ON THE SIZE OF THE ALGEBRAIC DIFFERENCE OF TWO RANDOM CANTOR SETS

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Abstract. In this paper we consider some families of random Cantor sets on the line and investigate the question whether the condition that the sum of Hausdorff dimension is larger than one implies the existence of interior points in the difference set of two independent copies. We prove that this is the case for the so called Mandelbrot percolation. On the other hand the same is not always true if we apply a slightly more general construction of random Cantor sets. We also present a complete solution for the deterministic case.

1. Introduction

Algebraic differences of Cantor sets occur naturally in the context of the dynamical behavior of diffeomorphisms. From these studies a conjecture by Palis([11]) originated, relating the size of the arithmetic difference $F_2 - F_1 = \{y - x : x \in F_1, y \in F_2\}$ to the Hausdorff dimensions of the two Cantor sets F_1 and F_2: if

$$\dim_{\mathrm{H}} F_1 + \dim_{\mathrm{H}} F_2 > 1$$

then *generically* it should be true that

$$F_2 - F_1 \text{ contains an interval.}$$

For generic dynamically generated *non-linear* Cantor sets this was proved in 2001 by de Moreira and Yoccoz ([10]). The problem is open for generic linear Cantor sets. The problem was put into a probabilistic context by Per Larsson in his thesis [7], (see also [8]). He considers a two parameter family of random Cantor sets $F_{a,b}$, and obtains that the Palis conjecture holds for a set of a and b of full Lebesgue measure. However Larsson's proof contains errors and significant gaps. In a forthcoming paper the authors of the present paper will correct these errors and fill the gaps in Larsson's

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proof. Here we will study Palis' conjecture for a natural class of random Cantor sets considered e.g. in [4], [1], [5] and [2]. A special member of this class was already considered in 1974 by Mandelbrot ([9]).

2. Random Cantor sets

Given are $M \geq 2$ and the vector $\mathbf{p} := (p_0, \dots, p_{M-1}) \in [0,1]^M$, in general *not* a probability vector and $p_i = 0$ or 1 are also allowed.
Let $\mathcal{T}$ be the M-adic tree. For each n $\mathcal{T}$ has M^n nodes at level n, which we denote by strings $\underline{i}_n = i_1 \dots i_n$, where $i_k \in \{0, \dots, M-1\}$ for $k = 1, \dots, n$. There is one node at level 0, the root, denoted $\emptyset$. We consider a probability measure $\mathbb{P}_\mathbf{p}$ on the space of labeled trees, i.e., each node $i_1 \dots i_n$ obtains a label $X_{i_1 \dots i_n}$ which will be 0 or 1. The probability measure is defined by requiring that the $X_{i_1 \dots i_n}$ are independent Bernoulli random variables, with $\mathbb{P}_\mathbf{p}(X_\emptyset = 1) = 1$, and for $n \geq 1$ and $i_1 \dots i_n \in \{0, \dots, M-1\}^n$

$$\mathbb{P}_\mathbf{p}(X_{i_1 \dots i_n} = 1) = p_{i_n}.$$

In particular, when the $X_{i_1 \dots i_n}$ are i.i.d.—i.e. $p_i = p$ for all i—then $\mathbb{P}_\mathbf{p}$ will generate Mandelbrot percolation.
The randomly labeled tree generates a random Cantor set in [0,1] in the following way. Define

$$I_{i_1 \dots i_n} := \left[\frac{i_1}{M} + \frac{i_2}{M^2} + \dots + \frac{i_n}{M^n}, \frac{i_1}{M} + \frac{i_2}{M^2} + \dots + \frac{i_n}{M^n} + \frac{i_n + 1}{M^n} \right].$$

The n-th level approximation F^n of the random Cantor set is a union of such n-th level M-adic intervals selected by the sets S_n defined by

$$S_n = \{ i_1 \dots i_n : X_{i_1} = X_{i_1 i_2} = \dots = X_{i_1 \dots i_n} = 1 \}.$$

The random Cantor set F is

$$F = \bigcap_{n=1}^{\infty} F^n = \bigcap_{n=1}^{\infty} \bigcup_{i_1 \dots i_n \in S_n} I_{i_1 \dots i_n}.$$

Let $Z_n = \mathrm{Card}(S_n)$ be the number of non-empty intervals $I_{i_1 \dots i_n}$ in F^n and let $Z_0 := 1$. Then $(Z_n)_{n \in \mathbb{N}}$ is a branching process with offspring distribution the law of Z_1. Namely, let $\xi_i^{(n)}$, for $i, n \geq 1$ be i.i.d. random variables such that $\xi_i^{(n)} \stackrel{\mathrm{d}}{=} Z_1$. Then

$$Z_{n+1} := \begin{cases} \xi_1^{(n+1)} + \dots + \xi_{Z_n}^{(n+1)}, & \text{if } Z_n > 0; \\ 0, & \text{if } Z_n = 0. \end{cases}$$

Note that

$$\mathbb{E}_\mathbf{p}(Z_1) = \mathbb{E}_\mathbf{p}(X_0 + \dots + X_{M-1}) = p_0 + \dots + p_{M-1}.$$

Therefore the branching process will almost surely die out—and F will be empty—if this expectation is smaller than 1. Hence we will assume from now on that

$$\sum_{k=0}^{M-1} p_k > 1. \tag{2.1}$$

The expectation also determines the Hausdorff dimension $\dim_{\mathrm{H}} F$ of F; it is well known ([4] or [6]) that:

Fact 1. $\dim_{\mathrm{H}} F = \log\left(\sum_{k=0}^{M-1} p_k\right) / \log M$ *almost surely on* $F \neq \emptyset$.

3. Differences of Random Cantor sets

Let F_1, F_2 be two independent copies of the random Cantor set F above. From now on $\mathbb{P}$ will denote the product probability $\mathbb{P}_{\mathbf{p}} \times \mathbb{P}_{\mathbf{p}}$. Let F_1^n and F_2^n be the corresponding n-th level approximants of F_1 and F_2, so

$$F_i := \bigcap_{n=1}^{\infty} F_i^n, \text{ for } i = 1, 2.$$

Our aim here is to investigate whether the difference set

$$F_2 - F_1 = \{y : \exists\, x_i \in F_i, y = x_2 - x_1\}$$

contains an interval. It is immediate that

Fact 2. *For a set $A \subset \mathbb{R}^2$ we denote the projection of A on the y axis along lines having a 45° angle with the x axis by $Proj_{45^\circ}(A)$. Then*

$$F_2 - F_1 = Proj_{45^\circ}(F_1 \times F_2).$$

In this way, if $\dim_{\mathrm{H}} F < \frac{1}{2}$ then $\dim_{\mathrm{H}}(F_2 - F_1) < 1$, so it does not contain any interval. By Fact 1, this happens if and only if $\sum_{k=0}^{M-1} p_k < \sqrt{M}$. So, we may hope to find an interval in $F_2 - F_1$ only if the following condition holds:

$$\dim_{\mathrm{H}} F_1 + \dim_{\mathrm{H}} F_2 > 1, \text{ that is } \sum_{k=0}^{M-1} p_k > \sqrt{M}. \tag{3.1}$$

Define $p_{M+j} = p_j$ for $j = 0, 1, \ldots, M$. Now we can define the *cyclic autocorrelations* γ_k by

$$\gamma_k := \sum_{j=0}^{M-1} p_j p_{j+k} \quad \text{for} \quad k = 0, \ldots, M.$$

Theorem 1. *Conditional on $F_1, F_2 \neq \emptyset$, we have*

- **(a):** *If $\gamma_k > 1$ for all k then $F_2 - F_1$ contains an interval almost surely.*
- **(b):** *If there exists an $k \in \{0, \dots, M-1\}$ such that γ_k and γ_{k+1} are both less than 1 then $F_2 - F_1$ almost surely does not contain any intervals.*

In the case of the Mandelbrot percolation all $p_i = p$ for some $0 \leq p \leq 1$. In this case $\gamma_k = Mp^2$ for all k. With Fact 1 we obtain the following corollary.

Corollary 1. *The Palis conjecture holds for Mandelbrot percolation. That is, if F is Mandelbrot percolation, then $\dim_{\mathrm{H}} F_1 + \dim_{\mathrm{H}} F_2 < 1$ implies that $F_2 - F_1$ almost surely contains no interval, and $\dim_{\mathrm{H}} F_1 + \dim_{\mathrm{H}} F_2 > 1$ implies that $F_2 - F_1$ almost surely does contain an interval (conditional on F_1, F_2 being non-empty).*

4. Comments on Theorem 1

4.1. **Exceptional behaviour.** It can happen that Condition (3.1) holds but almost surely, $F_2 - F_1$ does not contain an interval. Let $M = 3$ and for a small number $\varepsilon > 0$ (say, $\varepsilon < 1/4$) let $p_0 = 1, p_1 = 0, p_2 = 1 - \varepsilon$. (This is almost the triadic Cantor set with the difference that the second interval is chosen with probability less than one.) Then Condition (3.1) holds, but $\gamma_1 = \gamma_2 = 1 - \varepsilon < 1$, so almost surely there is no interval in $F_2 - F_1$. That is, the so-called Palis Conjecture does not hold.

4.2. **Scope of the theorem.** In the general case it can happen that for some k, $\gamma_k < 1$ but $\gamma_{k+1} > 1$. In this case our theorem is inconclusive (see Section 7 for a further discussion). However, if $M = 3$ then $\gamma_0 \geq \gamma_1 = \gamma_2$. Thus, if $p_0p_1 + p_1p_2 + p_2p_0 > 1$ then $F_2 - F_1$ almost surely contains an interval given that F_1, F_2 are non empty. On the other hand, if $p_0p_1 + p_1p_2 + p_2p_0 < 1$ then $F_2 - F_1$ does not contain any interval almost surely.

4.3. **The deterministic case.** In the case that all $p_i \in \{0, 1\}$ we have a complete answer to the question whether $F_2 - F_1$ contains an interval or not. This will be given in Section 8.

4.4. **A generalisation.** Theorem 1 remains true when we consider the difference set of two independent Cantor sets F_1 and F_2 generated by two different p-vectors of the same length; the autocorrelations simply have to be replaced by cross correlations. Assume that the probabilities for F_1 are $p_0, \dots, p_{M-1}$ and for F_2 the probabilities are $q_0, \dots, q_{M-1}$. Then to get

$$\dim_{\mathrm{H}} F_1 + \dim_{\mathrm{H}} F_2 > 1$$

we need to assume that

$$\sum_{i=0}^{M-1} p_i \cdot \sum_{j=0}^{M-1} q_j > M.$$

The cross correlations are:

$$\gamma_k := \sum_{j=0}^{M-1} q_j p_{j+k}$$

With this all calculations will be the same except for a small adaptation of the proof of Lemma 2. The obvious generalization of Corollary 1 remains true.

5. Counting triangles

Before we start the proof of Theorem 1 we would like to introduce some notation. Since it is easier to study 90° projections we rotate the $[0,1]\times[0,1]$ square by 45° in the positive direction and translate it, so that its horizontal diagonal, let us call it J, is the $\left[-\frac{1}{2}\sqrt{2}, \frac{1}{2}\sqrt{2}\right]$ interval on the x axis. Let this transformation be called φ, and let

$$Q := \varphi([0,1]\times[0,1]), \quad \Lambda^n := \varphi(F_1^n \times F_2^n), \quad \Lambda := \varphi(F_1 \times F_2).$$

In this way instead of the 45° degree projection Proj_{45° of $F_1 \times F_2$ to the y axis, it is equivalent to consider the orthogonal projection of Λ to J.
The image under φ of the square $I_{i_1\dots i_n}\times I_{j_1\dots j_n}$ is denoted $Q_{i_1\dots i_n, j_1\dots j_n}$. Every n-th level square $Q_{i_1\dots i_n, j_1\dots j_n}$ is divided into two congruent triangles by its vertical diagonal. The one which is on the left side is denoted $L_{\underline{i}_n, \underline{j}_n}$. We call $L_{i_1\dots i_n, j_1\dots j_n}$ an n-th level L-triangle. The other part of the square $Q_{\underline{i}_n, \underline{j}_n}$ is denoted $R_{\underline{i}_n, \underline{j}_n}$. In the same way, we divide the square Q into two triangles L and R , as in Figure 1. Note that Λ satisfy a symmetry property: if we replace (x,y) by $(-x,y)$ (i.e. $(\underline{i}_n, \underline{j}_n)$ by $(\underline{j}_n, \underline{i}_n)$ at level n) then $L \cap \Lambda$ is mapped to $R \cap \Lambda$ and vice versa. Moreover, since this corresponds to replacing $F_1 \times F_2$ by $F_2 \times F_1$, $\mathbb{P}$ is invariant for this mirroring. It follows that properties that we deduce for $R \cap \Lambda$ will also hold for $L \cap \Lambda$. For this reason, and to simplify the statements, several of the following results are formulated for the R-triangle only.
The orthogonal projection (any projection from now on will be meant to be orthogonal) of the n-th level L- and R-triangles to $\left[0, \frac{1}{2}\sqrt{2}\right]$ are M^n intervals of length $\frac{1}{2}\sqrt{2} \cdot M^{-n}$. We denote them in the following way:

$$J_{k_1\dots k_n} := \frac{1}{2}\sqrt{2} \cdot I_{k_1\dots k_n}.$$

The intervals in $\left[-\frac{1}{2}\sqrt{2}, 0\right]$ will be denoted

$$J^-_{k_1\dots k_n} := J_{k_1\dots k_n} - \tfrac{1}{2}\sqrt{2}.$$

Now we introduce the appropriate vertical columns intersecting triangle L respectively R: we write

$$C^-_{k_1\dots k_n} := J^-_{k_1\dots k_n} \times \mathbb{R}, \qquad C_{k_1\dots k_n} := J_{k_1\dots k_n} \times \mathbb{R}.$$

We are going to count the number of L- and R-triangles in these columns. The idea is that as long as there are L- and R-triangles in $C_{k_1\dots k_n}$, then the interval $J_{k_1\dots k_n}$ is in the projection of Λ^n. Let for $U, V \in \{L, R\}$ the number

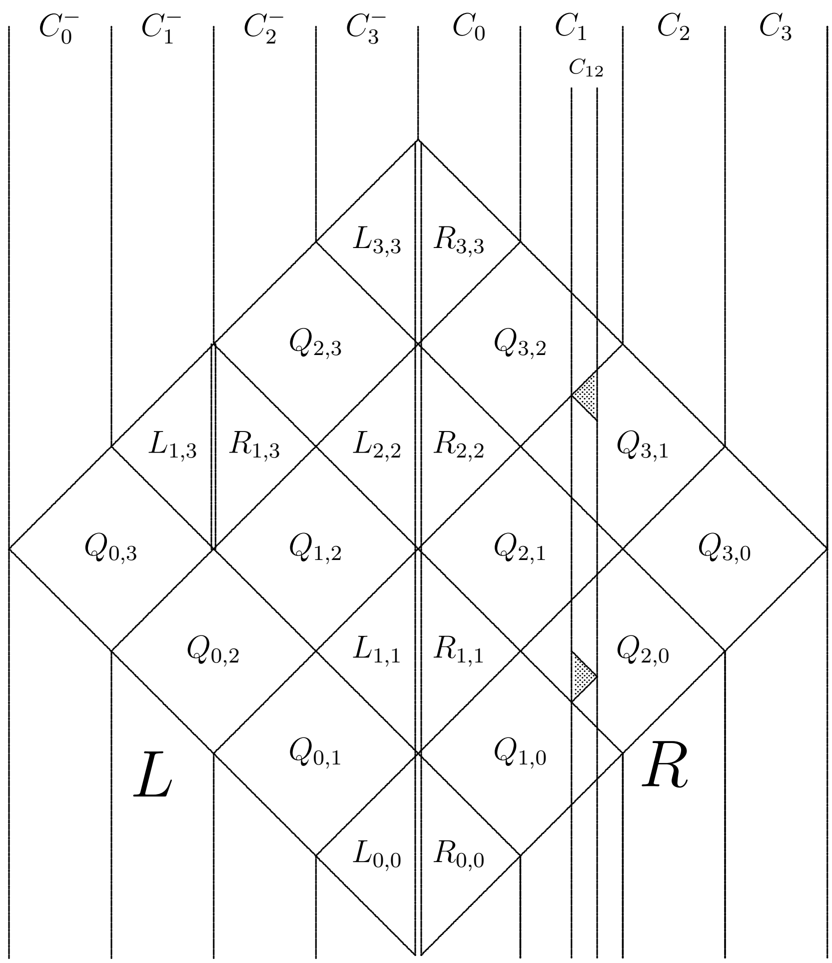

FIGURE 1. The case $M = 4$. The square Q split into two triangles L and R with level 1 squares, triangles and columns. Also shown is level 2 column C_{12} with a Δ-pair.

of level 1 V-triangles in $\Lambda^1 \cap C_k^-$ (if $U = L$), respectively $\Lambda^1 \cap C_k$ (if $U = R$), generated by a level 0 U-triangle be denoted by $Z^{UV}(k)$. So for instance

$$Z^{LR}(k) = \#\left\{(i,j) : Q_{i,j} \subset \Lambda^1, R_{i,j} \subset C_k^-\right\}.$$

More generally, we denote by $Z^{UV}(\underline{k}_n)$ the number of level n V-triangles in $\Lambda^n \cap C_{\underline{k}_n}^-$ (respectively $\Lambda^n \cap C_{\underline{k}_n}$) generated by a level 0 U-triangle. Let

$$\mathcal{M}(\underline{k}_n) := \begin{bmatrix} \mathbb{E}Z^{LL}(\underline{k}_n) & \mathbb{E}Z^{LR}(\underline{k}_n) \\ \mathbb{E}Z^{RL}(\underline{k}_n) & \mathbb{E}Z^{RR}(\underline{k}_n) \end{bmatrix}.$$

Then from the definition one can easily check that

$$\mathcal{M}(k_1 \dots k_n) = \mathcal{M}(k_1) \cdots \mathcal{M}(k_n). \tag{5.1}$$

In the context of branching processes this is an obvious property: for a fixed sequence $(k_1, k_2, \dots, k_n, \dots)$ the process $(Z^{UV}(\underline{k}_n))$ is a two type branching process in a varying environment with neighbour interaction. Actually we will be needing another process that is not a multi type branching process, but a superbranching process ([1]) in varying environment, which also has neighbour interaction. The reason we need this process is that an R-triangle will have very few offspring (next level L- and R-triangles) in the right most subcolumns in which it occurs, and possibly a lot in the left most columns. To balance this asymmetry we will pair L-triangles and R-triangles.

Let $n \geq 1$. Any pair $(L^n, R^n) = (L_{i_1 \dots i_n, j_1 \dots j_n}, R_{i'_1 \dots i'_n, j'_1 \dots j'_n})$ of *disjoint* n-th level L-triangles and R-triangles with $L^n \subset C_{\underline{k}_n}$ and $R^n \subset C_{\underline{k}_n}$ for some $\underline{k}_n$ is called a level n Δ-pair (cf. Figure 1). Disjoint means that they are not allowed to share an edge, so that their offspring distributions can not interact. We try to find as many Δ-pairs as possible in a column, where an L-triangle or R-triangle is allowed to belong to at most one Δ-pair. It can be proved by mathematical induction that for $m \geq 3$ we can find m Δ-pairs as soon as m L-triangles, and at least m R-triangles occur, or vice versa. We then say that m Δ-pairs occur in the column. To analyze the process of Δ-pairs we consider the number of V-triangles generated in columns $C_{\underline{k}_n}^-$ and $C_{\underline{k}_n}$ by the level 0 triangles L *and* R. This is equal to

$$Z^V(\underline{k}_n) := Z^{LV}(\underline{k}_n) + Z^{RV}(\underline{k}_n).$$

Note (cf. Figure 1) that for each $k \in \{0, \dots, M-1\}$:

$$L_{i,j} \in C_k \Leftrightarrow i - j = k + 1 \ (\mathrm{mod} M), \quad R_{i,j} \in C_k \Leftrightarrow i - j = k \ (\mathrm{mod} M),$$

and that this also holds for the C_k^- columns.

Since

$$\mathbb{P}(Q_{i,j} \in \Lambda^1) = \mathbb{P}(I_i \in F_1^1, I_j \in F_2^1) = \mathbb{P}_{\mathbf{p}}(I_i \in F_1^1)\mathbb{P}_{\mathbf{p}}(I_j \in F_2^1) = p_i p_j,$$

we obtain that for $k \in \{0, \dots, M-1\}$

$$\mathbb{E}Z^L(k) = \gamma_{k+1}, \qquad \mathbb{E}Z^R(k) = \gamma_k. \tag{5.2}$$

This is the reason that in the statement and the proof of Theorem 1 the number

$$\gamma := \min_{V \in \{L,R\}, 0 \le k \le M-1} \mathbb{E}Z^V(k) = \min_{0 \le k \le M-1} \gamma_k$$

has an important role.

We will need in the proof of Lemma 4 that there is a positive probability that the number of Δ-pairs grows exponentially fast in all columns. When all p_i are positive this is trivial. The following lemma deals with the case where some p_i may be zero.

Lemma 1. *For any n*

$$\mathbb{P}\left(Z^L(\underline{k}_n) \ge \gamma^n \text{ and } Z^R(\underline{k}_n) \ge \gamma^n \text{ for all } \underline{k}_n\right) > 0.$$

Proof. We want to know the maximal number of level n L-triangles and R-triangles that occur in the columns $C_{\underline{k}_n}$ and $C^-_{\underline{k}_n}$ with positive $\mathbb{P}$ probability. Let $p^*_j = p_j$ if $p_j = 0$, and $p^*_j = 1$ if $p_j > 0$. For $k = 0, \dots, M-1$ we denote the expectation matrices generated by the vector $(p^*_0, \dots, p^*_{M-1})$ by $\mathcal{M}^*(k)$. Note that if all the level n squares $Q_{i_1 \dots i_n, j_1 \dots j_n}$ for which $p_{i_1 \dots i_n} > 0$ and $p_{j_1 \dots j_n} > 0$ are selected (which happens with positive probability) then $Z^L(\underline{k}_n)$ ($Z^R(\underline{k}_n)$) is the sum of the two elements in the first column (second column) of $\mathcal{M}^*(\underline{k}_n)$ respectively. Now note that (with all inequalities componentwise)

$$\big(1\ 1\big)\mathcal{M}^*(\underline{k}_n) \ge \big(1\ 1\big)\mathcal{M}(\underline{k}_n) \ge \big(\gamma^n\ \gamma^n\big), \tag{5.3}$$

since the fact that the sum of the two elements of both of the columns of $\mathcal{M}(k)$ is greater than γ for every k implies that the sum of the two elements of both of the columns of $\mathcal{M}(\underline{k}_n)$ is larger than γ^n. The claim follows now directly from (5.3). $\square$

6. The proof of Theorem 1

First we show that we can start the Δ-pair process in R at level 2.

Lemma 2. *Let p_Δ be the probability that $C_{00} \cap \Lambda^2$ contains a level 2 Δ-pair. If $\gamma > 1$ then $p_\Delta > 0$.*

Proof. We know that $\gamma_{M-1} = p_0 p_{M-1} + p_1 p_0 + \cdots p_{M-2} p_{M-3} + p_{M-1} p_{M-2} > 1$. This means that at least one of the last $M-1$ terms is not zero. So, there is an $0 \le i \le M-2$ such that both $p_i > 0$ and $p_{i+1} > 0$. Then $Q_{i(i+1),ii}$ is selected with probability $p_i^3 p_{i+1}$ and $Q_{(i+1)(i+1),(i+1)(i+1)}$ is selected with probability p^4_{i+1}. Using that $L_{i(i+1),ii} = Q_{i(i+1),ii} \cap C_{00}$ and

$R_{(i+1)(i+1),(i+1)(i+1)} = Q_{(i+1)(i+1),(i+1)(i+1)} \cap C_{00}$ we obtain that with probability $p_i^3 p_{i+1} p_{i+1}^4$ we select the Δ-pair $(L_{i(i+1),ii}, R_{(i+1)(i+1),(i+1)(i+1)})$ in C_{00}. Thus $p_\Delta \geq p_i^3 p_{i+1}^5 > 0$. □

It follows from the self-similarity of the construction that the following fact is true.

Fact 3. *Let (L^n, R^n) be a n-th level Δ-pair in some column $C_{\underline{k}_n}$. Consider the following conditional probability:*

$$\mathbb{P}(Proj_{90^\circ}((L^n \cup R^n) \cap \Lambda) = J_{\underline{k}_n} |\, L^n \subset \Lambda^n, R^n \subset \Lambda^n).$$

Then this probability, denoted p_J, is independent of n, $\underline{k}_n$, and the choice of L^n and R^n.

Proposition 1. *Assume that $\gamma > 1$. Then $p_J > 0$.*

In the sequel we will denote

$$N(\underline{k}_n) := \min\left\{Z^L(00\underline{k}_n), Z^R(00\underline{k}_n)\right\}.$$

Note that $N(\underline{k}_n)$ counts the number of level $n+2$ Δ-pairs in subcolumns of C_{00}, and that by Lemma 2 we know that we start in C_{00} with a level 2 Δ-pair with positive probability.
Lemma 4 below will directly imply Proposition 1. In Lemma 4 we apply the large deviation theorem in the same way as in Falconer and Grimmett ([2]). Unfortunately in our case the appropriate random variables are not pairwise independent. To handle this problem we first prove a lemma which implies that $N(\underline{k}_n)$ level $n+2$ L-triangles and $N(\underline{k}_n)$ level $n+2$ R-triangles in column $C_{00\underline{k}_n}$ can be paired into Δ-pairs such that these $N(\underline{k}_n)$ Δ-pairs can be divided into three groups (of approximately the same cardinality) with the following property: any two triangles (left or right) from any two pairs from the same group are disjoint. This will provide the required independence. For each $\underline{k}_n$ we consider *all* the left and right triangles in column $C_{00\underline{k}_n}$. Let $K = K(\underline{k}_n)$ be their cardinality. We can naturally label these K triangles with $\{1, \ldots, K\}$ in the order in which they appear in the column, starting at the bottom. Then the odd numbers correspond to the level $n+2$ R-triangles of $C_{00\underline{k}_n} \cap \Lambda^{n+2}$ and the even numbers to the L-triangles. The assumption that an L and R triangle form a Δ-pair is equivalent to the assumption that the corresponding even and odd numbers are not consecutive. Through this identification the following combinatorial lemma ensures the division into three groups announced above.

Lemma 3. *We are given N distinct odd numbers $o_1, \ldots, o_N$ and N distinct even numbers $e_1, \ldots, e_N$. Then we can couple the odd numbers with the even numbers and we can color the N couples with three colors (say* `r`, `g` *and* `b`*) such that no two numbers in pairs of the same color are adjacent and all* *colors are used for at least* $\lfloor N/3 \rfloor$ *pairs. That is, there exists a permutation* π *of* $\{1,\dots,N\}$ *such that we can color the pairs*

$$(e_1, o_{\pi(1)}), \dots, (e_N, o_{\pi(N)})$$

with the three colors such that with each color we painted at least $\lfloor N/3 \rfloor$ *pairs and for any (also if* $\ell = k$*)* $(e_k, o_{\pi(k)})$ *and* $(e_\ell, o_{\pi(\ell)})$ *having the same color it is true that:*

$$|e_\ell - o_{\pi(k)}| > 1.$$

The proof of this three color lemma will be given in the appendix.
The following key lemma, and its proof, are very similar to the main result on orthogonal projections of random Cantor sets in [2].

Lemma 4. *Assume that* $\gamma > 1$*. Then*

$$\mathbb{P}\left(N(\underline{k}_n) > 0 \;\forall \underline{k}_n \in \{0,\dots,M-1\}^n \text{ for all } n\right) > 0$$

holds.

Proof. Using that $\mathbb{E}Z^V(k) \geq \gamma$ for $V \in \{L,R\}$ and $k \in \{0,\dots,M-1\}$, it follows from Large Deviation Theory that we can choose an η' with $1 < \eta' < \min\{2,\gamma\}$ and $0 < \delta < 1$ such that

$$\mathbb{P}\left(Z_1^V(k) + \cdots + Z_q^V(k) < q\eta'\right) \leq \delta^q \tag{6.1}$$

for all $q \geq 1$, whenever $Z_1^V(k), Z_2^V(k)\dots$ are independent random variables with the same distribution as $Z^V(k)$. Fix an $1 < \eta < \eta'$ and choose n_0 such that for all $n \geq n_0$

$$\eta \cdot \left(\left\lfloor \frac{\eta^n}{3} \right\rfloor + 1 \right) < \eta' \cdot \left\lfloor \frac{\eta^n}{3} \right\rfloor. \tag{6.2}$$

Let

$$A_n := \left\{ N(\underline{k}_n) \geq \eta^n : \forall \underline{k}_n \in \{0,\dots,M-1\}^n \right\}.$$

It follows from Lemma 1 and Lemma 2 that for all $n \geq 1$ we have

$$\mathbb{P}(A_n) > 0. \tag{6.3}$$

To continue the proof we have to get rid of possible dependence between Δ-pairs. Fix an arbitrary $\underline{k}_n$ and k. Let

$$N := 3 \cdot \left\lfloor \frac{N(\underline{k}_n)}{3} \right\rfloor.$$

Using Lemma 3 we can we can label and then pair the level $n+2$ left and right triangles of $C_{00\underline{k}_n} \cap \Lambda^{n+2}$ into N Δ-pairs $(L_1,R_1),\dots,(L_N,R_N)$ such that for every $i = 0,1,2$ we have that all the triangles

$$L_{iN/3+1}, R_{iN/3+1}, \dots, L_{(i+1)N/3}, R_{(i+1)N/3} \text{ are disjoint.} \tag{6.4}$$

For every $i = 0,1,2$ and $1 \leq j \leq N/3$ we denote the Δ-pairs $D_j^i := (L_{iN/3+j}, R_{iN/3+j})$ and we write $\widetilde{Z}^V_{iN/3+j}(k)$ for the number of level $n+3$ V triangles in $C_{00\underline{k}_n k} \cap D_j^i$. Note that for every i it follows from (6.4) that the $N/3$ random variables

$$\widetilde{Z}^V_{iN/3+1}(k), \dots, \widetilde{Z}^V_{(i+1)N/3}(k)$$

are independent and each of them has the same distribution as $Z^V(k)$. Now we define

$$S_i^V(00\underline{k}_n k) := \widetilde{Z}^V_{iN/3+1}(k) + \cdots + \widetilde{Z}^V_{(i+1)N/3}(k).$$

So, for any $\underline{k}_n$, k and $V \in \{L, R\}$ we have

$$Z^V(00\underline{k}_n k) \geq \sum_{i=0}^{2} S_i^V(00\underline{k}_n k).$$

This directly implies that

$$\begin{aligned}\mathbb{P}\left(Z^V(00\underline{k}_n k) < \eta^{n+1} | A_n\right) &\leq \sum_{i=0}^{2} \mathbb{P}\left(S_i^V(00\underline{k}_n k) < \eta\left(\left\lfloor \frac{\eta^n}{3} \right\rfloor + 1\right) \Big| A_n\right) \\ &\leq \sum_{i=0}^{2} \mathbb{P}\left(\widetilde{Z}^V_{iN/3+1}(k) + \cdots + \widetilde{Z}^V_{(i+1)N/3}(k) < \eta\left(\left\lfloor \frac{\eta^n}{3} \right\rfloor + 1\right) \Big| A_n\right),\end{aligned}$$

and thus, using that $N \geq \eta^n$ on A_n, that the $Z_i^V(k)$ are independent of A_n, and using (6.2) and (6.1) we obtain

$$\begin{aligned}\mathbb{P}\left(Z^V(00\underline{k}_n k) < \eta^{n+1} | A_n\right) &\leq 3 \cdot \mathbb{P}\left(Z_1^V(k) + \cdots + Z^V_{\lfloor \eta^n/3 \rfloor}(k) < \eta' \left\lfloor \frac{\eta^n}{3} \right\rfloor\right) \\ &\leq 3\delta^{\lfloor \eta^n/3 \rfloor}.\end{aligned}$$

So

$$\begin{aligned}\mathbb{P}\left(A_{n+1}^c | A_n\right) &= \mathbb{P}\left(\bigcup_{V \in \{L,R\}} \bigcup_{\underline{k}_n} \bigcup_{k} Z^V(00\underline{k}_n k) < \eta^{n+1} \Big| A_n\right) \\ &\leq \sum_{V \in \{L,R\}} \sum_{\underline{k}_n} \sum_{k} \mathbb{P}\left(Z^V(00\underline{k}_n k) < \eta^{n+1} \Big| A_n\right) \\ &\leq 6 \cdot M^n \cdot M \cdot \delta^{\lfloor \eta^n/3 \rfloor}.\end{aligned}$$

Using this and the fact that for any $r \leq n$, we have $\mathbb{P}\left(A_{n+1}^c | A_r \cap \cdots \cap A_n\right) = \mathbb{P}\left(A_{n+1}^c | A_n\right)$ we obtain that

$$\mathbb{P}\left(A_{n+1}^c | A_r \cap \cdots \cap A_n\right) \leq 6M^{n+1}\delta^{\lfloor \eta^n/3 \rfloor},$$

holds for all $r \leq n$. Therefore for all $r \leq n$,

$$\begin{aligned}\mathbb{P}\left(A_{n+1} \cap \cdots \cap A_r\right) &\geq \left(1-6M^{n+1}\delta^{\lfloor \eta^n/3 \rfloor}\right)\mathbb{P}\left(A_n \cap \cdots \cap A_r\right)\\ &\geq \mathbb{P}(A_r)\prod_{k=r}^{n}\left(1-6M^{k+1}\delta^{\lfloor \eta^k/3 \rfloor}\right).\end{aligned}$$

Choose $r \geq n_0$ such that $\prod_{n=r}^{\infty}\left(1-6M^{n+1}\delta^{\lfloor \eta^n/3 \rfloor}\right) > 0$. Using (6.3) this implies that $c := \mathbb{P}(A_r)\prod_{n=r}^{\infty}\left(1-6M^{n+1}\delta^{\lfloor \eta^n/3 \rfloor}\right) > 0$. Thus

$$\mathbb{P}\left(A_n \text{ holds for all } n \geq r\right) \geq c.$$

This immediately implies the statement of the Lemma. □

Corollary 2. *Let $\gamma > 1$. For every n and $(i_1 \dots i_n, j_1 \dots j_n)$ the conditional probability of the event that the projection of $\Lambda \cap Q_{i_1 \dots i_n, j_1 \dots j_n}$ to J contains an interval given that $Q_{i_1 \dots i_n, j_1 \dots j_n} \subset \Lambda^n$ is at least $p_\Delta \cdot p_J > 0$.*

Proof. By symmetry it suffices to prove this for squares such that $R_{i_1 \dots i_n, j_1 \dots j_n}$ is contained in R. Let $C_{\underline{k}_n}$ be the column that contains $R_{i_1 \dots i_n, j_1 \dots j_n}$. Then $C_{\underline{k}_n 00}$ will contain a Δ-pair of level $n+2$ with probability not less than p_Δ (see Lemma 2). Then by Fact 3 and Proposition 1 the probability that the projection of this Δ-pair intersected with Λ is the interval $J_{\underline{k}_n 00}$ is at least p_J. □

From here we can finish the proof of our theorem as in the proof of Theorem 1 in [2]. However, because there is a lot of dependence between the squares in Λ^n, our proof of part (a) is slightly more involved.

Proof of Theorem 1 (a). Here we assume that $\gamma > 1$. We call two squares $Q_{i_1 \dots i_n, j_1 \dots j_n}$ and $Q_{i'_1 \dots i'_n, j'_1 \dots j'_n}$ *unaligned* if both $i_1 \dots i_n \neq i'_1 \dots i'_n$ and $j_1 \dots j_n \neq j'_1 \dots j'_n$. For every n let $K(n)$ be the maximal number of pairwise unaligned squares of Λ^n. Then, by maximality of $K(n)$, we can cover Λ with $K(n) \cdot 2M^n$ squares of side M^{-n}. So, conditioned on $\Lambda \neq \emptyset$ the Hausdorff dimension of Λ almost surely satisfies

$$1 < \dim_{\mathrm{H}}(\Lambda) \leq \lim_{n \to \infty} \frac{\log\left(K(n) \cdot 2M^n\right)}{\log M^n}.$$

Here $\dim_{\mathrm{H}}(\Lambda) > 1$ follows from the hypothesis $\gamma_k > 1$ for all k (cf. condition (3.1), which is equivalent to $\gamma_0 + \cdots + \gamma_{M-1} > M$).
We obtained that

$$\{\Lambda \neq \emptyset\} \subset \{\lim_{n \to \infty} K(n) = \infty\}. \tag{6.5}$$

For every n we fix a system $\{Q_1^n, Q_2^n, \dots, Q_{K(n)}^n\}$ of pairwise unaligned n-squares contained in Λ^n which has cardinality $K(n)$. Let

$$\mathcal{C}_s^n := \{\mathrm{int}(\mathrm{Proj}_{90^\circ}(Q_s^n \cap \Lambda)) = \emptyset\},$$

and

$$\mathcal{C} := \{\text{int}(\text{Proj}_{90^\circ}(\Lambda)) = \emptyset\},$$

be the events that the unaligned squares $Q_s^n \cap \Lambda$ for $s = 1, \dots, K(n)$, and the total set Λ do not have an interval in their projection. Our goal is to prove that

$$\mathbb{P}\left(\mathcal{C} \,|\, \Lambda \neq \emptyset\right) = 0.$$

According to Corollary 2 it holds for all s that

$$\mathbb{P}\left(\mathcal{C}_s^n\right) < 1 - p_\Delta p_J =: t < 1.$$

By the definition it is clear that for every n, N we have

$$\begin{aligned}
\mathbb{P}(\mathcal{C}|\,\Lambda \neq \emptyset) &\leq \mathbb{P}(K(n) < N|\,\Lambda \neq \emptyset) + \mathbb{P}(\mathcal{C} \cap \{K(n) \geq N\}|\,\Lambda \neq \emptyset) \\
&\leq \mathbb{P}(K(n) < N|\,\Lambda \neq \emptyset) + \mathbb{P}(\mathcal{C}_1^n \cap \dots \cap \mathcal{C}_N^n|\,\Lambda \neq \emptyset) \\
&\leq \mathbb{P}(K(n) < N|\,\Lambda \neq \emptyset) + \frac{t^N}{\mathbb{P}(\Lambda \neq \emptyset)},
\end{aligned}$$

where we use that the branching processes which determine Λ in each of the unaligned squares run independently. Letting first $n \to \infty$, and then $N \to \infty$ we obtain from (6.5) and $t < 1$ that $\mathbb{P}(\mathcal{C}|\,\Lambda \neq \emptyset) = 0$ which completes our proof. □

Proof of Theorem 1 (b). If there is k such that both $\gamma_k, \gamma_{k+1} < 1$ then using (5.2) we obtain that both of the column sums of the matrix $\mathcal{M}(k)$ are less than one. This and (5.1) implies that for every $k_1 \dots k_n$ we have

$$\lim_{m \to \infty} \|\mathcal{M}(k_1 \dots k_n, \underbrace{k, k, \dots, k}_{m})\|_1 \to 0$$

Let Z_m be the total number of either left or right triangles of level $n+m$ in column $C_{k_1 \dots k_n, \underbrace{k, k, \dots, k}_{m}}$. Then

$$\mathbb{E}(Z_m) = \|\mathcal{M}(k_1 \dots k_n, \underbrace{k, k, \dots, k}_{m})\|_1$$

and by the Markov inequality, $\mathbb{P}(Z_m > 0) \leq \mathbb{E}(Z_m)$. So with probability one

$$\bigcap_{m=1}^{\infty} C_{\underline{k}_n, \underbrace{k, k, \dots, k}_{m}}$$

is *not* contained in the projection of Λ. Since by varying $\underline{k}_n$ we may obtain a dense set of such points, we conclude that in this case the projection of Λ does not contain any interval with probability one. □

7. Higher order Cantor sets and eigenvalues

Here we reconsider (cf. Subsection 4.2) the question of the scope of Theorem 1 by introducing higher order Cantor sets. We also discuss the connection with the eigenvalues of the matrices involved in the generation of $F_2 - F_1$. This will be illustrated by two examples. First we consider the family of random Cantor sets parametrised by ρ with $0 \le \rho \le 1$ given by $M = 4$ and $(p_0, \dots, p_3) = (1, 0, 1, \rho)$. Clearly this gives

$$\mathcal{M}(0) = \begin{bmatrix} \rho & 0 \\ \rho & 2+\rho^2 \end{bmatrix}, \mathcal{M}(1) = \begin{bmatrix} 1 & \rho \\ 1 & \rho \end{bmatrix}, \mathcal{M}(2) = \begin{bmatrix} \rho & 1 \\ \rho & 1 \end{bmatrix}, \mathcal{M}(3) = \begin{bmatrix} 2+\rho^2 & \rho \\ 0 & \rho \end{bmatrix}.$$

The cyclic autocorrelations are

$$\gamma_0 = 2+\rho^2,\ \gamma_1 = 2\rho,\ \gamma_2 = 2,\ \gamma_3 = 2\rho.$$

The Palis conjecture predicts that the difference set will contain an interval almost surely for all $\rho > 0$. Application of Theorem 1 gives no conclusion for $\rho < \frac{1}{2}$, and that for $\rho > \frac{1}{2}$ this is indeed the case. However, it is possible to get more out of the theorem by considering higher order Cantor sets.

The order 2 Cantor set associated to the set generated by $(p_0, \dots, p_{M-1})$ is the base M^2 Cantor set with vector

$$(p_0^{(2)}, \dots, p_{M^2-1}^{(2)})$$

given by

$$p_{Mi+j}^{(2)} = p_i p_j \quad \text{for} \quad i, j \in \{0, \dots, M-1\}.$$

We will denote the objects associated to $p^{(2)}$ all with a superindex (2), for instance $F^{(2)}$ is the random M^2-adic Cantor set generated by $p^{(2)}$, and $I_{k_1 \dots k_n}^{(2)}$ denotes an n-th level M^2-adic interval. The key fact is that for all $i_1 \dots i_n, j_1 \dots j_n \in \{0, \dots, M-1\}^n$ one has

$$I_{Mi_1+j_1, \dots, Mi_n+j_n}^{(2)} = I_{i_1 j_1 \dots i_n j_n}.$$

This implies that $F^{(2)}$ has the same distribution as $\bigcap_{n \ge 0} F^{2n}$, which equals the original Cantor set F. We can therefore obtain statements about F by applying Theorem 1 to $p^{(2)}$. Note that $\mathcal{M}^{(2)}(Mi+j) = \mathcal{M}(ij) = \mathcal{M}(i)\mathcal{M}(j)$. So in our example

$$\mathcal{M}^{(2)}(3) = \mathcal{M}(0)\mathcal{M}(3) = \begin{bmatrix} \rho & 0 \\ \rho & 2+\rho^2 \end{bmatrix} \begin{bmatrix} 2+\rho^2 & \rho \\ 0 & \rho \end{bmatrix} = \begin{bmatrix} 2\rho+\rho^3 & \rho^2 \\ 2\rho+\rho^3 & \rho^2+2\rho+\rho^3 \end{bmatrix}.$$

It follows that $\gamma_4^{(2)} = 4\rho + 2\rho^3$ and $\gamma_3^{(2)} = 2\rho + 2\rho^2 + \rho^3$. Clearly $\gamma_3^{(2)} < \gamma_4^{(2)}$, and the latter is smaller than 1 for all ρ smaller than the real root of $4\rho + 2\rho^3 = 1$, which is about 0.242. Theorem 1 now gives that $F_2 - F_1$ does almost surely not contain an interval for all $\rho < 0.242$. On the other hand we can also strengthen the conclusion for the opposite case: a straightforward

computation yields that for all ρ, $\gamma^{(2)} = 2\rho + 2\rho^2$. Hence $F_2 - F_1$ will contain an interval for all ρ larger than $(\sqrt{3}-1)/2 = 0.366\ldots$.
Note that for all positive ρ the Perron Frobenius eigenvalues of all the $\mathcal{M}(k)$ are larger than 1, but that still $F_2 - F_1$ does not contain an interval for a range of values of ρ. However, eigenvalues may be useful to prove the opposite case: the Perron Frobenius eigenvalue of $\mathcal{M}^{(2)}(3)$ is equal to

$$\left(\rho^2 + \rho/2 + 2 + 1/2\sqrt{4\rho^3 + \rho^2 + 8\rho}\right)\rho,$$

which is smaller than 1 when $\rho < 0.3221$. As in the proof of Theorem 1 part(b), this can be used to show that a dense set of points is not in $F_2 - F_1$. Using higher order Cantor sets (up to order 324), and Matlab we obtained that the critical point ρ_c where $F_2 - F_1$ changes from empty to non empty interior, satisfies $0.3222 < \rho_c < 0.3226$.
We consider a second parametrized family which has a very different behaviour. Let $M = 5$, and $(p_0, \ldots, p_4) = (1, 0, \rho, 0, 1)$ for $0 \le \rho \le 1$. (See Figure 2.) One finds

$$\mathcal{M}(0) = \begin{bmatrix} 1 & 0 \\ 0 & 2+\rho^2 \end{bmatrix}, \mathcal{M}(1) = \begin{bmatrix} 0 & 1 \\ 2\rho & 0 \end{bmatrix}, \mathcal{M}(2) = \begin{bmatrix} 2\rho & 0 \\ 0 & 2\rho \end{bmatrix},$$

and $\mathcal{M}(3), \mathcal{M}(4)$ are obtained from $\mathcal{M}(1)$, respectively $\mathcal{M}(0)$ by interchanging R and L. Since $\gamma_1^{(n)} = 1$ for all n, Theorem 1 is not applicable, even if one considers higher order Cantor sets. However, since the 5 matrices are either diagonal or anti-diagonal, it is not hard to prove that the Perron-Frobenius eigenvalues $\lambda(k_1 \ldots k_n)$ of the matrices $\mathcal{M}(k_1 \ldots k_n)$ satisfy

$$\lambda(k_1 \ldots k_n) \ge (\sqrt{2\rho})^n.$$

This seems to suggest that the critical ρ for this family is equal to 1/2. Surprisingly, we here have $\rho_c = 1$. It follows from Theorem 2 that $F_2 - F_1$ contains an interval when $\rho = 1$. To see that $F_2 - F_1$ has empty interior for $\rho < 1$, consider column $C_{i_1 \ldots i_n 44 \cdots 42}$ of level $n+m+1$ for each $i_1 \ldots i_n$ such that $C_{i_1 \ldots i_n}$ has only R-triangles (these are in fact all the columns where $i_1, \ldots, i_n$ are all even numbers). Then if there are K R-triangles in $C_{i_1 \ldots i_n}$ this will be also true for all columns $C_{i_1 \ldots i_n 44 \cdots 4}$ of level $n+k$, where $k = 1, 2, \ldots, m$, and moreover, column $C_{i_1 \ldots i_n 44 \cdots 42}$ will be empty with probability

$$\left[(1-\rho)^2\right]^K$$

for all m. It follows as at the end of the proof of Theorem 1 b) that there is a dense set of points in the complement of the projection of Λ.
An alternative would be to adapt the proof (to our setting which has much more dependence) of the main result on fractal percolation of [2], or rather its supplement from [3]. The crucial observation here is that the first level columns C_k^- and C_k split in pairs $(C_0^-, C_1^-), \ldots, (C_3, C_4)$, that do not interact

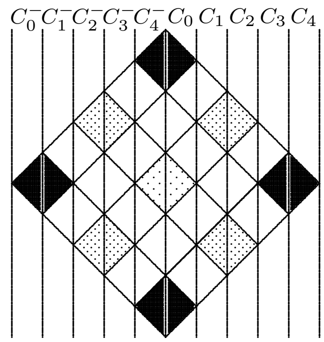

FIGURE 2. The square Q for the $(1, 0, \rho, 0, 1)$ Cantor set; the shades indicate the probabilities with which the $Q_{i,j}$ do occur.

with each other, since the $Q_{i,j}$ only occur with positive probability when $i-j$ is even. Redefining each pair of columns as a single new column, and tilting the $Q_{i,j}$'s, the question of empty interior could then be resolved by the (extended) results of [2] and [3].

8. The deterministic case

In the deterministic case each p_i is either 0 or 1, and the matrices $\mathcal{M}(k)$ simply count the number of level 1 L-triangles and R-triangles in the columns C_k^- and C_k. The crux to the solution in this case is that we can reduce the problem to a finite problem by observing that to have a non-empty projection in a certain column we only have to know whether there is *at least one* L- or R-triangle in that column. We relax the Δ-pair condition: now a pair of adjacent R- and L-triangles is also allowed since independence is no longer an issue.

For a non-negative integer matrix A, let its *reduction* A^∇ be defined by $a_{ij}^\nabla = 0$ if $a_{i,j} = 0$, and $a_{ij}^\nabla = 1$ if $a_{i,j} \geq 1$. Note that the reduction of the product of two matrices equals the product of their reductions. It follows that the reduction of $\mathcal{M}(k_1)^\nabla \cdots \mathcal{M}(k_n)^\nabla$ describes the presence or absence of n-th order L- and R-triangles in columns $k_1 \dots k_n$ of order n. Let $\mathcal{T}$ be the set of 2×2 matrices with entries 0 and 1. For convenience we denote these matrices by their natural binary coding:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = T_j \quad \Leftrightarrow \quad j = a + 2b + 4c + 8d.$$

Define the map $G : 2^\mathcal{T} \to 2^\mathcal{T}$ by $G(\emptyset) = \emptyset$, and for $\mathcal{C} \neq \emptyset$

$$G(\mathcal{C}) = \{(TT')^\nabla : T \in \mathcal{C}, T' \in \mathcal{C}\}.$$

Then there is an empty column of order n in Λ^n if and only if

$$T_0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in G^n(\{\mathcal{M}(0)^\triangledown, \cdots, \mathcal{M}(M-1)^\triangledown\}),$$

where G^n is the n-th iterate of G. Since G is acting on a finite set, the orbit of any point becomes eventually periodic. We call that periodic sequence an *attractor*, denoted by $\mathcal{A}$. Examples are fixed points of G, as e.g., $\mathcal{A} = \{T_0\}$ and $\mathcal{A} = \{T_6, T_9\}$. Assisted by the computer we can show that actually *all* attractors are fixed points (the proof below can be adapted to a proof which does not explicitly use this result).

Theorem 2. *Let the Cantor set F be generated by a 0-1-vector $(p_0, \dots, p_{M-1})$. Then $F_2 - F_1$ does not contain any intervals if and only if $T_0 \in \mathcal{A}$, where $\mathcal{A}$ is the fixed point of the map G starting from $\{\mathcal{M}(0)^\triangledown, \cdots, \mathcal{M}(M-1)^\triangledown\}$.*

Proof. $\Leftarrow$) If $T_0 \in \mathcal{A}$ then an empty column has to occur in some column of order n, where $n \leq 2^{16}$ (actually a computer analysis shows that $n \leq 3$). The proof that $F_2 - F_1$ does not contain any intervals, is then finished as the proof of Theorem 1, part b).

$\Rightarrow$) Suppose $T_0 \notin \mathcal{A}$. We split into two cases.

Case 1. For some $n \geq 1$ a Δ-pair of order n occurs.

Suppose that this happens in column $C_{\underline{k}_n}$ or in $C^-_{\underline{k}_n}$. For arbitrary m and $l_1 \dots l_m$ the fact that T_0 does not occur in $\mathcal{A}$ implies that $\mathcal{M}(l_1 \dots l_m) \neq T_0$, and hence that all subcolumns $C_{\underline{k}_n \underline{l}_m}$ will contain at least one order $n+m$ triangle for all m, and so the complete interval $J_{\underline{k}_n}$ respectively $J^-_{\underline{k}_n}$ will lie in the projection of Λ.

Case 2. A Δ-pair never occurs.

Then $\mathcal{A}$ can not contain a matrix with a row of two 1's. This means that

$$\mathcal{A} \subset \{T_1, T_2, T_4, T_5, T_6, T_8, T_9, T_{10}\}.$$

But since $T_2^2 = T_4^2 = T_0$, these two matrices can also not occur, and hence

$$\mathcal{A} \subset \{T_1, T_5, T_6, T_8, T_9, T_{10}\}.$$

Now suppose that $T_1 \in \mathcal{A}$. Then, since $T_1 T_8 = T_0$, $T_1 T_6 = T_4$ and $T_1 T_{10} = T_0$, it follows (using again that $T_4^2 = T_0$) that

$$\mathcal{A} \subset \{T_1, T_5, T_9\}, \qquad \text{given that } T_1 \in \mathcal{A}.$$

Now note that all three matrices T_1, T_5, and T_9 have a 0 in the LR position. This implies that for a certain n (actually it is not hard to show that one can take $n = 1$) the n-th order Cantor set Λ^n has the property that there are no R-triangles in its intersection with the L triangle. This happens only if at most one $p_i^{(n)} \neq 0$, which contradicts (2.1). Conclusion: $T_1 \notin \mathcal{A}$.

Analogously (replacing L by R), it follows that $T_8 \notin \mathcal{A}$. So we find that necessarily

$$\mathcal{A} \subset \{T_5, T_6, T_9, T_{10}\}.$$

But the matrices T_5, T_6, T_9, and T_{10} each have at least one 1 in each row. It follows that for *all* n *all* columns of order n contain at least one triangle, i.e., that $\mathrm{Proj}_{90^\circ}(\Lambda)$ is the whole interval $[-\frac{1}{2}\sqrt{2}, \frac{1}{2}\sqrt{2}]$. □

9. Appendix: proof of Lemma 3

Proof. Let $S_0 := \{o_1, \ldots, o_N, e_1, \ldots, e_N\}$. We say that S_0 is a *3C set* if the assertion of the Lemma holds for S_0. First we prove that:

(*) If S_0 consists of $2N$ consecutive numbers $S_0 := \{u_1, \ldots, u_{2N}\}$ then S_0 is a 3C set.

To see this we write $N = 3p + r$ where $r = 0, 1$ or 2. Then we couple and color the first $6p$ numbers of S_0 as follows:

$$\begin{aligned}(u_{3k+1}, u_{3k+4}) &= (\mathtt{r}, \mathtt{r})\\ (u_{3k+2}, u_{3k+5}) &= (\mathtt{g}, \mathtt{g})\\ (u_{3k+3}, u_{3k+6}) &= (\mathtt{b}, \mathtt{b}),\end{aligned}$$

for $0 \le k \le p-1$. Since $\lfloor N/3 \rfloor = \lfloor p+r/3 \rfloor = p$ we have verified the assertion of (*) in this way (without having actually colored the last $2r$ numbers, an option which we will leave open till the end of the proof).

A subset $B \subset \mathbb{N}$ is *connected* if $n_1, n_2 \in B$ and $n_1 \le k \le n_2, k \in \mathbb{N}$ implies that $k \in B$. We say that a subset $I \subset S_0$ is an *interval* of S_0 if I is a maximal connected subset of S_0 (it is allowed that I consists of one element). In particular if J_1 and $J_2 \neq J_1$ are intervals of S_0 then there exists an $\ell \notin S_0$ such that ℓ separates J_1 and J_2. Let $\mathcal{I}_0^{eo}$ be the family of the intervals of S_0 for which the left endpoint is an even number and the right endpoint is an odd number. Analogously we define the family of intervals $\mathcal{I}_0^{ee}$, $\mathcal{I}_0^{oe}$ and $\mathcal{I}_0^{oo}$. Let $\mathcal{I}_0$ be the family of all of these intervals. So,

$$S_0 = \bigcup_{I \in \mathcal{I}_0^{ee}} I \cup \bigcup_{I \in \mathcal{I}_0^{eo}} I \cup \bigcup_{I \in \mathcal{I}_0^{oe}} I \cup \bigcup_{I \in \mathcal{I}_0^{oo}} I = \bigcup_{I \in \mathcal{I}_0} I.$$

Let us define a "gluing and shifting" operation Φ on $\mathcal{I}_0$ as follows: if there exist two intervals $J_i = [k_i, \ell_i] \in \mathcal{I}_0$, $i = 1, 2$ such that $\ell_1 + k_2 = 1 \mod 2$ then we select the two left most intervals with this property and we form the interval

$$J := (J_1 + n) \cup (J_2 + \ell_1 - k_2 + 1 + n),$$

where $n \in \mathbb{N}$ is the smallest number such that J is separated by a distance of at least 2 from any intervals of $\mathcal{I}_0 \setminus \{J_1, J_2\}$. In this case we define

$$\mathcal{I}_1 := \Phi(\mathcal{I}_0) := \{J\} \cup \mathcal{I} \setminus \{J_1, J_2\}.$$

If there are no such J_1, J_2 then let $\mathcal{I}_1 := \Phi(\mathcal{I}_0) := \mathcal{I}_0$. By induction we define $\mathcal{I}_k$ for every k. We obtain also by induction that the set

$$S_k := \bigcup_{I \in \mathcal{I}_k} I$$

consists of N odd numbers and N even numbers and if S_k is a 3C set then S_{k-1} is also a 3C set for all $k \geq 1$. We claim that

$$\text{if } \#(\mathcal{I}_k) \geq 3 \text{ then } \#(\mathcal{I}_{k+1}) < \#(\mathcal{I}_k). \tag{9.1}$$

We argue by contradiction. If $\#(\mathcal{I}_{k+1}) = \#(\mathcal{I}_k)$ then $\mathcal{I}_{k+1} = \mathcal{I}_k$. Observe that

$$\mathcal{I}_{k+1} = \mathcal{I}_k \text{ implies that } \mathcal{I}_k^{ee} = \mathcal{I}_k^{oo} = \emptyset. \tag{9.2}$$

Namely, the cardinality of $\mathcal{I}_k^{ee}, \mathcal{I}_k^{oo}$ is the same since we have N odd numbers and N even numbers in S_k and if this cardinality is not 0 then we can form an interval J like above by selecting a J_1 from $\mathcal{I}_k^{oo}$ and a J_2 from $\mathcal{I}_k^{ee}$. Further, if either $\mathcal{I}_k^{eo} \neq \emptyset$ or $\mathcal{I}_k^{oe} \neq \emptyset$ then their cardinality is at most one. This is so because otherwise choosing $J_1 \neq J_2$ from the same family, we could form J like above. In this way we have verified (9.1). Let k_0 be then smallest number for which $\mathcal{I}_{k_0} = \mathcal{I}_{k_0+1}$. Then it follows from (9.2) that

$$\mathcal{I}_{k_0} = \mathcal{I}_{k_0}^{eo} \cup \mathcal{I}_{k_0}^{oe}$$

where both of the families on the right hand side consists of at most one interval. Let us call these intervals I_1 and I_2 with the possibility that one of them may be empty. That is

$$S_{k_0} = I_1 \cup I_2.$$

Since both of the intervals I_1, I_2 consist of an even number of consecutive numbers, it follows from (*) that they are both 3C sets. This implies that S_{k_0} is also a 3C set. The only thing to check is that

$$\text{we use all colors at least } \lfloor N/3 \rfloor \text{ times} \tag{9.3}$$

since I_1 and I_2 are separated by a distance of at least 2. To see that we can accomplish this, write $N_i = 3p_i + r_i$ where N_i is the cardinality of I_i, and $r_i = 0, 1$ or 2. Now if $r_1 + r_2 \leq 2$, then $N = N_1 + N_2 = 3(p_1 + p_2) + r_1 + r_2$ and $\lfloor N/3 \rfloor = p_1 + p_2$, so (9.3) is fulfilled. What remains are the cases $r_1 = 1, r_2 = 2$ (together with $r_1 = 2, r_2 = 1$ which is very similar), and $r_1 = 2, r_2 = 2$, in which cases $\lfloor N/3 \rfloor = p_1 + p_2 + 1$. In the first case we color the last two numbers of I_1 by $\mathtt{g}$ and $\mathtt{b}$, and the last four numbers of I_2 by $\mathtt{r}, \mathtt{b}, \mathtt{g}, \mathtt{r}$. Since the parities of these numbers are e, o, respectively o, e, o, e, we see that we can create the required extra pair for each color. In the case $r_1 = 2, r_2 = 2$ we color the last four numbers of both I_1 and I_2 by $\mathtt{r}, \mathtt{g}, \mathtt{b}, \mathtt{r}$, and again we can create an extra pair for each color. This proves (9.3). As we observed above, the fact that S_{k_0} is a 3C set implies that S_0 is also a 3C set, and the proof is complete. □

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