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Abstract

For non-atomic TU games ν satisfying suitable conditions, the core can be determined by computing appropriate derivatives of ν. Further, such computations yield one of two stark conclusions: either $core(\nu)$ is empty or it consists of a single measure that can be expressed explicitly in terms of derivatives of ν. In this sense, core theory for a class of games may be reduced to calculus.

1. INTRODUCTION

1.1. Outline

We show that for large (non-atomic) TU games satisfying suitable conditions, the core can be determined by computing appropriate derivatives of the characteristic function ν. Further, such computations yield one of two stark conclusions: either the core of ν is empty or it consists of a single measure. In the latter case, the core measure is expressed explicitly in terms of derivatives of ν. In this sense, core theory for a class of non-atomic games may be reduced to calculus.

There is simple intuition for a connection between $core(\nu)$, assumed for the moment to be nonempty, and the 'derivative' of ν. Denote by Σ the σ-algebra of

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feasible coalitions; thus $\nu : \Sigma \longrightarrow \mathbb{R}^1$. Let m be a measure in the core, so that

$$m(E) \geq \nu(E) \quad \text{for all coalitions } E,$$

and suppose that there exists a coalition A such that

$$m(A) = \nu(A). \tag{1.1}$$

Then

$$\min_{E \in \Sigma} \left(m(E) - \nu(E) \right) = 0 = m(A) - \nu(A). \tag{1.2}$$

In particular, A is a minimizer for $m(\cdot) - \nu(\cdot)$ over Σ. Analogy with calculus suggests that, if A is suitably 'interior', then the 'derivative' of the noted function should vanish at A. We propose a definition of derivative (or more accurately, differential) for characteristic functions ν. It satisfies the following natural property that might be expected of any notion of differentiation: There is a parallel between standard calculus (for real-valued functions defined on a Euclidean space) and the calculus proposed here (for real-valued functions defined on a σ-algebra of coalitions), in which additive set functions (measures) play the role of linear functions on a Euclidean space. Given such an analogy, then, because m is additive, it equals its differential. Consequently, if we denote the differential of ν at A by $\delta\nu(\cdot; A)$, then the first-order condition corresponding to (1.2) takes the form

$$m(\cdot) = \delta\nu(\cdot; A). \tag{1.3}$$

Finally, suppose that A satisfies (1.1) not only for a particular m in the core, but for *all* measures in the core. Conclude that the core of ν is $\{\delta\nu(\cdot; A)\}$ and we have a singleton core with an explicit representation in terms of ν.

There are two obvious questions regarding this informal argument. First, how restrictive is the assumption that there exists an 'interior' A that satisfies (1.1) for all m in the core? It turns out that this assumption is satisfied by many games of interest, for example in homogeneous measure games, including, in particular, market games.

The second and more difficult question is whether the first-order condition (1.3) can be justified. After all, the fact that the usual first-order condition takes the form of an equality relies heavily on the linear structure of Euclidean space. For example, using the obvious notation, if $\nabla f(x^*) \cdot y > 0$, then one can move from x^* in the *reverse* direction $-y$ and reduce thereby the value of f. By this reasoning, the stated inequality is ruled out if f has a minimum at x^*. However, in the present context where the domain is a σ-algebra, there is no counterpart of a 'reverse direction' and first-order conditions take the form of inequalities. Much of the work we do below is to identify added assumptions on games that lead to (1.3). Primarily, we define a class of so-called *coherent games*, for which the informal outline above can be made rigorous.

As a result, the core of any differentiable coherent game can be determined simply by computing the derivative of its characteristic function. Moreover, the latter task is arguably as routine as differentiating functions defined on a Euclidean space. That is because our derivative notion satisfies counterparts of the rules familar from calculus (the product and chain rules, for example). This is the sense in which core theory for differentiable coherent games is reduced to 'just calculus'.

The introduction of a new calculus into co-operative game theory is our main contribution; we hope that it will prove to be a fruitful tool in the study of non-atomic TU games. We demonstrate its usefulness here by means of the core theorem for abstract (coherent) games that was outlined above and also by means of applications of this general theorem to the more concrete class of measure games. In addition, we provide a new derivation of known results on market games and exchange economies (uniqueness of the core allocation in suitable exchange economies). This new derivation seems to us to be of value because of the radically different and simplifying perspective that it offers relative to the derivations in [2] and also [3].

1.2. Related Literature

We are not the first to adopt calculus techniques in order to study non-atomic TU games. Aumann and Shapley [2] develop a notion of derivative for non-additive set functions and apply it to study both the Shapley value and the core of non-atomic games. The primary intuition underlying their use of differentiability is that the Shapley value of each infinitesimal player is her *marginal contribution* to the worth of a representative coalition averaged over all such coalitions (see [2, (20.1)]). Because 'marginal contribution' is most naturally expressed in terms of derivatives, a calculus approach is intuitive. Note that this intuition differs substantially from that underlying our analysis; in particular, it focuses on the Shapley value rather than the core.

One consequence of this difference in motivation is that Aumann and Shapley take a *class* of games as the primary object of study, while we focus on *individual* games. The first approach is natural for study of the Shapley value which is defined by its properties over a class; but not so for study of the core, where it is desirable to have an approach that is applicable to any given single game. Our notion of derivative is such a game-by-game tool.

There are other (related) differences. Roughly, the Aumann-Shapley definition begins with the subspace pNA of games (the closed subspace of BV spanned by all powers of positive nonatomic measures measures). The authors show (Theorem G) that each game or non-additive set function ν in pNA admits a suitable extension to an 'integral' ν^*. Because an integral can be viewed as a functional on the linear space of integrands (real-valued random variables), a Gateaux-like derivative notion can be defined in the usual way and this is adequate for characterization of the Shapley value (Theorem H). However, in order to check whether a given game ν is differentiable and then to compute its derivative, one must determine whether ν lies in pNA and if so, determine the corresponding integral ν^*. Neither of these steps is routine in general. The above analysis admits generalizations - for example, domains larger than pNA can be accommodated (Section 22 and [7], for example) and the Gâteaux derivative can be related to a Fréchet-type notion (Section 24) - but the above noted difficulties persist.

Even more important than tractability is that at the level of abstract games, the Aumann-Shapley approach leads to the formulation of assumptions about games through restrictions on their extensions. However, it is often difficult to understand the meaning of such restrictions in terms of the assumptions they embody about the underlying game.[1] In contrast, we define derivative *explicitly* in terms of the given ν. Then we formulate coherence and other assumptions about games in terms of these derivatives. This approach permits interpretation much as restrictions on derivatives in ordinary calculus are often readily interpreted.

Other related literature includes work by Rosenmuller [11], who defines differentiability for large games. Our derivative notion is inspired by his, particularly by his use of partitions, but the two notions differ. Though Rosenmuller is also concerned with the core of TU games (albeit only convex games), neither our results nor the underlying intuition are apparent in [11].

The notion of derivative that we use here originates in [4] in a decision theory context, where instead of being a game, the set function ν is a 'non-additive probability' as in [13]. A related notion is developed and applied to decision theory in [6].

[1] Examples of such assumptions include homogeneity ($\nu^*(\alpha 1_E) = \alpha \nu^*(1_E)$) [2, p. 167] and concavity of $\nu^*(\cdot)$ [3].

2. PRELIMINARIES

The set of players is Ω and the σ-algebra Σ denotes the set of admissible coalitions. Subsets of Ω are understood to be in Σ even where not stated explicitly and they are referred to both as sets and as coalitions.

A set function $\nu : \Sigma \to \mathbb{R}$ is a *game* if $\nu(\varnothing) = 0$. A game ν is

positive if $\nu(A) \geq 0$ for all A,

monotone if $\nu(A) \geq \nu(B)$ whenever $B \subset A$,

superadditive if $\nu(A \cup B) \geq \nu(A) + \nu(B)$ for all pairwise disjoint sets A and B,

continuous at A if $\lim_{n\to\infty} \nu(A_n) = \nu(A)$ whenever $A_n \uparrow A$,

continuous if it is continuous at every A,

additive (or a charge) if $\nu(A \cup B) = \nu(A) + \nu(B)$ for all pairwise disjoint sets A and B,

countably additive if $\nu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \nu(A_i)$ for all countable collections of pairwise disjoint sets $\{A_i\}_{i=1}^{\infty}$.

The set of all additive (countably additive) games that are bounded with respect to the variation norm is denoted FA (CA). An additive game m is *convex-ranged* if for all $\alpha \in (0,1)$ and all $A \in \Sigma$ with $|m|(A) > 0$, there exists $B \subset A$ such that $|m|(B) = \alpha\,|m|(A)$. Denote by NA the set of all additive convex-ranged games. If $m = (m_1, ..., m_N)$, where each m_i is an additive convex-ranged game, then by a version of the Lyapunov Theorem [9, Theorem 11.4.9], the range $R(m) = \{m(A) : A \in \Sigma\}$ is a convex subset of $\mathbb{R}^N$. Throughout the paper when we refer to 'the Lyapunov Theorem', the intention is to this version.

The *core* of $\nu \in V$ is

$$core(\nu) = \{m \in FA : m(\Omega) = \nu(\Omega) \text{ and } m(A) \geq \nu(A) \text{ for all } A \in \Sigma\}.$$

We fix also some terminology and notation for functions defined on a Euclidean space. Let U be an open subset of $\mathbb{R}^N$. A function $g : U \to \mathbb{R}$ is differentiable at $x \in U$ if there is a linear map $Dg(x) : \mathbb{R}^N \to \mathbb{R}$ such that

$$\lim_{h\to 0} \frac{|g(x+h) - g(x) - Dg(x)(h)|}{|h|} = 0,$$

where $|\cdot| : \mathbb{R}^N \to \mathbb{R}$ is any norm on $\mathbb{R}^N$. Call $Dg(x)$ the *derivative* of g at x. If $Dg(x)$ exists for all $x \in U$, say that g is differentiable on U. Since $Dg(x)(\cdot)$ is linear, $Dg(x)(h) = \nabla g(x) \cdot h$ for all $x \in U$, where $\nabla g(x) \in \mathbb{R}^N$ is the gradient of g at x. If A is an arbitrary subset of $\mathbb{R}^N$, not necessarily open, say that the function $g : A \to \mathbb{R}$ is differentiable on A if it can be extended to a differentiable function on some open set U containing A.

3. DIFFERENTIABLE GAMES

For any $A \in \Sigma$, let $\left\{A^{j,\lambda}\right\}_{j=1}^{n_\lambda}$ be a finite partition of A. Denote by $\left\{A^{j,\lambda}\right\}_\lambda$ the net of all finite partitions of A, where $\lambda' > \lambda$ implies that the partition corresponding to λ' refines that corresponding to λ.

Definition 3.1. *A game $\nu : \Sigma \to \mathbb{R}$ is differentiable at $E \in \Sigma$ if there exists a bounded and convex-ranged measure $\delta\nu(\cdot; E)$ on Σ such that*

$$\sum_{j=1}^{n_\lambda} \left|\nu\left(E \cup F^{j,\lambda} - G^{j,\lambda}\right) - \nu(E) - \delta\nu\left(F^{j,\lambda}; E\right) + \delta\nu\left(G^{j,\lambda}; E\right)\right| \underset{\lambda}{\longrightarrow} 0, \tag{3.1}$$

for all $F \subset E^c$ and $G \subset E$.[2]

For all the results to follow, we could adopt a weaker 'one-sided' definition of derivative. Define an *outer derivative* at E, denoted $\delta^+\nu(\cdot; E)$, by

$$\sum_{j=1}^{n_\lambda} \left|\nu\left(E \cup F^{j,\lambda}\right) - \nu(E) - \delta^+\nu\left(F^{j,\lambda}; E\right)\right| \underset{\lambda}{\longrightarrow} 0 \tag{3.2}$$

and an *inner derivative* at E, denoted $\delta^-\nu(\cdot; E)$, by

$$\sum_{j=1}^{n_\lambda} \left|\nu\left(E - G^{j,\lambda}\right) - \nu(E) + \delta^-\nu\left(G^{j,\lambda}; E\right)\right| \underset{\lambda}{\longrightarrow} 0, \tag{3.3}$$

[2] $A - B$ denotes $A \cap B^c = \{\omega \in \Omega : \omega \in A, \omega \notin B\}$.

Because a difference quotient is not apparent in the defining condition, it may be comforting to make the following observation: For a function $\varphi : \mathcal{R}^1 \longrightarrow \mathcal{R}^1$ that is differentiable at some x in the usual sense, elementary algebraic manipulation of the definition of the derivative $\varphi'(x)$ yields the following expression paralleling (3.1):
$\Sigma_{i=1}^N \mid \varphi(x + N^{-1}) - \varphi(x) - N^{-1}\varphi'(x) \mid \longrightarrow 0$ as $N \longrightarrow \infty$.

where the convergence is required to hold for each F and G as above. Given the existence of $\delta^{+}\nu\,(\cdot;E)$ and $\delta^{-}\nu\,(\cdot;E)$, one could define $\delta\nu\,(\cdot;E)$ as the sum $\delta^{+}\nu\,(\cdot;E)+\delta^{-}\nu\,(\cdot;E)$, in which case (3.1) is satisfied when E is perturbed by either F or G, but not necessarily when perturbed by both simultaneously.[3]

Turn to interpretation. Roughly, $\delta\nu\,(\cdot;E)$ approximates the marginal value of coalitions relative to the base coalition E, in such a way that the approximation is additive (across disjoint incremental coalitions) and becomes exact in the limit for 'small' coalitions.[4] To elaborate, for F disjoint from E, approximate the incremental value of F as follows: Partition F into arbitrarily small subcoalitions $F^{j,\lambda}$ and compute the marginal value $\nu(E\cup F^{j,\lambda})\,-\,\nu(E)$ of each $F^{j,\lambda}$ relative to the base E. Then the sum of these marginal values equals $\delta\nu\,(F;E)$, which therefore represents the *total marginal value* of F relative to the base coalition E. A similar interpretation applies when $G\subset E$, in which case $\delta\nu\,(G;E)$ is the sum of the marginal values $\nu(E)\,-\,\nu(E-G^{j,\lambda})$ for an arbitrarily fine partition $\{G^{j,\lambda}\}$ of G. Finally, any coalition A can be written uniquely in the form $A\,=\,E\cup F-G$, where $F\subset E^{c}$ and $G\subset E$, so that

$$\delta\nu\,(A;E)\;=\;\delta\nu\,(F;E)\;+\;\delta\nu\,(G;E)\,,$$

which provides the sense in which $\delta\nu\,(A;E)$ represents the total marginal value of A relative to the base coalition E.[5]

A crucial feature of our notion of differentiability for games is that it satisfies many of the properties familiar from calculus, including a form of the Chain Rule as well as 'sum' and 'product' rules (see Appendix A).[6] Bounded and convex-ranged measures play the role of linear functions in calculus; in particular, if m is such a measure, then it is differentiable at any E and $\delta m\,(\cdot;E)\;=\;m\,(\cdot)$. Modulo this translation, formulae familiar from calculus are valid also for games. For example, if $\nu(\cdot)$ equals the product $p(\cdot)\,q(\cdot)$ of two convex-ranged measures, then

$$\delta\nu\,(\cdot;E)\;=\;p(E)\,q(\cdot)\;+\;q(E)\,p(\cdot),$$

for all E. More general formulae arise in the context of measure games.[7]

[3] In the context of measure games $g(P)$, this would permit weakening our assumptions on g to require only that it have one-sided derivatives.

[4] 'Smallness' is measured via fineness of partitions, which relies on the richness of Σ; indeed, if Σ is finite, then only additive games are differentiable.

[5] The counterparts for finite games are $\Sigma_{i\in F}\,(\nu(E\cup\{i\})-\nu(E))$ and $\Sigma_{i\in G}\,(\nu(E)-\nu(E-\{i\}))$ for $\delta\nu\,(F;E)$ and $\delta\nu\,(G;E)$ respectively.

[6] The appendix also contains some information about the connection with the Aumann-Shapley style derivative.

[7] See Appendix A.3 for an example of a differentiable game that is not a measure game.

3.1. Measure Games

The game $\nu : \Sigma \to \mathbb{R}$ is a *measure game* if there exist a vector charge (that is, a finitely additive vector measure) $P = (P_1, ..., P_N) : \Sigma \to \mathbb{R}^N_+$, with $P(\Omega) \neq 0$ and with each $P_i : \Sigma \to \mathbb{R}_+$ bounded and convex-ranged, and a function $g : R(P) \to \mathbb{R}$ such that

$$\nu(\cdot) = g(P(\cdot)) \quad \text{on } \Sigma.$$

When $N = 1$, $\nu = g(P(\cdot))$ is called a *scalar measure game.*

Lemma 3.2. *Let $\nu = g(P) : \Sigma \to \mathbb{R}$ be a measure game. If g is differentiable at $P(E) \in \mathbb{R}^N_+$, then ν is differentiable at E and*

$$\delta\nu(\cdot; E) = \nabla g(P(E)) \cdot P(\cdot) = \sum_{i=1}^{N} g_i(P(E)) P_i(\cdot).$$

Though the proof is routine, we provide it here because it may help clarify for the reader the definition of differentiability and convince her of our claim that the calculus of games is analogous to the calculus of functions on Euclidean space.

Proof. For $x \in \mathbb{R}^N$, let $|x| = \max_{1\leq i\leq N} |x_i|$. Given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\frac{|g(P(E) + h) - g(P(E)) - \nabla g(P(E)) \cdot h|}{|h|} \leq \varepsilon \tag{3.4}$$

for all $|h| \leq \delta$.

Let $F \subset E^c$ and $G \subset E$. Since each P_i is convex-ranged on the σ-algebra Σ, by the Lyapunov Theorem there exist finite partitions $\{F^{j,\lambda_0}\}$ and $\{G^{j,\lambda_0}\}$ of F and G such that $|P(F^{j,\lambda_0})| + |P(G^{j,\lambda_0})| \leq \delta$. Hence, for all $\lambda > \lambda_0$,

$$\begin{aligned} \left|P\left(F^{j,\lambda}\right) - P\left(G^{j,\lambda}\right)\right| &\leq \left|P\left(F^{j,\lambda}\right)\right| + \left|P\left(G^{j,\lambda}\right)\right| \\ &\leq \left|P\left(F^{j,\lambda_0}\right)\right| + \left|P\left(G^{j,\lambda_0}\right)\right| \leq \delta. \end{aligned}$$

For convenience, set $\alpha^{j,\lambda} = P(F^{j,\lambda}) - P(G^{j,\lambda})$. Then, by (3.4),

$$\frac{\left|g\left(P(E) + \alpha^{j,\lambda}\right) - g(P(E)) - \nabla g(P(E))\, \alpha^{j,\lambda}\right|}{|\alpha^{j,\lambda}|} \leq \varepsilon$$

for all $\lambda > \lambda_0$. On the other hand,

$$\begin{aligned}
\sum_{j=1}^{n_\lambda} \left|\alpha^{j,\lambda}\right| &= \sum_{j=1}^{n_\lambda} \left|P\left(F^{j,\lambda}\right) - P\left(G^{j,\lambda}\right)\right| \leq \sum_{j=1}^{n_\lambda} \left|P\left(F^{j,\lambda}\right)\right| + \sum_{j=1}^{n_\lambda} \left|P\left(G^{j,\lambda}\right)\right| \\
&\leq \sum_{i=1}^{N} \sum_{j=1}^{n_\lambda} P_i\left(F^{j,\lambda}\right) + \sum_{i=1}^{N} \sum_{j=1}^{n_\lambda} P_i\left(G^{j,\lambda}\right) \leq 2 \sum_{i=1}^{N} P_i\left(\Omega\right),
\end{aligned}$$

and so, for all $\lambda > \lambda_0$,

$$\begin{aligned}
& \sum_{j=1}^{n_\lambda} \left|g\left(P\left(E\right) + \alpha^{j,\lambda}\right) - g\left(P\left(E\right)\right) - \nabla g\left(P\left(E\right)\right) \alpha^{j,\lambda}\right| \\
= & \sum_{j=1}^{n_\lambda} \frac{\left|g\left(P\left(E\right) + \alpha^{j,\lambda}\right) - g\left(P\left(E\right)\right) - \nabla g\left(P\left(E\right)\right) \alpha^{j,\lambda}\right|}{\left|\alpha^{j,\lambda}\right|} \left|\alpha^{j,\lambda}\right| \\
\leq & \; 2\varepsilon \sum_{i=1}^{N} P_i\left(\Omega\right), \quad \text{as desired.} \quad \blacksquare
\end{aligned}$$

4. LINEAR SETS AND CORE BOUNDS

Our focus in this paper is to relate $core(\nu)$ to the derivative of ν. We begin with an important preliminary relation and some immediate implications. The lemma describes inequalities that represent first-order conditions for (1.2).

Lemma 4.1. *Let $A \in \Sigma$ and $m \in core(\nu)$ be such that $m(A) = \nu(A)$. Then*

$$\begin{aligned}
\delta\nu\left(F; A\right) &\leq m\left(F\right) \quad \text{for all } F \subset A^c \text{ and} \\
\delta\nu\left(G; A\right) &\geq m\left(G\right) \quad \text{for all } G \subset A.
\end{aligned}$$

Proof. For any $F \subset A^c$,

$$\begin{aligned}
\delta\nu\left(F; A\right) &= \sum_{j=1}^{n_\lambda} \delta\nu\left(F^{j,\lambda}; A\right) = \sum_{j=1}^{n_\lambda} \left[\delta\nu\left(F^{j,\lambda}; A\right) - \nu\left(A \cup F^{j,\lambda}\right) + \nu\left(A\right)\right] \\
&\quad + \sum_{j=1}^{n_\lambda} \left[\nu\left(A \cup F^{j,\lambda}\right) - \nu\left(A\right)\right]
\end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{j=1}^{n_\lambda} \left|\delta\nu\left(F^{j,\lambda};A\right) - \nu\left(A\cup F^{j,\lambda}\right) + \nu\left(A\right)\right| + \sum_{j=1}^{n_\lambda}\left[\nu\left(A\cup F^{j,\lambda}\right) - \nu\left(A\right)\right] \\
&\leq \sum_{j=1}^{n_\lambda} \left|\delta\nu\left(F^{j,\lambda};A\right) - \nu\left(A\cup F^{j,\lambda}\right) + \nu\left(A\right)\right| + \sum_{j=1}^{n_\lambda}\left[m\left(A\cup F^{j,\lambda}\right) - m\left(A\right)\right] \\
&= \sum_{j=1}^{n_\lambda} \left|\delta\nu\left(F^{j,\lambda};A\right) - \nu\left(A\cup F^{j,\lambda}\right) + \nu\left(A\right)\right| + m\left(F\right).
\end{aligned}
$$

Thus $\lim_{\lambda\to 0}\sum_{j=1}^{n_\lambda}\left|\delta\nu\left(F^{j,\lambda};A\right) - \nu\left(A\cup F^{j,\lambda}\right) + \nu\left(A\right)\right| = 0$ implies $\delta\nu\left(F;A\right) \leq m\left(F\right)$.

For $G \subset A$,

$$
\begin{aligned}
\delta\nu\left(G;A\right) &= \sum_{j=1}^{n_\lambda}\delta\nu\left(G^{j,\lambda};A\right) = \sum_{j=1}^{n_\lambda}\left[\delta\nu\left(G^{j,\lambda};A\right) + \nu\left(A - G^{j,\lambda}\right) - \nu\left(A\right)\right] \\
&\quad + \sum_{j=1}^{n_\lambda}\left[\nu\left(A\right) - \nu\left(A-G^{j,\lambda}\right)\right] \\
&\geq -\sum_{j=1}^{n_\lambda}\left|\delta\nu\left(G^{j,\lambda};A\right) + \nu\left(A-G^{j,\lambda}\right) - \nu\left(A\right)\right| + \sum_{j=1}^{n_\lambda}\left[\nu\left(A\right) - \nu\left(A-G^{j,\lambda}\right)\right] \\
&\geq -\sum_{j=1}^{n_\lambda}\left|\delta\nu\left(G^{j,\lambda};A\right) + \nu\left(A-G^{j,\lambda}\right) - \nu\left(A\right)\right| + \sum_{j=1}^{n_\lambda}\left[m\left(A\right) - m\left(A-G^{j,\lambda}\right)\right] \\
&= -\sum_{j=1}^{n_\lambda}\left|\delta\nu\left(G^{j,\lambda};A\right) + \nu\left(A-G^{j,\lambda}\right) - \nu\left(A\right)\right| + m\left(G\right).
\end{aligned}
$$

Thus $\lim_{\lambda\to 0}\sum_{j=1}^{n_\lambda}\left|\delta\nu\left(G^{j,\lambda};A\right) + \nu\left(A-G^{j,\lambda}\right) - \nu\left(A\right)\right| = 0$ implies $\delta\nu\left(G;A\right) \geq m\left(G\right)$. ■

We make heavy use of this lemma. Also important is the notion of a linear coalition (or set).

Definition 4.2. *The coalition $A \in \Sigma$ is* linear *with respect to the game ν if $\nu\left(A\right) + \nu\left(A^c\right) = \nu\left(\Omega\right)$.*

Given the binary partition of Ω into the linear coalitions A and A^c, there is no gain (or loss) in forming the grand coalition.[8] Denote by $\mathcal{A}$ the collection of all linear coalitions; $\mathcal{A}$ contains both $\emptyset$ and Ω. The importance of linear coalitions for our purposes is that they deliver (1.1) which is an important part of our approach to determining the core. This fact and more are established in the next lemma.

Lemma 4.3. *If $\{A_i\}_{i\in I}$ is a countable partition of Ω satisfying $\sum_{i\in I}\nu(A_i) = \nu(\Omega)$, then $m(A_i) = \nu(A_i)$ for all $i \in I$ and $m \in$ core(ν). If, in addition, ν is superadditive and either (i) ν is continuous, or (ii) the partition is finite, then $A_i \in \mathcal{A}$ for each i in I.*

Proof. Evidently,

$$0 = \nu(\Omega) - \nu(\Omega) = \sum_{i\in I} m(A_i) - \sum_{i\in I} \nu(A_i) = \sum_{i\in I}[m(A_i) - \nu(A_i)],$$

so that $m(A_i) = \nu(A_i)$ for all i because $m(A_i) \geq \nu(A_i)$ for all i. Next, suppose that ν is superadditive and continuous (the finite partition case is trivial). For convenience, consider A_1. We have

$$\begin{aligned}\nu(\Omega) &\geq \nu(A_1) + \nu(A_1^c) = \nu(A_1) + \lim_{n\to\infty}\nu\left(\bigcup_{i=2}^{n} A_i\right) \\ &\geq \nu(A_1) + \lim_{n\to\infty}\sum_{i=2}^{n}\nu(A_i) = \nu(\Omega),\end{aligned}$$

and so $\nu(A_1) + \nu(A_1^c) = \nu(\Omega)$. ■

For linear sets, Lemma 4.1 takes the following stronger form that is a direct consequence of the two preceding lemmas.

Lemma 4.4. *Let $A \in \mathcal{A}$. For any suitably differentiable ν and $m \in$ core(ν),*

$$\begin{aligned}\delta\nu(F;A) &\leq m(F) \leq \delta\nu(F;A^c) \quad \text{for all } F \subset A^c \text{ and} \\ \delta\nu(G;A^c) &\leq m(G) \leq \delta\nu(G;A) \quad \text{for all } G \subset A.\end{aligned}$$

[8] A (nonbinary) partition of S with the corresponding property (see the next lemma) is called an efficient coalition structure by Aumann and Dreze [1].

For any linear set A, these inequalities define bounds on elements of $core(\nu)$, that is, for any E,

$$\delta\nu\left(E\cap A^c;A\right)\,+\delta\nu\left(E\cap A;A^c\right)\leq m\left(E\right)\leq\delta\nu\left(E\cap A^c;A^c\right)\,+\delta\nu\left(E\cap A;A\right).$$

In particular, because the empty set is linear, one obtains the bound

$$\delta\nu\left(\cdot;\emptyset\right)\;\leq\;m(\cdot)\;\leq\delta\nu\left(\cdot;\Omega\right). \tag{4.1}$$

An implication is that every measure in $core(\nu)$ is absolutely continuous with respect to $\delta\nu\left(\cdot;\Omega\right)$. Further implications constitute necessary conditions for nonemptiness of the core; for example, nonemptiness requires that (for all linear A and for all E)

$$\delta\nu\left(E\cap A^c;A\right)+\delta\nu\left(E\cap A;A^c\right)\leq\;\delta\nu\left(E\cap A^c;A^c\right)\,+\delta\nu\left(E\cap A;A\right).$$

5. COHERENT GAMES

Definition 5.1. *A game $\nu:\Sigma\to\mathbb{R}$ is coherent at $A\in\Sigma$ if*

$$\delta\nu\left(A;A\right)\;>\;\nu(A)\quad\Longrightarrow\quad\delta\nu\left(A;A^c\right)\;\geq\;\nu(A)\quad\textit{and} \tag{5.1}$$

$$\delta\nu\left(A^c;A^c\right)\;>\;\nu(A^c)\quad\Longrightarrow\quad\delta\nu\left(A^c;A\right)\;\geq\;\nu(A^c). \tag{5.2}$$

Defer interpretation for a moment and observe that: (i) ν is coherent at A if and only if it is coherent at A^c; and (ii) ν is coherent at Ω (and at $\emptyset$) if and only if

$$\delta\nu\left(\Omega;\Omega\right)\leq\nu(\Omega)\quad\text{or}\quad\delta\nu\left(\Omega;\emptyset\right)\geq\nu(\Omega). \tag{5.3}$$

As an example, consider the measure game $g\left(P\right)$, where g is differentiable at 0 and at $P(\Omega)$. Then $g\left(P\right)$ is coherent at Ω if

$$\sum_{i=1}^{N}g_i\left(P\left(\Omega\right)\right)\;P_i(\Omega)\leq g\left(P\left(\Omega\right)\right)\quad\text{or}\quad\sum_{i=1}^{N}g_i\left(0\right)\;P_i(\Omega)\;\geq\;g\left(P\left(\Omega\right)\right).$$

In particular, this is true if g is nonnegative-valued and homogeneous of degree $k\in[0,1]$ (by Euler's Theorem) or if g is concave, in which case

$$\delta\nu\left(\Omega;\Omega\right)=\sum_{i=1}^{N}g_i\left(P\left(\Omega\right)\right)P_i\left(\Omega\right)\leq g\left(P\left(\Omega\right)\right)=\nu\left(\Omega\right).$$

To understand the meaning of coherence, recall the intuition sketched in the introduction surrounding the optimization problem (1.2) and the need to convert the (first-order) inequalities in Lemma 4.1 into an equality such as (1.3). Roughly, the obstacle is the apparent lack of a notion of 'reversal in direction'. The sense in which a reversal *is* possible is that the *subtraction* of any $G \subset A$ from A is tantamount to the *addition* of G to A^c. This trivial observation can be exploited if $\delta\nu(\cdot; A)$ and $\delta\nu(\cdot; A^c)$ are suitably related - and that is the role of coherence. Thus, for example, (5.1) requires that if $\delta\nu(A; A)$ is large in the sense of being larger than $\nu(A)$, then (ignoring the distinction between strict and weak inequalities) so is $\delta\nu(A; A^c)$. In terms of the interpretation offered earlier for derivatives, if the value of A is less than the total marginal value of A relative to the base coalition A, corresponding to the effect of *removing* the players in A, then the same must be true when the total marginal value is computed relative to A^c as the base coalition, corresponding to the effect of *adding* the players in A to A^c. (The second condition (5.2) requires the same for A^c.) The consistent relative evaluation of the total marginal contribution of A suggests the name consistency; we employ 'coherence' because consistency has other meanings in co-operative game theory.

At a formal level, one might wonder about the variations of (5.1) and (5.2) obtained by reversing the directions of all inequalities; after all, these also express a form of consistency in the evaluation of the total marginal contribution of A. In fact, these conditions are implied (even without coherence) if ν has a nonempty core, because then, by Lemma 4.4,

$$\delta\nu\,(A; A^c) \,\leq\, \nu(A) \quad \text{and} \quad \delta\nu\,(A^c; A) \,\leq\, \nu(A^c).$$

The intuition in the introduction relies also on a coalition A having the property (1.1), which is true for linear sets, by Lemma 4.3. Call the game ν *coherent* if it is coherent at some linear set A (which presumes differentiability at A and A^c). For coherent games, we do not obtain (1.3); it is not surprising that the calculus intuition is imperfect. However, we can prove that the core is either empty or that it is a singleton having an explicit representation in terms of the derivatives of ν.

Theorem 5.2. *Let the game $\nu : \Sigma \to \mathbb{R}$ be differentiable at some linear coalition A and suppose that core$(\nu) \neq \emptyset$. Then the following statements are* equivalent*:*
(i) ν is coherent at A.
(ii) core$\,(\nu)$ equals the singleton $\{m\}$, where m has the form: There exist scalar

coefficients a, b, c and d such that

$$a + b = 1 = c + d, \;\; ab = cd = 0 \;\; \text{and} \tag{5.4}$$

$$m(E) = a\,\delta\nu\,(E \cap A; A) + b\,\delta\nu\,(E \cap A; A^c) + c\,\delta\nu\,(E \cap A^c; A) + d\,\delta\nu\,(E \cap A^c; A^c) \tag{5.5}$$

for all E in Σ.
(iii) $m \in core\,(\nu)$ for some measure m having the form (5.5).

In particular, coherence at some linear coalition implies that the core is either empty or equals a singleton. Conversely, coherence at A is also necessary for the representation (5.5).

In the latter representation, two coefficients equal 1 and two equal 0, which leads to 4 possibilities in total, corresponding to the fact that there are 4 distinct ways in which coherence at A can be satisfied. These and their corresponding representations are:[9]

1. $\nu\,(A) \geq \delta\nu\,(A; A)$ and $\nu\,(A^c) \leq \delta\nu\,(A^c; A) \Longleftrightarrow$
$$core(\nu) = \{\delta\nu\,(\cdot; A)\}. \tag{5.6}$$

2. $\nu\,(A) \geq \delta\nu\,(A; A)$ and $\nu\,(A^c) \geq \delta\nu\,(A^c; A^c) \Longleftrightarrow$
$$core(\nu) = \; \{\delta\nu\,(\cdot \cap A; A) + \delta\nu\,(\cdot \cap A^c; A^c)\}\,. \tag{5.7}$$

3. $\nu\,(A) \leq \delta\nu\,(A; A^c)$ and $\nu\,(A^c) \leq \delta\nu\,(A^c; A) \Longleftrightarrow$
$$core(\nu) = \; \{\delta\nu\,(\cdot \cap A; A^c) + \delta\nu\,(\cdot \cap A^c; A)\}\,. \tag{5.8}$$

4. $\nu\,(A) \leq \delta\nu\,(A; A^c)$ and $\nu\,(A^c) \geq \delta\nu\,(A^c; A^c) \Longleftrightarrow$
$$core(\nu) = \{\delta\nu\,(\cdot; A^c)\}. \tag{5.9}$$

Note that the first representation corresponds to (1.3), but that the surrounding intuition described in the introduction is consistent also with the three other representations.

[9] These elaborations on the theorem are established in proving the theorem. Naturally, the stated hypotheses, including nonemptiness of the core, are assumed.

Proof. (i) $\Longrightarrow$ (ii): We use repeatedly the following elementary Fact: Given $E \in \Sigma$, if p and q are two measures satisfying

$$p(B) \geq q(B) \text{ for all } B \subset E \text{ and } p(E) = q(E),$$

then $p(B) = q(B)$ for all $B \subset E$.

Let $m \in core\,(\nu)$. Then $\nu\,(A) = m\,(A)$ and $\nu\,(A^c) = m\,(A^c)$ by Lemma 4.3. Therefore, by Lemma 4.4, we have both

$$\begin{aligned} \delta\nu\,(F; A) &\leq m\,(F) \leq \delta\nu\,(F; A^c) \quad \text{for all } F \subset A^c \text{ and} \\ \delta\nu\,(G; A^c) &\leq m\,(G) \leq \delta\nu\,(G; A) \quad \text{for all } G \subset A. \end{aligned} \tag{5.10}$$

and, taking $F = A^c$ and $G = A$,

$$\begin{aligned} \delta\nu\,(A^c; A) &\leq \nu\,(A^c) \leq \delta\nu\,(A^c; A^c)\,, \\ \delta\nu\,(A; A^c) &\leq \nu\,(A) \leq \delta\nu\,(A; A)\,. \end{aligned} \tag{5.11}$$

Now consider in turn each of the possibilities enumerated above.

For #1, deduce from (5.11) that

$$\delta\nu\,(A; A) = m(A) \text{ and } \delta\nu\,(A^c; A) = m(A^c). \tag{5.12}$$

From (5.10), $\delta\nu\,(\cdot; A) \geq m\,(\cdot)$ within A and $\delta\nu\,(\cdot; A) \leq m\,(\cdot)$ within A^c. Then (5.12) and the Fact imply that $\delta\nu\,(\cdot; A) = m\,(\cdot)$ within A and $\delta\nu\,(\cdot; A) = m\,(\cdot)$ within A^c, that is, $m\,(\cdot) = \delta\nu\,(\cdot; A)$ within Ω (that is, on Σ).

For #2, deduce from (5.11) that

$$\delta\nu\,(A; A) = m(A) \text{ and } \delta\nu\,(A^c; A^c) = m(A^c). \tag{5.13}$$

Argue as above, using (5.10) and the Fact, that $\delta\nu\,(\cdot; A) = m\,(\cdot)$ within A and $\delta\nu\,(\cdot; A^c) = m\,(\cdot)$ within A^c. Possibilities #3 and #4 are similar.

(iii) $\Longrightarrow$ (i): Once again, argue case by case. If $\delta\nu\,(\cdot; A) \in core(\nu)$, then Lemma 4.3 implies $\delta\nu\,(A; A) = \nu(A)$ and $\delta\nu\,(A^c; A) = \nu(A^c)$, which in turn implies coherence at A. Similarly for the other cases.

The implication (ii) $\Longrightarrow$ (iii) is obvious. ■

Several applications of the theorem are provided below in the context of more concrete (e.g., measure or market) games. Conclude this section with an application at the level of abstract games.

Corollary 5.3. *Let the game $\nu : \Sigma \to \mathbb{R}$ be differentiable at A and A^c, where $A \in \mathcal{A}$ and $0 < \nu(A) < \nu(\Omega)$. Assume that core $(\nu) \neq \emptyset$ and that ν is* <u>*either*</u> *(i) monotone* <u>*or*</u> *(ii) superadditive and differentiable at $\emptyset$. Suppose finally that ν satisfies*

$$\delta\nu(\cdot; A^c) = \kappa\, \delta\nu(\cdot; A) \text{ for some } \kappa \neq 0. \tag{5.14}$$

Then

$$core\,(\nu) = \{\delta\nu(\cdot; A)\} = \{\delta\nu(\cdot; A^c)\}. \tag{5.15}$$

The condition (5.14) is a simple formal condition that relates the derivatives at A and A^c, thus connecting with our central intuition. As a simple illustration, consider the scalar measure game $\nu(\cdot) = g(P(\cdot))$, where g is monotone and both $g'(P(A))$ and $g'(P(\Omega) - P(A))$ exist and are nonzero. Then condition (5.14) is satisfied because

$$\delta\nu(\cdot; A) = g'(P(A))\, P(\cdot) = \frac{g'(P(A))}{g'(P(A^c))} g'(P(A^c))\, P(\cdot) = \frac{g'(P(A))}{g'(P(A^c))} \delta\nu(\cdot; A^c).$$

Suppose that A is a linear set for ν. It follows from the Corollary that if $0 < g(P(A)) < P(\Omega)$, then $core(\nu)$ is empty unless $g(x) \leq x$ for all x in $[0, P(\Omega)]$, in which case the core is just $\{P\}$.

Proof of Corollary 5.3: (i) From $0 < \nu(A) < \nu(\Omega)$ and Lemmas 4.3 and 4.4, deduce that each of $\delta\nu(A; A)$, $\delta\nu(A^c; A)$, $\delta\nu(A; A^c)$ and $\delta\nu(A^c; A^c)$ is positive.

Next prove that $\kappa = 1$: By Lemma 4.4,

$$\begin{aligned} \delta\nu(A^c; A) &\leq m(A^c) = \nu(A^c) \leq \delta\nu(A^c; A^c) = \kappa\delta\nu(A^c; A), && (5.16) \\ \kappa\delta\nu(A; A) &= \delta\nu(A; A^c) \leq m(A) = \nu(A) \leq \delta\nu(A; A). && (5.17) \end{aligned}$$

Hence, (5.16) implies $\kappa \geq 1$ and (5.17) implies $\kappa \leq 1$, so that $\kappa = 1$ and

$$\delta\nu(\cdot; A^c) = \delta\nu(\cdot; A). \tag{5.18}$$

Finally, (5.16) and (5.17) imply that $\delta\nu(A^c; A) = \nu(A^c)$ and $\nu(A) = \delta\nu(A; A)$. Hence ν is coherent at A and representation (5.6) applies.

(ii) By Lemma A.5,

$$\nu'(E) \equiv |\delta\nu|(E; \varnothing) + \nu(E) \geq -\delta\nu(E; \varnothing) + \nu(E) \geq 0$$

for all $E \in \Sigma$. Thus the game ν' is nonnegative. Furthermore,

$$\begin{aligned}
\nu'(A) + \nu'(A^c) &= \nu(A) + \nu(A^c) + |\delta\nu|(A;\varnothing) + |\delta\nu|(A^c;\varnothing) \\
&= \nu(\Omega) + |\delta\nu|(\Omega;\varnothing) = \nu'(\Omega), \\
\nu'(A) &= \nu(A) + |\delta\nu|(A;\varnothing) < \nu(\Omega) + |\delta\nu|(\Omega;\varnothing), \\
\nu'(A^c) &= \nu(A^c) + |\delta\nu|(A^c;\varnothing) < \nu(\Omega) + |\delta\nu|(\Omega;\varnothing).
\end{aligned}$$

Hence, A is linear for ν' and $0 < \nu'(A) < \nu'(\Omega)$.

Let $m(\cdot) \in core(\nu)$. Then $m(\cdot) + |\delta\nu|(\cdot;\varnothing) \in core(\nu')$, and so, by the preceding argument for monotone games,

$$m(\cdot) + |\delta\nu|(\cdot;\varnothing) = \delta\nu'(\cdot;A) = \delta\nu(\cdot;A) + |\delta\nu|(\cdot;\varnothing),$$

which evidently leads to (5.15). ■

6. APPLICATIONS

Thus far we have been dealing with abstract games. We turn now to applications to more concrete specializations.

6.1. Market Games and Exchange Economies

Definition 6.1. *A market game is a positive superadditive measure game $g(P) : \Sigma \to \mathbb{R}$, where $g : R(P) \to \mathbb{R}$ is homogeneous of degree one.*[10]

Market games play a fundamental role in co-operative game theory (see [2, Ch. 6], for example). As an immediate consequence of Theorem 5.2, we can prove the following result for such games:

Corollary 6.2. *Let $\nu = g(P) : \Sigma \to \mathbb{R}$ be a market game such that g is differentiable at $P(\Omega)$. Then*

$$core(\nu) = \left\{ \sum_{i=1}^{N} g_i(P(\Omega)) P_i(\cdot) \right\}. \tag{6.1}$$

[10] Notice that we do not require countable additivity of P.

Proof. By Euler's Theorem, $\delta\nu(\Omega;\Omega) = \sum_{i=1}^{N} g_i(P(\Omega))\,P_i(\Omega) = g(P(\Omega)) = \nu(\Omega)$. Hence, ν is coherent at Ω. By Theorem 5.2, see (5.6) in particular, $core(\nu) = \{\delta\nu(\cdot;\Omega)\}$ if $\delta\nu(\cdot;\Omega) \in core(\nu)$. Therefore, to complete the proof it suffices to show that $\delta\nu(\cdot;\Omega) \in core(\nu)$.

We know that $\delta\nu(\Omega;\Omega) = \nu(\Omega)$. Let $E \in \Sigma$ with $P(E) \neq 0$. By the Lyapunov Theorem, for each $t \in (0,1)$ there exists B_t such that $P(B_t) = tP(E)$. Hence, by superadditivity,

$$g(P(\Omega) - P(B_t)) = g(P(B_t^c)) \leq g(P(\Omega)) - g(P(B_t)) = g(P(\Omega)) - g(tP(E)),$$

and by homogeneity,

$$\begin{aligned}
\nu(E) &= g(P(E)) = \lim_{t\downarrow 0} \frac{g(P(\Omega)) + g(tP(E)) - g(P(\Omega))}{t} \\
&\leq \lim_{t\downarrow 0} \frac{g(P(\Omega)) - g(P(\Omega) - tP(E))}{t} = \nabla g(P(\Omega))\,P(E) = \delta\nu(E;\Omega).
\end{aligned}$$

Finally, $P(E) = 0$ implies $\nu(E) = 0 = \nabla g(P(\Omega))\,P(E) = \delta\nu(E;\Omega)$. Conclude that $\delta\nu(\cdot;\Omega) \in core(\nu)$. ■

An important special case of a market game is provided by an exchange economy. An exchange economy of finite type consists of:[11]

1. a measure space (Ω, Σ, μ) of agents, where μ is a non-atomic probability measure;

2. a partition $\{\Omega_i\}_{i=1}^{K} \subset \Sigma$ of Ω;

3. a space $\mathbb{R}_+^N$ of goods;

4. a nondecreasing and concave utility function $u(\cdot,\omega) : \mathbb{R}_+^N \rightarrow \mathbb{R}_+$ for each $\omega \in \Omega$ such that $u(\cdot,\omega) = u(\cdot,\omega')$ for all ω, ω' belonging to the same element of the partition;

5. an endowment $e : \Omega \longrightarrow \mathbb{R}_+^N$ such that $\int_\Omega e\,d\mu \in \mathbb{R}_{++}$.

[11] See [2, Ch. 6] for further details and terminology.

Consider the game defined by

$$\nu(E) = \max\left\{\int_E u(x(\omega), \omega)\, d\mu : x(\omega) \in \mathbb{R}_+^N \text{ and } \int_E x(\omega)\, d\mu \le \int_E e(\omega)\, d\mu\right\}, \tag{6.2}$$

where the maximum is taken over all μ-integrable allocations x. By [2, Theorem J], the game is well defined. In particular, concavity of the utility functions justifies the restriction to type-symmetric allocations x, that is, x such that $x(\omega) = x(\omega')$ if ω and ω' belong to the same element of the partition [2, p. 235].

Because of type-symmetry of the relevant allocations, the above game is a market game. Indeed, $\nu(E) = g(P(E))$ for all $E \in \Sigma$, where:

$$\begin{aligned}
\eta_i(E) &= \mu(E \cap \Omega_i) \quad \text{for } 1 \le i \le K, \\
\zeta_j(E) &= \int_E e^j(\omega)\, d\mu \quad \text{for } 1 \le j \le N, \\
P(E) &= (\zeta(E), \eta(E)), \ f_i(\cdot) = u(\cdot, \omega) \text{ for } \omega \text{ in } \Omega_i, \\
g(z, y) &= \max\left\{\sum_{i=1}^K y^i f_i(x_i) : x \in \mathbb{R}_+^N \text{ and } \sum_{i=1}^K y^i x_i \le z\right\}, \ (z, y) \in \mathbb{R}_+^{N+K}.
\end{aligned}$$

Under suitable conditions, spelled out in [2, pp. 234-41], g is differentiable at $P(\Omega)$. Under those conditions, therefore, the core of a market game ν is the singleton as in (6.1). This is essentially a result of [2, Ch. 6] and it plays a key role in their analysis of exchange economies. Our contribution is to show how it can be derived from an approach based primarily on the elementary and familiar perspective provided by calculus.

6.2. The Core of Measure Games

As Hart and Neyman observe [5, p. 32] "in many applications, one usually encounters games ... that depend on finitely many measures." Therefore, we provide some results that apply to a broad class of measure games.

Definition 6.3. *The measure game $g(P) : \Sigma \to \mathbb{R}$ is a Dini measure game if $\liminf_{|x| \to 0} \frac{g(x)}{|x|} > -\infty$, where $|\cdot|$ is any norm on $\mathbb{R}^N$.*[12]

Special cases include

[12] We call them Dini games because $\liminf_{|x| \to 0} \frac{g(x)}{|x|}$ is the lower Dini derivative of g at 0.

(i) all positive measure games, since $g \geq 0$ implies that $\liminf_{|x| \to 0} \frac{g(x)}{|x|} \geq 0$;

(ii) all measure games $g(P)$ such that $g : \mathbb{R}^N \to \mathbb{R}$ is differentiable at 0.

We make use also of special categories of sets or coalitions, called diagonal and pivotal respectively, that we now introduce. Let $P : \Sigma \to \mathbb{R}^N$ be a finitely additive vector measure with each measure P_i convex-ranged. Given any two distinct sets $E', E'' \in \Sigma$, with $P(E') \leq P(E'')$, the set

$$\langle E', E'' \rangle \equiv \{E \in \Sigma : P(E) = tP(E') + (1-t) P(E'') \text{ for some } t \in (0,1)\},$$

is called the *segment* joining E' and E''; it is nonempty by the Lyapunov Theorem.[13] Call E *diagonal* if $E \in \langle \emptyset, \Omega \rangle$, that is, if

$$P(E) = t\, P(\Omega) \quad \text{for some } \; 0 < t < 1,$$

which expresses a sense in which E is a representative subcoalition of Ω. Say that E is *pivotal* if there exist linear sets A' and A'' such that $A' \in \langle \emptyset, E \rangle$ and $A'' \in \langle E, \Omega \rangle$.

Remarks: There exists a linear and diagonal set if and only if Ω is pivotal.[14] Thus, assuming that there exists a (linear and) pivotal set is weaker than assuming that there exists a (linear and) diagonal set; see Theorems 6.4 and 6.5 below. It is easy to see that Ω is pivotal if and only if $\emptyset$ is pivotal. For scalar measure games, Ω is pivotal iff there exists a linear set A such that $0 < P(A) < P(\Omega)$.

Our first result exploits the fact that the coherence required by Theorem 5.2 is implied by the existence of a linear and diagonal set.

Theorem 6.4. *Let $\nu = g(P) : \Sigma \to \mathbb{R}$ be a Dini measure game with P countably additive and g bounded below. Let A be a linear and diagonal set and suppose that ν is differentiable at A and at A^c. Then*

$$core(\nu) = \emptyset \quad or \quad core(\nu) = \{\delta\nu(\cdot; A)\}.$$

[13] Adopt the convention that $\langle E, E \rangle = \{\emptyset\}$ for all $E \in \Sigma$.

[14] In fact, $E \in \langle \emptyset, \Omega \rangle$ and $\emptyset \in \langle \Omega, \Omega \rangle = \langle \emptyset, \emptyset \rangle$, so that we can take $A' = E$ and $A'' = \emptyset$.

The Theorem applies in particular to games $g(P)$ where $g : R(P) \to \mathbb{R}$ is homogeneous of degree one and $g(0,...,0) = 0$, because such games admit many sets that are linear and diagonal. In fact, by the Lyapunov Theorem, for each $\alpha \in (0,1)$ there exists $B_\alpha \in \Sigma$ such that $P_i(B_\alpha) = \alpha P_i(\Omega)$ for $1 \leq i \leq N$. Therefore,

$$\begin{aligned}
\nu(B_\alpha) &= g(P_1(B_\alpha),...,P_N(B_\alpha)) = g(\alpha P_1(\Omega),...,\alpha P_N(\Omega)) \\
&= \alpha g(P_1(\Omega),...,P_N(\Omega)) = \alpha\nu(\Omega); \\
\nu(B_\alpha^c) &= g(P_1(B_\alpha^c),...,P_N(B_\alpha^c)) = g((1-\alpha)P_1(\Omega),...,(1-\alpha)P_N(\Omega)) \\
&= (1-\alpha)g(P_1(\Omega),...,P_N(\Omega)) = (1-\alpha)\nu(\Omega).
\end{aligned}$$

Hence, each B_α is linear and diagonal. Conclude from the Theorem that if the core is nonempty (and given differentiability of g at $P(\Omega)$), then

$$core(\nu) = \{ \Sigma_i \, g_i(P(\Omega)) \, P_i(\cdot) \}.$$

As noted above, there exists a linear and diagonal set if and only if Ω is pivotal. We can weaken this requirement and assume only that there exists *some* pivotal set, not necessarily Ω, if we restrict attention to games $g(P)$ such that either g is convex or $g(P)$ is totally balanced.[15]

Theorem 6.5. *Let $\nu = g(P) : \Sigma \to \mathbb{R}$ be a Dini measure game with P countably additive. Let A be a linear and pivotal set and suppose that ν is differentiable at A and at A^c.*

(i) If g is convex and bounded below, then

$$core(\nu) = \emptyset \quad or \quad core(\nu) = \{\delta\nu(\cdot; A)\}.$$

(ii) If ν is totally balanced, then

$$core(\nu) = \{m\}, \text{ where } m(E) = \delta\nu(E \cap A; A) + \delta\nu(E \cap A^c; A^c), \; E \in \Sigma.$$

[15] Recall that a game is totally balanced iff all its subgames have nonempty cores.
For a scalar measure game, $g(P)$ is convex if and only if $g : [0, P(\Omega)] \to \mathbb{R}$ is convex. This is not true for $N > 1$. For example, if $g(x_1, x_2) = \sqrt{x_1^2 + x_2^2}$, then by Theorem 5.2, $core(g(P)) \subseteq \left\{\frac{1}{\sqrt{2}}(P_1(\cdot) + P_2(\cdot))\right\}$. Hence, $g(P)$ has at most a singleton core and so it is not convex as, otherwise, it would be additive. In fact, $core(g(P)) = \emptyset$.

We describe one more result for Dini measure games. Say that the measure game $g(P)$ is *super-homogeneous of degree $k > 0$ at the set E* if $g(\alpha P(E)) \geq \alpha^k g(P(E))$ for all $\alpha \in (0,1)$. For example, if $g : R(P) \to \mathbb{R}_+$ is concave with $g(0) = 0$, then $g(P)$ is homogeneous of degree $k \geq 1$ at every E. This property implies (given auxiliary assumptions) that $g(P)$ is coherent at Ω and hence permits application of Theorem 5.2. More precisely, we can prove:

Theorem 6.6. *Let $\nu = g(P) : \Sigma \to \mathbb{R}$ be a Dini measure game with g differentiable at $P(\Omega)$.*

(a) Suppose that ν is super-homogeneous of degree $k \in [0,1]$ at $P(\Omega)$ and that either g is bounded below or that ν is superadditive.

- *(i) If $k < 1$, then core $(\nu) = \emptyset$.*
- *(ii) If $k = 1$, then core $(\nu) \subset \{\delta\nu(\cdot;\Omega)\}$.*

(b) Suppose that g is bounded below and that ν is super-homogeneous of degree 1 at every $E \in \Sigma$. Then the following statements are equivalent:

- *(i) core $(\nu) \neq \emptyset$.*
- *(ii) core $(\nu) = \{\delta\nu(\cdot;\Omega)\}$.*
- *(iii) $\nu(\Omega) = \delta\nu(\Omega;\Omega)$ and $\nu(E) + \nu(E^c) \leq \nu(\Omega)$ for all $E \in \Sigma$.*

It is noteworthy that this theorem requires differentiability of g only at the single point $P(\Omega)$. This feature makes possible the following generalization: By the Rademacher Theorem, locally Lipschitzian functions (a large class that includes convex functions) are differentiable everywhere outside a Lebesgue measure zero subset $D \subseteq \mathbb{R}^N$; for convex functions, the set D is first category in $\mathbb{R}^N$ (see [8, Theorem 1.18] and [10, Theorem 25.4]). Consequently, even without any differentiabilty requirement on g, Theorem 6.6 holds if g is locally Lipschitzian and if $P(\Omega)$ lies outside the corresponding 'small' set D. (A similar remark applies to Corollary 6.2.)

A. APPENDIX: DERIVATIVES

A.1. Basic Properties

Proposition A.1. *For any $E \in \Sigma$, when it exists, the derivative $\delta\nu(\cdot;E)$ of a game ν is unique.*

Proof. Let $\delta_1\nu(\cdot;E)$ and $\delta_2\nu(\cdot;E)$ be derivatives. For any $G\subset E$, we have:

$$\begin{aligned}
|\delta_1\nu(G;E)-\delta_2\nu(G;E)| &= \left|\sum_{j=1}^{n_\lambda}\delta_1\nu\left(G^{j,\lambda};E\right)-\delta_2\nu\left(G^{j,\lambda};E\right)\right| \\
&\leq \sum_{j=1}^{n_\lambda}\left|\delta_1\nu\left(G^{j,\lambda};E\right)-\delta_2\nu\left(G^{j,\lambda};E\right)\right| \\
&= \sum_{j=1}^{n_\lambda}\left|\delta_1\nu\left(G^{j,\lambda};E\right)-\nu\left(E-G^{j,\lambda}\right)+\nu(E)\right. \\
&\quad \left.+\nu\left(E-G^{j,\lambda}\right)-\nu(E)-\delta_2\nu\left(G^{j,\lambda};E\right)\right| \\
&\leq \sum_{j=1}^{n_\lambda}\left|\delta_1\nu\left(G^{j,\lambda};E\right)-\nu\left(E-G^{j,\lambda}\right)+\nu(E)\right| \\
&\quad +\sum_{j=1}^{n_\lambda}\left|\nu\left(E-G^{j,\lambda}\right)-\nu(E)-\delta_2\nu\left(G^{j,\lambda};E\right)\right| \underset{\lambda}{\longrightarrow} 0
\end{aligned}$$

and so $\delta_1\nu(G;E)=\delta_2\nu(G;E)$. A similar argument holds for all $F\subset E^c$. By additivity, it then follows that $\delta_1\nu(\cdot;E)=\delta_2\nu(\cdot;E)$. ■

Proposition A.2. *Suppose $\nu_1,\nu_2\in V$ are two games differentiable at $E\in\Sigma$. Then both $\nu_1+\nu_2$ and $\nu_1\nu_2$ are differentiable at E, and*

(i) $\delta(\nu_1+\nu_2)(\cdot;E)=\delta\nu_1(\cdot;E)+\delta\nu_2(\cdot;E)$,

(ii) $\delta(\nu_1\nu_2)(\cdot;E)=\nu_2(E)\,\delta\nu_1(\cdot;E)+\nu_1(E)\,\delta\nu_2(\cdot;E)$.

Moreover, if $\nu_2(E)\neq 0$, then ν_1/ν_2 is differentiable at E and

(iii) $\delta\left(\dfrac{\nu_1}{\nu_2}\right)(\cdot;E)=\dfrac{\nu_2(E)\ \delta\nu_1(\cdot;E)\ -\ \nu_1(E)\ \delta\nu_2(\cdot;E)}{[\nu_2(E)]^2}$.

Proof. We provide the proof of (ii): By adding and subtracting $\nu_1(E)\,\nu_2\left(E\cup F^{j,\lambda}-G^{j,\lambda}\right)$ we obtain:

$$\sum_{j=1}^{n_\lambda}\left|\nu_1\left(E\cup F^{j,\lambda}-G^{j,\lambda}\right)\nu_2\left(E\cup F^{j,\lambda}-G^{j,\lambda}\right)-\nu_1(E)\,\nu_2(E)\right.$$

$$
\begin{aligned}
& -\nu_2(E)\,\delta\nu_1\left(F^{j,\lambda}-G^{j,\lambda};E\right)-\nu_1(E)\,\delta\nu_2\left(F^{j,\lambda}-G^{j,\lambda};E\right)\Big| \\
\leq\ & |\nu_1(E)|\sum_{j=1}^{n_\lambda}\left|\nu_2\left(E\cup F^{j,\lambda}-G^{j,\lambda}\right)-\nu_2(E)-\delta\nu_2\left(F^{j,\lambda}-G^{j,\lambda};E\right)\right| \\
& +\left|\sum_{j=1}^{n_\lambda}\nu_2\left(E\cup F^{j,\lambda}-G^{j,\lambda}\right)\left[\nu_1\left(E\cup F^{j,\lambda}-G^{j,\lambda}\right)-\nu_1(E)\right]\right. \\
& \left.-\nu_2(E)\,\delta\nu_1\left(F^{j,\lambda}-G^{j,\lambda};E\right)\right| \\
=\ & |\nu_1(E)|\sum_{j=1}^{n_\lambda}\left|\nu_2\left(E\cup F^{j,\lambda}-G^{j,\lambda}\right)-\nu_2(E)-\delta\nu_2\left(F^{j,\lambda}-G^{j,\lambda};E\right)\right| \\
& +\sum_{j=1}^{n_\lambda}\Big|\left[\nu_2\left(E\cup F^{j,\lambda}-G^{j,\lambda}\right)-\nu_2(E)+\nu_2(E)\right]\left[\nu_1\left(E\cup F^{j,\lambda}-G^{j,\lambda}\right)-\nu_1(E)\right] \\
& -\nu_2(E)\,\delta\nu_1\left(F^{j,\lambda}-G^{j,\lambda};E\right)\Big| \\
=\ & |\nu_1(E)|\sum_{j=1}^{n_\lambda}\left|\nu_2\left(E\cup F^{j,\lambda}-G^{j,\lambda}\right)-\nu_2(E)-\delta\nu_2\left(F^{j,\lambda}-G^{j,\lambda};E\right)\right| \\
& +\sum_{j=1}^{n_\lambda}\left|\nu_2\left(E\cup F^{j,\lambda}-G^{j,\lambda}\right)-\nu_2(E)\right|\left|\nu_1\left(E\cup F^{j,\lambda}-G^{j,\lambda}\right)-\nu_1(E)\right| \\
& +|\nu_2(E)|\sum_{j=1}^{n_\lambda}\left|\nu_1\left(E\cup F^{j,\lambda}-G^{j,\lambda}\right)-\nu_1(E)-\delta\nu_1\left(F^{j,\lambda}-G^{j,\lambda};E\right)\right|.
\end{aligned}
$$

By definition,

$$
\begin{aligned}
\sum_{j=1}^{n_\lambda}\left|\nu_2\left(E\cup F^{j,\lambda}-G^{j,\lambda}\right)-\nu_2(E)-\delta\nu_2\left(F^{j,\lambda}-G^{j,\lambda};E\right)\right| &\xrightarrow[\lambda]{} 0, \\
\sum_{j=1}^{n_\lambda}\left|\nu_1\left(E\cup F^{j,\lambda}-G^{j,\lambda}\right)-\nu_1(E)-\delta\nu_1\left(F^{j,\lambda}-G^{j,\lambda};E\right)\right| &\xrightarrow[\lambda]{} 0.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
& \sum_{j=1}^{n_\lambda}\left|\nu_2\left(E\cup F^{j,\lambda}-G^{j,\lambda}\right)-\nu_2(E)\right|\left|\nu_1\left(E\cup F^{j,\lambda}-G^{j,\lambda}\right)-\nu_1(E)\right| \\
\leq\ & \sum_{j=1}^{n_\lambda}\left|\nu_2\left(E\cup F^{j,\lambda}-G^{j,\lambda}\right)-\nu_2(E)-\delta\nu_2\left(F^{j,\lambda}-G^{j,\lambda};E\right)\right|\Big|\nu_1\left(E\cup F^{j,\lambda}-G^{j,\lambda}\right)
\end{aligned}
$$

$$
\begin{aligned}
&-\nu_1(E) - \delta\nu_1\left(F^{j,\lambda} - G^{j,\lambda}; E\right)\big| \\
&+\sum_{j=1}^{n_\lambda}\left|\delta\nu_2\left(F^{j,\lambda} - G^{j,\lambda}; E\right)\right|\left|\nu_1\left(E\cup F^{j,\lambda} - G^{j,\lambda}\right) - \nu_1(E) - \delta\nu_1\left(F^{j,\lambda} - G^{j,\lambda}; E\right)\right| \\
&+\sum_{j=1}^{n_\lambda}\left|\delta\nu_1\left(F^{j,\lambda} - G^{j,\lambda}; E\right)\right|\left|\nu_2\left(E\cup F^{j,\lambda} - G^{j,\lambda}\right) - \nu_2(E) - \delta\nu_2\left(F^{j,\lambda} - G^{j,\lambda}; E\right)\right| \\
&+\sum_{j=1}^{n_\lambda}\left|\delta\nu_2\left(F^{j,\lambda} - G^{j,\lambda}; E\right)\right|\left|\delta\nu_1\left(F^{j,\lambda} - G^{j,\lambda}; E\right)\right|.
\end{aligned}
$$

Let $\varepsilon > 0$. There exists λ_1 such that, for all $\lambda > \lambda_1$,

$$
\begin{aligned}
&\sum_{j=1}^{n_\lambda}\left|\nu_2\left(E\cup F^{j,\lambda} - G^{j,\lambda}\right) - \nu_2(E)\right|\left|\nu_1\left(E\cup F^{j,\lambda} - G^{j,\lambda}\right) - \nu_1(E)\right| \\
\leq\ &\varepsilon^2 + \varepsilon\sum_{j=1}^{n_\lambda}|\delta\nu_2|\left(F^{j,\lambda} - G^{j,\lambda}; E\right) + \varepsilon\sum_{j=1}^{n_\lambda}|\delta\nu_1|\left(F^{j,\lambda} - G^{j,\lambda}; E\right) \\
&+\sum_{j=1}^{n_\lambda}\left|\delta\nu_2\left(F^{j,\lambda} - G^{j,\lambda}; E\right)\right|\left|\delta\nu_1\left(F^{j,\lambda} - G^{j,\lambda}; E\right)\right|.
\end{aligned}
$$

Since the derivatives are convex-ranged, there exists λ_2 such that $|\delta\nu_2|\left(F^{j,\lambda} - G^{j,\lambda}; E\right) \leq \varepsilon$ for all $\lambda > \lambda_2$ and all $1 \leq j \leq n_\lambda$. Hence, for all $\lambda > \lambda_1 \vee \lambda_2$,

$$
\begin{aligned}
&\sum_{j=1}^{n_\lambda}\left|\nu_2\left(E\cup F^{j,\lambda} - G^{j,\lambda}\right) - \nu_2(E)\right|\left|\nu_1\left(E\cup F^{j,\lambda} - G^{j,\lambda}\right) - \nu_1(E)\right| \\
\leq\ &\varepsilon^2 + \varepsilon|\delta\nu_2|(F - G; E) + \varepsilon|\delta\nu_1|(F - G; E) + \varepsilon|\delta\nu_1|(F - G; E).
\end{aligned}
$$

Since the derivatives are bounded measures, it follows that

$$
\sum_{j=1}^{n_\lambda}\left|\nu_2\left(E\cup F^{j,\lambda} - G^{j,\lambda}\right) - \nu_2(E)\right|\left|\nu_1\left(E\cup F^{j,\lambda} - G^{j,\lambda}\right) - \nu_1(E)\right| \underset{\lambda}{\longrightarrow} 0,
$$

which completes the proof of (ii). ■

Proposition A.3. *Let ν be differentiable at $E \in \Sigma$. If ν is monotone, then $\delta\nu(\cdot; E)$ is a nonnegative measure.*

Proof. Let $F \subset E^c$. Then, by monotonicity,

$$\begin{aligned}
-\delta\nu(F;E) &= \sum_{j=1}^{n_\lambda} -\delta\nu\left(F^{j,\lambda};E\right) \le \sum_{j=1}^{n_\lambda}\left[-\delta\nu\left(F^{j,\lambda};E\right)+\nu\left(E\cup F^{j,\lambda}\right)-\nu(E)\right] \\
&\le \sum_{j=1}^{n_\lambda}\left|-\delta\nu\left(F^{j,\lambda};E\right)+\nu\left(E\cup F^{j,\lambda}\right)-\nu(E)\right| \underset{\lambda}{\longrightarrow} 0,
\end{aligned}$$

and so $-\delta\nu(F;E) \le 0$, i.e., $\delta\nu(F;E) \ge 0$. Now, let $G \subset E$. Again by monotonicity,

$$\begin{aligned}
-\delta\nu(G;E) &= \sum_{j=1}^{n_\lambda} -\delta\nu\left(G^{j,\lambda};E\right) \le \sum_{j=1}^{n_\lambda}\left[-\delta\nu\left(G^{j,\lambda};E\right)+\nu(E)-\nu\left(E-G^{j,\lambda}\right)\right] \\
& \sum_{j=1}^{n_\lambda}\left|-\delta\nu\left(G^{j,\lambda};E\right)+\nu(E)-\nu\left(E-G^{j,\lambda}\right)\right| \underset{\lambda}{\longrightarrow} 0,
\end{aligned}$$

and so $-\delta\nu(G;E) \le 0$, i.e., $\delta\nu(G;E) \ge 0$. By the additivity of $\delta\nu(\cdot;E)$, it follows that $\delta\nu(\cdot;E) \ge 0$. ■

A.2. Chain Rule

Proposition A.4. *Let $\nu : \Sigma \to \mathbb{R}$ be differentiable at $E \in \Sigma$, and $g : \mathbb{R} \to \mathbb{R}$ a function differentiable at $\nu(E)$. Then $g(\nu) : \Sigma \to \mathbb{R}$ is differentiable at E and*

$$\delta g(\nu(\cdot;E)) = g'(\nu(E))\ \delta\nu(\cdot;E).$$

Proof. We have

$$\begin{aligned}
& \sum_{j=1}^{n_\lambda}\left|g\left(\nu\left(E\cup F^{j,\lambda}-G^{j,\lambda}\right)\right)-g(\nu(E))-g'(\nu(E))\,\delta\nu\left(F^{j,\lambda}-G^{j,\lambda};E\right)\right| \\
=& \sum_{j=1}^{n_\lambda}\Big|g\left(\nu\left(E\cup F^{j,\lambda}-G^{j,\lambda}\right)\right)-g(\nu(E))-g'(\nu(E))\left[\nu\left(E\cup F^{j,\lambda}-G^{j,\lambda}\right)-\nu(E)\right] \\
& \quad +g'(\nu(E))\left[\nu\left(E\cup F^{j,\lambda}-G^{j,\lambda}\right)-\nu(E)\right]-g'(\nu(E))\,\delta\nu\left(F^{j,\lambda}-G^{j,\lambda};E\right)\Big| \\
\le& \sum_{j=1}^{n_\lambda}\left|g\left(\nu\left(E\cup F^{j,\lambda}-G^{j,\lambda}\right)\right)-g(\nu(E))-g'(\nu(E))\left[\nu\left(E\cup F^{j,\lambda}-G^{j,\lambda}\right)-\nu(E)\right]\right| \\
& \quad +\left|g'(\nu(E))\right|\sum_{j=1}^{n_\lambda}\left|\nu\left(E\cup F^{j,\lambda}-G^{j,\lambda}\right)-\nu(E)-\delta\nu\left(F^{j,\lambda}-G^{j,\lambda};E\right)\right|
\end{aligned}$$

By definition of $\delta\nu(\cdot;E)$,

$$\sum_{j=1}^{n_\lambda}\left|\nu\left(E\cup F^{j,\lambda}-G^{j,\lambda}\right)-\nu(E)-\delta\nu\left(F^{j,\lambda}-G^{j,\lambda};E\right)\right|\underset{\lambda}{\longrightarrow}0. \tag{A.1}$$

Set $\alpha^{j,\lambda}=\nu\left(E\cup F^{j,\lambda}-G^{j,\lambda}\right)-\nu(E)$. Then

$$\begin{aligned}&\left|g\left(\nu\left(E\cup F^{j,\lambda}-G^{j,\lambda}\right)\right)-g(\nu(E))-g'(\nu(E))\left[\nu\left(E\cup F^{j,\lambda}-G^{j,\lambda}\right)-\nu(E)\right]\right|\\ =&\left|\frac{g\left(\alpha^{j,\lambda}+\nu(E)\right)-g(\nu(E))}{\alpha^{j,\lambda}}-g'(\nu(E))\right|\left|\alpha^{j,\lambda}\right|.\end{aligned}$$

In particular,

$$\begin{aligned}\left|\alpha^{j,\lambda}\right| &= \left|\nu\left(E\cup F^{j,\lambda}-G^{j,\lambda}\right)-\nu(E)\right|\\ &= \left|\nu\left(E\cup F^{j,\lambda}-G^{j,\lambda}\right)-\nu(E)-\delta\nu\left(F^{j,\lambda}-G^{j,\lambda};E\right)+\delta\nu\left(F^{j,\lambda}-G^{j,\lambda};E\right)\right|\\ &\le \left|\nu\left(E\cup F^{j,\lambda}-G^{j,\lambda}\right)-\nu(E)-\delta\nu\left(F^{j,\lambda}-G^{j,\lambda};E\right)\right|+\left|\delta\nu\left(F^{j,\lambda}-G^{j,\lambda};E\right)\right|.\end{aligned}$$

Let $\eta>0$. By (A.1), there exists λ_1 such that, for all $\lambda>\lambda_1$ and all $1\le j\le n_\lambda$,

$$\left|\nu\left(E\cup F^{j,\lambda}-G^{j,\lambda}\right)-\nu(E)-\delta\nu\left(F^{j,\lambda}-G^{j,\lambda};E\right)\right|\le\eta/2.$$

Moreover, since $\delta\nu(\cdot;E)$ is convex-ranged, there exists a finite partition of $F\cup G$, indexed by λ_2, such that, for all $1\le j\le n_{\lambda_2}$,

$$|\delta\nu|\left(F^{j,\lambda_2};E\right)\le\eta/2 \text{ and } |\delta\nu|\left(G^{j,\lambda_2};E\right)\le\eta/2,$$

and so

$$\begin{aligned}\left|\delta\nu\left(F^{j,\lambda}-G^{j,\lambda};E\right)\right| &\le \left|\delta\nu\left(F^{j,\lambda_2};E\right)\right|+\left|\delta\nu\left(G^{j,\lambda_2};E\right)\right|\\ &\le |\delta\nu|\left(F^{j,\lambda_2};E\right)+|\delta\nu|\left(G^{j,\lambda_2};E\right)\le\eta.\end{aligned}$$

Hence, by setting $\lambda_{\delta^*}=\lambda_1\vee\lambda_2$, we have that for all $\lambda>\lambda_{\delta^*}$,

$$\left|\alpha^{j,\lambda}\right|\le\left|\nu\left(E\cup F^{j,\lambda}-G^{j,\lambda}\right)-\nu(E)-\delta\nu\left(F^{j,\lambda}-G^{j,\lambda};E\right)\right|+\left|\delta\nu\left(F^{j,\lambda}-G^{j,\lambda};E\right)\right|\le\eta$$

for all $1\le j\le n_\lambda$. On the other hand, since the derivative $g'(\nu(E))$ exists, given any $\varepsilon>0$ there exists $\eta_\varepsilon>0$ such that

$$\left|\frac{g\left(\alpha^{j,\lambda}+\nu(E)\right)-g(\nu(E))}{\alpha^{j,\lambda}}-g'(\nu(E))\right|\le\varepsilon$$

for all $\left|\alpha^{j,\lambda}\right| \leq \eta_\varepsilon$. Therefore, taking $\eta = \eta_\varepsilon$, we have

$$\begin{aligned} &\sum_{j=1}^{n_\lambda} \left| \frac{g\left(\alpha^{j,\lambda} + \nu\left(E\right)\right) - g\left(\nu\left(E\right)\right)}{\alpha^{j,\lambda}} - g'\left(\nu\left(E\right)\right) \right| \left|\alpha^{j,\lambda}\right| \\ \leq\ & \varepsilon \sum_{j=1}^{n_\lambda} \left|\alpha^{j,\lambda}\right|, \quad \text{for all } \lambda \geq \lambda_{\eta_\varepsilon}. \end{aligned}$$

But, by the definition of $\delta\nu\left(\cdot; E\right)$, given any $\varepsilon > 0$, there exists λ_0 such that $\lambda > \lambda_0$ implies

$$\begin{aligned} \Sigma_{j=1}^{n_\lambda} \left|\alpha^{j,\lambda}\right| &\leq \varepsilon + \Sigma_{j=1}^{n_\lambda} \left|\delta\nu\left(F^{j,\lambda}; E\right) - \delta\nu\left(G^{j,\lambda}; E\right)\right| \\ &\leq \varepsilon + \Sigma_{j=1}^{n_\lambda} \left[\left|\delta\nu\left(F^{j,\lambda}; E\right)\right| + \left|\delta\nu\left(G^{j,\lambda}; E\right)\right|\right] \leq K, \end{aligned}$$

for some $K < \infty$ that is independent of λ, F and G, as provided by the boundedness of the measure $\delta\nu(\cdot; E)$. Conclude that, given any $\varepsilon > 0$,

$$\sum_{j=1}^{n_\lambda} \left| \frac{g\left(\alpha^{j,\lambda} + \nu\left(E\right)\right) - g\left(\nu\left(E\right)\right)}{\alpha^{j,\lambda}} - g'\left(\nu\left(E\right)\right) \right| \left|\alpha^{j,\lambda}\right| \leq K\varepsilon,$$

and so

$$\sum_{j=1}^{n_\lambda} \left|g\left(\nu\left(E \cup F^{j,\lambda} - G^{j,\lambda}\right)\right) - g\left(\nu\left(E\right)\right) - g'\left(\nu\left(E\right)\right)\delta\nu\left(F^{j,\lambda} - G^{j,\lambda}; E\right)\right| \underset{\lambda}{\longrightarrow} 0,$$

as desired. ■

A.3. Miscellaneous Facts

Lemma A.5. *Let $\nu : \Sigma \to \mathbb{R}$ be a superadditive game differentiable at $\varnothing$. Then*

$$\delta\nu(E; \varnothing) \leq \nu\left(E\right)$$

for all $E \in \Sigma$.

Remark. This lemma generalizes the fact that for superadditive real-valued functions $g : [0, 1] \to \mathbb{R}$, we have

$$g'\left(0\right) x \leq g\left(x\right)$$

for all $x \in [0,1]$, whenever $g'(0) = \lim_{x \downarrow 0} g(x)/x$ exists. In fact, let $\nu = g(P) : \Sigma \to \mathbb{R}$ be the scalar measure game defined through g. The Chain Rule (Proposition A.4), and Lemma A.5 imply that

$$g'(0) P(E) = \delta\nu(E;\varnothing) \leq \nu(E) = g(P(E)).$$

Proof. For any $E \in \Sigma$,

$$\delta\nu(E^{j,\lambda};\varnothing) \leq \nu\left(E^{j,\lambda}\right) + \left|\delta\nu(E^{j,\lambda};\varnothing) - \nu\left(E^{j,\lambda}\right)\right|,$$

and so

$$\begin{aligned} \delta\nu(E;\varnothing) &= \sum_{j=1}^{n_\lambda} \delta\nu(E^{j,\lambda};\varnothing) \leq \sum_{j=1}^{n_\lambda} \nu\left(E^{j,\lambda}\right) + \sum_{j=1}^{n_\lambda} \left|\delta\nu(E^{j,\lambda};\varnothing) - \nu\left(E^{j,\lambda}\right)\right| \\ &\leq \nu(E) + \sum_{j=1}^{n_\lambda} \left|\delta\nu(E^{j,\lambda};\varnothing) - \nu\left(E^{j,\lambda}\right)\right|. \end{aligned}$$

Since ν is differentiable at $\varnothing$, $\lim_\lambda \sum_{j=1}^{n_\lambda} \left|\delta\nu(E^{j,\lambda};\varnothing) - \nu\left(E^{j,\lambda}\right)\right| = 0$, and so $\delta\nu(E;\varnothing) \leq \nu(E)$. ■

Next we describe an example of a differentiable game that is not a measure game: Fix any ordered sequence $(p_1, p_2, ..., p_n, ...)$ of countably additive probability measures and define the game ν by

$$\nu(A) = \Sigma_{k=1}^{\infty} \left(\Pi_{i=1}^{k} p_i(A)\right) / k!.$$

Then each summand $\left(\Pi_{i=1}^{k} p_i(\cdot)\right) / k!$ is differentiable by the product rule and the sum of these derivatives yields $\delta\nu(\cdot;E)$. The game ν is not a measure game if there is no finite linear basis for $\{p_n(\cdot)\}$.

Suppose that all the measures p_n assign the same probability $p(A)$ to some specific A. Then A is a linear set for the transformed game $\nu^* = \log(1+\nu)$ $(\nu^*(A) = p(A))$; and ν^* is differentiable by the Chain Rule.

A.4. Automorphisms

Derivatives are invariant to a 'renaming' of agents in the following sense: Call the function $\theta : \Omega \to \Omega$ an *automorphism* of Ω if it is Σ-measurable and bijective, that is, if it is a permutation over Ω. Let Θ be the group of all Σ-measurable automorphisms of Ω. Given $\nu \in V$, each $\theta \in \Theta$ induces a game $\theta_* \nu \in V$ defined by

$$\theta_* \nu (E) = \nu (\theta (E))$$

for all $E \in \Sigma$ [2, p. 15]. Evidently, for any F and partition $\{F^{j,\lambda}\}$, $\{\theta(F^{j,\lambda})\}$ is a partition of $\theta(F)$. Furthermore, $\left\{\theta\left(F^{\lambda,j}\right)\right\}_\lambda$ is a subnet of the net of all finite partitions of $\theta(F)$: Let $\{A^j\}$ be any finite partition of $\theta(F)$. We want to show that there is some partition of $\theta(F)$ of the form $\left\{\theta\left(F^{\lambda,j}\right)\right\}$ which refines $\{A^j\}$. The collection $\left\{\theta^{-1}\left(A^j\right)\right\}$ is a finite partition of F, and since $\left\{F^{\lambda,j}\right\}_\lambda$ is the set of all finite partitions of F, there is some λ' such that $F^{j,\lambda'} = \theta^{-1}\left(A^j\right)$ for all j. For all $\lambda > \lambda'$, $\left\{F^{\lambda,j}\right\}$ refines $\left\{F^{\lambda',j}\right\}$, and so the finite partition $\left\{\theta\left(F^{\lambda,j}\right)\right\}$ of $\theta(F)$ refines $\left\{\theta\left(F^{\lambda',j}\right)\right\} = \{A^j\}$.

Lemma A.6. *If θ is an automorphism and if ν is differentiable at $\theta(E)$, then $\theta_*(\nu)$ is differentiable at E and*

$$\delta\left(\theta_* \nu\right)(\cdot; E) \;=\; \theta_*\left(\delta\nu(\cdot;\theta(E)\right) \;=\; \delta\nu(\,\theta(\cdot);\,\theta(E)\,).$$

Proof.

$$\sum_{j=1}^{n_\lambda} \left|(\theta_*\nu)\left(E \cup F^{j,\lambda} - G^{j,\lambda}\right) - (\theta_*\nu)(E) - \delta\nu\left(\theta(F^{j,\lambda});\theta(E)\right) + \delta\nu\left(\theta(G^{j,\lambda});\theta(E)\right)\right| \;=$$

$$\sum_{j=1}^{n_\lambda} \left|\nu\left(\theta(E \cup F^{j,\lambda} - G^{j,\lambda})\right) - \nu(\theta(E)) - \delta\nu\left(\theta(F^{j,\lambda});\theta(E)\right) + \delta\nu\left(\theta(G^{j,\lambda});\theta(E)\right)\right| \;=$$

$$\sum_{j=1}^{n_\lambda} \left|\nu\left(\theta(E) \cup \theta(F^{j,\lambda}) - \theta(G^{j,\lambda})\right) - \nu(\theta(E)) - \delta\nu\left(\theta(F^{j,\lambda});\theta(E)\right) + \delta\nu\left(\theta(G^{j,\lambda});\theta(E)\right)\right| \;.$$

The latter converges to 0 if ν is differentiable at $\theta(E)$ because $\left\{\theta\left(F^{\lambda,j}\right)\right\}$ and $\left\{\theta\left(G^{\lambda,j}\right)\right\}$ are subnets of the nets of all finite partitions of $\theta(F)$ and $\theta(G)$ respectively. ■

A.5. Connection with Aumann-Shapley

Let $\mathbf{B}_1$ denote the set $\{f \in B(\Omega,\Sigma) : 0 \leq f \leq 1\}$. Aumann and Shapley show (Theorem G, p. 144) that each ν in pNA has an extension to a unique 'integral' $\nu^* : B_1 \longrightarrow \mathbb{R}^1$ satisfying specified properties. In particular, for any measure game $\nu = g(P)$ where g is a polynomial function and $P = (P_1, ..., P_N)$ is nonatomic, then

$$\nu^*(f) = g(P^*(f)),$$

where $P^*(f) = (P_1^*(f), ..., P_N^*(f))$ and $P_i^*(f) = \int_\Omega f\, dP_i$ for $f \in \mathbf{B}_1$. It follows from the sum and product rules for differentiation that for all such polynomial measure games, $G \subset E$ and $F \subset E^c$,

$$\delta\nu(F; E) = \frac{d}{d\tau}\nu^*(1_E + \tau\, 1_F)\ |_{\tau=0} \quad \text{and}$$

$$\delta\nu(G; E) = -\frac{d}{d\tau}\nu^*(1_E - \tau\, 1_G)\ |_{\tau=0}.$$

and both derivatives exist. More generally, for any $0 < t \leq 1$ and set E_t satisfying $P(E_t) = tP(E)$,

$$\delta\nu(F; E_t) = \frac{d}{d\tau}\nu^*(t1_E + \tau\, 1_F)\ |_{\tau=0},$$

provided that the range of $P(\cdot)$ is full-dimensional. Gateaux derivatives like those on the right are the calculus tool used by Aumann and Shapley.

We do not know how our derivative $\delta\nu_n(\cdot;\cdot)$ behaves along a suitably convergent sequence of games $\{\nu_n\}$. Thus it is not clear how the above connection made for polynomial games might extend to games in pNA.

B. APPENDIX

This appendix proves Theorems 6.4 and 6.5, beginning with some common lemmas.

B.1. Common Lemmas

Lemma B.1. *Let $\nu = g(P) : \Sigma \to \mathbb{R}$ be a Dini measure game with core $(\nu) \neq \varnothing$. Suppose that either ν is superadditive or that g is bounded below. Then there exists a positive integer k such that the game $\nu^* : \Sigma \to \mathbb{R}$ defined by $\nu^* = \nu + k\sum_{i=1}^N P_i$ is positive and $\nu^*(\Omega) > 0$.*

Proof. Let $|\cdot|_p$ be the norm defined by $|x|_p = \sum_{i=1}^N |x_i|$ for all $x \in \mathbb{R}^N$. Since all norms in $\mathbb{R}^N$ are equivalent, it is easy to see that, for any norm $|\cdot|$,

$$\lim_{|x|\to 0}\inf \frac{g(x)}{|x|} > -\infty \Longleftrightarrow \lim_{|x|_p\to 0}\inf \frac{g(x)}{|x|_p} > -\infty.$$

Therefore, $\liminf_{|x|_p \to 0} \frac{g(x)}{|x|_p} > -\infty$ for any Dini game $g(P)$. Assume first that g is bounded below, i.e., $\inf_{x \in R(P)} g(x) > -\infty$. Given $M > 0$, there exists $\varepsilon > 0$ such that

$$\frac{g(x)}{|x|_p} > -M \quad \text{if } x \in R(P) \text{ and } 0 < |x|_p < \varepsilon. \tag{B.1}$$

Let $x \in R(P)$, $|x|_p \geq \varepsilon$ and $g(x) < 0$. Then $\frac{g(x)}{|x|_p} \geq \frac{g(x)}{\varepsilon} \geq \frac{\inf_{x \in R(P)} g(x)}{\varepsilon}$. Let k' be a positive integer such that $k' \geq \frac{\left|\inf_{x \in R(P)} g(x)\right|}{\varepsilon} + M$. Then $\frac{g(x)}{|x|_p} > -k'$ for all $x \in R(P) \setminus \{0\}$ such that either $0 < |x|_p < \varepsilon$ or $g(x) < 0$, implying that $g(x) + k' |x|_p > 0$ for all such x. But $g(x) + k' |x|_p > 0$ also for all $x \in R(P) \setminus \{0\}$ such that $g(x) > 0$. Conclude that $\nu(E) + k' \sum_{i=1}^N P_i(E) \geq 0$ for all $E \in \Sigma$.

Next consider the case where ν is superadditive. Given M, choose ε as above satisfying (B.1). For $x \in R(P) \setminus \{0\}$, let $E \in \Sigma$ be such that $P(E) = x$. By the Lyapunov Theorem there exists a partition $\{E_l\}_{l=1}^L \subseteq \Sigma$ of E such that $0 < |P(E_l)|_p < \varepsilon$ for each $1 \leq l \leq L$. Hence,

$$\nu(E) + k' \sum_{i=1}^N P_i(E) \geq \sum_{l=1}^L \left(\nu(E_l) + k' \sum_{i=1}^N P_i(E_l) \right) > 0$$

since $\nu(E_l) + k' \sum_{i=1}^N P_i(E_l) > 0$ for each $1 \leq l \leq L$. Conclude that also in this case $\nu(E) + k' \sum_{i=1}^N P_i(E) \geq 0$ for all $E \in \Sigma$.

Finally, assume that $P(\Omega) \neq 0$, so that $|P(\Omega)|_p > 0$. Let $k > \max\left\{k', \frac{-\nu(\Omega)}{|P(\Omega)|_p}\right\}$ and set $\nu^* = \nu + k \sum_{i=1}^N P_i$. If $\nu(\Omega) > 0$, then $\nu^*(\Omega) > 0$ and we are done. If $\nu(\Omega) \leq 0$, then $\nu^*(E) \geq \nu(E) + k' \sum_{i=1}^N P_i(E) \geq 0$ for all $E \in \Sigma$ and
$\nu^*(\Omega) = \nu(\Omega) + k \sum_{i=1}^N P_i(\Omega) > \nu(\Omega) + \frac{-\nu(\Omega)}{|P(\Omega)|_p} \sum_{i=1}^N P_i(\Omega) = 0$. ■

Lemma B.2. *Let ν and ν^* be as in Lemma B.1. Then:*

1. *core* $(\nu^*) = \left\{ m + k \sum_{i=1}^N P_i(E) : m \in \textit{core}(\nu) \right\}$,
2. *A is linear with respect to ν^* if and only if it is linear with respect to ν.*
3. *ν is coherent at $A \in \mathcal{A}$ if and only if ν^* is coherent at A.*

Proof. The first two assertions are easily proven. For coherence, it suffices to see that

$$\begin{aligned}
\delta\nu^*(A;A) &\leq \nu^*(A) \Longleftrightarrow \delta\nu(A;A) \leq \nu(A), \\
\delta\nu^*(A^c;A) &\geq \nu^*(A^c) \Longleftrightarrow \delta\nu(A^c;A) \leq \nu(A^c), \\
\delta\nu^*(A;A^c) &\geq \nu^*(A) \Longleftrightarrow \delta\nu(A;A^c) \geq \nu(A), \\
\delta\nu^*(A^c;A^c) &\leq \nu^*(A^c) \Longleftrightarrow \delta\nu(A^c;A^c) \leq \nu(A^c).
\end{aligned}$$

■

The subset of $core(\nu)$ consisting of countably additive measures is the *σ-core* and is denoted by

$$core^{\sigma}(\nu) = \{m \in CA : m(\Omega) = \nu(\Omega), \text{ and } m(A) \geq \nu(A) \text{ for all } A \in \Sigma\}.$$

Lemma B.3. *Given a positive game $\nu : \Sigma \to \mathbb{R}_+$, every measure $m \in core^{\sigma}(\nu)$ is non-atomic if at least one of the following two conditions is satisfied:*

(i) there is some $A \in \mathcal{A}$, with $\nu(A) = \nu(\Omega)$, such that ν is differentiable at A;

(ii) there is some $A \in \mathcal{A}$, with $0 < \nu(A) < \nu(\Omega)$, such that ν is differentiable at both A and A^c.

Proof. For case (ii), suppose that $0 < \nu(A) < \nu(\Omega)$. Then $m(A) > 0$ and $m(A^c) > 0$. By definition, both $\delta\nu(\cdot;A)$ and $\delta\nu(\cdot;A^c)$ are convex-ranged. Since ν is positive, each $m \in core(\nu)$ is positive. Hence, by Lemma 4.4, $\delta\nu(\cdot;A)$ and $\delta\nu(\cdot;A^c)$ are positive measures on A and A^c, respectively. Again by Lemma 4.4, for all $G \subseteq A$, $\delta\nu(G;A) = 0 \Rightarrow m(G) = 0$, i.e., m is absolutely continuous with respect to the finitely additive measure $\delta\nu(\cdot;A)$ on A. This implies that that m has no atoms in A as we proceed to show.

Let $m(E) > 0$ with $\Sigma \ni E \subseteq A$. By above, $\delta\nu(E;A) > 0$. Since $\delta\nu(\cdot;A)$ is convex-ranged, there exists a partition E^1, B^1 of E such that $\delta\nu(E^1;A) = \delta\nu(B^1;A) = \frac{1}{2}\delta\nu(E;A)$. If $0 < m(E^1) < m(E)$ or $0 < m(B^1) < m(E)$, we are done. Suppose, in contrast, that either $m(E^1) = m(E)$ or $m(B^1) = m(E)$. Wlog, let $m(E^1) = m(E)$. Again, there exists a partition E^2 and B^2 of E^1 such that $\delta\nu(E^2;A) = \delta\nu(B^2;A) = \frac{1}{2}\delta\nu(E^1;A)$. If $0 < m(E^2) < m(E^1)$ or $0 < m(B^2) < m(E^1)$, we are done. Suppose, in contrast, that either $m(E^2) = m(E^1)$ or $m(B^2) = m(E^1)$. Wlog, let $m(E^2) = m(E^1)$. Proceeding in this way, either we find a set $B \subseteq E$ such that $0 < m(B) < m(E)$ or we can construct a chain

$\{E^n\}_{n\geq 1}$ such that $\delta\nu(E^n;A) = \frac{1}{2^n}\delta\nu(E;A)$ and $m(E^n) = m(E)$ for all $n \geq 1$. Hence, because $\bigcap_{n\geq 1} E^n \in \Sigma$ and $\bigcap_{n\geq 1} E^n \subseteq E$, we have $\delta\nu\left(\bigcap_{n\geq 1} E^n; A\right) = 0$ and $m\left(\bigcap_{n\geq 1} E^n\right) = m(E) > 0$, a contradiction. Conclude that there exists some set $B \subseteq E$ such that $0 < m(B) < m(E)$, and that m has no atoms in A.

Next replace A by A^c and use the convex range of $\delta\nu(\cdot;A^c)$ to deduce in a similar fashion that m has no atoms in A^c. Now, suppose that E is an atom for m in Σ. By definition, either $m(E\cap A) = 0$ or $m(E\cap A) = m(E)$. If $m(E\cap A) = m(E)$, then $E \cap A$ is an atom in A, which was ruled out above. Hence, $m(E\cap A) = 0$. A similar argument shows that $m(E\cap A^c) = 0$ and so $m(E) = 0$, a contradiction. Hence, there are no atoms in Σ. Because m is countably additive and non-atomic, it is also convex-ranged.

Finally, for case (i), suppose that $\nu(A) = \nu(\Omega)$. Then $m(A^c) = 0$. Then if E is an atom in Σ, it must be the case that $m(E\cap A) = m(E) > 0$, and so $E\cap A$ is an atom on A. This can be ruled out as above. ■

Lemma B.4. *Let $\nu = g(P) : \Sigma \to \mathbb{R}_+$ be a positive measure game with P countably additive. Then core $(\nu) =$ core$^\sigma(\nu)$ given at least one of conditions (i) and (ii) of Lemma B.3.*

Proof. Let $m \in core(\nu)$. Suppose (i) is true. Since ν is differentiable at A, it is continuous at A. Suppose that $E_n \uparrow \Omega$. Then $E_n \cap A \uparrow A$ and so $m(E_n) = m(E_n\cap A) \uparrow m(A) = m(\Omega)$, that is, $m \in core^\sigma(\nu)$.

Next suppose (ii) is true. Since ν is differentiable at A and A^c, ν is continuous at both A and A^c. Because both are linear sets, this implies that, for all $m \in core(\nu)$, $m(E'_n) \uparrow m(A)$ and $m(E''_n) \uparrow m(A^c)$ if $E'_n \uparrow A$ and $E''_n \uparrow A^c$. Therefore, if $E_n \uparrow A$, then

$$m(E_n) = m(E_n\cap A) + m(E_n\cap A^c) \uparrow m(A) + m(A^c) = m(\Omega),$$

as desired. ■

B.2. Proof of Theorem 6.4

Prove the result first for $\nu \geq 0$. Since the game is differentiable at A and A^c, by Lemmas B.3 and B.4 every $m \in core(\nu)$ is non-atomic and countably additive.

Suppose that $core(\nu) \neq \emptyset$. By Lemma B.3, every all measure in $core(\nu)$ is convex-ranged. For convenience, assume that $P(\Omega) = 1 \in \mathbb{R}^N$. Set $x = P(A) \in [0,1]^N$ and let $m \in core(\nu)$.

Since $A \in \langle \emptyset, \Omega \rangle$, $P(A) \notin \{0, P(\Omega)\}$. Hence, by reversing the roles of A and A^c if necessary, we can assume that $\alpha P(A) = P(A^c)$ for some $\alpha \in (0,1)$. By the Lyapunov Theorem applied to the $N+1$ dimensional vector measure (P, m), there exists $\Sigma \ni E \subset A$ such that $P(E) = \alpha P(A)$ and $m(E) = \alpha m(A)$. Hence,

$$g(\alpha x) = g(P(E)) \leq m(E) = \alpha m(A) = \alpha g(P(A)) = \alpha g(x). \tag{B.2}$$

By Lemma 4.1, $\delta\nu(A^c; A) \leq m(A^c) = \nu(A^c)$. Since g is differentiable at x, this implies that $\nabla g(x) \cdot (1-x) \leq g(1-x)$. But $\alpha x = (1-x)$. Hence, using (B.2),

$$\nabla g(x) \cdot (\alpha x) \leq g(\alpha x) \leq \alpha g(x),$$

which implies $\nabla g(x) \cdot x \leq g(x)$, that is, $\delta\nu(A; A) \leq \nu(A)$. On the other hand, by Lemma 4.1, $\delta\nu(A; A) \geq \nu(A)$, and so $\delta\nu(A; A) = \nu(A)$. Then $\nabla g(x) \cdot x = g(x)$, so that

$$\nabla g(x) \cdot (1-x) = \nabla g(x) \cdot (\alpha x) = \alpha g(x) \geq g(\alpha x) = g(1-x),$$

that is, $\nabla g(x) \cdot (1-x) \geq g(1-x)$ and hence $\delta\nu(A^c; A) \geq \nu(A^c)$. Conclude that ν is efficient at A.

This completes the proof for $\nu \geq 0$. Let ν^* be the positive game provided by Lemma B.1. By Lemma B.2, *core* $(\nu) \neq \emptyset$ implies *core* $(\nu^*) \neq \emptyset$. Moreover, it is easy to check that $A \in \langle \emptyset, \Omega \rangle$ with respect to ν^* and that ν^* is differentiable at A and A^c. Hence ν^* is coherent at A when *core* $(\nu) \neq \emptyset$; and ν is coherent at A by Lemma B.2. In particular, it is now easy to see that *core* $(\nu) \subset \{\delta\nu(\cdot; A)\}$. ■

B.3. Proof of Theorem 6.5

Prove statement (i) for $\nu \geq 0$. By Lemmas B.3 and B.4, every $m \in$ *core* (ν) is countably additive and non-atomic.

There exists $\alpha \in (0,1)$ such that $P(A') = \alpha P(A)$. Apply the Lyapunov Theorem to (P, m), where $m \in$ *core* (ν). There exists $\Sigma \ni A_\alpha \subset A$ be such that $P(A_\alpha) = \alpha P(A)$ and $m(A_\alpha) = \alpha m(A)$. Because $P(A') = P(A_\alpha)$ and consequently, $P((A')^c) = P(A_\alpha^c)$, we have $A_\alpha \in \mathcal{A}$. Hence, $m(A_\alpha) = \nu(A_\alpha)$ for all $m \in$ *core* (ν). In particular,

$$g(\alpha x) = g(P(A_\alpha)) = m(A_\alpha) = \alpha m(A) = \alpha g(x).$$

We want to show that $g(\beta x) = \beta g(x)$ for all $1 \geq \beta \geq \alpha$. Suppose, *per contra*, that $g(\beta x) < \beta g(x)$ for some $1 \geq \beta > \alpha$. Since g is convex, $g(\beta x) \leq \beta g(x)$ and

$$g(\alpha x) = g\left(\frac{\alpha}{\beta}\beta x\right) \leq \frac{\alpha}{\beta} g(\beta x) < \frac{\alpha}{\beta}\beta g(x) = \alpha g(x),$$

a contradiction. Set $P(A) = x$. For t small enough, $(1-t) \geq \alpha$, so that

$$\lim_{t \downarrow 0} \frac{g(x) - g(x - tx)}{t} = \lim_{t \downarrow 0} \frac{tg(x)}{t} = g(x),$$

and so $\nabla g(x)\, x = g(x)$. Hence $\delta\nu(A; A) = \nu(A)$.

There exists $\alpha \in (0,1)$ such that $P(A'') = \alpha P(\Omega) + (1-\alpha) P(A)$. By the Lyapunov Theorem applied to (P, m), where $m \in \mathit{core}(\nu)$, there exists $\Sigma \ni A_\alpha \subset A$ such that $P(A_\alpha) = \alpha P(\Omega) + (1-\alpha) P(A)$ and $m(A_\alpha) = \alpha m(\Omega) + (1-\alpha) m(A)$. Because $P(A'') = P(A_\alpha)$ and $P((A'')^c) = P(A_\alpha^c)$, conclude that $A_\alpha \in \mathcal{A}$. Hence, $m(A_\alpha) = \nu(A_\alpha)$ for all $m \in \mathit{core}(\nu)$. In particular,

$$\begin{aligned} g(\alpha P(\Omega) + (1-\alpha) x) &= g(P(A_\alpha)) = m(A_\alpha) \\ &= \alpha m(\Omega) + (1-\alpha) m(A) = \alpha g(P(\Omega)) + (1-\alpha) g(x). \end{aligned}$$

We want to show that $g(\beta P(\Omega) + (1-\beta) x) = \beta g(P(\Omega)) + (1-\beta) g(x)$ for all $0 \leq \beta \leq \alpha$. If not, there exists some $0 \leq \beta < \alpha$ such that $g(\beta P(\Omega) + (1-\beta) x) < \beta g(P(\Omega)) + (1-\beta) g(x)$. Then

$$\begin{aligned} g(\alpha P(\Omega) + (1-\alpha) x) &= g\left(\frac{1-\alpha}{1-\beta} (\beta P(\Omega) + (1-\beta) x) + \left(\frac{\alpha - \beta}{1-\beta}\right) P(\Omega)\right) \\ &\leq \frac{1-\alpha}{1-\beta} g(\beta P(\Omega) + (1-\beta) x) + \left(\frac{\alpha-\beta}{1-\beta}\right) g(P(\Omega)) \\ &< \frac{1-\alpha}{1-\beta} (\beta g(P(\Omega)) + (1-\beta) g(x)) + \left(\frac{\alpha-\beta}{1-\beta}\right) g(P(\Omega)) \\ &= \alpha g(P(\Omega)) + (1-\alpha) g(x), \end{aligned}$$

a contradiction. Therefore,

$$\begin{aligned} \frac{g(x + t(P(\Omega) - x)) - g(x)}{t} &= \frac{g((1-t)x + tP(\Omega)) - g(x)}{t} \\ &= \frac{(1-t) g(x) + tg(P(\Omega)) - g(x)}{t} \\ &= \frac{tg(P(\Omega)) - tg(x)}{t} \\ &= g(P(\Omega)) - g(x) = g(P(A^c)). \end{aligned}$$

Since g is differentiable at x, this implies that $\nabla g(x)\, P(A^c) = g(P(A^c))$, that is, $\delta\nu(A^c; A) = \nu(A^c)$. Conclude that ν is coherent at A.

This completes the proof for the case $\nu \geq 0$. Let ν^* be the positive game provided by Lemma B.1 (such a ν^* exists because g is bounded below). By Lemma B.2, $core(\nu) \neq \emptyset$ implies $core(\nu^*) \neq \emptyset$. Hence, by what has been proven above, ν^* is coherent at A if $core(\nu) \neq \emptyset$. Again by Lemma B.2, this implies that ν is coherent at A if $core(\nu) \neq \emptyset$. Consequently, $core(\nu) = \{\delta\nu(\cdot; A)\}$.

Next prove statement (ii). We first prove the following claim:

Claim. Let $\nu = g(P) : \Sigma \to \mathbb{R}$ be a totally balanced Dini measure game. Then $g(\beta x) \leq \beta g(x)$ for all $\beta \in [0,1]$.

Proof of the Claim. Suppose that $\nu \geq 0$. Let $E \in \Sigma$ be such that $P(E) = x$. It is easy to see that the subgame $\nu_{|E}$ is superadditive. In addition, its dual $\overline{\nu}_{|E}$ is subadditive. In fact, for any pair E_1, E_2 of disjoint sets in Σ_E, there exists a measure m such that $m(E_1 \cup E_2) = \overline{\nu}_{|E}(E_1 \cup E_2)$ and $m(E) \leq \overline{\nu}_{|E}(E)$ for all $\Sigma \ni E \subseteq E_1 \cup E_2$. Hence, $\overline{\nu}_{|E}(E_1 \cup E_2) = m(E_1 \cup E_2) = m(E_1) + m(E_2) \leq \overline{\nu}_{|E}(E_1) + \overline{\nu}_{|E}(E_2)$. Therefore, for all $\{x_i\}_{i=1}^n \subset R(P)$ such that $\sum_{i=1}^n x_i \leq x$, we have

$$\begin{aligned} g\left(\sum_{i=1}^n x_i\right) &\geq \sum_{i=1}^n g(x_i) \quad \text{and} \\ g\left(x - \sum_{i=1}^n x_i\right) &\geq \sum_{i=1}^n g(x - x_i) + (1-n)\, g(x). \end{aligned}$$

Moreover, for all $\alpha, \beta \in [0,1]$, $g(\alpha x) \leq g(\beta x)$ if $\alpha \leq \beta$: Let $E \in \Sigma$ and $P(E) = \beta x$. By the Lyapunov Theorem there exists $\Sigma \ni E' \subset E$ such that $P(E') = \frac{\alpha}{\beta} P(E) = \alpha x$. Since ν is positive and superadditive, it is monotone. Thus $g(\alpha x) = \nu(E') \leq \nu(E) = g(\beta x)$.

We can now use a simple variation of an induction argument of Wasserman and Kadane [14, p. 1729] that we repeat here for the sake of completeness. Let $y = \frac{1}{k}x$, for some positive integer k. Then $g(y) \leq \frac{1}{k} g(x)$, as otherwise, $g(x) = g(ky) \geq kg(y) > k\frac{1}{k} g(x) = g(x)$, a contradiction. Similarly, $g(x-y) \leq \left(1 - \frac{1}{k}\right) g(x)$, since otherwise, we have the following contradiction:

$$\begin{aligned} g(y) &= g\left(x - \frac{k-1}{k}x\right) \geq (k-1)\, g\left(x - \frac{1}{k}x\right) + (2-k)\, g(x) \\ &> (k-1)\left(1 - \frac{1}{k}\right) g(x) + (2-k)\, g(x) = \frac{1}{k} g(x) \geq g(y). \end{aligned}$$

Suppose now that for some positive integer $h < k$, $g\left(\frac{r}{k}x\right) \geq \frac{r}{k}g\left(x\right)$ and $g\left(\left(1-\frac{r}{k}\right)x\right) \geq \left(1-\frac{r}{k}\right)g\left(x\right)$ for all $r \in \{1, ..., h\}$. Consider $r = h+1$. Then $g\left(\frac{r}{k}x\right) \leq \frac{r}{k}g\left(x\right)$: If not, by setting $s = \frac{k-h}{r}$, we reach the contradiction

$$g\left(\left(1-\frac{h}{k}\right)x\right) = g\left(s\frac{r}{k}x\right) \geq sg\left(\frac{r}{k}x\right) > s\frac{h+1}{k}g\left(x\right) = \left(1-\frac{h}{k}\right)g\left(x\right).$$

Conclude by induction that $g\left(\beta x\right) \leq \beta g\left(x\right)$ for all rational numbers $\beta \in (0,1)$. Next, let β be any real number in $(0,1)$. Given $\varepsilon > 0$ and a rational number $q \in (0,1)$ such that $q-\varepsilon < \beta \leq q$, then $g\left(\beta x\right) \leq g\left(qx\right) \leq qg\left(x\right) \leq \left(\beta+\varepsilon\right)g\left(x\right)$ for all $\varepsilon > 0$. Hence $g\left(\beta x\right) \leq \beta g\left(x\right)$.

If ν is not positive, let ν^* be the positive game provided by Lemma B.2. This game is totally balanced if ν is totally balanced. Thus

$$g\left(\beta x\right) = g^*\left(\beta x\right) - k\sum_{i=1}^{N}\beta x_i \leq \beta g^*\left(\beta x\right) - \beta k\sum_{i=1}^{N}x_i = \beta g\left(x\right),$$

which completes the proof of the Claim.

Assume that $\nu \geq 0$. By Lemmas B.3 and B.4, all measures in $core\left(\nu\right)$ are non-atomic and countably additive. There exists $\alpha \in (0,1)$ such that $P\left(A'\right) = \alpha P\left(A\right)$. By the Lyapunov Theorem applied to (P, m), where $m \in core\left(\nu\right)$, there exists $\Sigma \ni A_\alpha \subset A$ be such that $P\left(A_\alpha\right) = \alpha P\left(A\right)$ and $m\left(A_\alpha\right) = \alpha m\left(A\right)$. Because $P\left(A'\right) = P\left(A_\alpha\right)$ and $P\left(\left(A'\right)^c\right) = P\left(A_\alpha^c\right)$, we have $A_\alpha \in \mathcal{A}$. Hence, $m\left(A_\alpha\right) = \nu\left(A_\alpha\right)$ for all $m \in core\left(\nu\right)$. In particular,

$$g\left(\alpha x\right) = g\left(P\left(A_\alpha\right)\right) = m\left(A_\alpha\right) = \alpha m\left(A\right) = \alpha g\left(x\right).$$

We want to show that $g\left(\beta x\right) = \beta g\left(x\right)$ for all $\beta \geq \alpha$. If not, then $g\left(\beta x\right) < \beta g\left(x\right)$ for some $\beta > \alpha$ and

$$g\left(\alpha x\right) = g\left(\frac{\alpha}{\beta}\beta x\right) \leq \frac{\alpha}{\beta}g\left(\beta x\right) < \frac{\alpha}{\beta}\beta g\left(x\right) = \alpha g\left(x\right),$$

a contradiction. Set $P\left(A\right) = x$. For t small enough, $(1-t) \geq \alpha$, so that

$$\nabla g\left(x\right)x = \lim_{t\downarrow 0}\frac{g\left(x\right) - g\left(x-tx\right)}{t} = \lim_{t\downarrow 0}\frac{tg\left(x\right)}{t} = g\left(x\right).$$

Hence $\delta\nu\left(A; A\right) = \nu\left(A\right)$.

There exists $\alpha \in (0,1)$ such that $P(A'') = \alpha P(A) + (1-\alpha) P(\Omega)$. Then, $P((A'')^c) = \alpha P(A^c)$. Proceeding as before we get $\delta\nu(A^c; A^c) = \nu(A^c)$. Hence, ν is coherent at A and, for all $m \in \mathit{core}(\nu)$,

$$m(E) = \delta\nu(E \cap A; A) + \delta\nu(E \cap A^c; A^c).$$

This completes the proof for $\nu \geq 0$. Since ν is superadditive, we can use Lemma B.1, and thus the extension to Dini games is evident. ■

B.4. Proof of Theorem 6.6

Part (a): We first prove the result for $\nu \geq 0$. Under (ii), it is easy to see that $\nabla g(P(\Omega)) P(\Omega) \leq g(P(\Omega))$, that is $\delta\nu(\Omega; \Omega) \leq \nu(\Omega)$. Hence, by (5.3) ν is coherent at Ω, so that by Theorem 5.2, $\mathit{core}(\nu) \neq \emptyset$ if and only if $\mathit{core}(\nu) = \{\delta\nu(\cdot; \Omega)\}$. As to (i), suppose that $k < 1$. By the Lyapunov Theorem, for each $\alpha \in (0,1)$ there is $A_\alpha \in \Sigma$ such that $P(A_\alpha) = \alpha P(\Omega)$. We have,

$$\begin{aligned}\lim_{t\downarrow 0} \frac{g(P(\Omega)) - g(P(\Omega) - tP(A_\alpha))}{t} &\leq \lim_{t\downarrow 0} \frac{g(P(\Omega)) - (1-t\alpha)^k g(P(\Omega))}{t} \\ &< \lim_{t\downarrow 0} \frac{g(P(\Omega)) - (1-t\alpha) g(P(\Omega))}{t} = \alpha g(P(\Omega)),\end{aligned}$$

and so $\nabla g(P(\Omega)) P(A_\alpha) < \alpha g(P(\Omega))$, which implies $\nabla g(P(\Omega)) P(\Omega) < g(P(\Omega))$. Hence, ν is coherent at Ω but $\delta\nu(\cdot; \Omega) \notin \mathit{core}(\nu)$. By Theorem 5.2, $\mathit{core}(\nu) = \emptyset$. This completes the proof for positive measure games. Now, let ν be any Dini game and let ν^* be the positive game provided by Lemma B.1. It is easy to check that ν^* is super-homogeneous of degree $k \in [0,1]$ at $P(\Omega)$. Hence, $\mathit{core}(\nu^*) \subset \{\delta\nu^*(\cdot; \Omega)\}$ if $k = 1$ and $\mathit{core}(\nu^*) = \emptyset$ if $k < 1$. By Lemma B.2, $\mathit{core}(\nu) \subset \{\delta\nu(\cdot; \Omega)\}$ if $k = 1$ and $\mathit{core}(\nu) = \emptyset$ if $k < 1$.

Part (b): The only non-trivial part is that (iii) implies (ii). By Theorem 6.6, it suffices to show that $\delta\nu(\cdot; \Omega) \in \mathit{core}(\nu)$. Since $\delta\nu(\Omega; \Omega) = \nu(\Omega)$, this can be shown by proceeding as in the proof of Corollary 6.2, which needs only that $\nu(E) + \nu(E^c) \leq \nu(\Omega)$ for all $E \in \Sigma$. ■

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