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▶ To cite this version:

Olivier Bouré, Nazim A. Fatès, Vincent Chevrier. First steps on asynchronous lattice-gas models with an application to a swarming rule. Natural Computing, 2013, 12 (4), pp.551-560. 10.1007/s11047-013-9389-2 . hal-00790561

HAL Id: hal-00790561

https://inria.hal.science/hal-00790561v1

Submitted on 20 Feb 2013

First steps on asynchronous lattice-gas models with an application to a swarming rule

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Abstract

Lattice-gas cellular automata are often considered as a particular case of cellular automata in which additional constraints apply, such as conservation of particles or spatial exclusion. But what about their updating? How to deal with non-perfect synchrony? Novel definitions of asynchronism are proposed that respect the specific hypotheses of lattice-gas models. These definitions are then applied to a swarming rule in order to explore the robustness of the global emergent behaviour. In particular, we compare the synchronous and asynchronous case, and remark that anti-alignment of particles is no longer observed when a small critical amount of asynchronism is added.

Keywords: Asynchronous cellular automata, Lattice-Gas cellular automata, Robustness, Swarming behaviour

1 Introduction

In the field of discrete dynamical systems, cellular automata (CA) are widely regarded as tools for the simulation of natural phenomena and models of parallel computing. Indeed, their discrete and spatially-extended structure makes them computationally simple models yet capable of displaying a wide range of complex behaviours. However, the use of discrete models may also induce peculiar behaviours that are rather related to the regularity of the system than to the pure interactions of composing entities [15]. For this reason, it is necessary to take into account the robustness of models, that is, to evaluate the stability of their behaviour when external disturbances and structural changes are applied. If various perturbations have been studied for binary CA [5, 11], the study of robustness for more elaborate models, such as lattice-gas cellular automata, is still a largely fallow issue.

Lattice-gas cellular automata (LGCA) are a particular case of CA, historically designed to simulate fluid dynamics, used in numerous applications from crowds dynamics [17] to cellular motion [8]. They rely on the representation of particles that travel through the lattice according to the successive application of the interaction step (I) – which reorganises particles within cells – and the propagation step (P) – which displaces particles towards the corresponding neighbours. LGCA differ from classical CA as their structure is based on additional principles, such as the conservation or the spatial exclusion of particles. In order to study their robustness, one must thus ask: are CA tools for studying robustness compatible with the LGCA principles? And if not, which perturbations to consider that take into account the specificity of LGCA? As a start, we propose in Sec. 2 to investigate this question in the scope of a perturbation of the updating scheme, by discussing the characteristics of both LGCA and asynchronism with a view to introduce a first definition of asynchronous LGCA.

1.0.1 The robustness to asynchronism.

As far as discrete models are concerned, questioning the synchronous updating scheme is a case in point. While initial studies only considered a simultaneous updating of all components, recent years have seen an increasing interest in asynchronous CA models, where the perfect synchrony hypothesis is relaxed. Robustness boils down to exploring how much of the CA behaviour is related to the synchronous scheme, and how much comes intrinsically from the individual rule [16]. The introduction of asynchronism led to the observation of a wider variety of behaviours that arise from different updating schemes, including phase transitions [5, 13] and thus a better understanding of CA models. Similarily, studying LGCA models in the light of asynchronism may exhibit novel behaviour, display the models limit or even reveal synchrony-related issues. In Sec. 3 we give an example of such approach for a well-known LGCA model with a swarming rule introduced by Deutsch [10], and compare the behaviour under synchronous and asynchronous updating scheme.

1.0.2 Synchronous singularities.

The perturbations of the updating scheme exhibited a phenomenon of interest: the existence of *synchronous singularities*, that is, behaviours for which the synchronicity of the model is a necessary condition. Examples of such phenomena include periodic patterns in the *Game of Life* [4, 12], several Elementary CA [13] and even the spatially-extended Prisoner's Dilemma [14]. From those examples, one may wonder that such observations can be only made in systems with an ad hoc deterministic transition rule, or whether it can also occur for models of natural phenomena, such as LGCA.

To illustrate this question, we focus on Deutsch's LGCA model of swarming [10]. This model, inspired by the biological phenomenon of swarming, uses a stochastic interaction rule which favours the local alignment of particles with the neighbouring cells. Starting from a random inital configuration, it has been observed that the behaviour generally converges to a global consensus in the directions of particles (provided that the alignment strength is high enough). However, a high density of particles led to a paradoxical observation [6]: configurations where most particles have opposite directions with the neighbouring

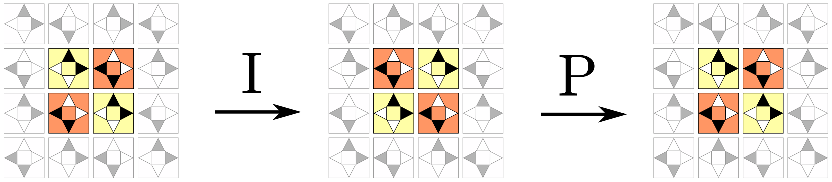

Figure 1: Phenomenon occurring in a synchronous LGCA with an alignment interaction rule. A black triangle represents one particle present in a channel, and white an empty channel.

cells. If the stability of this configuration can be explained in the synchronous case, as displayed in the Fig. 1, one may note that the behaviour is intuitively dependent on the synchronous updating scheme. Thus the question: to which extent is this behaviour robust to an asynchronous mode? Can it be considered as synchronous singularity?

We study these questions for the LGCA model of swarming, by using quantitative and analytical approaches in Sec. 4. We then discuss the role of asynchronism in discrete dynamical systems in Sec. 5, based on our definitions and experimental results.

2 Defining asynchronous lattice-gas cellular automata

There exists a wide range of interpretations and definitions given to asynchronism [2, 5], and even more diverse resulting behaviours. The problem we encounter here is that the LGCA system we want to “make asynchronous” is a lattice-gas cellular automaton which, in spite of its compatibility with the classical CA definition, includes additional hypotheses that needs to be conserved when changing the updating scheme. Let us now present how we tackle this problem formally.

2.1 Definition of lattice-gas cellular automata

By definition, lattice-gas cellular automata are particular CA, where:

- the *cellular space* $\mathcal{L} = \{\mathbb{Z}/L{\cdot}\mathbb{Z}\}^2$ is a 2-dimensional square lattice of length L with periodic boundary conditions.
- the *neighbourhood* $(n_i)_{i\,\in[\![1,4]\!]} = \{(1,0);(0,1);(-1,0);(0,-1)\}$ associates to a cell $c \in \mathcal{L}$ the set of its four nearest neighbours.

In a LGCA, each cell is connected to its neighbours via *channels* through which particles can travel. Cell configurations $\vec{x}_c = (x_i(c))_{i\,\in[\![1,4]\!]}$ are therefore represented as a vector of the numbers of particles contained by each of its channels

$x_i(c) \in \mathbb{N}$. The dynamics of LGCA is determined by the successive application of two global transitions:

1. The *interaction step* reorganises particles within cells according to a transition function f_{I}, which applies to the configuration of a cell and its neighbourhood:

$$\forall c \in \mathcal{L},\ \vec{x}^{\mathrm{I}}_c = f_{\mathrm{I}}(\vec{x}_c, \vec{x}_{c+n_1}, \ldots, \vec{x}_{c+n_4}). \tag{1}$$

2. The *propagation step* displaces particles from the channels of a cell to the corresponding neighbours:

$$\forall c \in \mathcal{L},\ \forall i \in [\![1,4]\!],\ x^{\mathrm{P}}_i(c) = x^{\mathrm{I}}_i(c - n_i). \tag{2}$$

In addition, we consider two fundamental constraints which we want to keep valid under an asynchronous updating scheme: the *particle exclusion* – channels may contain at most one particle, and are therefore considered as either empty (state `0`) or full (`1`) – and the *particle conservation* – particles must always be conserved when they interact and propagate[1].

2.2 Which asynchronism?

For this work, we will consider *α-synchronism* [13, 5] which follows the general definition: at each time step, each component updates according to the regular transition function with probability α, or remains unchanged using the identity function with probability $1-\alpha$. This updating scheme provides us with a useful control parameter, the *synchrony rate* α, allowing for a continuous control of the updating scheme from evolution in perfect synchrony ($\alpha = 1$) to quasi-sequential updates ($\alpha \to 0$).

The global transition function of LGCA from a time t to $t+1$ is constituted of two steps, interaction and propagation, applied successively to all cells. When we want to apply asynchronism, we need to properly define the meaning of "updating" and "remaining unchanged" in terms of computation. Does asynchronism apply to the sole interaction? To the propagation? Shall we consider these transitions as correlated, independent?

To address this issue, we propose as a starting point three possible implementations of asynchronism:

(1) asynchronous interaction : the interaction is applied with a probability α_{I}, called the *interaction rate*. The propagation is always applied.

(2) asynchronous propagation : the interaction is always applied, but the propagation is applied with a probability α_{P}, called the *propagation rate*.

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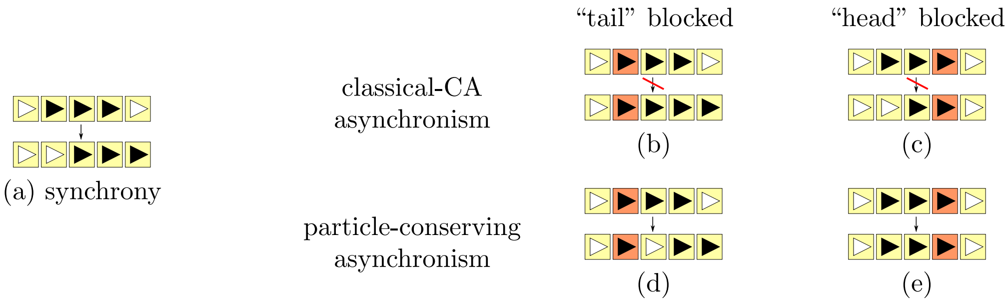
[1] Note that our particle-oriented interpretation of the system resembles *Totally Asymmetric Simple Exclusion Processes* (TASEP) [9].



(a) synchrony (b) (c) (d) (e)

Figure 2: Representation of the propagation step along one channel direction in a sample of five cells, for different updating schemes and situations of particles. Darker cells are not selected for update. (a) is the synchronous classical case, (b,c) the asynchronous case without the particle-conserving changes and (d,e) our proposition for an asynchronous propagation.

(3) correlated interaction and propagation : for each cell, interaction and propagation are both applied with a probability α_{C}, otherwise none of them is applied.

However, if the asynchronous interaction (1) can be implemented without problem as the interaction rule itself is particle-conserving, desynchronising the propagation (2, 3) is not a straightforward operation and requires further reflection.

2.2.1 Conserving particles during the propagation.

Let us first consider the case where asynchronous propagation is simply determined as the application of the transition rule with probability α_{P}, and identity function with probability $1 - \alpha_{\mathrm{P}}$. In LGCA models, there is no actual means for a given cell to know whether its neighbours are selected for update or not. This is problematic as we need this type of information to decide whether to propagate the state of a channel or not (see Fig. 2-b and -c).

In other contexts, similar issues have often been solved by adapting the model with special constructs: for instance, Adachi *et al.* were able to perform universal computation asynchronously by introducing additional states [1]. Similarly, several models of agent-based CA introduced additional steps to take into account asynchronous updates (*e.g.* transactional CA [19], influence-reaction [3]). In the case of LGCA, our proposition here consists in modifying the point of view of the asynchronous propagation function by considering no longer the cells but the channels as the base components of the system. We first select randomly the cells which are to be updated, and then determine from non-selected cells which channels are *free*, and thus updated, and which are *blocked* and remaining unchanged. We now specify the formal definitions of asynchronous interaction and propagation using selection functions.

2.3 Definition of an asynchronous LGCA

2.3.1 Interaction step.

We introduce a *selection function* $\Delta_{\mathrm{I}} : \mathbb{N} \to \mathcal{P}(\mathcal{L})$ which associates to each time t the subset of cells to be updated during the interaction step, where each cell has a probability α_{I} to be selected. The interaction transition function becomes : $\forall t \in \mathbb{N},\ \forall c \in \mathcal{L},\ \mathcal{N} = \{n_1, \ldots, n_4\}$:

$$\vec{x}_c^{\mathrm{I}} = f_{\Delta_{\mathrm{I}}}(\vec{x}_c, \vec{x}_{c+n_1}, \ldots, \vec{x}_{c+n_4}) = \begin{cases} f_{\mathrm{I}}(\ldots) & \text{if } c \in \Delta_{\mathrm{I}}(t) \\ \vec{x}_c & \text{otherwise.} \end{cases} \tag{3}$$

Note that when $\alpha_{\mathrm{I}} = 1$, the system is equivalent to the synchronous version, whereas when $\alpha_{\mathrm{I}} = 0$ the system corresponds to a ballistic propagation, that is, particles never change direction.

2.3.2 Propagation step.

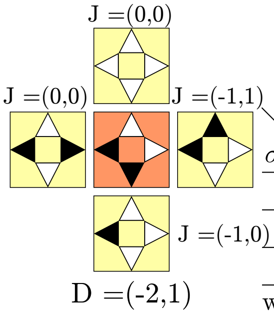
First, we need to determine which cells update, and which remain unchanged. Similarly to asynchronous interaction, we introduce a *selection function* $\Delta_{\mathrm{P}} : \mathbb{N} \to \mathcal{P}(\mathcal{L})$ which returns for time t the subset of cells to be updated during the propagation step, where each cell has a probability α_{P} to be selected. In addition, we define $\mathcal{B}^t \subset [\![1, 4]\!] \times \mathcal{L}$ as the set of *blocked channels*, which will remain unchanged between times t and $t + 1$. To build this set, we state that a channel $(i, c)^t$ (channel i of cell c at time t) is *blocked* if it contains a particle ($x_i^t(c) > 0$), and if one of the two conditions is true: ($C1$) its containing cell c is not selected for update, or ($C2$) its destination channel is blocked. We thus have:

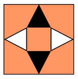
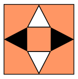
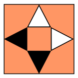
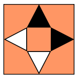
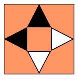
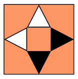
$$(i, c)^t \in \mathcal{B}^t \text{ if } \big(x_i^t(c) > 0\big) \text{ and } \begin{cases} c \notin \Delta_{\mathrm{P}}(t) & (C1) \\ \quad \text{or} & \\ (i, c + n_i)^t \in \mathcal{B}^t & (C2) \end{cases}$$

The construction of $\mathcal{B}^t$ describes a general relation between particles. It can be implemented with a recursive algorithm: once a channel is blocked because of condition ($C1$), the chain of the channels that "point" to this one must be also blocked if they contain a particle because of condition ($C2$) (see Fig. 2-e). The propagation transition function therefore becomes:

$$x_i^{\mathrm{P}}(c) = \begin{cases} x_i^{\mathrm{I}}(c) & \text{if } (i, c) \in \mathcal{B}^t \\ 0 & \text{if } (i, c) \notin \mathcal{B}^t \text{ and } (i, c - n_i) \in \mathcal{B}^t \\ x_i^{\mathrm{I}}(c - n_i) & \text{otherwise.} \end{cases} \tag{4}$$

Note that when $\alpha_{\mathrm{P}} = 1$, the system is equivalent to the synchronous version, whereas when $\alpha_{\mathrm{P}} = 0$ the particles will continue reorganising within cells but are never propagated and thus never leave their cell.



						
P / σ	$\frac{1}{Z}$	$\frac{1}{Z}$	$\frac{e^{\sigma}}{Z}$	$\frac{e^{-\sigma}}{Z}$	$\frac{e^{3\sigma}}{Z}$	$\frac{e^{-3\sigma}}{Z}$
0.5	0.25	0.25	0.2	10^{-2}	0.5	10^{-3}
1.5	10^{-2}	10^{-2}	10^{-2}	10^{-4}	~ 1	10^{-4}
4	10^{-5}	10^{-5}	10^{-3}	10^{-7}	~ 1	10^{-11}

with $Z = 1 + 1 + e^{\sigma} + e^{-\sigma} + e^{3\sigma} + e^{-3\sigma}$

Figure 3: Example of the application of the swarm interaction rule for the central cell. Left: typical states for a cell and its neighbours, with neighbouring fluxes and the resulting director field $D_c(\vec{x})$ of the center cell. Right: possible outputs and their associated alignment γ, along with a table of approximated probabilities for different values of σ.

3 Application to a swarming rule

We now want to apply our definitions on a particular LGCA model to observe the effects of asynchronism. We choose to consider the LGCA swarming model, borrowed from the works of Deutsch *et al.* [10], which describes a probabilistic swarming interaction rule in which each cell tend to reorganise its particles according to the neighbourhood predominant directions. The probability for a given cell configuration $\vec{x}_c$ to reorganise into $\vec{x}_c^{\mathrm{I}}$ is given by:

$$P(\vec{x}_c \rightarrow \vec{x}_c^I) = \frac{1}{Z}\exp\Big[\sigma \cdot \vec{J}_c(\vec{x}^{\mathrm{I}}) \cdot \vec{D}_c(\vec{x})\Big] \cdot \delta(\vec{x}_c, \vec{x}_c^{\mathrm{I}}) \tag{5}$$

where:

- Z is the normalisation factor, so that $\sum_{\vec{x}_c^I} P(\vec{x}_c \rightarrow \vec{x}_c^I) = 1$.
- $\sigma \in \mathbb{R}^+$ is the *alignment sensitivity*.
- $\vec{J}_c(\vec{x}^I)$ is the *flux* in cell c, the resulting vector of particle directions.
- $\vec{D}_c(\vec{x})$ is the *director field*, the sum of cell $\vec{x}_c$ neighbours' flux.
- $\delta(\vec{x}_c, \vec{x}_c^{\mathrm{I}}) = 1$ if cell configurations $\vec{x}_c$ and $\vec{x}_c^{\mathrm{I}}$ have same number of particles, 0 otherwise. This ensures conservation of particles.

Starting from a random initial configuration of density ρ, where ρ is the probability for each channel to contain a particle, we need only to set the parameter σ to determine the local interaction rule (see Fig. 3 for a practical application). This limits the parametric space to the (ρ, σ) plane, allowing for a complete exploration of the different qualitative behaviours that the model may display.

3.1 Experimental protocol

Mainly two monitoring approaches have been used to observe the emergence of ordered behaviours in this lattice-gas swarming model[2]:

- As a visualisation method, we monitor the *flux visualisation*, which intends to render the resulting flux of each cell by associating a colour to each direction. A zero-flux cell is represented in white, while other types of flux show a different color for each corresponding cardinal point: N (green), N-E (lime), E (yellow), S-E (orange), S (red), S-W (magenta), W (blue), N-W (cyan).
- As an order parameter, we use the *mean alignment* $\overline{\gamma}$ to express whether particles are in average aligned with the flux of their neighbours:

  $$\overline{\gamma}(\vec{x}) = \frac{1}{k(\vec{x})} \sum_{c \in \mathcal{L}} \frac{1}{\mathrm{card}(\mathcal{N})} J_c(\vec{x}) \cdot D_c(\vec{x}) \tag{6}$$

  where $k(\vec{x}) = \sum_{c \in \mathcal{L}} k(\vec{x}, c)$ is the total number of particles. Its value varies in $[-1, 1]$: $\overline{\gamma} = 1$ indicates that all particles are aligned, and for $\overline{\gamma} = -1$, all particles are antialigned.

In order to identify an organisation of the behaviour, these two monitoring devices work well together: the order parameter $\overline{\gamma}$ allows us to track and detect changes of behaviours over large samples, and the visualisation makes for a global and efficient reading of the spatial organisation of particles. In this manner, we intend to observe both quantitatively and qualitatively the changes of behaviour that may occur when applying the asynchronism to the lattice-gas swarming model.

3.2 Recapitulation of the synchronous behaviour

It has been previously observed that, using different values for the alignment sensitivity σ and the initial density ρ, the behaviour splits into two distinct phases, the first random and chaotic, the second organised and stable/periodic [10]. This transition has moreover been partially confirmed analytically via a mean-field approach [7].

However, the main interest of this particular model of swarming is that, in spite of a simple and stochastic updating rule, a wide variety of behaviour may arise. Indeed in a previous work, we studied experimentally the organisation phenomenon for a finite-size system and showed the organised phase may result in several distinct stable patterns of global particle organisation [6] (see Fig. 4). To be more precise, we proved that despite the system being ergodic – meaning that it evolves freely in the configuration space and will visit every configurations in asymptotic simulation times – the system quickly converge to metastable

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[2] for a more complete set of monitoring tools, see [6].

patterns, in which it will remain until random fluctuations allow the configuration to "escape" this behaviour and reach another pattern after relatively many iterations.

We identified, besides random configurations, three *minor patterns*, which we related to side-effects of small lattice sizes ($L < 50$), and three *major patterns* which can be observed for particular input parameters in most lattices. Therefore in the rest of the paper, we will suppose that the lattice size is at least $L = 50$ in order to minimise the appearance of minor patterns and concentrate our observations on major patterns, which are closer to the general behaviour of the system. The major patterns are:

Checkerboard (CB): local groups of alterning cells with two particles which are antialigned with the neighbours' fluxes[3]. This pattern typically occurs for high densities ($\rho \approx [0.4, 0.5]$) and is associated with a strongly negative mean alignment ($\overline{\gamma} \sim -1$).

Diagonal stripe (DS): a moving diagonal band of particles pointing in two orthogonal directions, that loops over the lattice boundaries. This pattern is mostly observed for "intermediate" values of the parameters (for example $\sigma = 1.5, \rho = 0.2$), and is associated with a strongly positive alignment ($\overline{\gamma} \sim 1$).

Clusters (CL): several distinct clusters of aligned particles, traveling in all directions. This pattern occurs mostly for high sensitivities ($\sigma \approx [3, 4]$) and is associated with a positive alignment ($\overline{\gamma} \in [0, 1]$).

Interestingly, each of these major patterns behaviours can be associated with a particular region of the parametrical diagram (ρ, σ) shown in Fig. 4. This means that in practice, for a given set of parameters and a large lattice, only one type of pattern will usually be observed in reasonable simulation times (around $t \sim 1000$). As a result, we will now consider that the definition of the system's behaviour in the form of these patterns stands as a good approximation, which can be used to quantify the robustness of the behaviour to asynchrony. More precisely, we are interested in knowing whether their appearance depends on a perfectly synchronous updating, or if it is also valid for small perturbations of the updating scheme.

3.3 Exploration of the asynchronous behaviour

With two control parameters (density ρ and sensitivity σ) and three types of asynchronism (α_{I}, α_{P}, α_{C}), the task of describing the effects of asynchronism on the behaviour of a LGCA is difficult, due to the size of the configuration space. As a preliminary study, we therefore propose to focus our observations on three parametrical points, each corresponding to typical settings for the three major patterns previously introduced. Using a square lattice of size $L = 50$

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[3] This pattern resembles the checkerboard-like configurations observed *e.g.* in asynchronous binary CA with a minority rule and a Von Neumann neighbourhood [18].

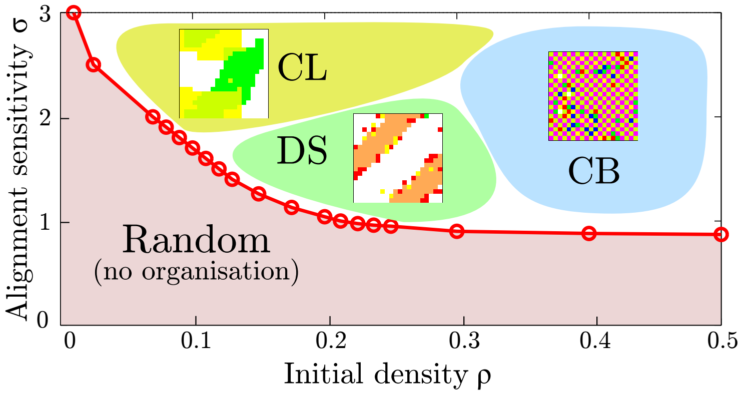


Figure 4: Spatial distribution of the different patterns in the sensitivity-density parametric plane. The red circles and line display the measured transition for the ordering phenomena. Each pattern (CL for clusters, CB for checkerboards, DS for diagonal stripes) is associated to a typical configuration and an approximate region of appearance.

over $t = 5000$ iterations, we want to observe whether the expected pattern still emerges from a random initial configuration, and compare the result for each of the different updating schemes.

3.3.1 Asynchronous interaction.

Three observations can be made (see Fig. 5-a):

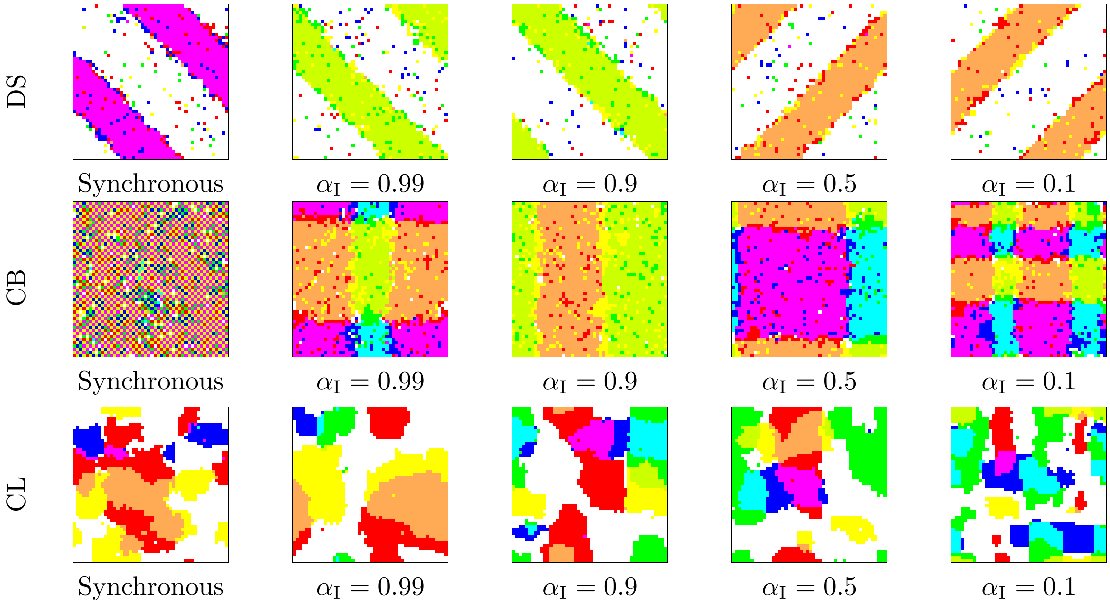
1. For the diagonal stripe, the asynchronous interaction ($\alpha_{\mathrm{I}} < 1$) has no apparent effects on the pattern after a long simulation time.
2. On the contrary, the checkerboard pattern is very sensitive to asynchronism. Indeed, a small amount of asynchronism ($\alpha_{\mathrm{I}} \approx 0.99$) is sufficient to switch the behaviour to a novel pattern, which we call *tartan pattern* (in reference to the Scottish fabric), formed of crossing vertical and horizontal stripes of aligned particles that loops over the boundaries of the lattice.
3. Finally, asynchronous interaction has little to no effects on clusters patterns.

As a consequence, the asynchronous interaction is interesting in the sense that while the aligned patterns appear completely robust, the checkerboard is extremely unstable for $\alpha_{\mathrm{I}} < 1$ and exhibit a new unobserved pattern.

3.3.2 Asynchronous propagation.

The three tested patterns displayed a change of behaviour for a small amount of asynchronism (see Fig. 5-b):

(a) Asynchronous interaction



(b) Asynchronous propagation

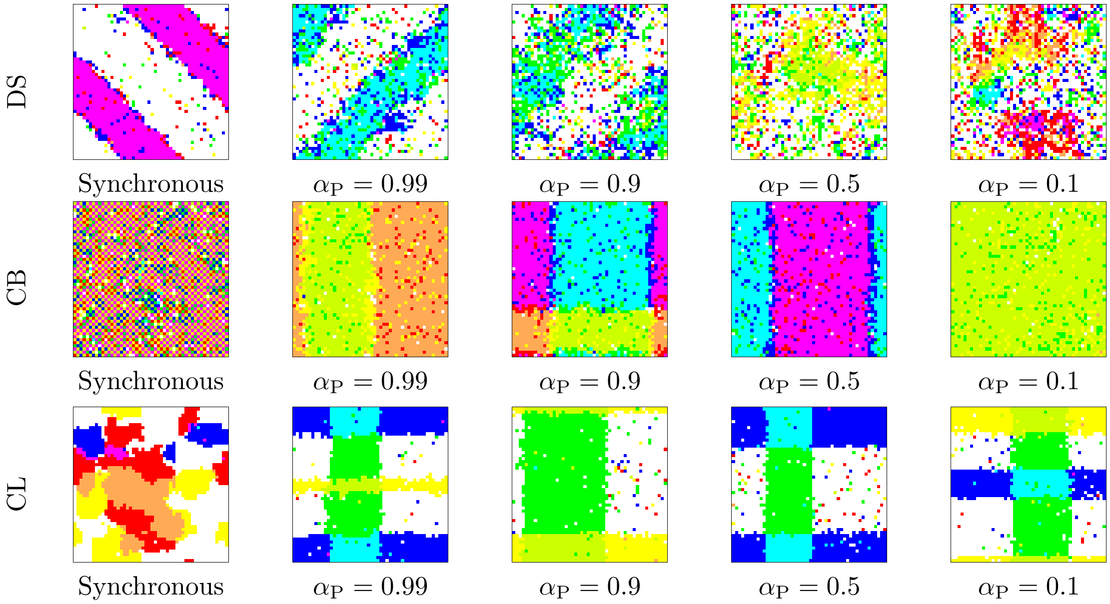


Figure 5: (Colours online) Evolution of the main patterns for different interaction (propagation) rate values α_{I} (α_{P}, respectively). Configuations are obtained from random initial configurations run for 5000 steps for typical values of σ and ρ (DS: $\{\sigma = 1.5; \rho = 0.2\}$, CB: $\{\sigma = 1.5; \rho = 0.45\}$, CL: $\{\sigma = 4; \rho = 0.2\}$). The simulations and visualisations are realised with *FiatLux*, a CA simulator in Java (`http://fiatlux.loria.fr`).

1. The diagonal stripe appears for high α_{P} but is soon ($\alpha_{\mathrm{P}} <\sim 0.9$) replaced by "semi-random" patterns, where some order regularly appears in some parts of the lattice and disappears, but never stabilises.
2. The checkerboard pattern is very unstable and quickly transforms into the tartan pattern, even for high synchrony rates α_{P}.
3. The clusters pattern is destabilised and transforms into the tartan pattern.

Contrary to the asynchronous interaction, asynchronous propagation has a notable effect on each pattern. But where the diagonal stripe show some degree of robustness for $\alpha_{\mathrm{P}} \in [0.9, 1]$, the clusters as well as the checkerboard both seem to disappear for high α_{P} in favour of the tartan pattern.

3.3.3 Correlated interaction and propagation.

Similar experiments have been applied to the correlated updating scheme: the behaviours are very close to the effects asynchronous propagation. Indeed, given that (1) the asynchronous interaction has no effect on either the diagonal stripe pattern or the clusters, (2) the changes observed for the checkerboard are close if not identical for both asynchronisms, this result is somehow "expected".

3.3.4 Synthesis.

Observations on the swarming model, which was thought rather robust in the first place due to the stochastic rule and self-organisation phenomena, highlighted the high dependence between the type of updating scheme and the stability of each pattern. For instance, we observed that there exist (1) differences between patterns in their robustness to asynchronism (*e.g.* diagonal stripe versus checkerboard under asynchronous interaction) and (2) differences between types of asynchronism in their effects on the system's behaviour (*e.g.* diagonal stripe under asynchronous interaction and propagation).

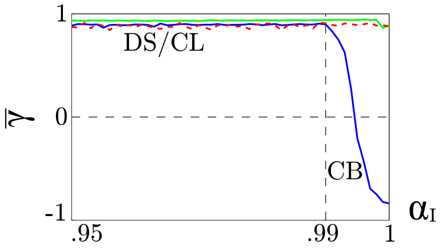
Moreover, these observations do not allow us to solve directly the question of whether there exists a fundamental change of behaviour between perfect synchrony ($\alpha = 1$) and quasi-synchrony ($\alpha = 1^{-}$).

4 Is the synchronous behaviour a singularity?

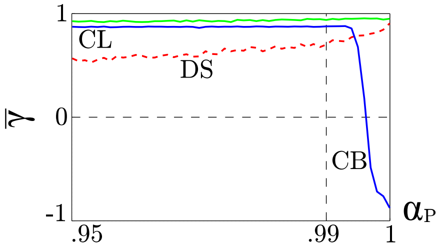
In order to investigate further the question of whether the checkerboard is a synchronous singularity, complementary approaches are necessary to give a estimation of the robustness of the behaviour for infinitesimally small perturbations ($\alpha < 1$).

4.1 Quantification of the transition

To quantify the behaviour around the perfect synchrony, we consider the following conditions: for each pattern, defined by their parameters (ρ, σ) given



(a) Asynchronous interaction



(b) Asynchronous propagation

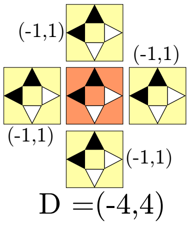
Figure 6: Mean alignment $\overline{\gamma}$ versus synchrony rate α for each of the major patterns: diagonal stripe (DS), checkerboard (CB) and clusters (CL).

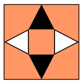
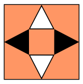
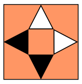
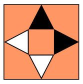
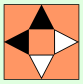
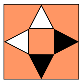
in Fig. 5, and for each value of $\alpha_{\mathrm{I}} \in [0.95, 1]$ (respectively α_{P}), we simulate a sample of size $L = 100$ for $t = 5000$ iterations, and compute the mean alignment $\overline{\gamma}$ for the next 100 steps. By averaging this process over 10 samples, we obtain an approximation of the behaviour of parameter $\overline{\gamma}$ as displayed in Fig. 6.

The plots of the mean alignment $\overline{\gamma}$ versus the synchrony rate α_{I} or α_{P} suggests several observations.

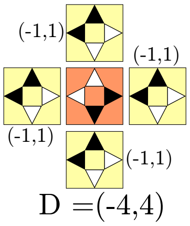
- First, as suggested by our visual observations, the diagonal stripe and clusters patterns do not seem to be affected by the asynchronous interaction.
- This is also true for the clusters pattern under asynchronous propagation, as $\overline{\gamma}(\alpha_{\mathrm{P}})$ do not reflect the distinction between clusters and tartan pattern.
- The diagonal stripe under asynchronous propagation shows a continuous decrease of the alignment $\overline{\gamma}$ as α_{P} goes down. However, the value converges to a value around $\overline{\gamma} \sim 0.5$ for $\alpha_{\mathrm{P}} \in [0.1, 0.9]$, which supports the observation that this behaviour is not completely random.
- Finally, the checkerboard pattern displays a drastic change of behaviour for a very high value of the synchrony rate α_{I} (respectively α_{P}). This suggests that the checkerboard behaviour is still robust to an infinitesimal amount of asynchronism, typically $\alpha_{\mathrm{I}} > 0.995$ (resp. $\alpha_{\mathrm{P}} > 0.995$).

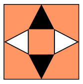
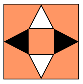
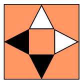
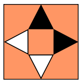
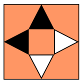
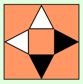
If these quantitative experiments confirm our manual observations, they also highlight a new result: quite surprisingly, even though the checkerboard pat-



						
$\alpha_{\mathrm{I}} = 1$	$\mathrm{e}^{-8\sigma}$	$\mathrm{e}^{-8\sigma}$	$\mathrm{e}^{-8\sigma}$	$\mathrm{e}^{-8\sigma}$	~ 1	$\mathrm{e}^{-16\sigma}$
$\alpha_{\mathrm{I}} < 1$	$\alpha_{\mathrm{I}}\mathrm{e}^{-8\sigma}$	$\alpha_{\mathrm{I}}\mathrm{e}^{-8\sigma}$	$\alpha_{\mathrm{I}}\mathrm{e}^{-8\sigma}$	$\alpha_{\mathrm{I}}\mathrm{e}^{-8\sigma}$	$\alpha_{\mathrm{I}} + \overline{\alpha_{\mathrm{I}}}$ ~ 1	$\alpha_{\mathrm{I}}\mathrm{e}^{-16\sigma}$

(a) Asynchronous interaction for the diagonal stripe pattern



						
$\alpha_{\mathrm{I}} = 1$	$\mathrm{e}^{-8\sigma}$	$\mathrm{e}^{-8\sigma}$	$\mathrm{e}^{-8\sigma}$	$\mathrm{e}^{-8\sigma}$	~ 1	$\mathrm{e}^{-16\sigma}$
$\overline{\alpha_{\mathrm{I}}} > \mathrm{e}^{-16\sigma}$	$\alpha_{\mathrm{I}}\mathrm{e}^{-8\sigma}$	$\alpha_{\mathrm{I}}\mathrm{e}^{-8\sigma}$	$\alpha_{\mathrm{I}}\mathrm{e}^{-8\sigma}$	$\alpha_{\mathrm{I}}\mathrm{e}^{-8\sigma}$	α_{I}	$\overline{\alpha_{\mathrm{I}}}$
$\overline{\alpha_{\mathrm{I}}} < \mathrm{e}^{-16\sigma}$	$\alpha_{\mathrm{I}}\mathrm{e}^{-8\sigma}$	$\alpha_{\mathrm{I}}\mathrm{e}^{-8\sigma}$	$\alpha_{\mathrm{I}}\mathrm{e}^{-8\sigma}$	$\alpha_{\mathrm{I}}\mathrm{e}^{-8\sigma}$	~ 1	$\alpha_{\mathrm{I}}\mathrm{e}^{-16\sigma}$

(b) Asynchronous interaction for the checkerboard pattern

Figure 7: Application of the swarm interaction rule for (a) the diagonal stripe and (b) the checkerboard patterns under asynchronous interaction. Right: possible outputs and their associated probabilities, in the synchronous ($\alpha_I = 1$) and the asynchronous case ($\alpha_I < 1$). The coloured column correspond to the case where the interaction is not applied.

tern is sensible to a change in the updating scheme, it does not seen to be a synchronous singularity, in the sense that it remains stable for values of asynchronism extremely close to perfect synchrony. We now explain this result by considering the effect of asynchronism on a single cell of the pattern.

4.2 Microscopic approach to asynchronism effects on patterns

Thanks to the simplicity of the local rule, it is possible to study the change of behaviour in asynchronous interaction for a single cell, by considering a typical pattern configuration and constructing the corresponding transition table. In particular, by comparing the synchronous case ($\alpha_{\mathrm{I}} = 1$) to the asynchronous case ($\overline{\alpha_{\mathrm{I}}} = 1 - \alpha_{\mathrm{I}} > 0$), changes appear in the order of magnitude in the probabilities of each output.

4.2.1 Asynchronous interaction in aligned patterns.

In the case of aligned patterns, *i.e.* the diagonal stripe or the clusters, the new probability table shown in Fig. 7a suggests that adding asynchronism in the interaction does not change fundamentally the relative order of magnitude between the probabilities of each output. Indeed, the case corresponding to skipping the interaction (the coloured column) is also the most probable output in the synchronous case. As a result, the order of the "most aligned" output is equal to $\alpha_{\mathrm{I}}+\overline{\alpha_{\mathrm{I}}} \approx 1$, while all other outputs have their probability multiplied by a factor α_{I}. As a consequence, this observation suggests that the asynchronous interaction improves the stability of aligned patterns.

4.2.2 Asynchronous interaction in the checkerboard pattern.

On the contrary, in the case of the checkerboard pattern, the probability table in Fig. 7b shows a notable change in the order of magnitude of the probability distribution of outputs. Therefore, for $\alpha_{\mathrm{I}} < 1$, while the probability of each output is multiplied by α_{I}, the case corresponding to skipping the interaction (the coloured column) is the least probable output in the synchronous case.

The first consequence of this change is that, for a sufficient amount of asynchrony ($\overline{\alpha_{\mathrm{I}}} >> e^{-16\sigma}$), the order of magnitude of the least probable output becomes the most probable one. This result supports the observation of the drastic change of behaviour for the checkerboard pattern.

Second, if we consider this time a small but non-infinitesimal synchrony rate $\overline{\alpha_{\mathrm{I}}} < e^{-16\sigma}$, the transition table of the interaction step is not fundamentally changed, which suggests that the behaviour should be invariant. This remark confirms the observation of the stability of the checkerboard pattern for a synchrony rate $\alpha_{\mathrm{I}} \sim 0.999$.

4.2.3 Asynchronous propagation.

A microscopic approach to study asynchronous propagation is more difficult. Indeed, if asynchronous interaction only effects on the transition probabilities, the propagation induces impredictable interactions between cells that prevent us from studying the changes on a single cell.

According to the quantitative data and a microscopic approach, we reckon that the checkerboard pattern may not be considered a synchronous singularity for either asynchronous interaction or propagation. This idea is supported by the argument that given that this pattern is stable despite a stochastic updating rule, an infinitesimal perturbation should not fundamentally change the stability of the behaviour. For now, the issue remains open and further high-scale experiments are needed to confirm this hypothesis.

5 Discussion

This paper presented a first definition of an asynchronous LGCA, by adapting the concept of asynchronism in classical CA to the principles of a lattice-gas model. We learned that although CA and LGCA share a common synchronous definition, the introduction of asynchronism revealed intrinsic differences. While

CA rely on cell-based transitions, LGCA intend to capture a transport of information between cells. Therefore the conservation and spatial exclusion of particles imposed us to change the modelling point of view and introduce *channels* as the new base components of the updating scheme.

We applied our definitions to a particular model of swarming, which displayed surprising results:

1. *All behaviours are not equally robust to asynchrony.* For instance, while "aligned" behaviours show some degree of robustness, the checkerboard pattern becomes highly unstable for a very small amount of asynchronism.

2. *Different definitions of asynchronism have different effects on the dynamics.* The asynchronous interaction, which can be seen as an "inertia effect", leaves aligned patterns unchanged. On the contrary, the asynchronous propagation alters drastically the mechanisms of the system and effects can be observed for all behaviours.

As observed before for binary CA [5], the study of the robustness to asynchronism may reveal the relations between the behaviour and the synchronous updating scheme, as well as exhibit new phenomena, such as novel patterns, and reveal singularities. In addition, the effects of asynchronism observed in this paper are based on the assumption of a finite-size lattice. However, the synchronous behaviours are known to be related to resonance effects, dependent on a finite-size square lattice and periodic boundaries [6]. Similarly, the asynchronous behaviours must be questioned by comparing finite and infinite lattices, in order to fully explore the robustness of the behaviour.

In a wider scope, the definitions we proposed here are far from being complete. For instance, in asynchronous propagation, a single particle may block an entire array of particles, which contradicts the locality of events intrinsic to spatially-distributed computing. A possible development of this approach consists in building a model where particles are considered as autonomous agents, with their own perception and decision process.

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