

The 3-rainbow index and connected dominating sets*

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Abstract

A tree in an edge-colored graph is said to be rainbow if no two edges on the tree share the same color. An edge-coloring of G is called 3-rainbow if for any three vertices in G , there exists a rainbow tree connecting them. The 3-rainbow index $rx_3(G)$ of G is defined as the minimum number of colors that are needed in a 3-rainbow coloring of G . This concept, introduced by Chartrand et al., can be viewed as a generalization of the rainbow connection. In this paper, we study the 3-rainbow index by using connected three-way dominating sets and 3-dominating sets. We shown that for every connected graph G on n vertices with minimum degree at least δ ($3 \leq \delta \leq 5$), $rx_3(G) \leq \frac{3n}{\delta+1} + 4$, and the bound is tight up to an additive constant; whereas for every connected graph G on n vertices with minimum degree at least δ ($\delta \geq 3$), we get that $rx_3(G) \leq n \frac{\ln(\delta+1)}{\delta+1} (1 + o_\delta(1)) + 5$. In addition, we obtain some tight upper bounds of the 3-rainbow index for some special graph classes, including threshold graphs, chain graphs and interval graphs.

Keywords: 3-rainbow index, connected dominating sets, rainbow paths

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1 Introduction

All graphs in this paper are undirected, finite and simple. We follow [1] for graph theoretical notation and terminology not described here. Let G be a nontrivial connected

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graph with an *edge-coloring* $c : E(G) \rightarrow \{1, 2, \dots, t\}, t \in \mathbb{N}$, where adjacent edges may be colored the same. A path is said to be a *rainbow path* if no two edges on the path have the same color. An edge-colored graph G is called *rainbow connected* if for every pair of distinct vertices of G there exists a rainbow path connecting them. The *rainbow connection number* of G , denoted by $rc(G)$, is defined as the minimum number of colors that are needed in order to make G rainbow connected. The *rainbow k -connectivity* of G , denoted by $rc_k(G)$, is defined as the minimum number of colors in an edge-coloring of G such that every two distinct vertices of G are connected by k internally disjoint rainbow paths. These concepts were introduced by Chartrand et al. in [9, 10]. Recently, there have been published a lot of results on the rainbow connections. The interested readers can see [16, 17] for a survey on this topic.

The (k, ℓ) -rainbow index was also introduced by Chartrand et al. in [11], which can be viewed as a generalization of the rainbow connection and rainbow connectivity. We call a tree T of an edge-colored graph G a *rainbow tree* if no two edges of T have the same color. For $S \subseteq V(G)$, a *rainbow S -tree* is a rainbow tree connecting the vertices of S . Suppose that $\{T_1, T_2, \dots, T_\ell\}$ is a set of rainbow S -trees. They are called *internally disjoint* if $E(T_i) \cap E(T_j) = \emptyset$ and $V(T_i) \cap V(T_j) = S$ for every pair of distinct integers i, j with $1 \leq i, j \leq \ell$ (Note that these trees are vertex-disjoint in $G \setminus S$). Given two positive integers k, ℓ with $k \geq 2$, the (k, ℓ) -rainbow index $rx_{k, \ell}(G)$ of G is the minimum number of colors needed in an edge-coloring of G such that for any set S of k vertices of G , there exist ℓ internally disjoint rainbow S -trees. In particular, for $\ell = 1$, we often write $rx_k(G)$ rather than $rx_{k, 1}(G)$ and call it the *k -rainbow index*. An edge-coloring of G is called a *k -rainbow coloring* if for any set S of k vertices of G , there exists a rainbow S -tree. A simple result for the k -rainbow index [11] is that $k - 1 \leq rx_k(G) \leq n - 1$. It is easy to see that $rx_{2, \ell}(G) = rc_\ell(G)$. In the sequel, we always assume $k \geq 3$. We refer to [2–4, 12, 15, 18] for more details about the (k, ℓ) -rainbow index.

Computing the rainbow connection number of a graph is NP-hard [7], so is computing the (k, ℓ) -rainbow index. For this reason, one of the most important goals for studying rainbow connection number and rainbow index is to obtain good upper and lower bounds. In the search toward good upper bounds, an idea that turned out to be successful more than once is considering the “strengthened” connected dominating set: find a suitable edge-coloring of the induced graph on such a set, and then extend it to the whole graph using a constant number of additional colors.

Given a graph G , a set $D \subseteq V(G)$ is called a *dominating set* if every vertex of $V \setminus D$ is adjacent to at least one vertex of D . Further, if the subgraph $G[D]$ of G induced by D is connected, we call D a *connected dominating set* of G . The *domination number* $\gamma(G)$ is the number of vertices in a minimum dominating set for G . Similarly, the *connected domination number* $\gamma_c(G)$ is the number of vertices in a minimum connected dominating set for G .

Let k be a positive integer. A dominating set D of G is called a *k-way dominating set* if $d(v) \geq k$ for every vertex $v \in V \setminus D$. In addition, if $G[D]$ is connected, we call D a *connected k-way dominating set*. A set $D \subseteq V(G)$ is called a *k-dominating set* of G if every vertex of $V \setminus D$ is adjacent to at least k distinct vertices of D . Furthermore, if $G[D]$ is connected, we call D a *connected k-dominating set*. Obviously, a (connected) k -dominating set is also a (connected) k -way dominating set, but the converse is not true.

There have been several results revealing the close relation between the dominating sets and the rainbow connection number and rainbow index.

Theorem 1. [8] *If D is a connected two-way dominating set of a connected graph G , then $rc(G) \leq rc(G[D]) + 3$.*

In [8], the authors employed Theorem 1 to get some tight upper bounds for the rainbow connection number of many special graph classes, which were otherwise difficult to obtain.

Theorem 2. [18] *Let G be a connected graph with minimal degree $\delta(G) \geq 3$. If D is a connected 2-dominating set of G , then $rx_3(G) \leq rx_3(G[D]) + 4$ and the bound is tight.*

From Theorem 2, the authors determined a tight upper bound for the 3-rainbow index of the complete bipartite graphs $K_{s,t}$ ($3 \leq s \leq t$).

The proofs of the above two theorems are similar. First color the edges in $G[D]$ using k different colors ($k = rc(G[D])$ or $rx_3(G[D])$). Then select a spanning tree in every connected component of $H = G - D$. So we construct a spanning forest F of H and choose X and Y as any one of the bipartitions defined by the forest F . Color the edges between X and D and the edges between Y and D as well as the edges between X and Y with suitable colors, which gives an edge-coloring we want. Note that in the process all the edges in $E(H) - E(F)$ are ignored.

In this paper, we will take the edges in $E(H) - E(F)$ into consideration to get a more subtle coloring strategy. We show that for a connected graph G , $rx_3(G) \leq rx_3(G[D]) + 6$, where D is a connected three-way dominating set of G . Moreover, this bound is tight. By using the results on spanning trees with many leaves, we obtain that $rx_3(G) \leq \frac{3n}{\delta+1} + 4$ for every connected graph G on n vertices with minimum degree at least δ ($3 \leq \delta \leq 5$), and the bound is tight up to an additive constant; whereas for every connected graph G on n vertices with minimum degree at least δ ($\delta \geq 3$), we get that $rx_3(G) \leq n \frac{\ln(\delta+1)}{\delta+1} (1 + o_\delta(1)) + 5$. In addition, when considering a connected 3-dominating set D of G , we prove that $rx_3(G) \leq rx_3(G[D]) + 3$, and the bound is tight. The farthest we can get with this idea is some tight upper bounds for some special graph classes, including threshold graphs, chain graphs and interval graphs.

2 Preliminaries

For a graph G , we use $V(G)$, $E(G)$, $|G|$, $\delta(G)$, and $diam(G)$ to denote its vertex set, edge set, order (number of vertices), minimum degree and the diameter (maximum distance between every pair of vertices) of G , respectively. For $D \subseteq V(G)$, let $\overline{D} = V(G) \setminus D$, and $G[D]$ be the subgraph of G induced on D . For $v \in V(G)$, let $N(v)$ denote the set of neighbors of v . For two disjoint subsets X and Y of $V(G)$, $E[X, Y]$ denotes the set of edges of G between X and Y .

Definition 1. *BFS (breadth-first search) is a strategy for searching in a graph. It begins at a root and inspects all its neighbors. Then for each of those neighbors in turn, it inspects their neighbors which were unvisited, and so on until all the vertices in the graph are visited.*

Definition 2. *A BFS-tree (breadth-first search tree) is a spanning rooted tree returned by BFS. Let T be a BFS-tree with r as its root. For a vertex v , the height of v is the distance between v and r . All the vertices of height k form the k th level of T . The ancestors of v are the vertices on the unique $\{v, r\}$ -path in T . The parent of v is its neighbor on the unique $\{v, r\}$ -path in T . Its other neighbors are called the children of v . The siblings of v are the vertices in the same level as v .*

Remark: *BFS-trees have a nice property: every edge of the graph joins vertices on the same or consecutive levels. It is not possible for an edge to skip a level. Thus the neighbor of a vertex v has three possibilities: (1) a sibling of v ; (2) the parent of v or a right sibling of the parent of v ; (3) a child of v or a left sibling of the children of v ; see Figure 1.*

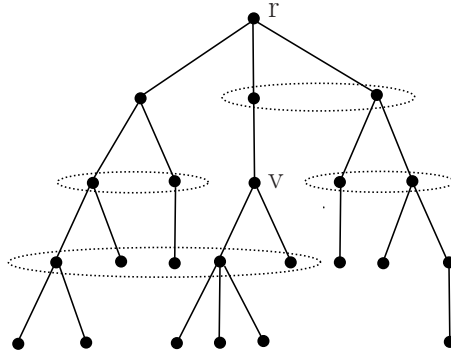


Figure 1: The vertices in the dotted circles are the potential neighbors of v .

Definition 3. *The Steiner distance $d(S)$ of a set S of vertices in a graph G is the minimum size of a tree in G containing S . The k -Steiner diameter $sdiam_k(G)$ of G is the maximum Steiner distance of S among all the sets S with k vertices in G . Obviously, $sdiam_2(G) = diam(G)$ and $sdiam_k(G) \leq sdiam_{k+1}(G)$.*

Definition 4. Let G be a graph, $D \subseteq V(G)$ and $v \in V(G) \setminus D$. we call a path $P = v_0v_1 \cdots v_k$ is a v - D path if $v_0 = v$ and $V(P) \cap D = \{v_k\}$. Two or more paths are called internally disjoint if none of them contains an inner vertex of another.

Definition 5. An edge-colored graph is rainbow if no two edges in the graph share the same color.

Definition 6. Let D be a dominating set of a graph G . For $v \in \overline{D}$, its neighbors in D are called foots of v , and the corresponding edges are called legs of v .

Definition 7. A graph G is called a threshold graph, if there exists a weight function $w : V(G) \rightarrow \mathbb{R}$ and a real constant t such that two vertices $u, v \in V(G)$ are adjacent if and only if $w(u) + w(v) \geq t$. We call t the threshold for G .

Definition 8. A bipartite graph $G(A, B)$ is called a chain graph, if the vertices of A can be ordered as $A = (a_1, a_2, \dots, a_k)$ such that $N(a_1) \subseteq N(a_2) \subseteq \dots \subseteq N(a_k)$.

Definition 9. An intersection graph of a family \mathcal{F} of sets is a graph whose vertices can be mapped to the sets in \mathcal{F} such that there is an edge between two vertices in the graph if and only if the corresponding two sets in \mathcal{F} have a non-empty intersection. An interval graph is an intersection graph of intervals on the real line.

3 Main results

Theorem 3. If D is a connected three-way dominating set of a connected graph G , then $rx_3(G) \leq rx_3(G[D]) + 6$. Moreover, the bound is tight.

The proof of Theorem 3 is given in Section 4. Let us first show how this implies the following results.

Corollary 4. For every connected graph G with $\delta(G) \geq 3$, $rx_3(G) \leq \gamma_c(G) + 5$.

Proof. In this case, every connected dominating set of G is a connected three-way dominating set. Now take a minimum connected dominating set D in G . Then $rx_3(G[D]) \leq |D| - 1 = \gamma_c(G) - 1$. It follows from Theorem 3 that $rx_3(G) \leq rx_3(G[D]) + 6 \leq \gamma_c(G) + 5$. \square

From the following lemma, we can get the next corollary.

Lemma 5. (1) [14] Every connected graph on n vertices with minimum degree $\delta \geq 3$ has a spanning tree with at least $\frac{1}{4}n + 2$ leaves;

(2) [13] Every connected graph on n vertices with minimum degree $\delta \geq 4$ has a spanning tree with at least $\frac{2}{5}n + \frac{8}{5}$ leaves;

(3) [13] Every connected graph on n vertices with minimum degree $\delta \geq 5$ has a spanning tree with at least $\frac{1}{2}n + 2$ leaves;

Corollary 6. (1) For every connected graph G on n vertices with $\delta(G) \geq 3$, $rx_3(G) \leq \frac{3}{4}n + 3$.

(2) For every connected graph G on n vertices with $\delta(G) \geq 4$, $rx_3(G) \leq \frac{3}{5}n + \frac{17}{5}$.

(3) For every connected graph G on n vertices with $\delta(G) \geq 5$, $rx_3(G) \leq \frac{1}{2}n + 3$.

Moreover, these bounds are tight up to an additive constant.

Proof. We only prove (1); (2) and (3) can be derived by the same arguments.

Clearly, we can take a connected dominating set consisting of all the non-leaves in the spanning tree. Thus by Lemma 5, for every connected graph G on n vertices with minimum degree $\delta(G) \geq 3$, $\gamma_c(G) \leq n - (\frac{1}{4}n + 2) = \frac{3}{4}n - 2$. Then it follows from Corollary 4 that $rx_3(G) \leq \frac{3}{4}n + 3$.

On the other hand, the factors in these bounds cannot be improved, since there exist infinitely many graphs G^* such that $rx_3(G^*) \geq \frac{3}{\delta+1}n - \frac{\delta+7}{\delta+1}$. We construct the graphs as follows (the construction was also mentioned in [5]): first take m copies of $K_{\delta+1}$, denoted by X_1, X_2, \dots, X_m and label the vertices of X_i with $x_{i,1}, \dots, x_{i,\delta+1}$. Then take two copies of $K_{\delta+2}$, denoted by X_0 and X_{m+1} and similarly label their vertices. Now join $x_{i,2}$ and $x_{i+1,1}$ for $i = 0, 1, \dots, m$ with an edge and delete the edges $x_{i,1}x_{i,2}$ for $i = 0, 1, \dots, m+1$. See Figure 2 for $\delta = 3$. It is easy to see that $diam(G^*) = \frac{3}{\delta+1}n - \frac{\delta+7}{\delta+1}$. The k -Steiner diameter of a graph is a trivial lower bound for its k -rainbow index [11], and so $rx_3(G^*) \geq sdiam_3(G^*) \geq diam(G^*) = \frac{3}{\delta+1}n - \frac{\delta+7}{\delta+1}$. For $\delta = 3$, $rx_3(G^*) \geq \frac{3}{4}n - \frac{5}{2}$; for $\delta = 4$, $rx_3(G^*) \geq \frac{3}{5}n - \frac{11}{5}$; for $\delta = 5$, $rx_3(G^*) \geq \frac{1}{2}n - 2$. Therefore, all these upper bounds are tight up to an additive constant. \square

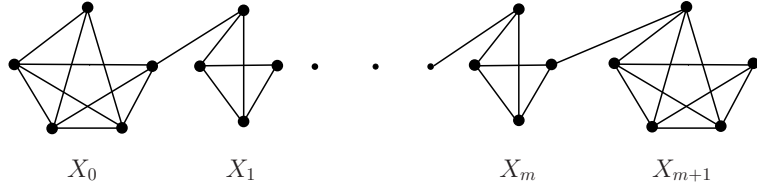


Figure 2: An example for $\delta = 3$.

As to general δ , Caro et. al. [6] proved that for every connected graph G on n vertices with minimum degree δ , $\gamma_c(G) = n \frac{\ln(\delta+1)}{\delta+1} (1 + o_\delta(1))$. Combining with Corollary 4, we get the following result.

Corollary 7. For every connected graph G on n vertices with minimum degree δ ($\delta \geq 3$), $rx_3(G) \leq n \frac{\ln(\delta+1)}{\delta+1} (1 + o_\delta(1)) + 5$.

The above bound is not believed to be optimal for $rx_3(G)$ in terms of δ . We pose the following conjecture, which has already been proved for $\delta = 3, 4, 5$ in Corollary 6. Note

that if the conjecture is true, it gives an upper bound tight up to an additive constant by the construction of the graph G^* .

Conjecture 1. *For every connected graph G on n vertices with minimum degree δ ($\delta \geq 3$), $rx_3(G) \leq \frac{3n}{\delta+1} + C$, where C is a positive constant.*

With regard to the graphs possessing vertices of degree 1 or 2, we obtain the following result.

Corollary 8. *For every connected graph G , $rx_3(G) \leq \gamma_c(G) + n_1 + n_2 + 5$, where n_1 and n_2 denote the number of vertices of degrees 1 and 2 in G , respectively.*

Proof. Obviously, adding all the vertices of degrees 1 and 2 into a minimum connected dominating set forms a connected three-way dominating set in G of size no more than $\gamma_c(G) + n_1 + n_2$. Consequently, by Theorem 3, $rx_3(G) \leq \gamma_c(G) + n_1 + n_2 + 5$. \square

We proceed with another upper bound for the 3-rainbow index of graphs concerning the connected 3-dominating set.

Theorem 9. *If D is a connected 3-dominating set of a connected graph G with $\delta(G) \geq 3$, then $rx_3(G) \leq rx_3(G[D]) + 3$. Moreover, the bound is tight.*

Proof. Since D is a connected 3-dominating set, every vertex in \overline{D} has at least three legs. Color one of them with 1, one of them with 2, and all the others with 3. Let $k = rx_3(G[D])$. Then we can color the edges in $G[D]$ with k different colors from $\{4, 5, \dots, k+3\}$ such that for every triple of vertices in D , there exists a rainbow tree in $G[D]$ connecting them. If there remain uncolored edges in G , we color them with 1.

Next we will show that this edge-coloring is a 3-rainbow coloring of G . For any triple $\{u, v, w\}$ of vertices in G , if $(u, v, w) \in D \times D \times D$, then there is already a rainbow tree connecting them in $G[D]$. If one of them is in \overline{D} , say $(u, v, w) \in \overline{D} \times D \times D$, join any leg of u (colored by 1, 2, or 3) with the rainbow tree connecting v, w and the corresponding foot of u in $G[D]$. If two of them are in \overline{D} , say $(u, v, w) \in \overline{D} \times \overline{D} \times D$, join one leg of u colored by 1, one leg of v colored by 2 with the rainbow tree connecting w and the corresponding foots of u, v in $G[D]$. If $(u, v, w) \in \overline{D} \times \overline{D} \times \overline{D}$, join one leg of u colored by 1, one leg of v colored by 2, one leg of w colored by 3 with the rainbow tree connecting the corresponding foots of u, v, w in $G[D]$.

The tightness of the bound can be seen from the next Corollary. \square

As immediate consequences of Theorem 3 and Theorem 9, we get the following:

Corollary 10. *Let G be a connected graph with $\delta(G) \geq 3$.*

(1) *if G is a threshold graph, then $rx_3(G) \leq 5$;*

(2) if G is a chain graph, then $rx_3(G) \leq 6$;

(3) if G is an interval graph, then $rx_3(G) \leq \text{diam}(G) + 4$. Thus $\text{diam}(G) \leq rx_3(G) \leq \text{diam}(G) + 4$;

Moreover, all these upper bounds are tight.

Proof. (1) Suppose that $V(G) = \{v_1, v_2, \dots, v_n\}$ where $w(v_1) \geq w(v_2) \geq \dots \geq w(v_n)$. Since the minimum degree of G is at least three, v_i ($1 \leq i \leq 3$) is adjacent to all the other vertices in G . Thus $D = \{v_1, v_2, v_3\}$ consists of a connected 3-dominating set of G . Note that D induces a K_3 , so $rx_3(G[D]) = 2$. It follows from Theorem 9 that $rx_3(G) \leq rx_3(G[D]) + 3 = 5$.

(2) Suppose that $G = G(A, B)$ and the vertices of A can be ordered as $A = (a_1, a_2, \dots, a_k)$ such that $N(a_1) \subseteq N(a_2) \subseteq \dots \subseteq N(a_k)$. Since the minimum degree of G is at least three, a_i ($k - 2 \leq i \leq k$) is adjacent to all the vertices in B , and $N(a_1)$ has at least three vertices, say $\{b_1, b_2, b_3\}$. Clearly b_i ($1 \leq i \leq 3$) is adjacent to all the vertices in A . Thus $D = \{a_{k-2}, a_{k-1}, a_k, b_1, b_2, b_3\}$ consists of a connected 3-dominating set of G . Note that D induces a $K_{3,3}$, so $rx_3(G[D]) = 3$ (see [12]). It follows from Theorem 9 that $rx_3(G) \leq rx_3(G[D]) + 3 = 6$.

(3) If G is isomorphic to a complete graph, then $rx_3(G) = 2$ or 3 (see [11]), the assertion holds trivially. Otherwise, it was showed in [8] that every interval graph G which is not isomorphic to a complete graph has a dominating path P of length at most $\text{diam}(G) - 2$. Since $\delta(G) \geq 3$, P consists of a connected three-way dominating set of G . It follows from Theorem 3 that $rx_3(G) \leq rx_3(P) + 6 \leq \text{diam}(G) + 4$. On the other hand, $rx_3(G) \geq \text{sdiam}_3(G) \geq \text{diam}(G)$. We conclude that for a connected interval graph G with $\delta \geq 3$, $\text{diam}(G) \leq rx_3(G) \leq \text{diam}(G) + 4$

Here we give examples to show the tightness of these upper bounds.

Example 1: A threshold graph G with $\delta(G) \geq 3$ and $rx_3(G) = 5$.

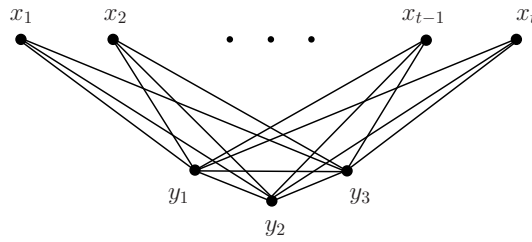


Figure 3: A threshold graph G with $\delta(G) \geq 3$ and $rx_3(G) = 5$.

Consider the graph in *Figure 3*, where $t \geq 2 \times 4^3 + 1$. It is easy to see that it is a threshold graph (y_1, y_2, y_3 can be given a weight 1, others a weight 0 and the threshold 1). By contradiction, we assume that G can be colored with 4 colors. Let S_i denote the star with x_i as its center and $E(S_i) = \{x_i y_1, x_i y_2, x_i y_3\}$. Every S_i can be colored in 4^3

different ways. Since $t \geq 2 \times 4^3 + 1$, there exist three completely identical edge-colored stars, say S_1, S_2 and S_3 . If two of the three edges in S_i ($1 \leq i \leq 3$) receive the same color, then there are no rainbow trees connecting x_1, x_2, x_3 , a contradiction. If the three edges in S_i ($1 \leq i \leq 3$) receive distinct colors, then the rainbow tree connecting x_1, x_2, x_3 must contain the vertices y_1, y_2, y_3 . Thus the tree has at least five edges, but only four different colors, a contradiction.

Example 2: A chain graph G with $\delta(G) \geq 3$ and $rx_3(G) = 6$.

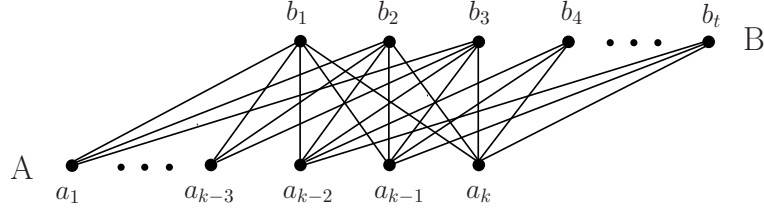


Figure 4: A chain graph G with $\delta(G) \geq 3$ and $rx_3(G) = 6$.

Consider the bipartite graph in *Figure 4*, where $N(a_1) = N(a_2) = \dots = N(a_{k-3}) = \{b_1, b_2, b_3\}$, $N(a_{k-2}) = N(a_{k-1}) = N(a_k) = \{b_1, b_2, \dots, b_t\}$, and $t \geq 2 \times 5^3 + 4$. By contradiction, we assume that G can be colored with 5 colors. Let S_i ($4 \leq i \leq t$) denote the star with b_i as its center and $E(S_i) = \{b_i a_{k-2}, b_i a_{k-1}, b_i a_k\}$. Every S_i can be colored in 5^3 different ways. Since $t - 3 \geq 2 \times 5^3 + 1$, among the $t - 3$ S_i 's there exist three completely identical edge-colored stars, say S_4, S_5 and S_6 . If two of the three edges in S_i ($4 \leq i \leq 6$) receive the same color, then there are no rainbow trees connecting b_4, b_5, b_6 , a contradiction. If the three edges in S_i ($4 \leq i \leq 6$) receive distinct colors, then the rainbow tree connecting b_4, b_5, b_6 must contain a_{k-2}, a_{k-1}, a_k and at least one vertex in $B \setminus \{b_4, b_5, b_6\}$ to connect a_{k-2}, a_{k-1}, a_k . Thus the tree has at least six edges, but only five different colors, a contradiction.

Example 3: An interval graph G with $\delta(G) \geq 3$ and $rx_3(G) = \text{diam}(G) + 4$.

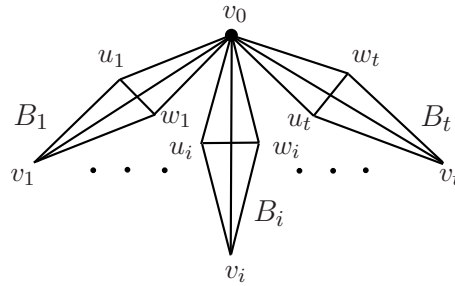


Figure 5 An interval graph G with $\delta(G) \geq 3$ and $rx_3(G) = \text{diam}(G) + 4$.

Consider the graph in *Figure 5* (it is known as a French Windmill), where $t \geq 2 \times 5^6 + 1$. It is easy to see that it is an interval graph with diameter 2. It follows from (3) that

$rx_3(G) \leq \text{diam}(G) + 4 = 6$. We will show that $rx_3(G) = 6$. By contradiction, we assume that G can be colored with 5 colors. Let B_i denote the K_4 induced by v_0, u_i, v_i, w_i . Obviously, each B_i can be colored in at most 5^6 different ways. Since $t \geq 2 \times 5^6 + 1$, there exist three completely identical edge-colored subgraphs, say B_1, B_2, B_3 . If two of the three edges incident with v_0 in B_i ($1 \leq i \leq 3$) receive the same color, say $c(v_0u_i) = c(v_0v_i) = 1$, then there are no rainbow trees connecting u_1, u_2, u_3 , a contradiction. If the three edges incident with v_0 in B_i ($1 \leq i \leq 3$) receive distinct colors, say $c(v_0u_i) = 1, c(v_0v_i) = 2, c(v_0w_i) = 3$, then $c(u_iv_i) \neq c(u_iw_i)$ and $c(u_iv_i), c(u_iw_i) \in \{4, 5\}$ because there exists a rainbow tree connecting $\{u_1, u_2, u_3\}$. Without loss of generality, suppose $c(u_iv_i) = 4$ and $c(u_iw_i) = 5$. Since there exists a rainbow tree connecting $\{v_1, v_2, v_3\}$, then $c(v_iw_i) = 5$. But then there exist no rainbow trees connecting $\{w_1, w_2, w_3\}$ in G , a contradiction. \square

4 Proof of Theorem 3

Let D be a connected three-way dominating set of a connected graph G . We want to show that $rx_3(G) \leq rx_3(G[D]) + 6$.

To start with, we introduce some definitions and notation that are used in the sequel. A set of rainbow paths $\{P_1, P_2, \dots, P_\ell\}$ is called *super-rainbow* if their union $\bigcup_{i=1}^{\ell} P_i$ is also rainbow. For a vertex v in \overline{D} , we call it *safe* if there are three internally disjoint super-rainbow $v - D$ paths. Otherwise, we call v *dangerous*. Let $c(e)$ be the color of an edge e , $c(H)$ the set of colors appearing on the edges in a graph H . For a vertex v in a *BFS*-tree, we denote the height of v by $h(v)$, the parent of v by $p(v)$, the child of v by $ch(v)$, the ancestor of v in the first level by $\pi(v)$.

Let us overview our idea: firstly, we aim to color the edges in $E[D, \overline{D}]$ and $E(G[\overline{D}])$ with six different colors. Our coloring strategy has two steps: in the first step, we give a periodical coloring on some edges in $E[D, \overline{D}]$ and $E(G[\overline{D}])$. And then most vertices in \overline{D} become safe; in the second step, we color the carefully chosen uncolored edges and recolor some colored edges intelligently to ensure that all the vertices in \overline{D} are safe. Then we extend the coloring to the whole graph by coloring the edges in $G[D]$ with $rx_3(G[D])$ fresh colors. Finally, we will show that this edge-coloring of G is a 3-rainbow coloring, which implies $rx_3(G) \leq rx_3(G[D]) + 6$.

4.1 Color the edges in $E[D, \overline{D}]$ and $E(G[\overline{D}])$

4.1.1 First step: a periodical coloring

Assume that C_1, C_2, \dots, C_q are the connected components of the subgraph $G - D$.

If C_i ($1 \leq i \leq q$) consists of an isolated vertex v , then v has at least three legs. We color one of them with 1, one of them with 2, and all the others with 3. Note that now v

is safe.

If C_i ($1 \leq i \leq q$) consists of an isolated edge uv , then u has at least two legs. We color one of them with 1, and all the others with 2. Similarly, v has at least two legs. We color one of them with 2, and all the others with 3. And color uv with 4. Note that now both u and v are safe.

If C_i ($1 \leq i \leq q$) consists of at least three vertices, then there exists a vertex v_0 in C_i possessing at least two neighbors in C_i . Starting from v_0 , we construct a *BFS*-tree T of C_i . Suppose the neighbors of v_0 in C_i are $\{v_1, v_2, \dots, v_k\}$ ($k \geq 2$), which forms the first level of T . For each vertex v in C_i , let e_v be one leg of v (if there are many legs, we pick one arbitrarily), $t(v)$ the corresponding foot of v , f_v the unique edge joining v and its parent in T .

Now we color the edges e_v and f_v as follows: $c(e_{v_0}) = 2$; $c(f_{v_i}) = 4$ and $c(e_{v_i}) = 1$ for $1 \leq i \leq k - 1$; $c(f_{v_k}) = 5$ and $c(e_{v_k}) = 3$; for each vertex v in $V(C_i) \setminus \{v_0, v_1, \dots, v_k\}$, if $\pi(v) = v_k$, then set $c(f_v) = 4$ and $c(e_v) = 2$ when $h(v) \equiv 0 \pmod{3}$, $c(f_v) = 5$ and $c(e_v) = 3$ when $h(v) \equiv 1 \pmod{3}$, $c(f_v) = 6$ and $c(e_v) = 1$ when $h(v) \equiv 2 \pmod{3}$; otherwise, if $\pi(v) = v_i$ ($1 \leq i \leq k - 1$), then set $c(f_v) = 6$ and $c(e_v) = 2$ when $h(v) \equiv 0 \pmod{3}$, $c(f_v) = 4$ and $c(e_v) = 1$ when $h(v) \equiv 1 \pmod{3}$, $c(f_v) = 5$ and $c(e_v) = 3$ when $h(v) \equiv 2 \pmod{3}$. In fact, this gives a periodical coloring as Figure 6.

We call the subtree of T rooted at v_i ($1 \leq i \leq k - 1$) of type *I* and the subtree of T rooted at v_k of type *II*. There may be many subtrees of type *I*, but only one subtree of type *II*. The subtrees of the same type are colored in the same way. More precisely, if two vertices u, v lie in the same level and belong to subtrees of the same type, then $c(e_u) = c(e_v)$ and $c(f_u) = c(f_v)$ after first step.

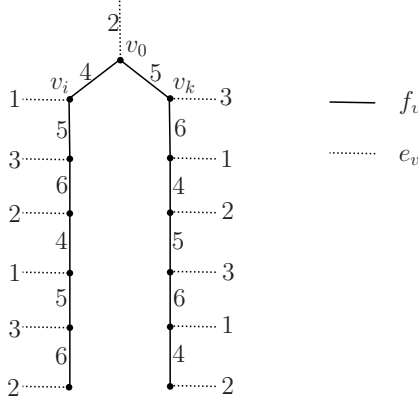


Figure 6: The left branch represents the coloring of subtrees of type *I* and the right branch represents the coloring of the subtree of type *II*.

Now each non-leaf vertex in T has three internally disjoint super-rainbow paths connecting it to D : for the root v_0 , $P_1^{v_0} = v_0, t(v_0)$; $P_2^{v_0} = v_0, v_1, t(v_1)$; $P_3^{v_0} = v_0, v_k, t(v_k)$. for other non-leaf vertex v in T , $P_1^v = v, t(v)$; $P_2^v = v, p(v), t(p(v))$; $P_3^v = v, ch(v), t(ch(v))$.

(Note that v may have many children u_1, u_2, \dots, u_ℓ , but all the e_{u_i} 's, f_{u_i} 's receive the same color. So they only contribute one path to the three internally disjoint super-rainbow $v-D$ paths.) In other words, after first step all the non-leaf vertices in T are safe .

As to each leaf v' in T , since v' has no children, it has exactly two internally disjoint super-rainbow $v' - D$ paths: $P_1^{v'} = v', t(v')$; $P_2^{v'} = v', p(v'), t(p(v'))$. In other words, after first step all the leaves in T are dangerous.

Example 4: The root v_0 is safe: $c(P_1^{v_0}) = \{2\}$, $c(P_2^{v_0}) = \{1, 4\}$, $c(P_3^{v_0}) = \{3, 5\}$.

If v_i ($1 \leq i \leq k-1$) is not a leaf of T , then v_i is safe: $c(P_1^{v_i}) = \{1\}$, $c(P_2^{v_i}) = \{2, 4\}$, $c(P_3^{v_i}) = \{3, 5\}$.

If v_k is not a leaf of T , then v_k is safe: $c(P_1^{v_k}) = \{3\}$, $c(P_2^{v_k}) = \{2, 5\}$, $c(P_3^{v_k}) = \{1, 6\}$.

Example 5: If v is a leaf of T in the second level with parent v_k , then v is dangerous: $c(P_1^v) = \{1\}$, $c(P_2^v) = \{3, 6\}$.

All the possible color sets of the three internally disjoint super-rainbow paths connecting a non-leaf vertex to D are: (the first part in every brace is the color of P_1 , the second is the color of P_2 , the third is the color of P_3)

$$\{1, 24, 35\}, \quad \{2, 36, 14\}, \quad \{3, 15, 26\}, \quad \{1, 36, 24\}, \quad \{2, 14, 35\}, \quad \{3, 25, 16\}.$$

Bearing in mind that D is a connected three-way dominating set, each leaf in T is incident with at least one uncolored edge. In second step, we will color such edges and recolor some colored edges suitably to ensure that all the vertices in C_i are safe.

4.1.2 Second step: more edges with a more intelligent coloring

Let v be a leaf in T and $g_v = vv'$ be one uncolored edge incident with v .

If g_v connects v to D , then we give g_v a smallest color from $\{1, 2, 3, 4, 5, 6\} \setminus (c(P_1^v) \cup c(P_2^v))$. For instance, $c(g_v) = 2$ for the vertex v in Example 5. Obviously, now v has three internally disjoint super-rainbow $v - D$ paths P_1^v, P_2^v, P_3^v , where $P_1^v = v, t(v)$; $P_2^v = v, v'$; $P_3^v = v, p(v), t(p(v))$. In other words, v is safe after second step. All the possible color sets of the three internally disjoint super-rainbow paths connecting v to D are: (the first part in every brace is the color of P_1 , the second is the color of P_2 , the third is the color of P_3)

$$\{1, 2, 36\}, \quad \{1, 3, 24\}, \quad \{1, 3, 25\}, \quad \{2, 3, 15\}, \quad \{2, 3, 14\}.$$

Now it remains to deal with the leaves in T whose incident uncolored edges all lie in C_i . Let A denote the set of such vertices. First, we flag all the vertices in $V(C_i) \setminus A$, which are already safe. Note that we only flag the safe vertices. Once one vertex gets flagged, it is always flagged. Next we arrange the vertices in A in a linear order by the following three rules:

(R1) for $u, v \in A$, let $\pi(u) = v_i$ and $\pi(v) = v_j$, if $i > j$, then u is before v in the ordering;

(R2) if $\pi(u) = \pi(v)$ and $h(v) > h(u)$, then u is before v in the ordering.

(R3) if $\pi(u) = \pi(v)$, $h(u) = h(v)$ and u is reached earlier than v in the *BFS*-algorithm, then u is before v in the ordering.

Assume the vertices in A are ordered as $A = (w_1, w_2, \dots, w_s)$. We will deal with them one by one. Suppose that now we go to the vertex w_i (w_1, w_2, \dots, w_{i-1} have been processed). If w_i is flagged, we go to the next vertex w_{i+1} ; otherwise, we distinguish the following four cases:

Case 1: $\pi(w_i) = v_k$ and there exists at least one uncolored edge connecting w_i to some subtree of type I . Then we choose one such edge $w_i v$ such that the height of v is as small as possible. Since T is a *BFS*-tree and the subtree of w_i is to the right of the subtree of v , then $h(v) = h(w_i)$ or $h(w_i) + 1$.

Fact 1. e_v is not recolored.

If $v \notin A$, then e_v never gets recolored; if $v \in A$, since $\pi(w_i) = v_k$ and $\pi(v) = v_j$ ($1 \leq j \leq k-1$), we have not dealt with v yet according to R1, thus e_v is not recolored.

We distinguish three subcases based on the height of w_i .

* *Subcase 1.1:* $h(w_i) \equiv 0 \pmod{3}$

If $h(v) = h(w_i)$, then color $w_i v$ with 5. We have $c(P_1^{w_i}) \cup c(P_2^{w_i}) \cup c(P_3^{w_i}) = \{2\} \cup \{1, 4\} \cup \{3, 5, 6\}$ and $c(P_1^v) \cup c(P_2^v) \cup c(P_3^v) = \{2\} \cup \{3, 6\} \cup \{1, 4, 5\}$.

If $h(v) = h(w_i) + 1$, then color $w_i v$ with 5. We have $c(P_1^{w_i}) \cup c(P_2^{w_i}) \cup c(P_3^{w_i}) = \{2\} \cup \{3, 4, 6\} \cup \{1, 5\}$ and $c(P_1^v) \cup c(P_2^v) \cup c(P_3^v) = \{1\} \cup \{3, 4, 6\} \cup \{2, 5\}$.

Now both w_i and v become safe. We flag w_i and v (if v is not flagged).

* *Subcase 1.2:* $h(w_i) \equiv 1 \pmod{3}$

If $h(v) = h(w_i)$, then color $w_i v$ with 6. We have $c(P_1^{w_i}) \cup c(P_2^{w_i}) \cup c(P_3^{w_i}) = \{3\} \cup \{2, 5\} \cup \{1, 6\}$ and $c(P_1^v) \cup c(P_2^v) \cup c(P_3^v) = \{1\} \cup \{2, 4\} \cup \{3, 6\}$.

If $h(v) = h(w_i) + 1$, then color $w_i v$ with 4 and recolor e_{w_i} with 6. In this way, we ensure that the parent of w_i is still safe. Now $c(P_1^{p(w_i)}) \cup c(P_2^{p(w_i)}) \cup c(P_3^{p(w_i)}) = \{2\} \cup \{1, 4\} \cup \{5, 6\}$. Moreover, we have $c(P_1^{w_i}) \cup c(P_2^{w_i}) \cup c(P_3^{w_i}) = \{6\} \cup \{2, 5\} \cup \{3, 4\}$ and $c(P_1^v) \cup c(P_2^v) \cup c(P_3^v) = \{3\} \cup \{1, 5\} \cup \{4, 6\}$.

Now both w_i and v become safe. We flag w_i and v (if v is not flagged).

* *Subcase 1.3:* $h(w_i) \equiv 2 \pmod{3}$

If $h(v) = h(w_i)$, then color $w_i v$ with 2 and recolor e_{w_i} with 4. In this way, we ensure that the parent of w_i is still safe. Now $c(P_1^{p(w_i)}) \cup c(P_2^{p(w_i)}) \cup c(P_3^{p(w_i)}) = \{3\} \cup \{2, 5\} \cup \{4, 6\}$. Moreover, we have $c(P_1^{w_i}) \cup c(P_2^{w_i}) \cup c(P_3^{w_i}) = \{4\} \cup \{3, 6\} \cup \{1, 2, 5\}$ and $c(P_1^v) \cup c(P_2^v) \cup c(P_3^v) = \{3\} \cup \{1, 5\} \cup \{2, 4\}$.

If $h(v) = h(w_i) + 1$, then color $w_i v$ with 5. We have $c(P_1^{w_i}) \cup c(P_2^{w_i}) \cup c(P_3^{w_i}) = \{1\} \cup \{3, 6\} \cup \{2, 5\}$ and $c(P_1^v) \cup c(P_2^v) \cup c(P_3^v) = \{2\} \cup \{3, 6\} \cup \{1, 5\}$.

Now both w_i and v become safe. We flag w_i and v (if v is not flagged).

Remarks: 1. When dealing with w_i , we just do two operations: (i) coloring w_iv ; (ii) recoloring e_{w_i} if necessary. Note that e_{w_i} is the only edge which may be recolored in this process. Furthermore, we recolor it in such a way that the parent of w_i is still safe. In fact, for that sake, we have no choice but to recolor e_{w_i} ($w_i \notin \{v_1, v_2, \dots, v_{k-1}\}$) with the unique color which is from $\{1, 2, 3, 4, 5, 6\}$ but does not appear on the three super-rainbow paths of $p(w_i)$ after the first step. For example, in *Subcase 1.2*, the color set of the three super-rainbow paths of $p(w_i)$ after first step is $\{2, 1, 4, 5, 3\}$, so we recolor e_{w_i} with 6. The exception that $w_i \in \{v_1, v_2, \dots, v_{k-1}\}$ will be discussed in *Subcase 3.2*.

2. One may wonder what is the effect of these operations. First of all, after the process, w_i becomes safe and gets flagged, and so does v if v is not flagged. In addition, the process guarantees that all the safe vertices remain safe. As mentioned above, $p(w_i)$ is still safe after this process. For every other safe vertex in $V(C_i) \setminus A$, obviously its three internally disjoint super-rainbow paths do not contain e_{w_i} , so it is still safe after this process. For each safe vertex v in A , if its three internally disjoint super-rainbow paths contain e_{w_i} , w_i is already safe and gets flagged before dealing with it. Then we go to w_{i+1} directly without doing this process. So we claim that the three internally disjoint super-rainbow paths of v do not contain e_{w_i} , and thus it is still safe after this process.

3. The three internally disjoint super-rainbow paths of w_i is one of the following three cases; see Figure 7.

- (i) $P_1^{w_i} = w_i, t(w_i), P_2^{w_i} = w_i, p(w_i), t(p(w_i)), P_3^{w_i} = w_i, v, t(v)$;
- (ii) $P_1^{w_i} = w_i, t(w_i), P_2^{w_i} = w_i, p(w_i), t(p(w_i)), P_3^{w_i} = w_i, v, p(v), t(p(v))$;
- (iii) $P_1^{w_i} = w_i, t(w_i), P_2^{w_i} = w_i, p(w_i), p(p(w_i)), t(p(p(w_i))), P_3^{w_i} = w_i, v, t(v)$.

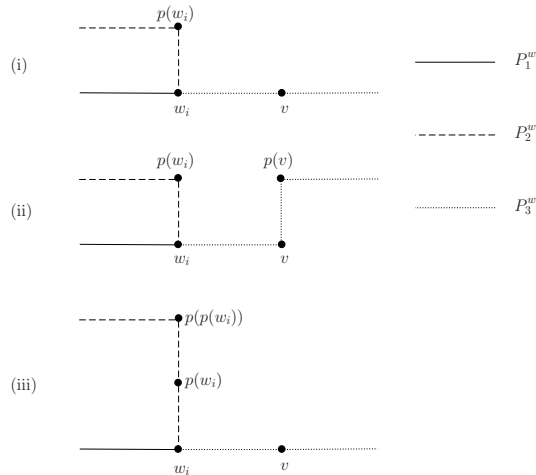


Figure 7: (3) of the Remarks.

Case 2: $\pi(w_i) = v_k$ and all the uncolored edges connect w_i to the subtree of type *II*. Then we choose one such edge w_iv such that the height of v is as small as possible.

Since T is a *BFS*-tree, we get $h(v) = h(w_i) - 1$, $h(w_i)$, or $h(w_i) + 1$. The following two facts are easy to see:

Fact 2. If $h(v) = h(w_i) - 1$, then v is already flagged.

If $v \notin A$, then v gets flagged at the very beginning; if $v \in A$, since $\pi(v) = \pi(w_i) = v_k$ and $h(v) < h(w_i)$, we have already dealt with v according to R2, thus v is flagged (note that e_v may be recolored).

Fact 3. If $h(v) = h(w_i) + 1$, then e_v is not recolored.

If $v \notin A$, then e_v never gets recolored; if $v \in A$, since $\pi(v) = \pi(w_i) = v_k$ and $h(v) > h(w_i)$, we have not dealt with v yet according to R2, thus e_v is not recolored.

We distinguish three subcases based on the height of w_i .

* *Subcase 2.1:* $h(w_i) \equiv 0 \pmod{3}$

If $h(v) = h(w_i) - 1$, by Fact 1 we know that v is already flagged. No matter whether e_v is recolored or not, we color w_iv with 5. Then $c(P_1^{w_i}) \cup c(P_2^{w_i}) \cup c(P_3^{w_i}) = \{2\} \cup \{1, 4\} \cup \{3, 5, 6\}$. Now w_i becomes safe. We flag w_i .

If $h(v) = h(w_i)$, then v may be flagged and e_v may be recolored. If e_v is not recolored ($c(e_v) = 2$), then color w_iv with 6 and recolor e_{w_i} with 5. The parent of w_i is still safe. Now $c(P_1^{w_i}) \cup c(P_2^{w_i}) \cup c(P_3^{w_i}) = \{1\} \cup \{3, 6\} \cup \{4, 5\}$. Moreover, $c(P_1^{w_i}) \cup c(P_2^{w_i}) \cup c(P_3^{w_i}) = \{5\} \cup \{1, 4\} \cup \{2, 6\}$ and $c(P_1^v) \cup c(P_2^v) \cup c(P_3^v) = \{2\} \cup \{1, 4\} \cup \{5, 6\}$, i.e. both w_i and v are safe. We flag w_i and v (if v is not flagged). If e_v is recolored ($c(e_v) = 5$), it implies $v \in A$ has been dealt with and got flagged. Then color w_iv with 6. Now $c(P_1^{w_i}) \cup c(P_2^{w_i}) \cup c(P_3^{w_i}) = \{2\} \cup \{1, 4\} \cup \{5, 6\}$, i.e. w_i becomes safe. We flag w_i .

If $h(v) = h(w_i) + 1$, by Fact 2 we know that e_v is not recolored ($c(e_v) = 3$). Then color w_iv with 6. We have $c(P_1^{w_i}) \cup c(P_2^{w_i}) \cup c(P_3^{w_i}) = \{2\} \cup \{1, 4\} \cup \{3, 6\}$ and $c(P_1^v) \cup c(P_2^v) \cup c(P_3^v) = \{3\} \cup \{2, 5\} \cup \{1, 4, 6\}$, i.e. both w_i and v are safe. We flag w_i and v (if v is not flagged).

* *Subcase 2.2:* $h(w_i) \equiv 1 \pmod{3}$

If $h(v) = h(w_i) - 1$, by Fact 1 we know that v is already flagged. No matter whether e_v is recolored or not, we color w_iv with 6. Then $c(P_1^{w_i}) \cup c(P_2^{w_i}) \cup c(P_3^{w_i}) = \{3\} \cup \{2, 5\} \cup \{1, 4, 6\}$. Now w_i becomes safe. We flag w_i .

If $h(v) = h(w_i)$, then v may be flagged and e_v may be recolored. If e_v is not recolored ($c(e_v) = 3$), then color w_iv with 4 and recolor e_{w_i} with 6. The parent of w_i is still safe. Now $c(P_1^{w_i}) \cup c(P_2^{w_i}) \cup c(P_3^{w_i}) = \{2\} \cup \{1, 4\} \cup \{5, 6\}$. Moreover, $c(P_1^{w_i}) \cup c(P_2^{w_i}) \cup c(P_3^{w_i}) = \{6\} \cup \{2, 5\} \cup \{3, 4\}$ and $c(P_1^v) \cup c(P_2^v) \cup c(P_3^v) = \{3\} \cup \{2, 5\} \cup \{4, 6\}$, i.e. both w_i and v are safe. We flag w_i and v (if v is not flagged). If e_v is recolored ($c(e_v) = 6$), it implies v is flagged. Then color w_iv with 4. Now $c(P_1^{w_i}) \cup c(P_2^{w_i}) \cup c(P_3^{w_i}) = \{3\} \cup \{2, 5\} \cup \{4, 6\}$, i.e. w_i becomes safe. We flag w_i .

If $h(v) = h(w_i) + 1$, by Fact 2 we know that e_v is not recolored ($c(e_v) = 1$). Then

color $w_i v$ with 4. We have $c(P_1^{w_i}) \cup c(P_2^{w_i}) \cup c(P_3^{w_i}) = \{3\} \cup \{2, 5\} \cup \{1, 4\}$ and $c(P_1^v) \cup c(P_2^v) \cup c(P_3^v) = \{1\} \cup \{3, 6\} \cup \{2, 4, 5\}$, i.e. both w_i and v are safe. We flag w_i and v (if v is not flagged).

* *Subcase 2.3: $h(w_i) \equiv 2 \pmod{3}$*

If $h(v) = h(w_i) - 1$, by Fact 1 we know that v is already flagged. No matter whether e_v is recolored or not, we color $w_i v$ with 4. Then $c(P_1^{w_i}) \cup c(P_2^{w_i}) \cup c(P_3^{w_i}) = \{1\} \cup \{3, 6\} \cup \{2, 4, 5\}$. Now w_i becomes safe. We flag w_i .

If $h(v) = h(w_i)$, then v may be flagged and e_v may be recolored. If e_v is not recolored ($c(e_v) = 1$), then color $w_i v$ with 5 and recolor e_{w_i} with 4. The parent of w_i is still safe. Now $c(P_1^{w_i}) \cup c(P_2^{w_i}) \cup c(P_3^{w_i}) = \{3\} \cup \{2, 5\} \cup \{1, 4, 6\}$. Moreover, $c(P_1^v) \cup c(P_2^v) \cup c(P_3^v) = \{4\} \cup \{3, 6\} \cup \{1, 5\}$ and $c(P_1^v) \cup c(P_2^v) \cup c(P_3^v) = \{1\} \cup \{3, 6\} \cup \{4, 5\}$, i.e. both w_i and v are safe. We flag w_i and v (if v is not flagged). If e_v is recolored ($c(e_v) = 4$), it implies v is flagged. Then color $w_i v$ with 5. Now $c(P_1^{w_i}) \cup c(P_2^{w_i}) \cup c(P_3^{w_i}) = \{1\} \cup \{3, 6\} \cup \{4, 5\}$, i.e. w_i becomes safe. We flag w_i .

If $h(v) = h(w_i) + 1$, by Fact 2 we know that e_v is not recolored ($c(e_v) = 2$). Then color $w_i v$ with 5. Now $c(P_1^{w_i}) \cup c(P_2^{w_i}) \cup c(P_3^{w_i}) = \{1\} \cup \{3, 6\} \cup \{2, 5\}$ and $c(P_1^v) \cup c(P_2^v) \cup c(P_3^v) = \{2\} \cup \{1, 4\} \cup \{3, 5, 6\}$, i.e. both w_i and v are safe. We flag w_i and v (if v is not flagged).

Case 3: $\pi(w_i) = v_j (1 \leq j \leq k - 1)$ and there exists at least one uncolored edge connecting w_i to some subtree of type I . Then we choose one such edge $w_i v$ such that the height of v is as small as possible. Since T is a *BFS*-tree, then $h(v) = h(w_i) - 1$, $h(w_i)$, or $h(w_i) + 1$. We have the following two facts, which are similar to *Fact 1* and *2*:

Fact 2'. If $h(v) = h(w_i) - 1$, then v is already flagged.

If $v \notin A$, then v gets flagged at the very beginning; if $v \in A$, let $\pi(v) = v_{j'}$, since $1 \leq j \leq j' \leq k - 1$ and $h(v) < h(w_i)$, we have already dealt with v according to R1 and R2, thus v is flagged (note that e_v may be recolored).

Fact 3'. If $h(v) = h(w_i) + 1$, then e_v is not recolored.

If $v \notin A$, then e_v never gets recolored; if $v \in A$, let $\pi(v) = v_{j'}$, since $1 \leq j' \leq j \leq k - 1$ and $h(v) > h(w_i)$, we have not dealt with v yet according to R1 and R2, thus e_v is not recolored.

We distinguish three subcases based on the height of w_i .

* *Subcase 3.1: $h(w_i) \equiv 0 \pmod{3}$*

If $h(v) = h(w_i) - 1$, by Fact 2' we know that v is already flagged. No matter whether e_v is recolored or not, we color $w_i v$ with 4. Then $c(P_1^{w_i}) \cup c(P_2^{w_i}) \cup c(P_3^{w_i}) = \{2\} \cup \{3, 6\} \cup \{1, 4, 5\}$. Now w_i becomes safe. We flag w_i .

If $h(v) = h(w_i)$, then v may be flagged and e_v may be recolored. If e_v is not recolored ($c(e_v) = 2$), then color $w_i v$ with 5 and recolor e_{w_i} with 4. The parent of w_i is still safe. Now $c(P_1^{w_i}) \cup c(P_2^{w_i}) \cup c(P_3^{w_i}) = \{3\} \cup \{1, 5\} \cup \{4, 6\}$. Moreover, $c(P_1^v) \cup c(P_2^v) \cup c(P_3^v) =$

$\{4\} \cup \{3, 6\} \cup \{2, 5\}$ and $c(P_1^v) \cup c(P_2^v) \cup c(P_3^v) = \{2\} \cup \{3, 6\} \cup \{4, 5\}$, i.e. both w_i and v are safe. We flag w_i and v (if v is not flagged). If e_v is recolored ($c(e_v) = 4$), it implies v is flagged. Then color w_iv with 5. Now $c(P_1^{w_i}) \cup c(P_2^{w_i}) \cup c(P_3^{w_i}) = \{2\} \cup \{3, 6\} \cup \{4, 5\}$, i.e. w_i becomes safe. We flag w_i .

If $h(v) = h(w_i) + 1$, by Fact 3' we know that e_v is not recolored ($c(e_v) = 1$). Then color w_iv with 5. Now $c(P_1^{w_i}) \cup c(P_2^{w_i}) \cup c(P_3^{w_i}) = \{2\} \cup \{3, 6\} \cup \{1, 5\}$ and $c(P_1^v) \cup c(P_2^v) \cup c(P_3^v) = \{1\} \cup \{2, 4\} \cup \{3, 5, 6\}$, i.e. both w_i and v are safe. We flag w_i and v (if v is not flagged).

* *Subcase 3.2: $h(w_i) \equiv 1 \pmod{3}$*

If $h(v) = h(w_i) - 1$, by Fact 2' we know that v is already flagged. No matter whether e_v is recolored or not, we color w_iv with 5. Then $c(P_1^{w_i}) \cup c(P_2^{w_i}) \cup c(P_3^{w_i}) = \{1\} \cup \{2, 4\} \cup \{3, 5, 6\}$. Now w_i becomes safe. We flag w_i .

If $h(v) = h(w_i)$, then v may be flagged and e_v may be recolored. If e_v is not recolored ($c(e_v) = 1$), then e_v never gets recolored in second step. For w_i not in the first level, color w_iv with 6 and recolor e_{w_i} with 5. The parent of w_i is still safe. Now $c(P_1^{p(w_i)}) \cup c(P_2^{p(w_i)}) \cup c(P_3^{p(w_i)}) = \{2\} \cup \{3, 6\} \cup \{4, 5\}$. Moreover, $c(P_1^{w_i}) \cup c(P_2^{w_i}) \cup c(P_3^{w_i}) = \{5\} \cup \{2, 4\} \cup \{1, 6\}$ and $c(P_1^v) \cup c(P_2^v) \cup c(P_3^v) = \{1\} \cup \{2, 4\} \cup \{5, 6\}$, i.e. both w_i and v are safe. We flag w_i and v (if v is not flagged). For w_i in the first level, the parent of w_i , namely v_0 , is already safe. Its three internally disjoint super-rainbow paths to D are $P_1^{v_0} = v_0, t(v_0)$, $P_2^{v_0} = v_0, v, t(v)$, $P_3^{v_0} = v_0, v_k, t(v_k)$, and $c(P_1^{v_0}) \cup c(P_2^{v_0}) \cup c(P_3^{v_0}) = \{2\} \cup \{1, 4\} \cup \{3, 5\}$ or $\{2\} \cup \{1, 4\} \cup \{5, 6\}$. Since the paths $P_1^{v_0}, P_2^{v_0}, P_3^{v_0}$ do not use e_{w_i} , we can recolor e_{w_i} with an arbitrary color from $\{1, 2, 3, 4, 5, 6\}$. In line with the previous case, we also color w_iv with 6 and recolor e_{w_i} with 5. Then again both w_i and v are safe. We flag w_i and v (if v is not flagged). If e_v is recolored ($c(e_v) = 5$), it implies v is flagged. Then color w_iv with 6. Now $c(P_1^{w_i}) \cup c(P_2^{w_i}) \cup c(P_3^{w_i}) = \{1\} \cup \{2, 4\} \cup \{5, 6\}$, i.e. w_i becomes safe. We flag w_i .

If $h(v) = h(w_i) + 1$, by Fact 3' we know that e_v is not recolored ($c(e_v) = 3$). Then color w_iv with 6. Now $c(P_1^{w_i}) \cup c(P_2^{w_i}) \cup c(P_3^{w_i}) = \{1\} \cup \{2, 4\} \cup \{3, 6\}$ and $c(P_1^v) \cup c(P_2^v) \cup c(P_3^v) = \{3\} \cup \{1, 5\} \cup \{2, 4, 6\}$, i.e. both w_i and v are safe. We flag w_i and v (if v is not flagged).

* *Subcase 3.3: $h(w_i) \equiv 2 \pmod{3}$*

If $h(v) = h(w_i) - 1$, by Fact 2' we know that v is already flagged. No matter whether e_v is recolored or not, we color w_iv with 6. Then $c(P_1^{w_i}) \cup c(P_2^{w_i}) \cup c(P_3^{w_i}) = \{3\} \cup \{1, 5\} \cup \{2, 4, 6\}$. Now w_i becomes safe. We flag w_i .

If $h(v) = h(w_i)$, then v may be flagged and e_v may be recolored. If e_v is not recolored ($c(e_v) = 3$), then color w_iv with 4 and recolor e_{w_i} with 6. The parent of w_i is still safe. Now $c(P_1^{p(w_i)}) \cup c(P_2^{p(w_i)}) \cup c(P_3^{p(w_i)}) = \{1\} \cup \{2, 4\} \cup \{5, 6\}$. Moreover, $c(P_1^{w_i}) \cup c(P_2^{w_i}) \cup c(P_3^{w_i}) = \{6\} \cup \{1, 5\} \cup \{3, 4\}$ and $c(P_1^v) \cup c(P_2^v) \cup c(P_3^v) = \{3\} \cup \{1, 5\} \cup \{4, 6\}$, i.e. both w_i and v are safe. We flag w_i and v (if v is not flagged). If e_v is recolored ($c(e_v) = 6$), it implies v is flagged. Then color w_iv with 4. Now $c(P_1^{w_i}) \cup c(P_2^{w_i}) \cup c(P_3^{w_i}) = \{3\} \cup \{1, 5\} \cup \{4, 6\}$, i.e. w_i becomes safe. We flag w_i .

If $h(v) = h(w_i) + 1$, by Fact 3' we know that e_v is not recolored ($c(e_v) = 2$). Then color w_iv with 4. Now $c(P_1^{w_i}) \cup c(P_2^{w_i}) \cup c(P_3^{w_i}) = \{3\} \cup \{1, 5\} \cup \{2, 4\}$ and $c(P_1^v) \cup c(P_2^v) \cup c(P_3^v) = \{2\} \cup \{3, 6\} \cup \{1, 4, 5\}$, i.e. both w_i and v are safe. We flag w_i and v (if v is not flagged).

Case 4: $\pi(w_i) = v_j (1 \leq j \leq k - 1)$ and all the uncolored edges connect w_i to the subtree of type II. Then we choose one such edge w_iv such that the height of v is as small as possible. Since T is a BFS-tree and the subtree of w_i is to the left of the subtree of v , then $h(v) = h(w_i) - 1$ or $h(w_i)$. We have the following fact:

Fact 4. v is already flagged.

If $v \notin A$, then v gets flagged at the very beginning; if $v \in A$, since $\pi(v) = v_k$ and $\pi(w_i) = v_j (1 \leq j \leq k - 1)$, we have already dealt with v according to R1, thus v is flagged (note that e_v may be recolored).

We distinguish three subcases based on the height of w_i .

* *Subcase 4.1:* $h(w_i) \equiv 0 \pmod{3}$

If $h(v) = h(w_i) - 1$, by Fact 4 we know that v is already flagged. If e_v is not recolored ($c(e_v) = 1$), then color w_iv with 5. We have $c(P_1^{w_i}) \cup c(P_2^{w_i}) \cup c(P_3^{w_i}) = \{2\} \cup \{3, 6\} \cup \{1, 5\}$. If e_v is recolored ($c(e_v) = 4$), then color w_iv with 5. We have $c(P_1^{w_i}) \cup c(P_2^{w_i}) \cup c(P_3^{w_i}) = \{2\} \cup \{3, 6\} \cup \{4, 5\}$. Now w_i becomes safe. We flag w_i .

If $h(v) = h(w_i)$, by Fact 4 we know that v is already flagged. If e_v is not recolored ($c(e_v) = 2$), then color w_iv with 5. We have $c(P_1^{w_i}) \cup c(P_2^{w_i}) \cup c(P_3^{w_i}) = \{2\} \cup \{3, 6\} \cup \{1, 4, 5\}$. If e_v is recolored ($c(e_v) = 5$), then color w_iv with 4. We have $c(P_1^{w_i}) \cup c(P_2^{w_i}) \cup c(P_3^{w_i}) = \{2\} \cup \{3, 6\} \cup \{4, 5\}$. Now w_i becomes safe. We flag w_i .

* *Subcase 4.2:* $h(w_i) \equiv 1 \pmod{3}$

If $h(v) = h(w_i) - 1$, by Fact 4 we know that v is already flagged. If e_v is not recolored ($c(e_v) = 2$), then color w_iv with 5. We have $c(P_1^{w_i}) \cup c(P_2^{w_i}) \cup c(P_3^{w_i}) = \{1\} \cup \{3, 4, 6\} \cup \{2, 5\}$. If e_v is recolored ($c(e_v) = 5$), then color w_iv with 6. We have $c(P_1^{w_i}) \cup c(P_2^{w_i}) \cup c(P_3^{w_i}) = \{1\} \cup \{2, 4\} \cup \{5, 6\}$. Now w_i becomes safe. We flag w_i .

If $h(v) = h(w_i)$, by Fact 4 we know that v is already flagged. If e_v is not recolored ($c(e_v) = 3$), then color w_iv with 6. We have $c(P_1^{w_i}) \cup c(P_2^{w_i}) \cup c(P_3^{w_i}) = \{1\} \cup \{2, 4\} \cup \{3, 6\}$. If e_v is recolored ($c(e_v) = 6$), then color w_iv with 3. We have $c(P_1^{w_i}) \cup c(P_2^{w_i}) \cup c(P_3^{w_i}) = \{1\} \cup \{2, 4\} \cup \{3, 6\}$. Now w_i becomes safe. We flag w_i .

* *Subcase 4.3:* $h(w_i) \equiv 2 \pmod{3}$

If $h(v) = h(w_i) - 1$, by Fact 4 we know that v is already flagged. If e_v is not recolored ($c(e_v) = 3$), then color w_iv with 4 and recolor e_{w_i} with 6. The parent of w_i is still safe. Now $c(P_1^{w_i}) \cup c(P_2^{w_i}) \cup c(P_3^{w_i}) = \{1\} \cup \{2, 4\} \cup \{5, 6\}$. Moreover, $c(P_1^{w_i}) \cup c(P_2^{w_i}) \cup c(P_3^{w_i}) = \{6\} \cup \{1, 5\} \cup \{3, 4\}$, i.e. w_i is safe. We flag w_i . If e_v is recolored ($c(e_v) = 6$), then color w_iv with 4. Now $c(P_1^{w_i}) \cup c(P_2^{w_i}) \cup c(P_3^{w_i}) = \{3\} \cup \{1, 5\} \cup \{4, 6\}$, i.e. w_i becomes safe. We flag w_i .

If $h(v) = h(w_i)$, by Fact 4 we know that v is already flagged. If e_v is not recolored ($c(e_v) = 1$), then color $w_i v$ with 3 and recolor e_{w_i} with 6. The parent of w_i is still safe. Now $c(P_1^{P(w_i)}) \cup c(P_2^{P(w_i)}) \cup c(P_3^{P(w_i)}) = \{1\} \cup \{2, 4\} \cup \{5, 6\}$. Moreover, $c(P_1^{w_i}) \cup c(P_2^{w_i}) \cup c(P_3^{w_i}) = \{6\} \cup \{2, 4, 5\} \cup \{1, 3\}$, i.e. w_i is safe. We flag w_i . If e_v is recolored ($c(e_v) = 4$), then color $w_i v$ with 6. Now $c(P_1^{w_i}) \cup c(P_2^{w_i}) \cup c(P_3^{w_i}) = \{3\} \cup \{1, 5\} \cup \{4, 6\}$, i.e. w_i becomes safe. We flag w_i .

Then we go to w_{i+1} and repeat the process, until all the vertices in A are visited. We do the same operation to all C_i 's. If there still exist uncolored edges in $E[D, \overline{D}] \cup E(G[\overline{D}])$, color them with 1. Now we have a coloring of all the edges in $E[D, \overline{D}] \cup E(G[\overline{D}])$ using six different colors from $\{1, 2, 3, 4, 5, 6\}$ such that all the vertices in \overline{D} are safe.

4.2 Color the edges in $E(G[D])$

Let $d := rx_3(G[D])$. Then we can color the edges in $G[D]$ with d fresh colors from $\{7, 8, \dots, d+6\}$ such that for each triple of vertices in D , there exists a rainbow tree in $G[D]$ connecting them. Hereto we obtain an edge-coloring $c : E(G) \rightarrow \{1, 2, \dots, d+6\}$.

4.3 Prove c is a 3-rainbow coloring

Next we will prove that this edge-coloring of G is a 3-rainbow coloring, which yields that $rx_3(G) \leq rx_3(G[D]) + 6$.

Claim 1. *Under this coloring, for any three vertices u, v, w in \overline{D} , there exists a rainbow $u - D$ path P^u , a rainbow $v - D$ path P^v and a rainbow $w - D$ path P^w such that $P^u \cup P^v \cup P^w$ is also rainbow.*

Before giving the proof of Claim 1, let us show how it implies our result. Let $S = \{u, v, w\} \subseteq V(G)$. If $|S \cap D| = 3$, i.e. $(u, v, w) \in D \times D \times D$, then there is already a rainbow S -tree in $G[D]$. If $|S \cap D| = 2$, say $(u, v, w) \in D \times D \times \overline{D}$, then let w' be the foot of w . The rainbow tree in $G[D]$ connecting u, v, w' together with the edge ww' forms a rainbow S -tree. If $|S \cap D| = 1$, say $(u, v, w) \in D \times \overline{D} \times \overline{D}$, then by Claim 1, there exists a rainbow $v - D$ path P^v and a rainbow $w - D$ path P^w such that $P^v \cup P^w$ is also rainbow. Assume the endvertex of P^v, P^w in D is v', w' respectively. Then the rainbow tree in $G[D]$ connecting u, v', w' together with the paths P^v and P^w forms a connected rainbow subgraph of G , denoted by H . Obviously, a spanning tree of H is a rainbow S -tree. If $|S \cap D| = 0$, i.e. $(u, v, w) \in \overline{D} \times \overline{D} \times \overline{D}$, then by Claim 1, there exists a rainbow $u - D$ path P^u , a rainbow $v - D$ path P^v and a rainbow $w - D$ path P^w such that $P^u \cup P^v \cup P^w$ is also rainbow. Assume the endvertex of P^u, P^v, P^w in D is u', v', w' respectively. Then the rainbow tree in $G[D]$ connecting u', v', w' together with the paths P^u, P^v and P^w forms a connected rainbow subgraph of G , denoted by H' . Obviously, a spanning tree

of H' is a rainbow S -tree. So we come to the conclusion that the edge-coloring c is a 3-rainbow coloring.

Proof of Claim 1: For any three vertices u, v, w in \overline{D} , u, v, w are safe under this coloring. That is, there exist three internally-disjoint super-rainbow $u - D$ paths P_1^u, P_2^u, P_3^u , three internally-disjoint super-rainbow $v - D$ paths P_1^v, P_2^v, P_3^v and three internally-disjoint super-rainbow $w - D$ paths P_1^w, P_2^w, P_3^w . If we can pick out P_i^u, P_j^v and P_k^w ($1 \leq i, j, k \leq 3$) from these paths satisfying $P_i^u \cup P_j^v \cup P_k^w$ is also rainbow, we are done. But unfortunately in some cases, we can not do that. For example, if $c(P_1^u) \cup c(P_2^v) \cup c(P_3^w) = \{1\} \cup \{2, 4\} \cup \{5, 6\}$, $c(P_1^v) \cup c(P_2^w) \cup c(P_3^u) = \{1\} \cup \{2, 5\} \cup \{4, 6\}$, $c(P_1^w) \cup c(P_2^u) \cup c(P_3^v) = \{1\} \cup \{2, 6\} \cup \{4, 5\}$, then one can check that $P_i^u \cup P_j^v \cup P_k^w$ is not rainbow for each $1 \leq i, j, k \leq 3$. Here we will show a sufficient and necessary condition for the situation in which we can pick out suitable P_i^u, P_j^v and P_k^w . (Note that P_1^u, P_1^v, P_1^w contains exactly one edge.)

There exist $i, j, k \in \{1, 2, 3\}$ satisfying $P_i^u \cup P_j^v \cup P_k^w$ is rainbow if and only if

(C1) $c(P_1^u), c(P_1^v), c(P_1^w)$ are not the same or

(C2) there exist two distinct vertices $x, y \in \{u, v, w\}$ and two integers $s, t \in \{2, 3\}$ (s may equal to t) such that $c(P_s^x) \cap c(P_t^y) = \emptyset$.

If (C1) is true, without loss of generality, we assume $c(P_1^u) = 1$ and $c(P_1^v) = 2$. If $c(P_1^w) \in \{3, 4, 5, 6\}$, then $P_1^u \cup P_1^v \cup P_1^w$ is rainbow; If $c(P_1^w) \in \{1, 2\}$, without loss of generality let $c(P_1^w) = 1$. Since $P_1^w \cup P_2^w \cup P_3^w$ is rainbow, both P_2^w and P_3^w contain no edges colored by 1, and at least one of P_2^w and P_3^w contains no edges colored by 2, say P_2^w . Then $P_1^u \cup P_1^v \cup P_2^w$ is rainbow. If (C2) is true, without loss of generality, we assume that $c(P_2^u) \cap c(P_2^v) = \emptyset$. If $c(P_1^u), c(P_1^v), c(P_1^w)$ are not the same, then the assertion holds by (C1); otherwise, without loss of generality let $c(P_1^u) = c(P_1^v) = c(P_1^w) = 1$. Then P_2^u and P_2^v contain no edges colored by 1. Bearing in mind that $c(P_2^u) \cap c(P_2^v) = \emptyset$, we get that $P_1^w \cup P_2^u \cup P_2^v$ is rainbow. For the other direction, assume that (C1) is not true, we will show (C2) holds by contradiction. Suppose that $c(P_1^u) = c(P_1^v) = c(P_1^w)$, and for any two distinct vertices $x, y \in \{u, v, w\}$ and any two integers $s, t \in \{2, 3\}$, $c(P_s^x) \cap c(P_t^y) \neq \emptyset$. Since $P_i^u \cup P_j^v \cup P_k^w$ is rainbow, we know that at most one of i, j, k is equal to 1, say $j, k \in \{2, 3\}$. Then by hypothesis, $c(P_j^v) \cap c(P_k^w) \neq \emptyset$, a contradiction to the fact that $P_i^u \cup P_j^v \cup P_k^w$ is rainbow.

From the above assertion, we can see that the colors of the three internally disjoint super-rainbow paths connecting a vertex in \overline{D} to D plays a crucial role. Here we list out all the possible color sets of these paths under this coloring. For the sake of brevity, we write $\{1, 24, 35\}$ instead of $c(P_1^u) \cup c(P_2^v) \cup c(P_3^w) = \{1\} \cup \{2, 4\} \cup \{3, 5\}$ and $c(P_1^v) \cup c(P_2^u) \cup c(P_3^w) = \{1\} \cup \{3, 5\} \cup \{2, 4\}$.

Class 0: $\{1, 2, 3\}, \{1, 2, 34\}, \{1, 2, 36\}, \{2, 3, 14\}, \{2, 3, 15\},$
 $\{1, 3, 24\}, \{1, 3, 25\}$

Class 1: $\{1, 24, 35\}$, $\{1, 36, 24\}$, $\{1, 36, 25\}$, $\{1, 24, 56\}$, $\{1, 36, 45\}$,
 $\{1, 36, 245\}$, $\{1, 24, 356\}$, $\{1, 346, 25\}$.

Class 2: $\{2, 36, 14\}$, $\{2, 14, 35\}$, $\{2, 14, 56\}$, $\{2, 36, 15\}$, $\{2, 36, 45\}$,
 $\{2, 46, 35\}$, $\{2, 36, 145\}$, $\{2, 14, 356\}$, $\{2, 346, 15\}$.

Class 3: $\{3, 15, 26\}$, $\{3, 25, 16\}$, $\{3, 15, 46\}$, $\{3, 25, 46\}$, $\{3, 15, 24\}$,
 $\{3, 25, 14\}$, $\{3, 25, 146\}$, $\{3, 15, 246\}$.

Class 4: $\{4, 36, 15\}$, $\{4, 36, 25\}$, $\{4, 36, 125\}$.

Class 5: $\{5, 14, 26\}$, $\{5, 24, 16\}$.

Class 6: $\{6, 25, 34\}$, $\{6, 15, 34\}$, $\{6, 15, 24\}$, $\{6, 245, 13\}$.

For every triple $\{u, v, w\}$ of vertices in \overline{D} , if $c(P_1^u)$, $c(P_1^v)$ and $c(P_1^w)$ are not the same, we are done. Now suppose $c(P_1^u) = c(P_1^v) = c(P_1^w)$. If there exists one vertex satisfying at least two of its three paths are of length 1, without loss of generality, we assume that $c(P_1^u) = c(P_1^v) = c(P_1^w) = 1$, P_2^u is of length 1, and $c(P_2^u) = 2$. Since $P_1^u \cup P_2^u \cup P_3^u$ is rainbow, we can find out one path, say P_2^u , which contains no edges colored by 1 or 2. Then $P_2^u \cup P_2^v \cup P_1^w$ is rainbow. Again we are done. Thus to prove *Claim 1*, it suffices to check whether (C2) holds for every three color sets in Class i ($1 \leq i \leq 6$). Since the number of color sets in one class is no more than 9, the checking work can be done in a short time and the answer in turn is affirmative. We complete the proof of *Claim 1*.

To end the section, we illustrate the tightness of the bound $rx_3(G) \leq rx_3(G[D]) + 6$ with the graph in *Figure 5*. It is easy to see that $D = \{v_0\}$ is a connected three-way dominating set. By Theorem 3, $rx_3(G) \leq rx_3(G[D]) + 6 = 6$. On the other hand, we have already proved that $rx_3(G) = 6$. So the bound is tight.

5 Concluding remarks

To sum up, as for the 3-rainbow index of a graph, we can consider the following three strengthened connected dominating sets:

Let G be a connected graph and D be a connected dominating set of G .

(a) if every vertex in \overline{D} is adjacent to at least three distinct vertices of D , then $rx_3(G) \leq rx_3(G[D]) + 3$ (Theorem 9);

(b) if every vertex in \overline{D} is of degree at least three and adjacent to at least two distinct vertices of D , then $rx_3(G) \leq rx_3(G[D]) + 4$ (Theorem 2);

(c) if every vertex in \overline{D} is of degree at least three, then $rx_3(G) \leq rx_3(G[D]) + 6$ (Theorem 3).

From (a) to (c), we loosen the restrictions on the connected dominating sets, while the additive constant increases. We cannot tell which bound is the best. For example,

for a French Windmill in Figure 5, (c) is better than (a) and (b), whereas for a threshold graph with $\delta \geq 3$, (a) and (b) which imply $rx_3(G) \leq 5$ are better than (c) which implies $rx_3(G) \leq 6$. Given a connected graph G , we can calculate three upper bounds for the 3-rainbow index of G using (a), (b), (c) respectively (some of them may be the same), and then choose the smallest one of them.

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