bmatrix} \mathbf{J}_{11}|_{\varepsilon=0} & \mathbf{J}_{12}|_{\varepsilon=0} \\ \mathbf{J}_{21}|_{\varepsilon=0} & \mathbf{J}_{22}|_{\varepsilon=0} \end{bmatrix} = \begin{bmatrix} \mathbf{J}_0 & 0 \\ 0 & \mathbf{J}_0 \end{bmatrix}, \tag{25}$$

where

$$\mathbf{J}_0 = \mathrm{D}^T \otimes I_{\bar{n}} + I_{\bar{n}} \otimes \mathrm{D}^T + \sum_{p=1}^{N} A_{p0}^T \otimes A_{p0}^T,$$
$$\mathrm{D} := \textbf{block diag} \begin{pmatrix} D_{11} & D_{22} \end{pmatrix},\ A_{p0} := \textbf{block diag} \begin{pmatrix} A_{p11} & A_{p22} \end{pmatrix}.$$

Obviously, D_{11} and D_{22} are nonsingular under Assumption 2. Thus, det$\mathbf{J} \neq 0$, i.e., $\mathbf{J}$ is nonsingular at $\varepsilon = 0$. The conclusion of Theorem 3 is obtained directly by using the implicit function theorem. On the other hand, taking into account the fact that $\bar{P}_{ii}$ is the positive semidefinite matrix, for sufficiently small parameter ε, $P_{i\varepsilon}$ is also the positive semidefinite solution. ■

4 Newton's Method

In order to obtain the solution of CSAREs (16), the following useful algorithm is given.

$$\begin{aligned}
&P_{1\varepsilon}^{(k+1)} \left(A_\varepsilon - S_{1\varepsilon}P_{1\varepsilon}^{(k)} - S_{2\varepsilon}P_{2\varepsilon}^{(k)}\right) + \left(A_\varepsilon - S_{1\varepsilon}P_{1\varepsilon}^{(k)} - S_{2\varepsilon}P_{2\varepsilon}^{(k)}\right)^T P_{1\varepsilon}^{(k+1)} \\
&\quad + \sum_{p=1}^{N} A_{p\varepsilon}P_{1\varepsilon}^{(k+1)}A_{p\varepsilon} - P_{2\varepsilon}^{(k+1)}S_{2\varepsilon}P_{1\varepsilon}^{(k)} - P_{1\varepsilon}^{(k)}S_{2\varepsilon}P_{2\varepsilon}^{(k+1)} + \varepsilon P_{2\varepsilon}^{(k+1)}S_{12\varepsilon}P_{2\varepsilon}^{(k)} + \varepsilon P_{2\varepsilon}^{(k)}S_{12\varepsilon}P_{2\varepsilon}^{(k+1)} \\
&\quad + P_{1\varepsilon}^{(k)}S_{2\varepsilon}P_{2\varepsilon}^{(k)} + P_{2\varepsilon}^{(k)}S_{2\varepsilon}P_{1\varepsilon}^{(k)} + P_{1\varepsilon}^{(k)}S_{1\varepsilon}P_{1\varepsilon}^{(k)} - \varepsilon P_{2\varepsilon}^{(k)}S_{12\varepsilon}P_{2\varepsilon}^{(k)} + Q_{1\varepsilon} = 0, \qquad (26a)\\
&P_{2\varepsilon}^{(k+1)} \left(A_\varepsilon - S_{1\varepsilon}P_{1\varepsilon}^{(k)} - S_{2\varepsilon}P_{2\varepsilon}^{(k)}\right) + \left(A_\varepsilon - S_{1\varepsilon}P_{1\varepsilon}^{(k)} - S_{2\varepsilon}P_{2\varepsilon}^{(k)}\right)^T P_{2\varepsilon}^{(k+1)} \\
&\quad + \sum_{p=1}^{N} A_{p\varepsilon}P_{2\varepsilon}^{(k+1)}A_{p\varepsilon} - P_{1\varepsilon}^{(k+1)}S_{1\varepsilon}P_{2\varepsilon}^{(k)} - P_{2\varepsilon}^{(k)}S_{1\varepsilon}P_{1\varepsilon}^{(k+1)} + \varepsilon P_{1\varepsilon}^{(k+1)}S_{21\varepsilon}P_{1\varepsilon}^{(k)} + \varepsilon P_{1\varepsilon}^{(k)}S_{21\varepsilon}P_{1\varepsilon}^{(k+1)} \\
&\quad + P_{2\varepsilon}^{(k)}S_{1\varepsilon}P_{1\varepsilon}^{(k)} + P_{1\varepsilon}^{(k)}S_{1\varepsilon}P_{2\varepsilon}^{(k)} + P_{2\varepsilon}^{(k)}S_{2\varepsilon}P_{2\varepsilon}^{(k)} - \varepsilon P_{1\varepsilon}^{(k)}S_{21\varepsilon}P_{1\varepsilon}^{(k)} + Q_{2\varepsilon} = 0, \qquad (26b)
\end{aligned}$$

with the initial conditions

$$P_{i\varepsilon}^{(0)} = \bar{P}_i. \qquad (27)$$

The following theorem indicates that the proposed algorithm (26) that is based on the Newton's method attains the quadratic convergence.

Theorem 4 *Under Assumption 2, there exists the small constant $\bar{\sigma}$ such that for all $\varepsilon \in (0,\ \bar{\sigma})$, $\bar{\sigma} \leq \sigma^*$, the iterative algorithm* (26) *converges to the exact solution of $P_{i\varepsilon}^*$ with the rate of the quadratic convergence, where $P_{i\varepsilon}^{(k)}$ is positive semidefinite matrix. Moreover, the convergence solutions attain a local unique solution $P_{i\varepsilon}^*$ of the CSAREs* (16) *in the neighborhood of the initial condition $P_{i\varepsilon}^{(0)} = \bar{P}_i$. That is, the following conditions are satisfied.*

$$\|P_{i\varepsilon}^{(k)} - P_{i\varepsilon}^*\| = O(\varepsilon^{2^k}),\ i = 1,\ 2,\ k = 0,\ 1,\ \ldots\ . \qquad (28)$$

Proof: The proof is given directly by applying the Newton-Kantorovich theorem [10] for the CSAREs (16). It is immediately obtained from the CSAREs (16) that there exists a positive scalar γ such that for any $P_{i\varepsilon}^a$ and $P_{i\varepsilon}^b$

$$\begin{aligned}
&\|\nabla\mathcal{G}(\varepsilon,\ P_{1\varepsilon}^a,\ P_{2\varepsilon}^a) - \nabla\mathcal{G}(\varepsilon,\ P_{1\varepsilon}^b,\ P_{2\varepsilon}^b)\| \\
&\leq \gamma\|([\mathrm{vec}P_{1\varepsilon}^a]^T,\ [\mathrm{vec}P_{2\varepsilon}^a]^T) - ([\mathrm{vec}P_{1\varepsilon}^b]^T,\ [\mathrm{vec}P_{2\varepsilon}^b]^T)\|,
\end{aligned}$$

where $\mathcal{G} := \begin{bmatrix} \mathcal{G}_1 & \mathcal{G}_2 \end{bmatrix}^T$ and $\gamma := 6(\|S_{1\varepsilon}\| + \|S_{2\varepsilon}\|) + 2\varepsilon(\|S_{12\varepsilon}\| + \|S_{21\varepsilon}\|)$.

Moreover, it is easy to verify that $\nabla\mathcal{G}(\varepsilon,\ P_{1\varepsilon}^{(0)},\ P_{2\varepsilon}^{(0)}) = \nabla\mathcal{G}(0,\ \bar{P}_1,\ \bar{P}_2) = \mathbf{J} + O(\varepsilon)$ is nonsingular because for small ε, using (23) and $\mathbf{J}$ is also nonsingular. Therefore, there exists β such that $\beta = \|[\nabla\mathcal{G}(\varepsilon,\ \bar{P}_1,\ \bar{P}_2)]^{-1}\|$.

On the other hand, since $\|\mathcal{G}(\varepsilon,\ \bar{P}_1,\ \bar{P}_2)\| = O(\varepsilon)$, there exists η such that $\eta = \|[\nabla\mathcal{G}(\varepsilon,\ \bar{P}_1,\ \bar{P}_2)]^{-1}\| \cdot \|\mathcal{G}(\varepsilon,\ \bar{P}_1,\ \bar{P}_2)\| = O(\varepsilon)$. Thus, there exists θ such that $\theta = \beta\eta\gamma < 2^{-1}$ because $\eta = O(\varepsilon)$. Finally, the Newton-Kantorovich theorem results in the desired results (28).

Second, the local uniqueness of the solution is discussed. Now, let us define $t^* \equiv [1 - \sqrt{1-2\theta}]/(\gamma\beta)$. Clearly, $S \equiv \{\ P_{i\varepsilon}\ :\ \|P_{i\varepsilon} - P_{i\varepsilon}^{(0)}\| \le t^*\ \}$ is in the certain convex set D. In the sequel, since $\|P_{i\varepsilon} - P_{i\varepsilon}^{(0)}\| = O(\varepsilon)$ holds for a small ε, the local uniqueness of $P_{i\varepsilon}^*$ is guaranteed in the neighbourhood of $\varepsilon = 0$ for a subset S by applying the Newton-Kantorovich theorem. ■

5 A Numerical Algorithm for Solving the CSALE

When the cross-coupled stochastic algebraic Lyapunov equation (CSALE) (26) is solved, the existence of the cross-coupled term in CSALE (26) makes it difficult to solve this equation directly. Thus, in order to avoid the cross-coupled term, a new decoupling algorithm that is based on the fixed-point algorithm is established. Taking into account the fact that $S_{j\varepsilon}P_{i\varepsilon}^{(k)} = O(\varepsilon),\ i \ne j$, let us consider CSALE (29) in its general form.

$$X_{i\varepsilon}\Lambda_{i\varepsilon} + \Lambda_{i\varepsilon}^T X_{i\varepsilon} + \sum_{p=1}^{N} A_{p\varepsilon}X_{i\varepsilon}A_{p\varepsilon} + \varepsilon(X_{j\varepsilon}\Phi_{j\varepsilon} + \Phi_{j\varepsilon}^T X_{j\varepsilon}) + U_{i\varepsilon} = 0,\ i,\ j = 1,\ 2,\ i \ne j, \tag{29}$$

where

$$X_{1\varepsilon} := \begin{bmatrix} X_{111} & \varepsilon X_{112} \\ \varepsilon X_{112}^T & \varepsilon X_{122} \end{bmatrix},\ X_{2\varepsilon} := \begin{bmatrix} \varepsilon X_{211} & \varepsilon X_{212} \\ \varepsilon X_{212}^T & X_{222} \end{bmatrix},$$
$$\Lambda_{i\varepsilon} := \begin{bmatrix} \Lambda_{i11} & \varepsilon\Lambda_{i12} \\ \varepsilon\Lambda_{i21} & \Lambda_{i22} \end{bmatrix},\ \Phi_{i\varepsilon} := \begin{bmatrix} \Phi_{i11} & \varepsilon\Phi_{i21} \\ \varepsilon\Phi_{i21} & \Phi_{i22} \end{bmatrix},$$
$$U_{1\varepsilon} := \begin{bmatrix} U_{111} & \varepsilon U_{112} \\ \varepsilon U_{112}^T & \varepsilon U_{122} \end{bmatrix},\ U_{2\varepsilon} := \begin{bmatrix} \varepsilon U_{211} & \varepsilon U_{212} \\ \varepsilon U_{212}^T & U_{222} \end{bmatrix}.$$

It should be noted that

$$P_{i\varepsilon}^{(k+1)} \Rightarrow X_{i\varepsilon},\ A_\varepsilon - S_{1\varepsilon}P_{1\varepsilon}^{(k)} - S_{2\varepsilon}P_{2\varepsilon}^{(k)} + M_{i\varepsilon}P_{i\varepsilon}^{(k)} \Rightarrow \Lambda_{i\varepsilon},$$
$$-S_{i\varepsilon}P_{j\varepsilon}^{(k)} + \varepsilon S_{ji\varepsilon}P_{i\varepsilon}^{(k)} \Rightarrow \varepsilon\Phi_{i\varepsilon},$$
$$P_{i\varepsilon}^{(k)}S_{j\varepsilon}P_{j\varepsilon}^{(k)} + P_{j\varepsilon}^{(k)}S_{j\varepsilon}P_{i\varepsilon}^{(k)} + P_{i\varepsilon}^{(k)}S_{i\varepsilon}P_{i\varepsilon}^{(k)} - \varepsilon P_{j\varepsilon}^{(k)}S_{ij\varepsilon}P_{j\varepsilon}^{(k)} - P_{i\varepsilon}^{(k)}M_{i\varepsilon}P_{i\varepsilon}^{(k)} + Q_{i\varepsilon} \Rightarrow U_{i\varepsilon},$$

where $\Rightarrow$ represents the replacement.

Without loss of generality, the following condition is assumed for CSALE (29).

Assumption 3 $(\Lambda_{i\varepsilon},\ A_{1\varepsilon})$, *$i = 1,\ 2$ are stable.*

Algorithm (30) for solving CSALE (29) is given as follows:

$$X_{i\varepsilon}^{(n+1)}\Lambda_{i\varepsilon} + \Lambda_{i\varepsilon}^T X_{i\varepsilon}^{(n+1)} + \sum_{p=1}^{N} A_{p\varepsilon}X_{i\varepsilon}^{(n+1)}A_{p\varepsilon} + \varepsilon(X_{j\varepsilon}^{(n)}\Phi_{j\varepsilon} + \Phi_{j\varepsilon}^T X_{j\varepsilon}^{(n)}) + U_{i\varepsilon} = 0, \tag{30}$$
$$i,\ j = 1,\ 2,\ i \ne j,\ n = 0,\ 1,\ \ldots\ ,$$

where $X_{i\varepsilon}^{(0)} = 0$.

It should be noted that the numerical computation of (30) can be carried out independently for each solution. The following theorem indicates the convergence of algorithm (30).

Theorem 5 *Under Assumption 3, the fixed-point algorithm* (30) *converges to an exact solution $X_{i\varepsilon}$ with a rate of*

$$\|X_{i\varepsilon}^{(n)} - X_{i\varepsilon}\| = O(\varepsilon^n),\ n = 1,\ 2,\ \ldots\ . \tag{31}$$

In order to prove Theorem 5, the following Lemma will be used.

Lemma 1 *If* $dz(t) = Az(t)dt + \sum_{p=1}^{N} A_p z(t)dw_p(t)$ *is exponentially mean-square stable and* $Q = Q^T \geq 0$, $z^T(0)Pz(0) = \int_0^\infty z^T(t)Qz(t)dt$, *where* P *satisfies the stochastic algebraic Lyapunov equation (SALE)* : $A^T P + PA + \sum_{p=1}^{N} A_p^T PA_p + Q = 0$.

Proof : The proof of Theorem 5 can be derived by using mathematical induction. When $n = 0$ for algorithm (30), it is easy to verify that the first order approximations $X_{i\varepsilon}$ corresponding to ε are $X_{i\varepsilon}^{(1)}$. When $n = h,\ h \geq 2$, it is assumed that

$$\|X_{i\varepsilon}^{(h)} - X_{i\varepsilon}\| = O(\varepsilon^h). \tag{32}$$

By subtracting (29) from (30) and substituting h into n, the following equations hold:

$$\begin{aligned}
&(X_{i\varepsilon}^{(h+1)} - X_{i\varepsilon})\Lambda_{i\varepsilon} + \Lambda_{i\varepsilon}^T(X_{i\varepsilon}^{(h+1)} - X_{i\varepsilon}) + \sum_{p=1}^{N} A_{p\varepsilon}^T(X_{i\varepsilon}^{(h+1)} - X_{i\varepsilon})A_{p\varepsilon} \\
&= -\varepsilon\left[(X_{j\varepsilon}^{(h)} - X_{j\varepsilon})\Phi_{j\varepsilon} + \Phi_{j\varepsilon}^T(X_{j\varepsilon}^{(h)} - X_{j\varepsilon})\right].
\end{aligned} \tag{33}$$

Using the assumption (32), the following equations are satisfied:

$$(X_{i\varepsilon}^{(h+1)} - X_{i\varepsilon})\Lambda_{\varepsilon} + \Lambda_{\varepsilon}^T(X_{i\varepsilon}^{(h+1)} - X_{i\varepsilon}) + \sum_{p=1}^{N} A_{p\varepsilon}^T(X_{i\varepsilon}^{(h+1)} - X_{i\varepsilon})A_{p\varepsilon} = O(\varepsilon^{h+1}). \tag{34}$$

Hence, using Lemma 1, it is easy to verify that

$$\|X_{i\varepsilon}^{(h+1)} - X_{i\varepsilon}\| = O(\varepsilon^{h+1}). \tag{35}$$

Consequently, error equations (31) hold for all $n \in \mathbf{N}$. This completes the proof of Theorem 5. ■

When CSALE (30) is solved, a large computational dimension $\bar{n} := n_1 + n_2$ is required to be compared with the small computational dimensions n_i. Thus, in order to reduce the computational dimension, a fixed-point algorithm can be applied again.

Let us consider the SALE (36) in the following general form.

$$Y_\varepsilon E_\varepsilon + E_\varepsilon^T Y_\varepsilon + \sum_{p=1}^{N} A_{p\varepsilon}^T Y_\varepsilon A_{p\varepsilon} + H_\varepsilon = 0, \tag{36}$$

where

$$\begin{aligned}
&E_\varepsilon := \begin{bmatrix} E_{11} & \varepsilon E_{12} \\ \varepsilon E_{21} & E_{22} \end{bmatrix},\ H_\varepsilon := \begin{bmatrix} H_1 & \varepsilon H_{12} \\ \varepsilon H_{12}^T & H_2 \end{bmatrix},\ Y_\varepsilon = \begin{bmatrix} Y_1 & \varepsilon Y_{12} \\ \varepsilon Y_{12}^T & Y_2 \end{bmatrix}, \\
&E_{ii},\ A_{pii} \in \mathbf{R}^{n_i \times n_i},\ Y_i = Y_i^T \geq 0 \in \mathbf{R}^{n_i \times n_i},\ H_i = H_i^T \in \mathbf{R}^{n_i \times n_i},\ i = 1,\ 2.
\end{aligned}$$

It should be noted that for the SALE (36),

$$A_\varepsilon - S_\varepsilon P_\varepsilon^{(n)} \Rightarrow E_\varepsilon,\ A_{p\varepsilon} \Rightarrow A_{p\varepsilon},\ P_\varepsilon^{(n)} S_\varepsilon P_\varepsilon^{(n)} + Q \Rightarrow H_\varepsilon, P_\varepsilon^{(n+1)} \Rightarrow Y_\varepsilon,$$

where $\Rightarrow$ stands for the replacement.

Substituting Y_ε into the SALE (36), the following set of three linear equations (37) is satisfied.

$$\begin{aligned}
&E_{11}^T Y_1 + Y_1 E_{11} + \varepsilon^2(E_{21}^T Y_{12}^T + Y_{12}E_{21}) \\
&\quad + \sum_{p=1}^{N}\left[A_{p11}^T Y_1 A_{p11} + \varepsilon^2(A_{p21}^T Y_2 A_{p21} + A_{p21}^T Y_{12}^T A_{p11} + A_{p11}^T Y_{12} A_{p21})\right] + H_1 = 0, \qquad (37a) \\
&E_{11}^T Y_{12} + Y_1 E_{12} + E_{21}^T Y_2 + Y_{12} E_{22}
\end{aligned}$$

$$+\sum_{p=1}^{N}\left[A_{p11}^TY_1A_{p12}+A_{p21}^TY_2A_{p22}+\varepsilon^2A_{p21}^TY_{12}^TA_{p12}+A_{p11}^TY_{12}A_{p22}\right]+H_{12}=0, \tag{37b}$$

$$E_{22}^TY_2+Y_2E_{22}+\varepsilon^2(E_{12}^TY_{12}+Y_{12}^TE_{12})$$
$$+\sum_{p=1}^{N}\left[A_{p22}^TY_2A_{p22}+\varepsilon^2(A_{p12}^TY_1A_{p12}+A_{p22}^TY_{12}^TA_{p12}+A_{p12}^TY_{12}A_{p22})\right]+H_2=0. \tag{37c}$$

First, in order to guarantee the existence of the unique solution of the set of the SALE (37), the following assumption is supposed.

Assumption 4 $\det F_{ii}\neq 0$, $\det F_{12}\neq 0$, *where*

$$F_{ii}:=E_{ii}^T\otimes I_{n_i}+I_{n_i}\otimes E_{ii}^T+\sum_{p=1}^{N}A_{pii}^T\otimes A_{pii}^T,\ i=1,\ 2,$$

$$F_{12}:=E_{22}^T\otimes I_{n_1}+I_{n_2}\otimes E_{11}^T+\sum_{p=1}^{N}A_{p22}^T\otimes A_{p11}^T.$$

It should be noted that the Assumption 4 is satisfied automatically for sufficiently small ε because of Assumption 2.

By considering the form of equation (37), the following algorithm in equation (38) to solve the SALE (36) is given.

$$E_{11}^TY_1^{(m+1)}+Y_1^{(m+1)}E_{11}+\sum_{p=1}^{N}A_{p11}^TY_1^{(m+1)}A_{p11}$$
$$+\varepsilon^2\left[E_{21}^TY_{12}^{(m)T}+Y_{12}^{(m)}E_{21}+\sum_{p=1}^{N}(A_{p21}^TY_2^{(m)}A_{p21}+A_{p21}^TY_{12}^{(m)T}A_{p11}+A_{p11}^TY_{12}^{(m)}A_{p21})\right]+H_1=0, \tag{38a}$$

$$E_{22}^TY_2^{(m+1)}+Y_2^{(m+1)}E_{22}+\sum_{p=1}^{N}A_{p22}^TY_2^{(m+1)}A_{p22}$$
$$+\varepsilon^2\left[E_{12}^TY_{12}^{(m)}+Y_{12}^{(m)T}E_{12}+\sum_{p=1}^{N}(A_{p12}^TY_1^{(m)}A_{p12}+A_{p22}^TY_{12}^{(m)T}A_{p12}+A_{p12}^TY_{12}^{(m)}A_{p22})\right]+H_2=0, \tag{38b}$$

$$E_{11}^TY_{12}^{(m+1)}+Y_{12}^{(m+1)}E_{22}+\sum_{p=1}^{N}A_{p11}^TY_{12}^{(m+1)}A_{p22}+Y_1^{(m+1)}E_{12}+E_{21}^TY_2^{(m+1)}$$
$$+\sum_{p=1}^{N}\left[A_{p11}^TY_1^{(m+1)}A_{p12}+A_{p21}^TY_2^{(m+1)}A_{p22}+\varepsilon^2A_{p21}^TY_{12}^{(m)T}A_{p12}\right]+H_{12}=0, \tag{38c}$$
$$Y_i^{(0)}=0,\ i=1,\ 2,\ Y_{12}^{(0)}=0.$$

The following theorem indicates the convergence of the algorithm in equation (38).

Theorem 6 *Suppose that Assumption 4 is satisfied. There exists a small* $\hat{\sigma}$ *such that for all* $\varepsilon\in(0,\ \hat{\sigma})$, $0<\hat{\sigma}$, *the iterative algorithm in equation* (38) *converges to the exact solutions of* Y_i *and* Y_{21} *with a rate equal to that of linear convergence. Subsequently, the following equations hold.*

$$\|Y_i^{(m)}-Y_i\|=O(\varepsilon^{2m}),\ i=1,\ 2, \tag{39a}$$
$$\|Y_{12}^{(m)}-Y_{12}\|=O(\varepsilon^{2m}),\ m=1,\ 2,\ \dots\ . \tag{39b}$$

Proof: The proof of Theorem 6 can be done by mathematical induction. Subtracting equation (37) from equation (38), the following equations (40) are obtained.

$$E_{11}^T\tilde{Y}_1^{(m+1)}+\tilde{Y}_1^{(m+1)}E_{11}+\sum_{p=1}^N A_{p11}^T\tilde{Y}_1^{(m+1)}A_{p11}$$
$$+\varepsilon^2\left[E_{21}^T\tilde{Y}_{12}^{(m)T}+\tilde{Y}_{12}^{(m)}E_{21}+\sum_{p=1}^N(A_{p21}^T\tilde{Y}_2^{(m)}A_{p21}+A_{p21}^T\tilde{Y}_{12}^{(m)T}A_{p11}+A_{p11}^T\tilde{Y}_{12}^{(m)}A_{p21})\right]=0, \quad (40a)$$
$$E_{22}^T\tilde{Y}_2^{(m+1)}+\tilde{Y}_2^{(m+1)}E_{22}+\sum_{p=1}^N A_{p22}^T\tilde{Y}_2^{(m+1)}A_{p22}$$
$$+\varepsilon^2\left[E_{12}^T\tilde{Y}_{12}^{(m)}+\tilde{Y}_{12}^{(m)T}E_{12}+\sum_{p=1}^N(A_{p12}^T\tilde{Y}_1^{(m)}A_{p12}+A_{p22}^T\tilde{Y}_{12}^{(m)T}A_{p12}+A_{p12}^T\tilde{Y}_{12}^{(m)}A_{p22})\right]=0, \quad (40b)$$
$$E_{11}^T\tilde{Y}_{12}^{(m+1)}+\tilde{Y}_{12}^{(m+1)}E_{22}+\sum_{p=1}^N A_{p11}^T\tilde{Y}_{12}^{(m+1)}A_{p22}+\tilde{Y}_1^{(m+1)}E_{12}+E_{21}^T\tilde{Y}_2^{(m+1)}$$
$$+\sum_{p=1}^N\left[A_{p11}^T\tilde{Y}_1^{(m+1)}A_{p12}+A_{p21}^T\tilde{Y}_2^{(m+1)}A_{p22}+\varepsilon^2A_{p21}^T\tilde{Y}_{12}^{(m)T}A_{p12}\right]=0, \quad (40c)$$

where

$$\tilde{Y}_i^{(m)}:=Y_i^{(m)}-Y_i,\ i=1,\ 2,\ \tilde{Y}_{12}^{(m)}:=Y_{12}^{(m)}-Y_{12}.$$

Setting $k=0$, the following equations (41) hold.

$$E_{ii}^T\tilde{Y}_i^{(1)}+\tilde{Y}_i^{(1)}E_{ii}+\sum_{p=1}^N A_{pii}^T\tilde{Y}_i^{(1)}A_{pii}+O(\varepsilon^2)=0,\ i=1,\ 2. \quad (41)$$

Hence, taking into account the nonsingularity of $F_i,\ i=1,\ 2$ and F_{12} and Lemma 1, $\|Y_i^{(1)}-Y_i\|=O(\varepsilon^2)$ is satisfied. Moreover, substituting these equations into equation (40c), $\|Y_{12}^{(1)}-Y_{12}\|=O(\varepsilon^2)$ is also satisfied. Thus, the relation holds true for $k=1$. Assume that the relation is true for $k=l$ for some $l\geq 2$.

$$\|X_i^{(l)}-Y_i\|=O(\varepsilon^{2l}),\ i=1,\ 2, \quad (42a)$$
$$\|X_{12}^{(l)}-Y_{12}\|=O(\varepsilon^{2l}). \quad (42b)$$

Using the above assumptions, it is not difficult to obtain (43).

$$E_{ii}^T\tilde{Y}_i^{(l+1)}+\tilde{Y}_i^{(l+1)}E_{ii}+\sum_{p=1}^N A_{pii}^T\tilde{Y}_i^{(l+1)}A_{pii}+O(\varepsilon^{2l+2})=0,\ i=1,\ 2. \quad (43)$$

After simplification, since $F_{ii},\ i=1,\ 2$ are nonsingular, $\|Y_i^{(l+1)}-Y_i\|=O(\varepsilon^{2l+2})$ holds by using Lemma 1. Similarly, substituting these equations into equation (40c), $\|Y_{12}^{(l+1)}-Y_{12}\|=O(\varepsilon^{2l+2})$ is also obtained. Thus, the relation is also true for $k=l+1$. Consequently, the error equation (39) holds for all $k\in\mathbf{N}$ by using mathematical induction. ■

An algorithm which solves the CSAREs (16) with the small positive parameter ε is given below.

Step 1. *Solve the SAREs* (22) *that is given as the initial conditions of the Newton's method* (26)*. It should also be noted that the solutions of SAREs* (22) *can be obtained by applying the Newton' method.*

Step 2. *In order to carry out the Newton's method* (26)*, apply the new proposed algorithm* (30)*.*

Step 3. *In order to reduce the dimension of the workspace for solving the CSALE* (30)*, apply the new proposed algorithm* (38)*. As a result, the sequence of solution of the Newton's method* (26) *is obtained.*

Step 4. *If the new combined algorithm converges, then $P_{i\varepsilon}$ is the solution of the CSAREs* (16)*, STOP. Otherwise, increment $n\to n+1$ and go to Step 2.*

6 High-Order Approximate Stochastic Nash Strategies

The attention is focused on the design of the high-order approximate stochastic Nash strategies. Such strategy is obtained by using the iterative solution (26).

$$u_i^{(k)*}(t) = -R_{ii}^{-1} B_{i\varepsilon}^T P_{i\varepsilon}^{(k)} x(t),\ i = 1,\ 2. \tag{44}$$

The degradation of the cost functional via new high-order approximate stochastic Nash strategies (44) is given as follows.

Theorem 7 *Under Assumption 2, the use of the high-order approximate stochastic Nash strategies* (44) *results in* (45)

$$J_i(u_1^{(k)*},\ u_2^{(k)*},\ x(0)) = J_i(u_1^*,\ u_2^*,\ x(0)) + O(\varepsilon^{2^k+1}),\ i = 1,\ 2. \tag{45}$$

Proof. When $u_i^{(k)*}(t)$ is used, the equilibrium values of the cost functional are

$$J_i(u_1^{(k)*},\ u_2^{(k)*},\ x(0)) = x^T(0) Z_{i\varepsilon} x(0), \tag{46}$$

where $Z_{i\varepsilon}$ is a positive semidefinite solution of the following SALE

$$\begin{aligned} &Z_{i\varepsilon}\left(A_\varepsilon - S_{1\varepsilon}P_{1\varepsilon}^{(k)} - S_{2\varepsilon}P_{2\varepsilon}^{(k)}\right) + \left(A_\varepsilon - S_{1\varepsilon}P_{1\varepsilon}^{(k)} - S_{2\varepsilon}P_{2\varepsilon}^{(k)}\right)^T Z_{i\varepsilon} + \sum_{p=1}^{N} A_{p\varepsilon}^T Z_{i\varepsilon} A_{p\varepsilon} \\ &+P_{i\varepsilon}^{(k)} S_{i\varepsilon} P_{i\varepsilon}^{(k)} + \varepsilon P_{j\varepsilon}^{(k)} S_{ij\varepsilon} P_{j\varepsilon}^{(k)} + Q_{i\varepsilon} = 0,\ i,\ j = 1,\ 2,\ i \neq j. \end{aligned} \tag{47}$$

Subtracting (16) from (47), $V_\varepsilon = Z_{i\varepsilon} - P_{i\varepsilon}$ satisfies the following SALE

$$\begin{aligned} &V_{i\varepsilon}\left(A_\varepsilon - S_{1\varepsilon}P_{1\varepsilon}^{(k)} - S_{2\varepsilon}P_{2\varepsilon}^{(k)}\right) + \left(A_\varepsilon - S_{1\varepsilon}P_{1\varepsilon}^{(k)} - S_{2\varepsilon}P_{2\varepsilon}^{(k)}\right)^T V_{i\varepsilon} + \sum_{p=1}^{N} A_{p\varepsilon}^T V_{i\varepsilon} A_{p\varepsilon} \\ &+P_{i\varepsilon} S_{j\varepsilon}\left(P_{j\varepsilon} - P_{j\varepsilon}^{(k)}\right) + \left(P_{j\varepsilon} - P_{j\varepsilon}^{(k)}\right) S_{j\varepsilon} P_{i\varepsilon} \\ &+\left(P_{i\varepsilon}^{(k)} - P_{i\varepsilon}\right) S_{i\varepsilon} \left(P_{i\varepsilon}^{(k)} - P_{i\varepsilon}\right) + \varepsilon P_{j\varepsilon}^{(k)} S_{ij\varepsilon} P_{j\varepsilon}^{(k)} - \varepsilon P_{j\varepsilon} S_{ij\varepsilon} P_{j\varepsilon} = 0. \end{aligned} \tag{48}$$

By using the result of (28), it is easy to verify that

$$V_{i\varepsilon}\left(A_\varepsilon - S_{1\varepsilon}P_{1\varepsilon}^{(k)} - S_{2\varepsilon}P_{2\varepsilon}^{(k)}\right) + \left(A_\varepsilon - S_{1\varepsilon}P_{1\varepsilon}^{(k)} - S_{2\varepsilon}P_{2\varepsilon}^{(k)}\right)^T V_{i\varepsilon} + \sum_{p=1}^{N} A_{p\varepsilon}^T V_{i\varepsilon} A_{p\varepsilon} + O(\varepsilon^{2^k+1}) = 0. \tag{49}$$

Therefore, $V_{i\varepsilon} = O(\varepsilon^{2^k+1})$ because of Lemma 1. Hence

$$\begin{aligned} &x(0)^T V_{i\varepsilon} x(0) = x(0)^T Z_{i\varepsilon} x(0) - x^T(0) P_{i\varepsilon} x(0) \\ =\ &J_i(u_1^{(k)*},\ u_2^{(k)*},\ x(0)) - J_i(u_1^*,\ u_2^*,\ x(0)) = O(\varepsilon^{2^k+1}) \end{aligned} \tag{50}$$

results in (45). ■

7 Computational Example

In order to demonstrate the efficiency of the proposed hybrid algorithm, a computational example is given. The system matrices are given as follows.

$$N = 1,\ A_{11} = \begin{bmatrix} 0 & 1 & -0.266 & -0.009 \\ -2.75 & -2.78 & -1.36 & -0.037 \\ 0 & 0 & 0 & 1 \\ -4.95 & 0 & -55.5 & -0.039 \end{bmatrix},\ A_{12} = \begin{bmatrix} 0.0024 & 0 & -0.087 & 0.002 \\ -0.185 & 0 & 1.11 & -0.011 \\ 0 & 0 & 0 & 0 \\ 0.222 & 0 & 8.17 & 0.004 \end{bmatrix},$$

$$\bar{P}_1 = \textbf{block diag}\begin{pmatrix} \bar{P}_{111} & 0 \end{pmatrix},\ \bar{P}_{111} = \begin{bmatrix} 1.7123e-01 & 4.3365e-03 & 7.1775e-01 & -1.7063e-02 \\ 4.3365e-03 & 2.6823e-03 & 1.8830e-02 & 4.3122e-05 \\ 7.1775e-01 & 1.8830e-02 & 8.0187 & -4.4513e-02 \\ -1.7063e-02 & 4.3122e-05 & -4.4513e-02 & 1.4427e-01 \end{bmatrix},$$

$$\bar{P}_2 = \textbf{block diag}\begin{pmatrix} 0 & \bar{P}_{222} \end{pmatrix},\ \bar{P}_{222} = \begin{bmatrix} 1.2736e-01 & 1.5610e-03 & 6.6965e-01 & -1.7821e-02 \\ 1.5610e-03 & 1.2588e-03 & 8.5483e-03 & -1.1701e-04 \\ 6.6965e-01 & 8.5483e-03 & 1.2265e+01 & -5.7345e-02 \\ -1.7821e-02 & -1.1701e-04 & -5.7345e-02 & 2.2051e-01 \end{bmatrix},$$

$$P_{1\varepsilon}^{(4)} = \left[\begin{array}{cccccc} 1.7116e-01 & 4.3347e-03 & 7.1692e-01 & -1.6549e-02 & 3.0879e-04 & -8.7559e-07 \\ 4.3347e-03 & 2.6822e-03 & 1.8789e-02 & 5.5618e-05 & 2.7774e-05 & 2.2816e-07 \\ 7.1692e-01 & 1.8789e-02 & 7.9996 & -4.4580e-02 & 8.0812e-03 & 4.9144e-05 \\ -1.6549e-02 & 5.5618e-05 & -4.4580e-02 & 1.4391e-01 & 5.8779e-03 & 7.3899e-05 \\ 3.0879e-04 & 2.7774e-05 & 8.0812e-03 & 5.8779e-03 & 5.1657e-04 & 6.3888e-06 \\ -8.7559e-07 & 2.2816e-07 & 4.9144e-05 & 7.3899e-05 & 6.3888e-06 & 7.9906e-08 \\ -1.0598e-02 & 1.1305e-04 & -2.0946e-02 & 1.0910e-01 & 9.1598e-03 & 1.1510e-04 \\ -1.0024e-02 & -2.6288e-04 & -1.0992e-01 & 7.9670e-04 & -1.9330e-04 & -9.4303e-07 \end{array} \right.$$

$$\left. \begin{array}{cc} -1.0598e-02 & -1.0024e-02 \\ 1.1305e-04 & -2.6288e-04 \\ -2.0946e-02 & -1.0992e-01 \\ 1.0910e-01 & 7.9670e-04 \\ 9.1598e-03 & -1.9330e-04 \\ 1.1510e-04 & -9.4303e-07 \\ 1.6808e-01 & 3.1479e-04 \\ 3.1479e-04 & 3.0621e-03 \end{array} \right],$$

$$P_{2\varepsilon}^{(4)} = \left[\begin{array}{cccccc} 4.7354e-04 & 1.2072e-05 & 3.9670e-03 & -2.1963e-04 & 1.4892e-03 & 1.7818e-05 \\ 1.2072e-05 & 3.1989e-07 & 1.0766e-04 & -2.8842e-06 & 3.3793e-05 & 4.3778e-07 \\ 3.9670e-03 & 1.0766e-04 & 6.5371e-02 & -4.8525e-04 & -2.3526e-02 & -2.8743e-04 \\ -2.1963e-04 & -2.8842e-06 & -4.8525e-04 & 1.1785e-03 & -8.6911e-04 & -1.4425e-05 \\ 1.4892e-03 & 3.3793e-05 & -2.3526e-02 & -8.6911e-04 & 1.2729e-01 & 1.5602e-03 \\ 1.7818e-05 & 4.3778e-07 & -2.8743e-04 & -1.4425e-05 & 1.5602e-03 & 1.2588e-03 \\ 2.1774e-02 & 4.9125e-04 & -4.0432e-01 & -2.2283e-02 & 6.6862e-01 & 8.5352e-03 \\ 1.4953e-03 & 5.2797e-05 & 2.3162e-02 & -7.0029e-03 & -1.7798e-02 & -1.1687e-04 \end{array} \right.$$

$$\left. \begin{array}{cc} 2.1774e-02 & 1.4953e-03 \\ 4.9125e-04 & 5.2797e-05 \\ -4.0432e-01 & 2.3162e-02 \\ -2.2283e-02 & -7.0029e-03 \\ 6.6862e-01 & -1.7798e-02 \\ 8.5352e-03 & -1.1687e-04 \\ 1.2249e+01 & -5.7512e-02 \\ -5.7512e-02 & 2.2015e-01 \end{array} \right].$$

Table 1: Error per iterations.

k	$\Vert\mathcal{G}^{(k)}(1.0e-01)\Vert$	$\Vert\mathcal{G}^{(k)}(1.0e-02)\Vert$	$\Vert\mathcal{G}^{(k)}(1.0e-03)\Vert$	$\Vert\mathcal{G}^{(k)}(1.0e-04)\Vert$
0	$1.8875e-01$	$1.8875e-02$	$1.8875e-03$	$1.8875e-04$
1	$1.3163e-02$	$7.7570e-05$	$7.7732e-07$	$7.7777e-09$
2	$4.1112e-04$	$1.0296e-08$	$1.0965e-12$	$4.0055e-14$
3	$3.2712e-09$	$1.3893e-13$		
4	$2.0230e-13$			

$$A_{21} = \begin{bmatrix} 0.021 & 0 & 0.121 & 0.003 \\ -1.1 & 0 & -1.62 & -0.015 \\ 0 & 0 & 0 & 0 \\ -2.43 & 0 & 1.37 & -0.034 \end{bmatrix},\ A_{22} = \begin{bmatrix} -0.21 & 1 & -1.6 & -0.005 \\ -1.9 & -1.8 & 9.3 & -0.12 \\ 0 & 0 & 0 & 1 \\ -3.1 & 0 & -56 & 0.032 \end{bmatrix},$$

$$A_{111} = 1.0e-02 \times \begin{bmatrix} 0.06 & 0 & 0.46 & 0.002 \\ -1 & 0 & 1.49 & -0.04 \\ 0 & 0 & 0 & 0 \\ 0.12 & 0 & 29.8 & -0.028 \end{bmatrix},\ A_{112} = 1.0e-02 \times \begin{bmatrix} -0.002 & 0 & 0.83 & 0 \\ -6.78 & 0 & -10.1 & 0.09 \\ 0 & 0 & 0 & 0 \\ -1.24 & 0 & 0.498 & -0.017 \end{bmatrix},$$

$$A_{121} = 1.0e-02 \times \begin{bmatrix} 0.011 & 0 & 0.22 & 0 \\ -2.1 & 0 & 1.7 & -0.123 \\ 0 & 0 & 0 & 0 \\ -0.07 & 0 & 6.38 & -0.011 \end{bmatrix},\ A_{122} = 1.0e-02 \times \begin{bmatrix} -0.197 & 1 & -1.2 & -0.003 \\ -54.5 & -20 & 70.1 & -2.37 \\ 0 & 0 & 0 & 1 \\ -3.4 & 0 & -21.0 & -0.017 \end{bmatrix},$$

$$B_{11} = \begin{bmatrix} 0 \\ 36.1 \\ 0 \\ 0 \end{bmatrix},\ B_{22} = \begin{bmatrix} 0 \\ 78.9 \\ 0 \\ 0 \end{bmatrix},\ B_{ij} = O,\ i \neq j,$$

$Q_1 = \textbf{block diag}\left(\begin{array}{cc} 0.1I_4 & O_{4\times4}\end{array}\right),\ Q_2 = \textbf{block diag}\left(\begin{array}{cc} O_{4\times4} & 0.1I_4\end{array}\right),$
$R_{11} = R_{22} = 0.1,\ R_{12} = 2,\ R_{21} = 3.$

First, the initial conditions (27) of Newton's method (26) are given in the top of this page.

The small parameter is chosen as $\varepsilon = 0.1$. It should be noted that the algorithm (26) converges to the exact solution with accuracy of $\|\mathcal{G}^{(k)}(\varepsilon)\| < 1.0e-10$ after four iterations, where $\|\mathcal{G}^{(k)}(\varepsilon)\| := \sum_{i=1}^{2}\|\mathcal{G}_i(\varepsilon,\ P_{1\varepsilon}^{(k)},\ P_{2\varepsilon}^{(k)})\|$. The convergence solutions $P_{i\varepsilon}^{(4)}$ are given in the top of this page.

In order to verify the exactitude of the solution, the remainder per iteration by substituting $P_{i\varepsilon}^{(k)}$ into the CSAREs (16) is computed. In Table 1, the results of the error $\|\mathcal{G}^{(k)}(\varepsilon)\|$ per iterations are given for several values ε. As a result, it can be seen that the algorithm (26) has the quadratic convergence.

Finally, the convergence of the fixed point algorithm in equation (38) is demonstrated. For $\varepsilon = 0.1$, Table 2 shows the errors per iteration for the algorithm for the first iteration of Newton's method, where the convergence condition is given by $\|\mathcal{H}(Y_\varepsilon^{(m)})\| = \|Y_\varepsilon^{(m)}E_\varepsilon + E_\varepsilon^T Y_\varepsilon^{(m)} + A_{1\varepsilon}^T Y_\varepsilon A_{1\varepsilon} + H_\varepsilon\| < 1.0e-10$.

Table 2.

m	$\|\mathcal{H}(Y_\varepsilon^{(m)})\|$
1	$1.79755e-01$
2	$1.30085e-03$
3	$1.75259e-05$
4	$1.89520e-07$
5	$2.19458e-09$
6	$2.48513e-11$

From Table 2, it can be verified that the proposed algorithm satisfies equation (39). That is, the proposed algorithm (38) has a linear rate of convergence. Hence, the combined algorithms in equations (26), (30) and (38) of this paper are very attractive. Furthermore, even if the weakly coupled large-scale systems (1) are composed of two four-dimensional subsystems, the required workspace is four. This feature is very useful from the practical viewpoint.

8 Conclusion

The infinite horizon stochastic Nash games have been discussed. First, the conditions for existence of Nash equilibria by utilizing the CSAREs have been established. Second, a numerical algorithm for solving the CSAREs that arose in the stochastic Nash games for weakly coupled large-scale systems has been studied. In order to solve CSARE, Newton's method and two fixed point algorithms have been combined. Using a new hybrid algorithm, it has been shown that the quadratic convergence and the reduced-order computation are both attained. Moreover, the local uniqueness of the solution has been proved for the first time. Thus, the proposed algorithm is expected to be very useful and reliable for a sufficiently small ε. As another important feature, the high-order approximate strategy such that the better cost performance is attained has been established. In fact, the cost degradation for using the proposed approximate strategy has been proved for the first time. Finally, the computational examples have shown excellent results that the quadratic convergence has been verified and the proposed algorithm has succeeded in reducing the computational workspace.

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