CHROMATIC THRESHOLDS IN DENSE RANDOM GRAPHS

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Abstract. The chromatic threshold $\delta_\chi(H,p)$ of a graph H with respect to the random graph $G(n,p)$ is the infimum over $d > 0$ such that the following holds with high probability: the family of H-free graphs $G \subseteq G(n,p)$ with minimum degree $\delta(G) \geqslant dpn$ has bounded chromatic number. The study of the parameter $\delta_\chi(H) := \delta_\chi(H,1)$ was initiated in 1973 by Erdős and Simonovits, and was recently determined for all graphs H. In this paper we show that $\delta_\chi(H,p) = \delta_\chi(H)$ for all fixed $p \in (0,1)$, but that typically $\delta_\chi(H,p) \neq \delta_\chi(H)$ if $p = o(1)$. We also make significant progress towards determining $\delta_\chi(H,p)$ for all graphs H in the range $p = n^{-o(1)}$. In sparser random graphs the problem is somewhat more complicated, and is studied in a separate paper.

1. Introduction

One of the most famous early applications of the probabilistic method is Erdős' proof [13] that there exist graphs with arbitrarily high girth and chromatic number. In 1973, Erdős and Simonovits [15] asked whether such constructions are still possible under the additional condition that the graph have high minimum degree. The *chromatic threshold* $\delta_\chi(H)$ of a graph H is defined to be the infimum over $d > 0$ such that there exists $C = C(H,d)$ with the following property: if G is an H-free graph on n vertices with minimum degree $\delta(G) \geqslant dn$, then $\chi(G) \leqslant C$. For example, it is easy to see that $\delta_\chi(H) = 0$ for all bipartite H, and it was proved by Thomassen [27, 28] that $\delta_\chi(K_3) = 1/3$ and that $\delta_\chi(C_{2k+1}) = 0$ for every $k \geqslant 2$.

Important breakthroughs in the study of chromatic thresholds of more general families of graphs were obtained by Lyle [22] and Łuczak and Thomassé [21]. Following [21], we say that a graph H is *near-acyclic* if $\chi(H) = 3$ and H admits a partition into a forest F and an independent set I such that every odd cycle of H meets I in at least two vertices. The family of near-acyclic graphs was introduced by Łuczak and Thomassé [21], who conjectured that they were exactly the 3-chromatic graphs with chromatic threshold zero. This was proved in [3], where moreover the chromatic threshold of every graph H was determined: If $\chi(H) = r \geqslant 3$ then

$$\delta_\chi(H) \in \left\{ \frac{r-3}{r-2}, \frac{2r-5}{2r-3}, \frac{r-2}{r-1} \right\},$$

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where the first possibility occurs exactly when it is possible to obtain a near-acyclic graph by removing $r-3$ independent sets from H, the third when H has no forest in its decomposition family[1], and the second otherwise.

In recent years, beginning with [16, 19], there has been a great deal of interest in *sparse random analogues* of results in extremal graph theory. For example, it can be proved[2] using the general methods introduced in [6, 8, 25, 26] that if $p \gg n^{-2/(r+1)}$ then, with high probability, every K_{r+1}-free subgraph of $G(n,p)$ with $\big(1-1/r+o(1)\big)p\binom{n}{2}$ edges can be made r-partite by removing $o(pn^2)$ edges. DeMarco and Kahn [11, 10] moreover proved that if $p \gg n^{-2/(r+1)}(\log n)^{2/(r+1)(r-2)}$ then with high probability the largest K_{r+1}-free subgraph of $G(n,p)$ is r-partite. For an excellent introduction to the area, see the recent survey [23].

In this paper we will study a sparse random analogue of the chromatic threshold. The following definition was first made in [3].

Definition 1.1. Given a graph H and a function $p = p(n) \in [0,1]$, define

$$\delta_\chi\big(H,p\big) \;:=\; \inf\Big\{d > 0 \,:\, \text{there exists } C > 0 \text{ such that the following holds with high probability: every } H\text{-free spanning subgraph } G \subseteq G(n,p) \text{ with } \delta(G) \geqslant dpn \text{ satisfies } \chi(G) \leqslant C\Big\}.$$

We call $\delta_\chi\big(H,p\big)$ the *chromatic threshold of H with respect to p.*

Note that $\delta_\chi(H) = \delta_\chi(H,1)$, so this definition generalises that of the chromatic threshold. We emphasise that the constant C is allowed to depend on the graph H, the function p and the number d, but not on the integer n. We also note that if, for some d, with high probability there is no spanning H-free subgraph of $G(n,p)$ whose average degree exceeds dpn, then vacuously we have $\delta_\chi(H,p) \leqslant d$.

1.1. **Our results.** Our first theorem shows that if $p \in (0,1]$ is constant, then the chromatic threshold does not depend on its value.

Theorem 1.2. *For each constant $p > 0$ and graph H, we have $\delta_\chi(H,p) = \delta_\chi(H)$.*

For functions $p = o(1)$, the situation is quite different. In this paper we will focus on the 'dense' regime, by which we mean the case in which $p = n^{-o(1)}$. In this regime, it is still trivially true that $\delta_\chi(H,p) = \delta_\chi(H) = 0$ for all bipartite H. We are also able to determine $\delta_\chi(H,p)$ in the case $\chi(H) \geqslant 4$, even for somewhat smaller values of p. Recall that the 2-density $m_2(H)$ is the maximum of $\frac{e(F)-1}{v(F)-2}$ over subgraphs $F \subseteq H$ with at least 3 vertices.

Theorem 1.3. *Let H be a graph with $\chi(H) \geqslant 4$, and let $p = p(n)$ be any function satisfying $\max\big\{n^{-1/m_2(H)}, n^{-1/2}\big\} \ll p = o(1)$. Then*

$$\delta_\chi(H,p) \;=\; \frac{\chi(H)-2}{\chi(H)-1}\,.$$

[1]Recall that the decomposition family of a graph H is the collection of bipartite graphs obtained from H by removing $\chi(H)-2$ independent sets.

[2]See [24] for a proof of this theorem using the method of [26].

Recall that $\delta_\chi(H) = \frac{\chi(H)-2}{\chi(H)-1}$ if and only if there is no forest in the decomposition family of H, so for all other graphs we have $\delta_\chi(H,p) > \delta_\chi(H)$ whenever $p \to 0$ sufficiently slowly. We remark that the upper bound in Theorem 1.3 is an immediate consequence of a theorem of Conlon and Gowers [8] and Schacht [26], while the lower bound is proved in [2].

Perhaps surprisingly, for graphs H with chromatic number $\chi(H) = 3$ the situation is significantly more complicated. In order to state our main theorem, which determines $\delta_\chi(H,p)$ for many (but not all) 3-chromatic graphs in the 'dense' range $p = n^{-o(1)}$, we will need the following definitions (see Figure 1.1).

Definition 1.4. A graph H is a *cloud-forest graph* if there is an independent set $I \subseteq V(H)$ (the *cloud*) such that $V(H) \setminus I$ induces a forest F, the only edges from I to F go to leaves or isolated vertices of F, and no two adjacent leaves in F send edges to I.

Moreover, H is a *thundercloud-forest graph* if there is a cloud $I \subseteq V(H)$, which witnesses that H is a cloud-forest graph, such that every odd cycle in H uses at least two vertices of I.

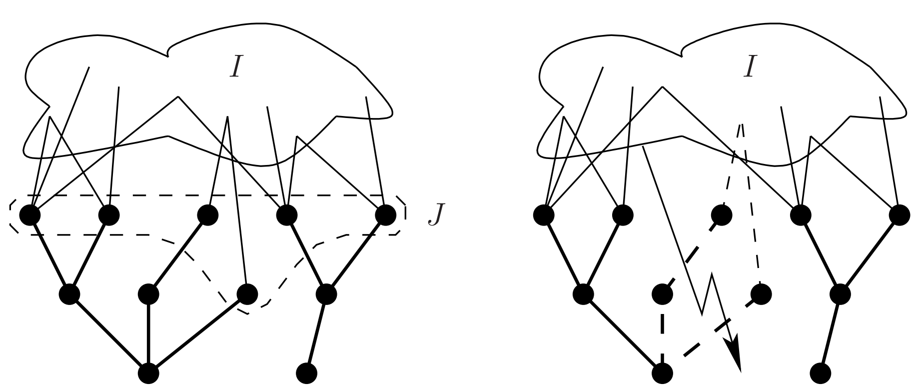

FIGURE 1. A cloud-forest graph (left), and a forbidden odd cycle for a thundercloud-forest graph (dashed, right).

An alternative definition of 'cloud-forest', which is used in our proofs, is the following: The vertex set can be partitioned into independent sets I and J, and a forest F', such that there are no edges from $V(F')$ to I and each vertex of J has at most one neighbour in $V(F')$. To obtain this partition from that of Definition 1.4, let J be the neighbours of vertices in I, as shown in Figure 1.1, and let F' be obtained by removing J from F.

For example, K_3 is not a cloud-forest graph, C_5 is a cloud-forest but not a thundercloud-forest graph, and C_{2k+1} is a thundercloud-forest graph for every $k \geqslant 3$. Note that every cloud-forest graph has a forest in its decomposition family, and similarly every thundercloud-forest graphs is near-acyclic, but in both cases the reverse inclusion does not hold (as K_3 and C_5 respectively demonstrate).

Our main theorem is the following partial characterisation of $\delta_\chi(H,p)$ for 3-chromatic graphs in the dense regime, i.e., for functions $p = p(n)$ that satisfy $p = o(1)$ and $p = n^{-o(1)}$. Together with Theorem 1.3, it determines $\delta_\chi(H,p)$ in this range for every H that is not a thundercloud forest graph.

Theorem 1.5. *Let H be a graph with $\chi(H) = 3$, and let $p = p(n)$ be a function satisfying $p = o(1)$ and $p = n^{-o(1)}$. Then*

$$\delta_\chi(H,p) \begin{cases} = \frac{1}{2} & \text{if } H \text{ is not a cloud-forest graph.} \\ = \frac{1}{3} & \text{if } H \text{ is a cloud-forest graph but not a thundercloud-forest graph.} \\ \leqslant \frac{1}{3} & \text{if } H \text{ is a thundercloud-forest graph.} \end{cases}$$

We consider this theorem (in particular, the upper bound $\delta_\chi(H,p) \leqslant 1/3$ for cloud-forest graphs) to be the main contribution of this paper. It is possible that $\delta_\chi(H,p) = 1/3$ for every function $\max\{n^{-1/m_2(H)}, n^{-1/2}\} \ll p \ll 1$ whenever H is a cloud-forest graph but not a thundercloud-forest graph, see [2, Question 6.3].

For thundercloud-forest graphs Theorem 1.5 only provides an upper bound on $\delta_\chi(H,p)$, which we do not believe to be sharp. We make the following conjecture, which would complete the characterisation of $\delta_\chi(H,p)$ in dense random graphs.

Conjecture 1.6. *If H is a thundercloud-forest graph, and $p = p(n)$ is a function satisfying $p = o(1)$ and $p = n^{-o(1)}$, then $\delta_\chi(H,p) = 0$.*

In [2, Theorem 1.5], we prove Conjecture 1.6 for odd cycles, that is, we prove that $\delta_\chi(C_{2k+1},p) = 0$ for every $k \geqslant 3$.

1.2. **Sparser random graphs.** In a companion paper [2] we study $\delta_\chi(H,p)$ for sparser random graphs, i.e., when $p = p(n)$ tends to zero faster than $n^{-\varepsilon}$ for some $\varepsilon > 0$. For example, in that paper we determine $\delta_\chi(H,p)$ for almost all values of p whenever $\chi(H) \geqslant 5$:

$$\delta_\chi(H,p) = \begin{cases} \delta_\chi(H) & \text{if} \quad p > 0 \text{ is constant,} \\ \frac{\chi(H)-2}{\chi(H)-1} & \text{if} \quad n^{-1/m_2(H)} \ll p \ll 1, \\ 1 & \text{if} \quad \frac{\log n}{n} \ll p \ll n^{-1/m_2(H)}. \end{cases} \tag{1}$$

Note that if $p \ll \frac{\log n}{n}$ then $G(n,p)$ is likely to have an isolated vertex, so trivially $\delta_\chi(H,p) = 0$. In the range $p = \Theta\big(n^{-1/m_2(H)}\big)$ we are not sure exactly what to expect, see [2, Problem 6.4].

As noted above, the situation is more complicated (and more interesting) for 3-chromatic graphs. In [2] we are able to determine $\delta_\chi(K_3,p)$ and $\delta_\chi(C_5,p)$ for most functions p, and $\delta_\chi(C_{2k+1},p)$ if either $p \gg n^{-1/2}$ or $\frac{\log n}{n} \ll p \ll n^{-(2k-3)/(2k-2)}$, for all $k \geqslant 3$. Perhaps most interestingly, we show that $\delta_\chi(C_5,p)$ has (at least) four different non-trivial regimes:

$$\delta_\chi(C_5,p) = \begin{cases} 0 & \text{if} \quad p > 0 \text{ is constant} \\ \frac{1}{3} & \text{if} \quad n^{-1/2} \ll p \ll 1 \\ \frac{1}{2} & \text{if} \quad n^{-3/4} \ll p \ll n^{-1/2} \\ 1 & \text{if} \quad \frac{\log n}{n} \ll p \ll n^{-3/4}. \end{cases}$$

We also show that (1) holds for a large class of 4-chromatic graphs (those with $m_2(H) > 2$), but we suspect (see [2, Conjecture 6.1]) that this is not the case for all 4-chromatic graphs.

All of the lower bound constructions in the range $p = o(1)$ are given in [2], but in Section 3 we will briefly describe those that are used in the proofs of Theorems 1.3 and 1.5. As noted above, the main contribution of this paper is the proof (see Section 4) that $\delta_\chi(H,p) \leqslant 1/3$ if H is a cloud-forest graph and $p = p(n)$ satisfies $p = o(1)$ and $p = n^{-o(1)}$.

1.3. **An approximate version of $\delta_\chi(H,p)$.** In the definition of $\delta_\chi(H,p)$, the H-free graph $G \subseteq G(n,p)$ is required to be C-partite (rather than 'close-to-C-partite'), and in this sense the theorems stated above have more in common with the theorem of DeMarco and Kahn [11, 10], stated earlier, than those of Conlon and Gowers [8] and Schacht [26] (see the discussion before Definition 1.1). The following 'approximate' random graph version of $\delta_\chi(H)$ was recently proposed by Conlon, Gowers, Samotij and Schacht [9].

Definition 1.7. For each graph H and function $p = p(n) \in (0,1]$, define

$$\delta^*_\chi\big(H,p\big) := \inf \Big\{ d > 0 \,:\, \text{there exists } C > 0 \text{ such that, for all } \varepsilon > 0 \text{, the following holds with high probability: every } H\text{-free spanning subgraph } G \subseteq G(n,p) \text{ with } \delta(G) \geqslant dpn \text{ can be made } C\text{-colourable by removing } \varepsilon pn^2 \text{ edges} \Big\}.$$

Conlon, Gowers, Samotij and Schacht [9] used the so-called Kohayakawa-Łuczak-Rödl conjecture [19], which was recently proved in [6, 9, 25], to deduce the following theorem, which determines $\delta^*_\chi(H,p)$ in terms of $\delta^*_\chi(H,1)$ for all H and essentially all p.

Theorem 1.8 (Conlon, Gowers, Samotij and Schacht [9]). *For every graph H,*

$$\delta^*_\chi(H,p) = \begin{cases} \delta^*_\chi(H) & \text{if} \quad p \gg n^{-1/m_2(H)} \\ 1 & \text{if} \quad \frac{\log n}{n} \ll p \ll n^{-1/m_2(H)}, \end{cases}$$

where $\delta^*_\chi(H) := \delta^*_\chi(H,1)$.

It is somewhat surprising that $\delta_\chi(H,p)$ and $\delta^*_\chi(H,p)$ have such different behaviour. Indeed, the threshold for an exact statement (such as that of DeMarco and Kahn) 'usually' differs from that for the asymptotic statement (such as that proved by Conlon-Gowers and Schacht) by only a poly-logarithmic factor. By contrast, for most graphs H there are at least two thresholds at which the value of $\delta_\chi(H,p)$ changes, and in the case $H = C_5$ there are (at least) three such thresholds. It seems reasonable to believe that multiple thresholds also exist for many other cloud-forest graphs.

Perhaps surprisingly, we do not have $\delta^*_\chi(H) = \delta_\chi(H)$ in general. However, we are able to determine $\delta^*_\chi(H)$ in terms of $\delta_\chi(H)$ for every graph H.

Theorem 1.9. *For every graph H,*

$$\delta^*_\chi(H) = \min \Big\{ \delta_\chi(H') \,:\, \textit{there exists a homomorphism from } H \textit{ to } H' \Big\}.$$

For example if H is obtained from C_5 by blowing up each vertex to a 2-vertex independent set then we have $\delta_\chi(H) = \frac{1}{2}$ but $\delta^*_\chi(H) = 0$. On the other hand, for some graphs H, such as K_3, we have $\delta^*_\chi(H) = \delta_\chi(H)$.

Finally, let us note that is interesting to ask how many edges one really needs to delete in order to obtain a graph with bounded chromatic number: perhaps one can replace εpn^2

in the definition of $\delta^*_\chi(H,p)$ by some asymptotically smaller function of n and p? For the specific case of K_3, Allen, Böttcher, Kohayakawa and Roberts [5] have shown that εpn^2 can be replaced by any function $f(n,p) \gg n/p$, but there exists $c > 0$ such that the function cn/p does not suffice.

1.4. **The structure of the paper.** In Section 2 we state the extremal and probabilistic tools we will use in the proofs of Theorems 1.2 and 1.5, together with the sparse random Erdős-Stone Theorem, which implies some of the upper bounds for Theorems 1.2 and 1.5, and the upper bound of Theorem 1.3. In Section 3 we describe the constructions that prove the lower bounds in Theorems 1.2, 1.3 and 1.5, and in Section 4 we prove the upper bound on $\delta_\chi(H,p)$ for cloud-forest graphs, which is our main new result and which completes the proof of Theorem 1.5. In Section 5 we show how to adapt the method of [3] in order to prove that $\delta_\chi(H,p) \leqslant \delta_\chi(H)$ for all fixed $p > 0$, and hence complete the proof of Theorem 1.2. Finally, in Section 6, we give a brief sketch of the proof of Theorem 1.9.

2. Preliminaries

In this section we state the sparse random Erdős-Stone theorem, which implies some of our claimed upper bounds, some basic probabilistic and graph-theoretic tools, and a sparse version of Szemerédi's Regularity Lemma.

2.1. **The sparse random Erdős-Stone theorem.** The following theorem was originally conjectured by Kohayakawa, Łuczak and Rödl [19], and was recently proved by Conlon and Gowers [8] (for strictly balanced graphs H) and Schacht [26] (in general), see also [6, 25].

Theorem 2.1. *For every graph H, every $\gamma > 0$, and every $p \gg n^{-1/m_2(H)}$, the following holds with high probability. For every H-free subgraph $G \subseteq G(n,p)$, we have*

$$e(G) \leqslant \left(1 - \frac{1}{\chi(H)-1} + \gamma\right) p \binom{n}{2}.$$

Theorem 2.1 has the following immediate corollary.

Corollary 2.2. *For every graph H, and every $p \gg n^{-1/m_2(H)}$,*

$$\delta_\chi(H,p) \leqslant 1 - \frac{1}{\chi(H)-1}.$$

Proof. Let $\gamma > 0$, and suppose that $G \subseteq G(n,p)$ is a spanning subgraph with $\delta(G) \geqslant \big(1 - \frac{1}{\chi(H)-1} + \gamma\big)pn$. Since $p \gg n^{-1/m_2(H)}$, it follows from Theorem 2.1 that, with high probability, $H \subseteq G$. Hence $\delta_\chi(H,p) \leqslant 1 - \frac{1}{\chi(H)-1}$, as claimed. □

2.2. **Probabilistic and graph-theoretic tools.** We will frequently use the following concentration bounds. Let $\mathrm{Bin}(n,p)$ denote the Binomial distribution, and let $\mathrm{Hyp}(n,m,s)$ denote the hypergeometric distribution, corresponding to respectively the number of elements of $[n]$ chosen if each is selected independently with probability p, and the number of elements of $[m]$ chosen if a set of s elements of $[n]$ is selected uniformly at random. The following relatively weak bounds on the large deviations of $\mathrm{Bin}(n,p)$ and $\mathrm{Hyp}(n,m,s)$, see for example [18, Theorems 2.1 and 2.10], will suffice for our purposes.

Chernoff bound. *Let $n \in \mathbb{N}$ and $p \in [0,1]$, and let $X \sim \mathrm{Bin}(n,p)$. Then*

$$\mathbb{P}\big(|X - \mathbb{E}[X]| \geqslant t\big) \leqslant e^{-\Omega(t)}$$

for every $t = \Omega\big(\mathbb{E}[X]\big)$.

Hoeffding's inequality. *Let $n \in \mathbb{N}$ and $m,s \in [n]$, and let $Y \sim \mathrm{Hyp}(n,m,s)$. Then*

$$\mathbb{P}\big(|Y - \mathbb{E}[Y]| \geqslant t\big) \leqslant e^{-\Omega(t)}$$

for every $t = \Omega\big(\mathbb{E}[Y]\big)$.

We will also need the following supersaturation theorem of Erdős and Simonovits [12].

Theorem 2.3 (Erdős and Simonovits)**.** *For every $s \in \mathbb{N}$ there exists $\beta > 0$ such that the following holds. Let G be a graph on n vertices, with $e(G) = \varrho n^2 \geqslant \beta^{-1} n^{2-1/s}$ edges, where $\varrho = \varrho(n)$. Then G contains at least $\beta \varrho^{s^2} n^{2s}$ copies of $K_{s,s}$.*

Finally, we will use the following straightforward and well-known fact.

Fact 2.4. *Let F be a forest and G be a graph on n vertices. If $e(G) \geqslant v(F)n$, then $F \subseteq G$.*

2.3. **Sparse regularity in $G(n,p)$.** One of our key tools in this paper will be the so-called 'sparse minimum degree form' of Szemerédi's Regularity Lemma. In order to state this result we need a little notation.

Definition 2.5 ((ε,p)-regular pairs and partitions, the reduced graph)**.** Given a graph G and $\varepsilon, d, p > 0$, a pair of disjoint vertex sets (A,B) is said to be (ε,p)-*regular* if

$$\left| \frac{e\big(G[A,B]\big)}{p|A||B|} - \frac{e\big(G[X,Y]\big)}{p|X||Y|} \right| < \varepsilon$$

for every $X \subseteq A$ and $Y \subseteq B$ with $|X| \geqslant \varepsilon|A|$ and $|Y| \geqslant \varepsilon|B|$. We say that the pair (A,B) is (ε,d,p)-*lower-regular* if $e\big(G[X,Y]\big) \geqslant (d-\varepsilon)p|X||Y|$ for every $X \subseteq A$ and $Y \subseteq B$ with $|X| \geqslant \varepsilon|A|$ and $|Y| \geqslant \varepsilon|B|$. Finally, we say that (A,B) is (ε,d,p)-*regular* if it is both (ε,p)-regular and (ε,d,p)-lower-regular.

A partition $V(G) = V_0 \cup \ldots \cup V_k$ is said to be (ε,p)-*regular* if $|V_0| \leqslant \varepsilon n$, $|V_1| = \ldots = |V_k|$, and at most εk^2 of the pairs (V_i,V_j) with $1 \leqslant i < j \leqslant k$ are not (ε,p)-regular.

The (ε,d,p)-*reduced graph* of an (ε,p)-regular partition is the graph R with vertex set $V(R) = \{1,\ldots,k\}$ and edge set

$$E(R) = \big\{ ij : (V_i,V_j) \text{ is an } (\varepsilon,d,p)\text{-regular pair} \big\}.$$

We will use the following form of the Regularity Lemma, see [7, Lemma 4.4] for a proof. Note that although the statement there only guarantees lower-regularity of pairs in the partition, the proof explicitly gives an (ε,p)-regular partition.

Szemerédi's Regularity Lemma (sparse minimum degree form). *Let $\delta, d, \varepsilon > 0$, $k_0 \in \mathbb{N}$ and $p = p(n) \gg (\log n)^4/n$. There exists $k_1 = k_1(\delta, d, \varepsilon, k_0) \in \mathbb{N}$ such that the following holds with high probability. If $G \subseteq G(n,p)$ has minimum degree $\delta(G) \geqslant \delta pn$, then there is an (ε, p)-regular partition of G into k parts, where $k_0 \leqslant k \leqslant k_1$, whose (ε, d, p)-reduced graph R has minimum degree at least $(\delta - d - \varepsilon)k$.*

When p is constant, we will use an associated 'Counting Lemma' (see [20, Theorem 3.1], for example) which says that if $H \subseteq R$ then G contains a positive fraction of all copies of H.

Lemma 2.6 (Counting Lemma)**.** *Given $p > 0$ and $d > 0$, if $\varepsilon > 0$ is sufficiently small the following holds for each $r_1 \in \mathbb{N}$ when $n \in \mathbb{N}$ is sufficiently large. If G is a graph on n vertices with (ε, d, p)-reduced graph R on $r \leqslant r_1$ vertices, such that $H \subseteq R$, then G contains at least $\frac{1}{2v(H)!}(dp)^{e(H)}(n/r)^{v(H)}$ copies of H.*

When $p = o(1)$, the proof of the Counting Lemma breaks down, since large subsets of an (ε, d, p)-regular pair no longer necessarily have sufficiently strong regularity properties. However, the following theorem shows that this desired 'inheritance of regularity' holds, with high probability, for almost all neighbourhoods in $G(n,p)$. This result follows with some work from Gerke, Kohayakawa, Steger and Rödl [17], for the details see [4].

Theorem 2.7. *For any $\varepsilon', d > 0$ there exist $\varepsilon_0 = \varepsilon_0(\varepsilon', d) > 0$ and $C = C(\varepsilon', d)$ such that, for any $0 < \varepsilon \leqslant \varepsilon_0$ and any function $p = p(n)$, the following holds with high probability for the graph $\Gamma = G(n,p)$. For any disjoint sets of vertices X and Y with*

$$\min\big\{|X|, |Y|\big\} \geqslant C \max\left\{\frac{1}{p^2}, \frac{\log n}{p}\right\},$$

and any (ε, d, p)-lower-regular bipartite subgraph $G \subseteq \Gamma[X,Y]$, there are at most

$$C \max\left\{\frac{1}{p^2}, \frac{\log n}{p}\right\},$$

vertices $w \in V(\Gamma)$ such that $\big(N_\Gamma(w) \cap X, N_\Gamma(w) \cap Y\big)$ is not (ε', d, p)-lower-regular in G.

Finally, let us state a useful (and straightforward) fact about regular pairs, known as the 'Slicing Lemma'.

Lemma 2.8 (Slicing Lemma)**.** *Let (U, W) be an (ε, d, p)-regular (respectively, lower-regular) pair and $U' \subseteq U$, $W' \subseteq W$ satisfy $|U'| \geqslant \alpha|U|$ and $|W'| \geqslant \alpha|W|$. Then (U', W') is $(\varepsilon/\alpha, d, p)$-regular (respectively, lower-regular).* □

3. The lower bounds: constructions

The aim of this short section is to state or adapt constructions from [2, 3, 21] which imply the lower bounds in Theorem 1.2, 1.3 and 1.5. We first state the lower bound construction from [2] which, together with Corollary 2.2, proves Theorem 1.3.

Theorem 3.1 ([2, Theorem 3.5]). *Let H be a graph with $\chi(H) \geqslant 4$, and suppose $p = p(n)$ satisfies $n^{-1/2} \ll p \ll 1$. Then*

$$\delta_\chi(H,p) \,\geqslant\, 1 - \frac{1}{\chi(H)-1}\,.$$

The next two constructions together give the lower bounds for Theorem 1.5.

Proposition 3.2 ([2, Proposition 4.2]). *Let H be a graph with $\chi(H) = 3$, and suppose that $n^{-1/2} \ll p(n) = o(1)$. If H is not a cloud-forest graph, then*

$$\delta_\chi(H,p) \geqslant \frac{1}{2}.$$

Proposition 3.3 ([2, Proposition 4.3]). *Let H be a graph with $\chi(H) = 3$, and suppose that $n^{-1/2} \ll p(n) = o(1)$. If H is not a thundercloud-forest graph, then*

$$\delta_\chi(H,p) \geqslant \frac{1}{3}.$$

All three of these constructions use the fact, proved in [2, Proposition 2.1], that if $n^{-1/2} \ll p = o(1)$, then $G(n,p)$ contains (with high probability) a subgraph F with $o(1/p)$ vertices, and arbitrarily high chromatic number and girth. This is not easy to prove for polynomially sparse random graphs: but it is worth noting that in the dense regime, when $p = n^{-o(1)}$, we can use the fact that for each constant t, with high probability $G(n,p)$ contains K_t, and appeal to Erdős' result [13] that there exist graphs with arbitrarily large girth and chromatic number to obtain F much more easily.

The first two constructions take the subgraph F, and (with high probability) find a partition of the remaining vertices of $G(n,p)$ into $\chi(H) - 1$ roughly equal parts $V_1, \dots, V_{\chi(H)-1}$, such that each vertex of F has about $\big(1 - \frac{1}{\chi(H)-1}\big)pn$ neighbours in V_1. For Theorem 3.1 we can then let G consist of F, together with the edges from F to V_1, and all edges between V_i and V_j for $i \neq j$. It is easy to check that this G has minimum degree close to $\big(1 - \frac{1}{\chi(H)-1}\big)pn$ and large chromatic number, while any $v(H)$-vertex subgraph of G intersects F in a forest, so that we can colour the vertices in each part V_i with colour i and the forest in F with colours 2 and 3. This gives a proper $\big(\chi(H)-1\big)$-colouring, so in particular $H \not\subseteq G$.

For Proposition 3.2 we modify this construction slightly, removing edges so that no two vertices of F have a common neighbour in V_1 in G. One can easily verify that this extra deletion does not significantly decrease the minimum degree of G, and again it is easy to check that any $v(H)$-vertex subgraph of G is a cloud-forest graph (the cloud consists of the vertices in V_2).

Finally, the proof of Proposition 3.3 is similar, but more complicated. In addition to the ideas above, it makes use of a construction of Łuczak and Thomassé [21], that was originally used to show $\delta_\chi(H) \geqslant \frac{1}{3}$ for graphs H that are not near-acyclic. For the details, and for proofs of all three statements, we refer the reader to [2].

It remains to prove the lower bound in Theorem 1.2, which is an immediate consequence of the following proposition.

Proposition 3.4. *Fix a graph H and a constant $p > 0$. Then $\delta_\chi(H,p) \geqslant \delta_\chi(H)$. That is, for each $C > 0$ and $\gamma > 0$, with high probability $G(n,p)$ contains a spanning H-free subgraph with minimum degree at least $\big(\delta_\chi(H) - \gamma\big)pn$ and chromatic number at least C.*

Proposition 3.4 follows by adapting the constructions from [3], which themselves are minor adaptations of constructions originally due to Łuczak and Thomassé [21]. We state only the features of these constructions we require in order to prove Proposition 3.4, and refer the reader to [3, 21] for further details. The following lemma was proved in [3] as Proposition 5, Theorem 16 or Proposition 35, depending on the value of $\delta_\chi(H)$.

Lemma 3.5. *For every graph H and constants $C, \gamma > 0$, there exists $K = K(H,\gamma,C) > 0$ such that the following holds. For all sufficiently large n, there exists an H-free graph G on n vertices with the following properties:*

(a) $\delta(G) \geqslant \big(\delta_\chi(H) - \gamma\big)n$.

(b) There exist disjoint sets $X, Y \subseteq V(G)$, with $|X| = K$ and $|Y| = n/K$, such that

$$\chi\big(G[X]\big) \geqslant C \qquad \textit{and} \qquad e\big(G[X,Y]\big) = e\big(G[Y]\big) = 0.$$

Note that for the application in [3] the important points are the high minimum degree and the subgraph $G[X]$ whose chromatic number is large. However for this paper we require in addition the existence of the set Y. It is easy to check that each of the constructions in [3] permits us to find such a set. For the constructions given there as Propositions 5 and 35, we need to choose in the construction not just any 'Erdős graph', that is, a graph with chromatic number at least C and girth at least $v(H) + 1$, but specifically one in which all but a fixed number K of vertices are independent; this is possible since graphs with chromatic number C and girth $v(H) + 1$ exist. We then let X be the K vertices which are not independent. For the construction given as Theorem 16, we always obtain the desired sets, taking X to be the vertex set of the Borsuk subgraph in that construction.

The next lemma allows us to find in $G(n,p)$ a graph corresponding to that described in Lemma 3.5.

Lemma 3.6. *Let $d, p, \gamma, K > 0$, and let G be a graph on n vertices satisfying:*

(a) $\delta(G) \geqslant dn$.

(b) There exist disjoint sets $X, Y \subseteq V(G)$, with $|X| = K$ and $|Y| = n/K$, such that

$$e\big(G[X,Y]\big) = e\big(G[Y]\big) = 0.$$

Then, with high probability, $G(n,p)$ contains as a spanning subgraph a subgraph of G with minimum degree at least $(d-\gamma)pn$ which includes all edges of $G[X]$.

Proof. We expose the edges of $G(n,p)$ in two rounds. First, we expose all the edges among the first $K + n/K$ vertices. It is easy to see that, with high probability, we will find a K-vertex clique in this set. Fix any injective map $\phi\colon V(G) \to [n]$ which takes X to the K-vertex clique and Y to the remaining n/K initial (i.e., already exposed) vertices.

We next expose the remaining edges, and claim that, with high probability, the intersection of $\phi(G)$ and $G(n,p)$ has minimum degree at least $(d-\gamma)pn$. Indeed, since $\delta(G) \geqslant dn$ and d,

p and γ are fixed, this follows easily by Chernoff's inequality and the union bound. Thus, by construction, we have found the desired subgraph of G. $\square$

Proposition 3.4 now follows immediately.

Proof of Proposition 3.4. Given H, C, p and γ, let n be sufficiently large, and let $K = K(H, \gamma, C) > 0$ and G be given by Lemma 3.5, so in particular G is H-free. Now, applying Lemma 3.6 with $d = \delta_\chi(H) - \gamma$, it follows that, with high probability, $G(n,p)$ contains a spanning subgraph $G' \subseteq G$ with minimum degree at least $\big(\delta_\chi(H) - 2\gamma\big)pn$ which includes all edges of $G[X]$, so $\chi(G') \geqslant C$. Since $\gamma > 0$ was arbitrary, this proves the proposition. $\square$

4. The upper bound for cloud-forest graphs

In this section we will prove the following proposition, which is our main new result.

Proposition 4.1. *If H is a cloud-forest graph, and $p = p(n)$ satisfies $p = o(1)$ and $p = n^{-o(1)}$, then*

$$\delta_\chi(H, p) \leqslant \frac{1}{3}.$$

This proposition, together with Corollary 2.2 and Propositions 3.2 and 3.3, proves Theorem 1.5. We begin by giving an outline of the proof of Proposition 4.1.

Let us fix a cloud-forest graph H with s vertices, and a function $p = p(n)$ such that $p = o(1)$ and $p = n^{-o(1)}$. Fix also a (small) constant $\gamma > 0$ and (with foresight) set $d = \gamma/6$. We will use the following definitions.

Definition 4.2. We define the (d,p)-*robust second neighbourhood* $N_2^*(v)$ of a vertex v in a graph G to be the set of vertices w of G such that v and w have at least dp^2n common neighbours in G. (Since d and p were fixed above, we suppress them from the notation.)

Given $u, v \in V(G)$, let us say that a set Z of size s is (u,v)-*completable* if Z has at least s common neighbours in each of $N(u)$ and $N(v)$.

Let G be an H-free spanning subgraph of $G(n,p)$ with $\delta(G) \geqslant \big(\frac{1}{3} + 2\gamma\big)pn$. Applying the sparse minimum degree form of Szemerédi's Regularity Lemma to G, we obtain a partition $V(G) = V_0 \cup V_1 \cup \cdots \cup V_k$, such that the (ε, d, p)-reduced graph R satisfies $\delta(R) \geqslant \big(\frac{1}{3} + \gamma\big)k$. For each $i \in [k]$, we let X_i be the set of vertices $v \in V(G)$ such that $N_2^*(v)$ covers at least a $\big(\frac{1}{2} + \gamma\big)$-fraction of V_i, and set $X_0 := V(G) \setminus \big(X_1 \cup \cdots \cup X_k\big)$. We will show that $\chi\big(G[X_i]\big) = O(1)$ for each $1 \leqslant i \leqslant k$, and that $X_0 = \emptyset$.

We will bound the chromatic number of $G[X_i]$ in three steps, as follows. First, we show (see Claim 2, below) that for every pair $u, v \in X_i$, there are $\Omega(n^s)$ sets $Z \subseteq V_i$ of size s that are (u,v)-completable. Second (see Claim 3), we show that if $\chi\big(G[X_i]\big) \gg 1$, then there exists a subgraph $E' \subseteq G[X_i]$ with arbitrarily large minimum degree. Using the pigeonhole principle, we can show that there exists a subgraph $E \subseteq E'$ with large average degree and an s-set $Z \subseteq V_i$ such that Z is (u,v)-completable for each $uv \in E$. Third (see Claim 4), a graph with large enough average degree contains all small forests, by Fact 2.4. Using the alternative definition of a cloud-forest graph, it follows that there exists a forest F' that is contained in E and which we can extend to a copy of H.

In order to complete the proof, we will show that $X_0 = \emptyset$, as follows. Suppose for a contradiction that there exists a vertex $u \in X_0$. Since the vertices of X_0 spread out their second neighbourhoods relatively evenly over the clusters of the regular partition, we will be able to find (see Claim 5) a pair (V_i, V_j) of clusters which form an (ε, d, p)-regular pair in G, and furthermore are such that there are at least $2dpn/3$ vertices in $N(u)$ with at least $dp|V_i|/3$ neighbours in each of V_i and V_j. We will then (see Claim 6) use Theorem 2.7 to show that for at least $dpn/3$ of those vertices $v \in N(u)$, the density of $\big(N(v) \cap V_i, N(v) \cap V_j\big)$ is at least $dp/2$ (this is 'inherited' from the (ε, d, p)-regular pair (V_i, V_j)). Now, by Theorem 2.3, any such bipartite graph contains many copies of $K_{s,s}$, which together with v gives us many copies of $K_{1,s,s}$. By an application of the pigeonhole principle we find a copy of $K_{s,s,s}$ in G. Since $H \subseteq K_{s,s,s}$, we thus obtain the desired contradiction.

In order to perform the first step (Claim 2) in the above sketch, we need the following lemma. It implies that if (W_1, Y) and (W_2, Y) are two 'sufficiently dense' pairs in a subgraph $G \subseteq G(n, p)$, and p is not too small, then we can find 'many' copies of $K_{s,2s}$ with the smaller part in Y, and the other part split equally between W_1 and W_2.

Lemma 4.3. *For every $\varepsilon > 0$ and $s \in \mathbb{N}$, there exists $\alpha > 0$ such that the following holds for all sufficiently large $n \in \mathbb{N}$. Let G be a graph on n vertices, and let $p = n^{-o(1)}$. Let $W_1, W_2, Y \subseteq V(G)$ be disjoint sets with $|Y| \geqslant 2pn$, with $\varepsilon pn \leqslant |W_i| \leqslant 2pn$, and with*

$$|N(y) \cap W_i| \geqslant \varepsilon p^2 n \qquad \text{and} \qquad \Big|\bigcap_{v \in S} N(v) \cap W_i\Big| \leqslant 2p^{s+1} n$$

for each $i \in \{1, 2\}$, $y \in Y$ and $S \subseteq Y$ of size s. Then there exist at least $\alpha |Y|^s$ sets $Z \subseteq Y$ of size s such that

$$\Big|\bigcap_{v \in Z} N(v) \cap W_i\Big| \geqslant s$$

for each $i \in \{1, 2\}$.

Proof. Fix $\varepsilon > 0$ and $s \in \mathbb{N}$, let $\beta > 0$ be the constant returned by Theorem 2.3, and set

$$\alpha := \frac{\beta}{2} \left(\frac{\varepsilon}{6s}\right)^s \left(\frac{\varepsilon^2}{32}\right)^{s^2}.$$

Let us assume without loss of generality that $|W_1| \geqslant |W_2|$, and say that a copy of $K_{s,2s}$ in the bipartite graph with parts Y and $W_1 \cup W_2$ is *balanced* if it has s vertices in each of W_1 and W_2. Our aim is to show that at least $\alpha |Y|^s$ sets $Z \subseteq Y$ of size s extend to a balanced copy of $K_{s,2s}$. We will do so in two steps: we will prove a lower bound on the total number of balanced copies of $K_{s,2s}$, and an upper bound on the number rooted at a given s-set in Y. By the pigeonhole principle, this will be enough to give the result.

To prove a lower bound on the number of balanced copies of $K_{s,2s}$, we will apply Theorem 2.3 to the following random bipartite graph $\mathbf{H}$. Let $\phi\colon W_2 \to W_1$ be a uniformly chosen random injective map, and let $Y' \subseteq Y$ be a uniformly chosen random subset of size $2pn$. Let $\mathbf{H}$ be the bipartite graph with parts Y' and W_1, and edge set

$$E(\mathbf{H}) = \big\{ yw \in G[Y', W_1] \,:\, y\phi^{-1}(w) \in G[Y', W_2] \big\}.$$

(If $\phi^{-1}(w)$ is not defined, we say the pair $y\phi^{-1}(w)$ is not in G.) Note that each copy of $K_{s,s}$ in $\mathbf{H}$ corresponds to a balanced copy of $K_{s,2s}$.

We claim first that, with high probability,

$$e(\mathbf{H}) \,\geqslant\, \frac{\varepsilon^2 p^4 n^2}{2} \,\geqslant\, \frac{\varepsilon^2 p^2}{32} \cdot v(\mathbf{H})^2. \tag{2}$$

To prove this, let $y \in Y'$, and recall that y has at least $\varepsilon p^2 n$ neighbours in each of W_1 and W_2, and that $|W_2| \leqslant |W_1| \leqslant 2pn$. Since ϕ is a random map, the degree $d_{\mathbf{H}}(y)$ of y in $\mathbf{H}$ is hypergeometrically distributed with mean at least $(\varepsilon p/2) \cdot \varepsilon p^2 n$, and it follows (by Hoeffding's inequality, and the fact that $p = n^{-o(1)}$) that

$$\mathbb{P}\bigg(d_{\mathbf{H}}(y) \leqslant \frac{\varepsilon^2 p^3 n}{4} \bigg) \,\ll\, \frac{1}{n}.$$

By the union bound, and recalling that $|Y'| = 2pn$, the claimed bound on $e(\mathbf{H})$ follows.

Let $\mathbf{K}$ denote the number of copies of $K_{s,s}$ in $\mathbf{H}$. We claim that

$$\mathbb{E}[\mathbf{K}] \,\geqslant\, \frac{\beta}{2} \bigg(\frac{\varepsilon^2 p^2}{32} \bigg)^{s^2} (2pn)^{2s}. \tag{3}$$

To prove this, observe that if (2) holds then Theorem 2.3, applied with $\varrho = \varepsilon^2 p^2/32$, implies that $\mathbf{K} \geqslant \beta \varrho^{s^2} v(H)^{2s}$. Since (2) holds with probability greater than $1/2$, and $v(H) \geqslant |Y'| = 2pn$, the bound (3) follows immediately.

We will next show that the number of balanced copies of $K_{s,2s}$ is at least

$$\bigg(\frac{|Y|}{3pn} \bigg)^{s} \bigg(\frac{\varepsilon pn}{2s} \bigg)^{s} \cdot \mathbb{E}[\mathbf{K}]. \tag{4}$$

To see this, simply note that for a given balanced copy K of $K_{s,2s}$, the s-set is contained in the randomly chosen Y' with probability

$$\binom{|Y| - s}{2pn - s} \binom{|Y|}{2pn}^{-1} \,\leqslant\, \bigg(\frac{2pn}{|Y| - s} \bigg)^{s} \,\leqslant\, \bigg(\frac{3pn}{|Y|} \bigg)^{s}.$$

If this event occurs, then K yields a copy of $K_{s,s}$ in $\mathbf{H}$ with probability

$$\frac{s! \binom{|W_1| - s}{|W_2| - s} \big(|W_2| - s \big)!}{\binom{|W_1|}{|W_2|} |W_2|!} = \frac{s! (|W_1| - s)!}{|W_1|!} \,\leqslant\, \frac{s!}{(\varepsilon pn - s)^s} \,\leqslant\, \bigg(\frac{2s}{\varepsilon pn} \bigg)^{s}.$$

where the first inequality uses the condition $|W_1| \geqslant \varepsilon pn$ of the lemma, and the second uses the facts that $p = n^{-o(1)}$ and n is sufficiently large. Thus $\mathbb{E}[\mathbf{K}]$ is at most the product of these two probabilities, times the number of choices for K, as claimed in (4).

Finally observe that, since no s-set of vertices of Y has more than $2p^{s+1}n$ neighbours in either W_1 or W_2, each s-set of vertices of Y extends to at most $(2p^{s+1}n)^{2s}$ balanced copies of $K_{s,2s}$. It follows from (3) and (4) that the number of s-sets in Y which extend to at least one balanced copy of $K_{s,2s}$ is at least

$$\left(\frac{|Y|}{3pn}\right)^s \left(\frac{\varepsilon pn}{2s}\right)^s \cdot \frac{\beta}{2}\left(\frac{\varepsilon^2 p^2}{32}\right)^{s^2} (2pn)^{2s} \cdot (2p^{s+1}n)^{-2s} \,=\, \alpha|Y|^s\,,$$

as required. $\square$

Before proving Proposition 4.1, let us note several properties of $\Gamma = G(n,p)$ that hold with high probability if $p = n^{-o(1)}$. In the proof we will assume that all of these properties hold:

(A1) For each $|S| = O(1)$ we have $\Big|\bigcap_{u\in S} N_\Gamma(u)\Big| = \big(1+o(1)\big)p^{|S|}n$.

(A2) For each $|U| = \Omega(pn)$ there are at most $p|U|^2$ edges in U, and there are at most $\frac{\log n}{p^2}$ vertices outside U with more than $2p|U|$ neighbours in U.

(A3) For any disjoint sets U and V of size $\Omega(pn)$ there are $\big(1+o(1)\big)p|U||V|$ edges between U and V.

In each case, the probability of failure can easily be shown to tend to zero using the Chernoff bound.

Proof of Proposition 4.1. Let H be a cloud-forest graph with s vertices, and let $\gamma > 0$. We claim that there exists $C = C(H,\gamma)$ such that the following holds with high probability: if G is a H-free spanning subgraph of $G(n,p)$ with $\delta(G) \geqslant \big(\frac{1}{3} + 2\gamma\big)pn$, then $\chi(G) \leqslant C$. Let $k_0 = 1/\gamma$, let $d = \gamma/6$, and let $\varepsilon' > 0$ be sufficiently small. Let $\varepsilon_0 < \varepsilon'$ and C' be the constants returned by Theorem 2.7 with inputs ε' and d, and choose $\varepsilon \leqslant \varepsilon_0/4$ sufficiently small.

Suppose that $\Gamma = G(n,p)$ satisfies assumptions (A1), (A2) and (A3), and the high probability events of Theorem 2.7 and of the sparse minimum degree form of Szemerédi's Regularity Lemma. Let $G \subseteq \Gamma$ be an H-free graph with minimum degree $\big(\frac{1}{3} + 2\gamma\big)pn$. We will write $N(u)$ for the neighbourhood of u in G, and $N_\Gamma(u)$ for the neighbourhood in Γ.

We begin by applying the sparse minimum degree form of Szemerédi's Regularity Lemma, with input $\delta = \frac{1}{3} + 2\gamma$, d, ε and k_0, to G. This gives us a partition of $V(G)$ into parts $V_0, V_1, \ldots, V_k$, where $k_0 \leqslant k \leqslant k_1$ and $k_1 = k_1(\delta, d, \varepsilon, k_0)$ does not depend on n, and a reduced graph R with

$$\delta(R) \geqslant \left(\frac{1}{3} + \gamma\right)k\,. \tag{5}$$

We now define sets $X_1, \ldots, X_k$ by

$$X_i := \left\{ v \in V(G) : \big|N_2^*(v) \cap V_i\big| \geqslant \left(\frac{1}{2} + d\right)|V_i| \right\} \tag{6}$$

and let $X_0 = V(G) \setminus \big(X_1 \cup \cdots \cup X_k\big)$. Let $\alpha' > 0$ be the constant returned by Lemma 4.3 with input $\varepsilon > 0$ and s, and set $\alpha = d^s\alpha'$.

Our first goal is to show that for each i we have $\chi\big(G[X_i]\big) = O(1)$. We break the proof up into a series of claims, the first of which gives us the set W_1 that we will use in our application of Lemma 4.3.

Claim 1. *For every $u \in V(G)$, there exists $W_1 \subseteq N(u)$ of size $d^2pn/6$ such that*

$$|N(w) \cap W_1| \geqslant d^3p^2n/24 \tag{7}$$

for every $w \in N_2^*(u)$, and

$$|N(w) \cap W_1| \leqslant 2d^2p^2n \tag{8}$$

for every $w \in V(G)$.

Proof of Claim 1. Choose a set $W_1 \subseteq N(u)$ of size $d^2pn/6$ uniformly at random. By the definition of $N_2^*(u)$, we have $|N(u) \cap N(w)| \geqslant dp^2n$ for every $w \in N_2^*(u)$. Moreover, by assumption (A1), and our bound on $\delta(G)$, we have $pn/3 \leqslant |N(u)| \leqslant 2pn$ and $|N(u) \cap N(w)| \leqslant 2p^2n$ for every $w \in V(G)$. It follows that $|N(w) \cap W_1|$ is a hypergeometrically distributed random variable with expected value at least $d^3p^2n/12$ for every $w \in N_2^*(u)$, and at most d^2p^2n for every $w \in V(G)$.

Since $p = n^{-o(1)}$, Hoeffding's inequality and the union bound tell us that with high probability $|N(w) \cap W_1|$ satisfies (7) for every $w \in N_2^*(u)$, and (8) for every $w \in V(G)$. Thus there must exist some such set $W_1 \subseteq N(u)$, as claimed. □

We now show that for each i and pair $u, v \in X_i$, there are many sets in V_i which are (u, v)-completable.

Claim 2. *For every $i \in [k]$, and every pair $u, v \in X_i$, there exist at least $\alpha|V_i|^s$ subsets $Z \subseteq V_i$ of size s that are (u, v)-completable.*

Proof of Claim 2. Let $W_1 \subseteq N(u)$ be given by Claim 1, and set

$$W_2 = N(v) \setminus W_1 \qquad \text{and} \qquad Y = N_2^*(u) \cap N_2^*(v) \cap V_i \setminus \big(W_1 \cup W_2 \cup \{u, v\}\big)\,.$$

We will use Lemma 4.3 to show that there exist at least $\alpha'|Y|^s$ sets $Z \subseteq Y$ of size s whose common neighbourhoods intersect each of W_1 and W_2 in at least s vertices, and which are therefore (u, v)-completable.

In order to apply Lemma 4.3, we need to check that W_1, W_2 and Y satisfy the various conditions of the lemma. To do so, note first that

$$|W_1| = \frac{d^2pn}{6} \qquad \text{and} \qquad \left(\frac{1}{3} - \frac{d^2}{6}\right)pn \leqslant |W_2| \leqslant |N(v)| \leqslant 2pn$$

by Claim 1, our lower bound on $\delta(G)$, and assumption (A1). Thus we have $\varepsilon pn \leqslant |W_i| \leqslant 2pn$ for $i \in \{1, 2\}$, as required.

Next, to bound $|Y|$, note that since $u, v \in X_i$, by inclusion-exclusion and (6) we have

$$\big|N_2^*(u) \cap N_2^*(v) \cap V_i\big| \;\geqslant\; 2\left(\frac{1}{2} + d\right)|V_i| - |V_i| \;=\; 2d|V_i|\,.$$

Since $|W_1| + |W_2| + 2 \leqslant 4pn + 2 \leqslant d|V_i|$ and $p = o(1)$, we obtain $|Y| \geqslant d|V_i| \geqslant 2pn$.

To bound $|N(y) \cap W_i|$, we use Claim 1 and the fact that $Y \subseteq N_2^*(u) \cap N_2^*(v)$. Indeed, by (7) we have $|N(y) \cap W_1| \geqslant d^3p^2n/24 \geqslant \varepsilon p^2 n$ for every $y \in Y \subseteq N_2^*(u)$, and by (8), together with the fact $W_2 = N(v) \setminus W_1$, we have

$$|N(y) \cap W_2| \geqslant |N(y) \cap N(v)| - |N(y) \cap W_1| \geqslant dp^2n - 2d^2p^2n \geqslant \varepsilon p^2 n$$

for every $y \in Y \subseteq N_2^*(v)$. Finally, since $W_1 \subseteq N(u)$, $W_2 \subseteq N(v)$ and $u, v \notin Y$, it follows from assumption (A1) that

$$\left| \bigcap_{w \in S} N(w) \cap W_i \right| \leqslant 2p^{s+1}n$$

for $i \in \{1,2\}$ and every set $S \subseteq Y$ of size s, as required.

Therefore, by Lemma 4.3, there exist at least $\alpha'|Y|^s \geqslant \alpha'(d|V_i|)^s = \alpha|V_i|^s$ sets $Z \subseteq Y \subseteq V_i$ of size s that are (u,v)-completable, as claimed. □

We now show that if $G[X_i]$ has large chromatic number then there is a single set $Z \subseteq V_i$ which is (u,v)-completable for many edges $uv \in G[X_i]$.

Claim 3. *For each $i \in [k]$, if $\chi\big(G[X_i]\big) > 2s/\alpha$ then there exists a set of edges $E \subseteq G[X_i]$ of average degree at least $2s$, and an s-set $Z \subseteq V_i$ which is (u,v)-completable for every $uv \in E$.*

Proof of Claim 3. Since $G[X_i]$ is not $2s/\alpha$-colourable, it is not $\big(2s/\alpha - 1\big)$-degenerate, and so contains a subgraph of minimum degree, and hence also average degree, at least $2s/\alpha$. Let E' be the edges of such a subgraph, and note that, by Claim 2, for each edge $uv \in E'$ there exist at least $\alpha|V_i|^s$ sets $Z \subseteq V_i$ of size s that are (u,v)-completable. Therefore, by the pigeonhole principle, there exists a subset $E \subseteq E'$ and a set $Z \subseteq V_i$ of size s, such that Z is (u,v)-completable for every $uv \in E$, and

$$|E| \geqslant \left(\frac{\alpha|V_i|^s}{\binom{|V_i|}{s}} \right) \cdot |E'| \geqslant \alpha|E'|\,.$$

Since E' has average degree at least $2s/\alpha$ it follows that E has average degree at least $2s$, as claimed. □

We can now bound $\chi\big(G[X_i]\big)$ for each $i \in [k]$.

Claim 4. $\chi\big(G[X_i]\big) \leqslant 2s/\alpha$ *for each $1 \leqslant i \leqslant k$.*

Proof of Claim 4. Suppose that $\chi\big(G[X_i]\big) > 2s/\alpha$ for some $i \in [k]$, and let $E \subseteq G[X_i]$ and $Z \subseteq V_i$ be given by Claim 3. Thus E has average degree at least $2s$, and for each $uv \in E$ the set Z is (u,v)-completable. Letting W be the set of vertices in edges of E, it follows that Z has at least s common neighbours in $N(u)$ for every vertex $u \in W$. We will show that $H \subseteq G$, which will contradict our assumption that G is H-free, and hence prove the claim.

Recall first that since H is a cloud-forest graph (using the alternative definition), its vertex set can be partitioned into independent sets I and J, and a forest F', such that there are no edges from $V(F')$ to I and each vertex of J has at most one neighbour in $V(F')$.

By Fact 2.4, E contains F'. We now construct an embedding of H into G as follows. We embed the copy of F' in H into that in E. We then embed I into vertices of Z outside the copy of F', and finally embed the vertices of J greedily, preserving the property of having a graph embedding. We can embed I to Z because Z has $s = v(H)$ vertices and there are no edges of H from $V(F')$ to I. Finally, each vertex of J has at most one neighbour in $V(F')$ and the rest of its neighbours are in I, so, since Z has at least $v(H)$ common neighbours in $N(u)$ for each u in the copy of F', the greedy embedding of J succeeds. $\square$

It remains to prove that $X_0 = \emptyset$. We start by showing that if this is false, then there is a dense regular pair (V_i, V_j) in G and a substantial number of vertices with many neighbours in each of V_i and V_j.

Claim 5. *If $X_0 \neq \emptyset$, then there exists an (ε, d, p)-regular pair (V_i, V_j) and at least $2dpn/3$ vertices with at least $dp|V_i|/3$ neighbours in each of V_i and V_j.*

Proof of Claim 5. Suppose that $u \in X_0$, and recall that therefore

$$\big|N_2^*(u) \cap V_j\big| < \left(\frac{1}{2} + d\right)|V_j|$$

for every $j \in [k]$. Since $\delta(G) \geqslant pn/3$, we can fix a set $U \subseteq N(u)$ of size $pn/3$, and for each $j \in [k]$, choose a subset $V_j' \subseteq V_j \setminus U$ of size $\big(\frac{1}{2} + d\big)|V_j|$ containing $N_2^*(u) \cap V_j \setminus U$.

We will first show (via some simple counting) that there exists $ij \in E(R)$ such that

$$e\big(G[U, V_i']\big) + e\big(G[U, V_j']\big) \;>\; (1 + 5d)p|U||V_i'|. \tag{9}$$

Indeed, let i be such that $e\big(G[U, V_i']\big)$ is maximised, and consider $j \in V(R)$, not necessarily adjacent to i. Note that if there exist $2k/3$ indices j such that the inequality (9) holds, then by (5), which bounds the minimum degree of R, at least one among them satisfies $ij \in E(R)$ and we are done. So let us suppose (for a contradiction) that there exist $k/3$ indices j such that (9) fails to hold. Then, by the maximality of $e\big(G[U, V_i']\big)$, we have

$$\sum_{j=1}^{k} e\big(G[U, V_j']\big) \;\leqslant\; \frac{k}{3} \cdot \Big((1 + 5d)p|U||V_i'| - e\big(G[U, V_i']\big)\Big) + \frac{2k}{3} \cdot e\big(G[U, V_i']\big), \tag{10}$$

We now establish a lower bound on the same sum. We have $\delta(G) \geqslant \big(\frac{1}{3} + 2\gamma\big)pn$, and

$$e\big(G[U, V(G) \setminus N_2^*(u)]\big) \;\leqslant\; dp^2n^2 \;=\; \frac{\gamma pn|U|}{2}$$

by the definition of $N_2^*(u)$ and since $|U| = pn/3$ and $d = \gamma/6$. Moreover,

$$e\big(G[U]\big) \leqslant p|U|^2 \ll pn|U| \qquad \text{and} \qquad e\big(G[U, V_0]\big) \leqslant 2p|U||V_0| \leqslant 2\varepsilon pn|U|$$

by our assumptions (A2) and (A3), and since $p = o(1)$. Now, since $N_2^*(u) \cap V_j \setminus U \subseteq V_j'$ for every $j \in [k]$, it follows from the above inequalities that

$$\sum_{j=1}^{k} e\big(G[U, V_j']\big) \;\geqslant\; e\big(G[U, N_2^*(u) \setminus U]\big) - e\big(G[U, V_0]\big) \;\geqslant\; \left(\frac{1}{3} + \gamma\right)pn|U|. \tag{11}$$

Now, combining (10) and (11), we have

$$\left(\frac{1}{3}+\gamma\right)pn|U| \;\leqslant\; \frac{k}{3}\cdot\Big(\big(1+5d\big)p|U||V_i'| + e\big(G[U,V_i']\big)\Big) \;\leqslant\; \left(\frac{1}{3}+2d\right)pn|U|,$$

where the second inequality follows as $e\big(G[U,V_i']\big) \leqslant \big(1+d\big)p|U||V_i'|$, by assumption (A3), and since $|V_i'| = \big(\frac{1}{2}+d\big)|V_i| \leqslant \big(\frac{1}{2}+d\big)n/k$. But recalling that $d=\gamma/6$, we see that this is a contradiction, and thus we have proved that there is a pair $ij \in E(R)$ for which (9) holds.

Let us fix such a pair, and set

$$U' \;=\; \left\{ w \in U : \min\big\{|N(w)\cap V_i'|, |N(w)\cap V_j'|\big\} \geqslant \frac{dp|V_i|}{3} \right\}.$$

In order to prove the claim, it will suffice to show that $|U'| \geqslant 2d|U| = 2dpn/3$. Set

$$c \;:=\; \frac{e\big(G[U,V_i']\big)}{p\cdot|U||V_i'|},$$

and observe that $c \leqslant 1+d$ by assumption (A3). We claim that

$$\Big|\big\{w \in U :\; |N(w)\cap V_i'| \geqslant dp|V_i'|\big\}\Big| \;\geqslant\; (c-2d)|U| \tag{12}$$

and

$$\Big|\big\{w \in U :\; |N(w)\cap V_j'| \geqslant dp|V_i'|\big\}\Big| \;\geqslant\; (1-c+4d)|U|, \tag{13}$$

from which it will follow (by inclusion-exclusion, and since $|V_i'| > |V_i|/3$) that we have $|U'| \geqslant 2d|U|$, as required. Suppose first that (12) fails to hold, and observe that therefore

$$e\big(G[U,V_i']\big) \;\leqslant\; (c-2d)|U|\cdot(1+d)p|V_i'| + (1-c+2d)|U|\cdot dp|V_i'| \;<\; cp|U||V_i'|$$

by assumption (A3), which contradicts the definition of c. Similarly, if (13) fails to hold, then

$$e\big(G[U,V_j']\big) \;\leqslant\; (1-c+4d)|U|\cdot(1+d)p|V_i'| + (c-4d)|U|\cdot dp|V_i'| \;=\; (1-c+5d)p|U||V_i'|,$$

again using assumption (A3), which contradicts (9). Hence, both (12) and (13) hold, and so the claim follows. □

We now use Claim 5 to show that if $X_0 \neq \emptyset$ then there exist a substantial number of vertices each of whose neighbourhoods is dense.

Claim 6. *If $X_0 \neq \emptyset$, then there exists an (ε,d,p)-regular pair (V_i,V_j) and a set W of size $dpn/3$ with the following property. For every $w \in W$, there exists a graph $G_w \subseteq G\big[N(w)\cap V_i, N(w)\cap V_j\big]$ with $dp|V_i|/2$ vertices and at least $2^{-6}d^3p^3|V_i|^2$ edges.*

Proof of Claim 6. By Claim 5 there is an (ε,d,p)-regular pair (V_i,V_j) and a set U' of $2dpn/3$ vertices, each of which has at least $dp|V_i|/3$ neighbours in each of V_i and V_j. We first prove that there exists a subset $W \subseteq U'$ of size $dpn/3$ such that the following hold for every $w \in W$:

(*a*) $|N_\Gamma(w)\cap V_i| \leqslant 2p|V_i|$ and $|N_\Gamma(w)\cap V_j| \leqslant 2p|V_j|$,

(*b*) the pair $\big(N_\Gamma(w)\cap V_i, N_\Gamma(w)\cap V_j\big)$ is (ε',d,p)-lower-regular, where $\varepsilon' = \varepsilon'(d) > 0$ was chosen earlier to be sufficiently small. This will be sufficient to prove the claim, because it follows, by the Slicing Lemma applied with $\alpha = d/6$, that $\big(N(w) \cap V_i, N(w) \cap V_j\big)$ is $(6\varepsilon'/d, d/2, p)$-lower-regular, and hence

$$e\big(G[X,Y]\big) \geqslant \frac{d^3p^3|V_i|^2}{2^6}$$

for every $X \subseteq N(w) \cap V_i$ and $Y \subseteq N(w) \cap V_j$ with $|X| = |Y| = dp|V_i|/4$.

We will obtain the set W by removing from U' those vertices which fail either condition (a) or (b), and then taking an arbitrary subset of the correct size. We claim that there are only $n^{o(1)}$ such 'bad' vertices in U'. To see that there are only $n^{o(1)}$ vertices $w \in U'$ such that $|N_\Gamma(w) \cap V_i| > 2p|V_i|$ or $|N_\Gamma(w) \cap V_j| > 2p|V_j|$, simply observe that assumption (A2) implies that the number of such vertices is at most $\frac{2\log n}{p^2} = n^{o(1)}$. To prove the corresponding statement for condition (b), we apply Theorem 2.7 with $X = V_i$ and $Y = V_j$. Recall that the pair (V_i, V_j) is (ε, d, p)-lower-regular, and therefore, by Theorem 2.7, there are at most $C' \max\big\{\frac{\log n}{p}, p^{-2}\big\} = n^{o(1)}$ vertices $w \in V(G)$ such that $\big(N_\Gamma(w) \cap V_i, N_\Gamma(w) \cap V_j\big)$ is not (ε', d, p)-lower-regular, as required. This completes the proof of the claim. □

We are finally ready to show that $X_0 = \emptyset$.

Claim 7. *$X_0 = \emptyset$.*

Proof of Claim 7. Suppose for a contradiction that $X_0 \neq \emptyset$, and let (V_i, V_j) and W be given by Claim 6. We will use Theorem 2.3 to find many copies of $K_{s,s}$ in G_w for each $w \in W$, which gives a lower bound on the number of copies of $K_{1,s,s}$ with parts in (respectively) W, V_i and V_j. This bound will be sufficiently large that, via an application of the pigeonhole principle, we can find a copy of $K_{s,s,s}$ in G. But $H \subseteq K_{s,s,s}$, so this gives the desired contradiction.

To spell out the details, let $\beta > 0$ be the constant returned by Theorem 2.3 with input s, and recall that, for every $w \in W$, the graph $G_w \subseteq G\big[N(w) \cap V_i, N(w) \cap V_j\big]$ has $dp|V_i|/2$ vertices and $d^3p^3|V_i|^2/2^6$ edges. By Theorem 2.3, it follows that G_w contains at least

$$\beta\left(\frac{dp}{16}\right)^{s^2}\left(\frac{dp|V_i|}{2}\right)^{2s}$$

copies of $K_{s,s}$, and hence, recalling that $|W| = dpn/3$, it follows that the number of pairs (w, K), where $w \in W$ and $K \subseteq G\big[N(w) \cap V_i, N(w) \cap V_j\big]$ is a copy of $K_{s,s}$, is at least

$$\frac{dpn}{3} \cdot \beta\left(\frac{dp}{16}\right)^{s^2}\left(\frac{dp|V_i|}{2}\right)^{2s} \geqslant \beta\left(\frac{dp}{16}\right)^{(s+1)^2}|V_i|^{2s}n \gg s|V_i|^{2s}, \tag{14}$$

since $p = n^{-o(1)}$.

Finally, note that there are at most $|V_i|^{2s}$ choices for the graph K, and so there must exist some K which is in the neighbourhood of at least s distinct $u \in W$. In particular, we have a copy of $K_{s,s,s}$ in G, and hence a copy of H. This contradiction proves the claim. □

By Claims 4 and 7, it follows that $\chi(G) \leqslant 2sk/\alpha$, which completes the proof of the proposition. □

5. Upper bounds for p constant

In this section we will prove the upper bound $\delta_\chi(H,p) \leqslant \delta_\chi(H)$ for all $p > 0$ constant. Together with the matching lower bound from Proposition 3.4, this completes the proof of Theorem 1.2. Recall that the bound $\delta_\chi(H,p) \leqslant 1 - \frac{1}{\chi(H)-1}$ for all H and any constant p was proved in Corollary 2.2, so that it remains to prove the following two propositions.

Recall that the decomposition family of a graph H is the collection of bipartite graphs obtained from H by removing $\chi(H) - 2$ independent sets. The following proposition generalises [3, Theorem 7], which proved it for $p = 1$.

Proposition 5.1. *Let H be a graph with $\chi(H) = r \geqslant 3$, and let $0 < p \leqslant 1$ be a constant. If H has a forest in its decomposition family, then*

$$\delta_\chi(H,p) \;\leqslant\; \delta_\chi(H) \;=\; \frac{2r-5}{2r-3}.$$

That is, for every $\gamma > 0$, there exists $C = C(H,p,\gamma) > 0$ such that, with high probability, every H-free spanning subgraph $G \subseteq G(n,p)$ with $\delta(G) \geqslant \big(\frac{2r-5}{2r-3} + \gamma\big)pn$ satisfies $\chi(G) \leqslant C$.

Recall that a graph H is near-acyclic if $\chi(H) = 3$ and H admits a partition into a forest F and an independent set I such that every odd cycle of H meets I in at least two vertices. It is r-near-acyclic if it is possible to obtain a near-acyclic graph by removing $\chi(H) - 3$ independent sets from H. The following proposition generalises [3, Theorem 34].

Proposition 5.2. *Let H be an r-near-acyclic graph with $\chi(H) = r \geqslant 3$, and let $0 < p \leqslant 1$ be a constant. Then we have*

$$\delta_\chi(H,p) \;\leqslant\; \delta_\chi(H) \;=\; \frac{r-3}{r-2}.$$

That is, for every $\gamma > 0$, there exists $C = C(H,p,\gamma) > 0$ such that, with high probability, every H-free spanning subgraph $G \subseteq G(n,p)$ with $\delta(G) \geqslant \big(\frac{r-3}{r-2} + \gamma\big)pn$ satisfies $\chi(G) \leqslant C$.

We emphasise that, in both propositions, the bound on the chromatic number is allowed to depend on p. Indeed, the construction used to prove [2, Theorem 1.3] shows that, for every $\gamma > 0$ and C, if $\chi(H) = r \geqslant 4$ and $p = p(H,\gamma) > 0$ is sufficiently small, then with high probability $G(n,p)$ contains spanning H-free subgraphs with minimum degree at least $\big(\frac{r-2}{r-1} - \gamma\big)pn$ and chromatic number at least C.

The proofs of both propositions are, with the exception of one key step in both proofs, relatively straightforward modifications of the proofs in [3] of the corresponding bounds in the case $p = 1$. For this reason we will emphasise the new ideas required in the random setting, and refer the reader to [3] for motivation of the lemmas we quote from [3]. We will first (in Section 5.1) state and prove a lemma which is the main new tool we will require; then (in Sections 5.2 and 5.3) we will describe how we combine this lemma with the method of [3] in order to prove the propositions.

5.1. **A bound on the number of pairs with few common neighbours.** Let G be a subgraph of $G(n,p)$, and suppose that U and W are sets of vertices with the property that every $u \in U$ has more than $\big(\frac{1}{2}+\gamma\big)p|W|$ neighbours in W. If $p=1$, then this implies that every pair of vertices has many common neighbours in W, but for $p<1$ there may exist some exceptional pairs with small common neighbourhood, even if W is quite large.

The following lemma, which is the key new step in the proof of both of the propositions above, says that (with high probability) few pairs in U have small common neighbourhood in W for every such pair (U,W) with $|W| = \Omega(n)$. As in the previous section, we will write $N_\Gamma(u)$ for the neighbourhood of u in $\Gamma = G(n,p)$ and $N(u)$ for the neighbourhood in G.

Lemma 5.3. *Given $p,\gamma,\alpha \in (0,1]$ there exists $C>0$ such that, with high probability, the following holds. For every subgraph $G \subseteq G(n,p)$, and every pair of vertex sets U and W satisfying $|W| \geqslant \alpha n$ and*

$$|N(u)\cap W| \geqslant \left(\frac{1}{2}+\gamma\right)p|W| \tag{15}$$

for every $u \in U$, we have

$$|N(u)\cap N(v)\cap W| \leqslant \gamma p^2 |W|$$

for at most $C|U|$ pairs $u,v \in U$.

The first step to proving this lemma is to find a bounded-size subset $X \subseteq U$ such that the Γ-neighbourhoods in W of subsets of $U \setminus X$ are all well-behaved. The following lemma is quite a bit more general (though no harder to prove) than the result we need. Note that for this lemma we require $p<1$.

Lemma 5.4. *Given $p \in (0,1)$ and $\delta \in (0,1]$, there exists $C>0$ such that, with high probability, the following holds. For every set of vertices $W \subseteq V = V\big(G(n,p)\big)$, there exists a set of vertices $X_W \subseteq V$ with $|X_W| \leqslant C$ such that*

$$\Big|\bigcap_{v\in S} N_\Gamma(v) \cap W\Big| \in p^{|S|}|W| \pm \delta n \tag{16}$$

for all $S \subseteq V \setminus X_W$.

Proof. Define $\ell = \ell(p,\delta)$ to be the least positive integer such that $p^\ell < \delta/2$. Given W, let $\mathcal{X}$ be a maximal family of disjoint vertex sets satisfying $|S| \leqslant \ell$ and

$$\Big|\bigcap_{v\in S} N_\Gamma(v) \cap W\Big| \notin p^{|S|}|W| \pm \frac{\delta n}{2} \tag{17}$$

for each $S \in \mathcal{X}$. We claim that $X_W = \bigcup_{S\in\mathcal{X}} S$ satisfies the required conditions. Indeed, for any subset $S \subseteq V \setminus X_W$ with $|S| \leqslant \ell$ it is immediate from the maximality of $\mathcal{X}$ that (16) holds. On the other hand, if $|S| > \ell$ then let S_0 be an arbitrary subset of S of cardinality ℓ, and observe that

$$0 \leqslant \Big|\bigcap_{v\in S} N_\Gamma(v) \cap W\Big| \leqslant \Big|\bigcap_{v\in S_0} N_\Gamma(v) \cap W\Big| \leqslant p^\ell|W| + \frac{\delta n}{2} \leqslant \delta n,$$

as required.

To show that, with high probability, $|X_W| \leqslant C$ for every set W, note first that, by the Chernoff bound,

$$\mathbb{P}\bigg(\Big| \bigcap_{v \in S} N_\Gamma(v) \,\cap\, W \Big| \notin p^{|S|}|W| \pm \frac{\delta n}{2} \bigg) \,\leqslant\, e^{-\Omega(\delta n)}.$$

These events are moreover independent for disjoint sets S, so if $\mathcal{X}$ contains t sets for some sufficiently large constant t, the probability that all of them satisfy (17) is at most e^{-n}. Any $\mathcal{X}$ with t sets gives X_W with at most $C = t\ell$ vertices. The number of ways to choose such an $\mathcal{X}$ is at most $n^{O(1)}$, and the number of ways to choose W is at most 2^n, so by the union bound we conclude that with high probability, for all W we have $|X_W| \leqslant C$ as desired. □

We can now prove Lemma 5.3.

Proof of Lemma 5.3. For $p = 1$, if U and W satisfy the conditions of the lemma, then for any $u, v \in U$ we have $|N(u) \cap N(v) \cap W| \geqslant 2\gamma|W| - 2 > \gamma p^2|W|$, so the lemma statement is true. We thus assume from now on that $p < 1$.

Fix $\delta = \delta(\alpha, \gamma, p) > 0$ sufficiently small, choose C sufficiently large for Lemma 5.4 to hold, and suppose that $\Gamma = G(n,p)$ has the property described in that lemma. Let us assume also that there do not exists sets $S, T \subseteq V(\Gamma)$ such that

$$|S| \geqslant \frac{\alpha\gamma p n}{5}, \qquad |T| \geqslant C \qquad \text{and} \qquad e\big(\Gamma[S,T]\big) \leqslant \Big(1 - \frac{\gamma}{10}\Big)\, p|S||T|,$$

and note that this also holds with high probability, by Chernoff's inequality.

Now let $G \subseteq \Gamma$ and let U and W satisfy the conditions of the lemma. Let X_W be the set given by Lemma 5.4. Recall that for each $u \in U$ we have $\big|N(u) \cap W\big| \geqslant \big(\frac{1}{2} + \gamma\big)p|W|$. For each $u \in U$ we define a set X_u of all vertices $v \in U \setminus X_W$ (so $N_\Gamma(u) \cap N_\Gamma(v) \cap W$ has about the expected size) such that $N(u) \cap N_\Gamma(v) \cap W$ is a bit smaller than expected:

$$X_u \,:=\, \left\{ v \in U \setminus X_W : |N(u) \cap N_\Gamma(v) \cap W| \leqslant \left(\frac{1}{2} + \frac{3\gamma}{5}\right) p^2 |W| \right\}. \tag{18}$$

Claim 1. *If $u \in U \setminus (X_W \cup X_v)$ and $v \in U \setminus (X_W \cup X_u)$, then*

$$|N(u) \cap N(v) \cap W| \,\geqslant\, \gamma p^2 |W|.$$

Proof of Claim 1. Since $u, v \notin X_W$, and assuming we chose $\delta < \alpha\gamma p^2/5$, we have

$$|N_\Gamma(u) \cap N_\Gamma(v) \cap W| \,\leqslant\, p^2|W| + \delta n \,\leqslant\, \left(1 + \frac{\gamma}{5}\right) p^2|W|.$$

Also, since $u \notin X_v$ and $v \notin X_u$, it follows that

$$\min\Big\{ |N(u) \cap N_\Gamma(v) \cap W|,\, |N(v) \cap N_\Gamma(u) \cap W| \Big\} \,\geqslant\, \left(\frac{1}{2} + \frac{3\gamma}{5}\right) p^2|W|.$$

Moreover, both are subsets of $N_\Gamma(u) \cap N_\Gamma(v) \cap W$, and so, by inclusion-exclusion,

$$|N(u) \cap N(v) \cap W| \,\geqslant\, 2\left(\frac{1}{2} + \frac{3\gamma}{5}\right) p^2|W| \,-\, \left(1 + \frac{\gamma}{5}\right) p^2|W| \,=\, \gamma p^2|W|,$$

as claimed. □

Claim 2. *$|X_u| \leqslant C$ for every $u \in U \setminus X_W$.*

Proof of Claim 2. We will count the paths of length two from u to X_u with first edge in G, second edge in $G(n,p)$, and middle vertex in W. By (18), the number of such paths is at most

$$\left(\frac{1}{2} + \frac{3\gamma}{5}\right)p^2|W| \cdot |X_u|. \tag{19}$$

On the other hand, defining

$$\hat{W}_u := \left\{ w \in W \,:\, |N_\Gamma(w) \cap X_u| \leqslant \left(1 - \frac{\gamma}{10}\right)p|X_u| \right\},$$

we obtain a lower bound on the number of such paths of

$$\big|N(u) \cap \big(W \setminus \hat{W}_u\big)\big| \cdot \left(1 - \frac{\gamma}{10}\right)p|X_u| \overset{(15)}{\geqslant} \left(\left(\frac{1}{2} + \gamma\right)p|W| - |\hat{W}_u|\right)\left(1 - \frac{\gamma}{10}\right)p|X_u|. \tag{20}$$

Now if $|\hat{W}_u| \leqslant \gamma p|W|/5$, then using (19) and (20), we obtain

$$\left(\frac{1}{2} + \frac{4\gamma}{5}\right)p|W| \cdot \left(1 - \frac{\gamma}{10}\right)p|X_u| \leqslant \left(\frac{1}{2} + \frac{3\gamma}{5}\right)p^2|W| \cdot |X_u|,$$

which is a contradiction. We conclude that $|\hat{W}_u| > \gamma p|W|/5$, so by definition of $\hat{W}_u$ and by our assumption on Γ (with $S = \hat{W}_u$ and $T = X_u$), we have $|X_u| \leqslant C$, as claimed. □

The lemma follows easily from the claims. Indeed, by Claim 1 the only pairs $u, v \in U$ with $|N(u) \cap N(v) \cap W| \leqslant \gamma p^2|W|$ are those with $u \in X_W \cup X_v$ or $v \in X_W \cup X_u$. Since $|X_W| + |X_u| = O(1)$ for every $u \in U$, by Lemma 5.4 and Claim 2 , it follows that the number of such pairs is at most $O(|U|)$, as required. □

5.2. **The proof of Proposition 5.1.** We need the following two lemmas from [3]. Let $K_\ell(t)$ be the t-blow-up of K_ℓ, that is, the graph obtained from K_ℓ by replacing each vertex with an independent set of size t, and each edge with a complete bipartite graph $K_{t,t}$. We write $F + H$ for the join of F and H, that is, the graph obtained from a disjoint union of F and H by adding all edges between F and H.

Lemma 5.5 (Lemma 9 of [3])**.** *Let $\alpha > 0$ and $3 \leqslant r, t \in \mathbb{N}$, let F be a forest, and let $H \subseteq F + K_{r-2}(t)$. Let G be a graph on n vertices, and $X \subseteq V(G)$. If $G[N(u)]$ contains at least $\alpha n^{(r-1)v(H)}$ copies of $K_{r-1}(v(H))$ for every $u \in X$, then either $H \subseteq G$ or $|X| \leqslant v(H)/\alpha$.*

Lemma 5.6 (Lemma 10 of [3])**.** *For every $\alpha > 0$ and $3 \leqslant r, t \in \mathbb{N}$, there exists $C = C(\alpha, r, t)$ such that for every graph $H \subseteq F + K_{r-2}(t)$, the following holds. Let G be an H-free graph on n vertices, and let $X \subseteq V(G)$ be such that every edge of $G[X]$ is contained in at least αn^{r-2} copies of K_r in G. Then $G[X]$ is $Cv(F)$-degenerate.*

We can now give the proof.

Proof of Proposition 5.1. Let F be a forest in the decomposition family of H, so in particular $H \subseteq F + K_{r-2}\big(v(H)\big)$. Let $\gamma > 0$, and let G be an H-free spanning subgraph of $G(n,p)$ with

$$\delta(G) \geqslant \left(\frac{2r-5}{2r-3} + 3\gamma\right) pn.$$

Applying the sparse minimum degree form of Szemerédi's Regularity Lemma, with $k_0 = r/\gamma$, $d = \gamma$ and ε sufficiently small, to G, we obtain a partition $V(G) = V_0 \cup V_1 \cup \dots \cup V_k$ with $k_0 \leqslant k \leqslant k_1 = O(1)$ such that the (ε, d, p)-reduced graph R satisfies

$$\delta(R) \geqslant \left(\frac{2r-5}{2r-3} + \gamma\right) k\,.$$

We now partition the vertices of $V(G)$ according to the sets V_i to which they send 'many' edges. More precisely, define for each $I_2 \subseteq I_1 \subseteq [k]$,

$$X_{I_1,I_2} := \Big\{ v \in V(G) \,:\, I_1 = \big\{ i \in [k] : |N(v) \cap V_i| \geqslant \gamma p |V_i| \big\}\,, \qquad I_2 = \big\{ i \in [k] : |N(v) \cap V_i| \geqslant \big(\tfrac{1}{2} + \gamma\big) p |V_i| \big\} \Big\}\,. \tag{21}$$

We remark that this is a refinement of the partition used in the proof of [3, Theorem 7]. Since the number of parts in this partition is at most 3^{k_1}, the following claim completes the proof of Proposition 5.1.

Claim. *$\chi\big(G[X_{I_1,I_2}]\big) = O(1)$ for every $I_2 \subseteq I_1 \subseteq [k]$.*

Proof of Claim. We partition the edges of $G[X_{I_1,I_2}]$ into graphs $J = J_{I_1,I_2}$ and $L = L_{I_1,I_2}$, where J consists of those edges whose endpoints have at least $\gamma p^2 |V_i|$ common neighbours in each of the clusters $\{V_i : i \in I_2\}$ and the remaining edges are in L. We will show that J and L are both $O(1)$-degenerate, which implies both are $O(1)$-colourable and hence that $G[X_{I_1,I_2}]$ is $O(1)$-colourable, as desired.

We start with L. For each $U \subseteq X_{I_1,I_2}$ and $i \in I_2$, consider the graph $L_i[U] \subseteq L[U]$ on vertex set U formed by the edges of G that have fewer than $\gamma p^2 |V_i|$ common neighbours in V_i. Setting $W = V_i$, note that the pair (U, W) satisfies (15), by the definition of X_{I_1,I_2} and since $i \in I_2$. By Lemma 5.3, it follows that $L_i[U]$ has at most $C|U|$ edges (for some constant $C = C(H, p, \gamma)$) and hence has bounded average degree. Since $L[U] \subseteq \bigcup_{i \in I_2} L_i[U]$, and $|I_2| \leqslant k$, it follows that the graph $L[U]$ has bounded average degree. Since this holds for every $U \subseteq X_{I_1,I_2}$, it follows that L is $O(1)$-degenerate, as claimed.

The proof that J is $O(1)$-degenerate is almost the same as in [3, Theorem 7], and so we shall only sketch the proof, skipping most of the details. Suppose first that

$$|I_1| \,\geqslant\, \left(\frac{2r-4}{2r-3}\right) k\,.$$

In this case we shall show that $|X_{I_1,I_2}| = O(1)$, and thus J is trivially $O(1)$-degenerate. We first claim that $R[I_1]$ contains a copy of K_{r-1}. Indeed, by our minimum degree condition on

R, we have

$$\delta\big(R[I_1]\big) \,\geqslant\, \delta(R) - \big(k - |I_1|\big) \,\geqslant\, |I_1| - \left(\frac{2}{2r-3} - \gamma\right) k \,\geqslant\, \left(\frac{r-3}{r-2} + \gamma\right) |I_1|\,,$$

so $R[I_1]$ contains a copy of K_{r-1}, as claimed. Let $\{W_1, \ldots, W_{r-1}\} \subseteq \{V_i : i \in I_1\}$ be the set of parts corresponding to this copy of K_{r-1}.

Now let $u \in X_{I_1,I_2}$, and recall that $|N(u) \cap W_i| \geqslant \gamma p|V_i|$ for each $i \in [r-1]$, by the definition of X_{I_1,I_2}. By the (dense) Slicing and Counting Lemmas, it follows that $G[N(u)]$ contains at least $\Omega\big(n^{(r-1)v(H)}\big)$ copies of $K_{r-1}(v(H))$. Since $H \not\subseteq G$, by Lemma 5.5 we have $|X_{I_1,I_2}| = O(1)$, as claimed.

So let us assume from now on that

$$|I_1| \,<\, \left(\frac{2r-4}{2r-3}\right) k\,,$$

and that $G(n,p)$ has the following property: for each vertex set S of size at least $n/(2k_1)$, the number of vertices that have more than $(1+\gamma)p|S|$ neighbours in S is at most a constant depending on p, γ and k_1 (for constant p this holds with high probability by the Chernoff bound). In particular this applies for $S = V_i$ for each $1 \leqslant i \leqslant k$. Now if there is no vertex of X_{I_1,I_2} which has at most $(1+\gamma)p|V_i|$ neighbours in each V_i, then this implies $|X_{I_1,I_2}| = O(1)$, in which case J is trivially $O(1)$-degenerate. So suppose $u \in X_{I_1,I_2}$ is a vertex with at most $(1+\gamma)p|V_i|$ neighbours in V_i for every $0 \leqslant i \leqslant k$.

We claim that $R[I_2]$ contains a copy of K_{r-2}. Indeed, since $\delta(G) \geqslant \big(\frac{2r-5}{2r-3} + 3\gamma\big)pn$, and since $|V_i| \leqslant n/k$ for each $i \in [k]$ and $|V_0| \leqslant \varepsilon n \leqslant \gamma n/2$, it follows that

$$\left(\frac{2r-5}{2r-3} + 3\gamma\right)pn \,\leqslant\, d(u) \,\leqslant\, \left((1+\gamma)|I_2| + \left(\frac{1}{2} + \gamma\right)\big(|I_1| - |I_2|\big) + 2\gamma k\right)\frac{pn}{k}.$$

Hence, using our bound on $|I_1|$, it follows that $|I_2| \geqslant \big(\frac{2r-6}{2r-3}\big)k$. Thus

$$\delta\big(R[I_2]\big) \,\geqslant\, \delta(R) - \big(k - |I_2|\big) \,\geqslant\, |I_2| - \left(\frac{2}{2r-3} - \gamma\right) k \,\geqslant\, \left(\frac{r-4}{r-3} + \gamma\right) |I_2|\,,$$

so $R[I_2]$ contains a copy of K_{r-2}, as claimed.

Finally, recall that the endpoints of each edge of J have at least $\gamma p^2|V_i|$ common neighbours in V_i for every $i \in I_2$. By the (dense) Slicing and Counting Lemmas, it follows that each edge of J is contained in $\Omega(n^{r-2})$ copies of K_r. Since $H \not\subseteq G$, Lemma 5.6 tells us that J is $O(1)$-degenerate. We have thus proved that both J and L are $O(1)$-degenerate, which implies that $\chi\big(G[X_{I_1,I_2}]\big) = O(1)$, as claimed. □

As noted earlier, the sets X_{I_1,I_2} partition the vertex set of G into at most 3^{k_1} parts, and hence the claim implies that $\chi(G) = O(1)$, as required. □

5.3. **The proof of Proposition 5.2.** We need the following lemma, which was essentially proved in [3, Section 7]. The difference here is that we have to explicitly assume (d) rather than deducing it from a degree condition.

Lemma 5.7. *For each $r \geqslant 3$ and $\gamma > 0$, and each r-near-acyclic graph H with $\chi(H) = r$, there exist $\varepsilon > 0$, $C > 0$ and $m_0 \in \mathbb{N}$ such that the following holds for every $m \geqslant m_0$. If G is a graph containing pairwise disjoint vertex sets $X, Y, Z_1, \ldots, Z_{r-3}$ with the following properties:*

(*a*) $\chi\big(G[X]\big) > C$,
(*b*) $|Y| = |Z_1| = \cdots = |Z_{r-3}| = m$,
(*c*) $|N(v) \cap Y| \geqslant \gamma m$ *for every* $v \in X$,
(*d*) $|N(u) \cap N(v) \cap Z_i| \geqslant \gamma m$ *for every* $uv \in E(G[X])$ *and every* $i \in [r-3]$,
(*e*) *each pair from* $Y, Z_1, \ldots, Z_{r-3}$ *forms an* ε*-regular pair in* G *of density at least* γ,

then $H \subseteq G$.

The proof of Lemma 5.7 is roughly as follows. We proceed as in the proof of [3, Theorem 34]: We apply [3, Proposition 26] (the so-called 'paired VC-dimension' argument), [3, Lemma 24] (an inductive double counting argument) and [3, Proposition 36] (which uses the Counting Lemma and the pigeonhole principle), followed by [3, Lemma 25] (which uses the fact that high degree graphs contain all trees). The only point where some extra care is needed is in the application of [3, Proposition 36], since this proposition requires that every vertex of the set X has at least $\big(\frac{1}{2} + \gamma\big)m$ neighbours in each set Z_i. However, the only use made of this condition is to deduce that each edge of $G[X]$ has a common neighbourhood of size at least γm in each Z_i, which is assumption (*d*) above, so the conclusion we need follows from exactly the same proof. For a complete proof of Lemma 5.7, see Appendix A.

The deduction of Proposition 5.2 from Lemma 5.7 follows the same outline as the proof of Proposition 5.1 above.

Proof of Proposition 5.2. Let H be r-near-acyclic, let $\gamma > 0$, and let G be an H-free spanning subgraph of $G(n,p)$ with

$$\delta(G) \geqslant \left(\frac{r-3}{r-2} + 2\gamma\right)pn\,. \tag{22}$$

Applying the sparse minimum degree form of Szemerédi's Regularity Lemma to G, with $k_0 = r/\gamma$, $d = \gamma$ and ε sufficiently small, we obtain a partition $V(G) = V_0 \cup V_1 \cup \cdots \cup V_k$ with $k_0 \leqslant k \leqslant k_1 = O(1)$ such that the reduced graph R satisfies

$$\delta(R) \geqslant \left(\frac{r-3}{r-2} + \gamma\right)k\,.$$

We define sets X_{I_1,I_2} for each $I_2 \subseteq I_1 \subseteq [k]$ exactly as in (21). Since, again, the number of parts X_{I_1,I_2} is at most 3^{k_1}, the following claim completes the proof of the proposition.

Claim. $\chi\big(G[X_{I_1,I_2}]\big) = O(1)$ *for every* $I_2 \subseteq I_1 \subseteq [k]$.

Proof of Claim. We partition the edges of $G[X_{I_1,I_2}]$ into graphs $J = J_{I_1,I_2}$ and $L = L_{I_1,I_2}$, exactly as in the proof of Proposition 5.1. That is, we let J consist of those edges whose endpoints have at least $\gamma p^2|V_i|$ common neighbours in each of the clusters $\{V_i : i \in I_2\}$. The proof that L is $O(1)$-degenerate (using Lemma 5.3) is exactly the same as before, since it does not use the minimum degree condition on G (and thus R), and we omit it.

The proof that J is $O(1)$-chromatic is somewhat different to that in Section 5.2, so let us give the details. As before, with high probability for each S of size at least $n/(2k_1)$, the number of vertices in $G(n,p)$ with more than $(1+\gamma)p|S|$ neighbours in S is $O(1)$ and in particular the number of vertices with more than $(1+\gamma)p|V_i|$ neighbours in any V_i with $0 \leqslant i \leqslant k$ is $O(1)$. Again, this implies that either $|X_{I_1,I_2}| = O(1)$ and thus J is trivially $O(1)$-degenerate, or there exists $u \in X_{I_1,I_2}$ with at most $(1+\gamma)p|V_i|$ neighbours in V_i for every $0 \leqslant i \leqslant k$. In this latter case we claim that $R[I_1]$ contains a pair of disjoint copies of K_{r-2}, each with at least $r-3$ vertices in I_2.

Indeed, counting neighbours of u, we obtain

$$\bigg((1+\gamma)|I_1| + \gamma\cdot\big(k-|I_1|\big) + 2\varepsilon k\bigg)\frac{pn}{k} \;\geqslant\; d(u) \;\geqslant\; \bigg(\frac{r-3}{r-2}+2\gamma\bigg)pn\,,$$

and hence $|I_1| \geqslant \big(\frac{r-3}{r-2}\big)k$. Similarly,

$$\bigg((1+\gamma)|I_2| + \bigg(\frac{1}{2}+\gamma\bigg)\big(k-|I_2|\big) + 2\varepsilon k\bigg)\frac{pn}{k} \;\geqslant\; \bigg(\frac{r-3}{r-2}+2\gamma\bigg)pn\,,$$

which implies that $|I_2| \geqslant \big(\frac{r-4}{r-2}\big)k$. By our minimum degree condition on R, it follows that we can choose (greedily) two disjoint cliques in $R[I_1]$ as claimed.

Let the clusters corresponding to the vertices of our two cliques be respectively $Y \in I_1$ and $Z_1,\ldots,Z_{r-3} \in I_2$ (for one), and $Y' \in I_1$ and $Z_1',\ldots,Z_{r-3}' \in I_2$ (for the other). Set

$$X_1 = X_{I_1,I_2} \setminus \big(Y \cup Z_1 \cup \cdots \cup Z_{r-3}\big),$$

and $X_2 = X_{I_1,I_2} \setminus X_1$, and suppose first that $\chi\big(J[X_1]\big) > C$, where $C = C(r,\gamma p^2)$ is the constant in Lemma 5.7. Note that X_1 is disjoint from $Y, Z_1,\ldots,Z_{r-3}$, by definition, and moreover we have

- (*a*) $\chi\big(J[X_1]\big) > C$, by assumption,
- (*b*) $|Y| = |Z_1| = \cdots = |Z_{r-3}| \geqslant n/2k$, since each is a part of the Szemerédi partition,
- (*c*) $|N(v) \cap Y| \geqslant \gamma p|V_i|$ for every $v \in X_1$, by the definition of X_{I_1,I_2},
- (*d*) $|N(u) \cap N(v) \cap Z_i| \geqslant \gamma p^2|V_i|$ for every $uv \in E(J[X])$, by the definition of J, and
- (*e*) each pair from $Y, Z_1,\ldots,Z_{r-3}$ forms an ε-regular pair in G of density at least γp, since $Z_1,\ldots,Z_{r-3}$ and Y form a clique in R, and $d=\gamma$.

Therefore, by Lemma 5.7, it follows that $H \subseteq G$, which is a contradiction. On the other hand, if $\chi\big(J[X_2]\big) > C$, then the same argument (with $Y, Z_1,\ldots,Z_{r-3}$ replaced by $Y', Z_1',\ldots,Z_{r-3}'$) gives the same contradiction. Hence $\chi(J) \leqslant 2C$, as required. □

As noted earlier, this completes the proof of Proposition 5.2, and hence of Theorem 1.2. □

6. Determining $\delta^*_\chi(H)$

In this section we sketch the proof of Theorem 1.9, which states that

$$\delta^*_\chi(H) = \min\Big\{\delta_\chi(H') \,:\, \text{there exists a homomorphism from } H \text{ to } H'\Big\} \tag{23}$$

for every graph H. We begin with the upper bound.

Proposition 6.1. *If H is any graph homomorphic to H', then $\delta^*_\chi(H) \leqslant \delta_\chi(H')$.*

We will use the following lemma, which follows from the result of Erdős [14] that the Turán density of any k-partite k-uniform hypergraph is zero.

Lemma 6.2. *For any graph H' and any $t \in \mathbb{N}$ and $c > 0$, if $n = v(G)$ is large enough and G contains $cn^{v(H')}$ copies of H', then G contains the t-blow-up of H'.*

Proof. Take a uniform random partition of $V(G)$ into $v(H')$ parts, and let F be a $v(H')$-uniform hypergraph on $V(G)$ whose edges correspond to copies of H' in G with the ith vertex of H' in the ith part of the partition for each i. In expectation, F contains at least $v(H')^{-v(H')}cn^{v(H')}$ edges, and by the result of Erdős [14] any F with so many edges contains a copy of the complete $v(H')$-partite hypergraph with parts of size t, giving the desired t-blow-up of H' in G. □

Proof of Proposition 6.1. Note that H is contained in the $v(H)$-blow-up of H'. Recall that, by the definition of $\delta_\chi(H')$, for each $\gamma > 0$ there exists $C = C(\gamma)$ such that any H'-free graph G' with minimum degree at least $\big(\delta_\chi(H') + \gamma\big)v(G')$ has chromatic number at most C. We claim that, for any $\varepsilon > 0$, every sufficiently large H-free graph G with minimum degree at least $\big(\delta_\chi(H') + 2\gamma\big)v(G)$ can be made C-partite by deleting at most $\varepsilon v(G)^2$ edges.

To see this, fix $\varepsilon > 0$ and choose $\mu > 0$ sufficiently small. By the Graph Removal Lemma (see, e.g., [20]), there exists $c > 0$ such that any n-vertex graph G either contains at least $cn^{v(H')}$ copies of H', or can be made H'-free by deleting at most μn^2 edges. By Lemma 6.2, we conclude that for all sufficiently large n, the graph G can be made H'-free by deleting at most μn^2 edges.

Let G' be obtained from G by deleting μn^2 edges in order to destroy all copies of H', and then sequentially vertices of degree less than $\big(\delta_\chi(H') + \gamma\big)n$ until no more remain. Since μ was chosen sufficiently small, this process terminates having deleted fewer than $\varepsilon n/2$ vertices. Thus G' is an H'-free graph with minimum degree at least $\big(\delta_\chi(H') + \gamma\big)v(G')$, and hence it has chromatic number at most C. Moreover, the total number of edges deleted from G to obtain G' is at most $\mu n^2 + \varepsilon n^2/2 < \varepsilon n^2$, as required. □

To complete the proof of (23), we will show that a simple modification of the constructions from [3] suffices to prove the claimed lower bound on $\delta^*_\chi(H)$. We will use the following variant of Lemma 3.5, which also follows from Propositions 5 and 35 and Theorem 16 of [3].

Lemma 6.3. *For every graph H', integer $s \in \mathbb{N}$ and constants $C, \gamma > 0$, there exists $K = K(H', s, \gamma, C) > 0$ such that the following holds. For all sufficiently large n, there exists a graph G on n vertices with the following properties:*

(*a*) *$\delta(G) \geqslant \big(\delta_\chi(H') - \gamma\big)n$.*
(*b*) *There exists a set $X \subseteq V(G)$, with $|X| = K$, such that $\chi\big(G[X]\big) > C$.*
(*c*) *$\delta_\chi(H^*) < \delta_\chi(H')$ for every subgraph of $H^* \subseteq G$ with at most s vertices.*

Given H, let H' be a graph which minimises $\delta_\chi(H')$ such that there exists a homomorphism from H to H'. We claim that $\delta^*_\chi(H) \geqslant \delta_\chi(H')$, i.e., that for every $\gamma > 0$ and $C > 0$, there exists $\varepsilon > 0$ and infinitely many H-free graphs G with $\delta(G) \geqslant \big(\delta_\chi(H') - 2\gamma\big)v(G)$ which cannot be made C-colourable by removing at most εn^2 edges.

To prove this, first let G' be the graph (on n vertices) given by Lemma 6.3 with inputs H', $s = v(H)$, C and γ. We construct a graph G by blowing up each vertex of X to size μn for some constant $\mu > 0$. Now, if $\mu < \gamma/K$, then

$$\delta(G) \geqslant \delta(G') \geqslant \big(\delta_\chi(H) - \gamma\big)n \geqslant \big(\delta_\chi(H) - 2\gamma\big)v(G),$$

since $v(G) \leqslant \big(1+\mu K\big)n$, and if $\mu \geqslant \sqrt{\varepsilon}$ then the chromatic number of G cannot be decreased by removing fewer than $\mu^2 n^2 \geqslant \varepsilon n^2$ edges. Thus it only remains to show that G is H-free.

Suppose that G is not H-free, and fix a copy of H in G. We can construct a subgraph $H^* \subseteq G'$ by taking all vertices of this copy of H lying outside the blow-up of X in G, and each vertex of X in G' whose blow-up in G contains one or more vertices of H. Note that H is homomorphic to H^*, by construction. But H^* has at most $v(H)$ vertices, and therefore $\delta_\chi(H^*) < \delta_\chi(H')$, by Lemma 6.3$(c)$, which contradicts our choice of H'. Hence G is H-free, and this completes the proof of Theorem 1.9.

Appendix A. The proof of Lemma 5.7

Before beginning, we should stress that the proof consists only of small and obvious modifications to the argument in [3]: in fact, the only changes are a modification to the statement and proof of Proposition A.7 below, and the proof of Lemma 5.7 itself which is essentially contained in the proof of 'Theorem 34' there. Most of this appendix is copied unchanged from [3], and it exists only to facilitate the sceptical reader's verification of Lemma 5.7.

Definition A.1 (Definition 19 of [3], Modified Zykov graphs)**.** Let $T_1, \dots, T_\ell$ be (disjoint) trees, and let T_j have bipartition $A_j \cup B_j$. We define $Z_\ell(T_1, \dots, T_\ell)$ to be the graph on vertex set

$$V\big(Z_\ell(T_1, \dots, T_\ell)\big) := \Bigg(\bigcup_{j\in[\ell]} A_j \cup B_j\Bigg) \cup \big\{u_I \colon I \subseteq [\ell]\big\}$$

and with edge set

$$E\big(Z_\ell(T_1, \dots, T_\ell)\big) := \bigcup_{j=1}^{\ell} \Bigg(E(T_j) \cup \bigcup_{j\in I\subseteq[\ell]} K\big(u_I, A_j\big) \cup \bigcup_{j\notin I\subseteq[\ell]} K\big(u_I, B_j\big)\Bigg).$$

For each $r \geqslant 3$ and $t \in \mathbb{N}$, the *modified Zykov graph* $Z_\ell^{r,t}(T_1, \dots, T_\ell)$ is the graph obtained from $Z_\ell(T_1, \dots, T_\ell)$ by performing the following two operations:

- (a) Add vertices $W = \{w_1, \dots, w_{r-3}\}$, and all edges with an endpoint in W.
- (b) Blow up each vertex u_I with $I \subseteq [\ell]$ and each vertex w_j in W to a set S_I or S'_j, respectively, of size t.

Finally, we shall write $Z_\ell^{r,t}$ for the modified Zykov graph obtained when each tree T_i, $i \in [\ell]$, is a single edge; that is, $Z_\ell^{r,t} = Z_\ell^{r,t}(e_1, \dots, e_\ell)$.

Observation A.2 (Observation 20 of [3])**.** *Let $\chi(H) = r$. Then H is r-near-acyclic if and only if there exist trees $T_1, \dots, T_\ell$ and $t \in \mathbb{N}$ such that H is a subgraph of $Z_\ell^{r,t}(T_1, \dots, T_\ell)$.*

It will be convenient for us to provide a compact piece of notation for the adjacencies in $Z_\ell^{r,t}$. For this purpose, given a graph G and a set $Y \subseteq V(G)$, and integers $\ell, t \in \mathbb{N}$ and $r \geqslant 3$, define $\mathcal{G}_\ell^{r,t}(Y)$ to be the collection of functions

$$S \,:\, 2^{[\ell]} \cup [r-3] \,\to\, \binom{Y}{t}.$$

It is natural to think of S as a family $\{S_I : I \subseteq [\ell]\} \cup \{S'_j : j \in [r-3]\}$ of subsets of Y of size t. We say that $S \in \mathcal{G}_\ell^{r,t}(Y)$ is *proper* if these sets are pairwise disjoint and $E(G)$ contains all edges xy with $x \in S_I \cup S'_j$ and $y \in S'_{j'}$, whenever $j \neq j'$. We shall write $\mathcal{F}_\ell^{r,t}(Y)$ for the collection of proper functions in $\mathcal{G}_\ell^{r,t}(Y)$. The idea behind this definition is that we will later want to consider a vertex set $Y \subseteq V(G)$ and a family of disjoint subsets $\{S_I \colon I \subseteq [\ell]\} \cup \{S'_j \colon j \in [r-3]\}$ of size t in Y that we want to extend to a copy of $Z_\ell^{r,t}$.

For an ordered pair (x, y) of vertices of G, a function $S \in \mathcal{F}_\ell^{r,t}(Y)$, and $i \in [\ell]$, we write $(x, y) \to_i S$, if $S'_j \subseteq N(x, y)$ for every $j \in [r-3]$ and

$$\bigcup_{I \,:\, i \in I} S_I \subseteq N(x) \qquad \text{and} \qquad \bigcup_{I \,:\, i \notin I} S_I \subseteq N(y)\,.$$

For an edge $e = xy \in E(G)$, we write $e \to_i S$ if either $(x, y) \to_i S$ or $(y, x) \to_i S$. Recall that $\mathbf{e}^\ell$ denotes the ℓ-tuple $(e_1, \ldots, e_\ell)$, with $\mathbf{e}^0$ the empty tuple. Define

$$\mathbf{e}^\ell \to S \qquad \Leftrightarrow \qquad e_i \to_i S \quad \text{for each } i \in [\ell]\,.$$

Observe that the graph $Z_\ell^{r,t}$ consists of a set of pairwise disjoint edges $e_1, \ldots, e_\ell$ and an $S \in \mathcal{F}_\ell^{r,t}(Y)$ such that $\mathbf{e}^\ell \to S$. An advantage of this notation is that we can write $\mathbf{e}^\ell \to S$ even if the edges in $\mathbf{e}^\ell$ are not pairwise disjoint.

We shall show how to find a well-structured set of many copies of $Z_\ell^{r,t}$ inside a graph with high minimum degree and high chromatic number. The following definition (in which we shall make use of the compact notation just defined) makes the concept of 'well-structured' precise. Given $X \subseteq V(G)$, we write $E(X)$ for the edge set of $G[X]$, and if $D \subseteq E(G)$, then $\delta(D)$ denotes the minimum degree of the graph $G[D]$.

In the following definition, the reader should think of the sets $D(\mathbf{e}^j)$ as a 'hierarchy' of graphs: we have a different graph $D(\mathbf{e}^{j+1})$ associated to each edge of $D(\mathbf{e}^j)$. Note that if the vector $\mathbf{e}^j$ and the edge e_{j+1} are contained in the same statement, then $\mathbf{e}^{j+1}$ is assumed to be their concatenation.

Definition A.3 (Definition 21 of [3], (C, α)-rich in copies of $Z_\ell^{r,t}$)**.** Let X and Y be disjoint vertex sets in a graph G, let $C \in \mathbb{N}$ and $\alpha > 0$, and let $s := (2^\ell + r - 3)t$. We say that (X, Y) is (C, α)-*rich* in copies of $Z_\ell^{r,t}$ if

$$\exists\, D = D(\mathbf{e}^0) \subseteq E(X)\, \forall\, e_1 \in D\, \exists\, D(\mathbf{e}^1) \subseteq E(X)\, \forall\, e_2 \in D(\mathbf{e}^1) \quad \ldots$$
$$\ldots \quad \forall\, e_{\ell-1} \in D(\mathbf{e}^{\ell-2})\, \exists\, D(\mathbf{e}^{\ell-1}) \subseteq E(X)\, \forall\, e_\ell \in D(\mathbf{e}^{\ell-1})$$

the following properties hold:

(*a*) $\delta(D), \delta\big(D(\mathbf{e}^1)\big), \ldots, \delta\big(D(\mathbf{e}^{\ell-1})\big) > C$, and
(*b*) $\big|\big\{S \in \mathcal{F}_\ell^{r,t}(Y) \,:\, \mathbf{e}^\ell \to S\big\}\big| \geqslant \alpha |Y|^s$.

The point of this definition is that since $Z_\ell(3,t)$ is quite a simple graph, we require only quite weak properties of (X,Y) in order to show that (X,Y) is (C,α)-rich in copies of $Z_\ell^{3,t}$. This is the content of the following proposition, whose proof is a 'paired VC-dimension' argument.

Proposition A.4 (Proposition 26 of [3])**.** *For every $\ell, t \in \mathbb{N}$ and $d > 0$, there exists $\alpha > 0$ such that, for every $C \in \mathbb{N}$, there exists $C' \in \mathbb{N}$ such that the following holds. Let G be a graph and let X and Y be disjoint subsets of $V(G)$, such that $|N(x) \cap Y| \geqslant d|Y|$ for every $x \in X$.*

Then either $\chi\big(G[X]\big) \leqslant C'$, or (X,Y) is (C,α)-rich in copies of $Z_\ell^{3,t}$.

Unfortunately, it is not easy to work with the concept of (C,α)-richness. The reason for this is the quantifier alternation in the definition: we would like to construct a tree T_1 in D, but different edges e_1 of D may give us entirely different sets $D(\mathbf{e}^1)$, and we need to construct a tree T_2 which is in $D(\mathbf{e}^1)$ for each $e_1 \in T_1$. The next definition eliminates this quantifier alternation. We write $\overline{d}(E)$ for the average degree of E, i.e. $2|E|$ divided by the number of vertices contained in some edge of E.

Definition A.5 (Definition 23 of [3], Good function, (C,α)-dense)**.** A function $S \in \mathcal{F}_\ell^{r,t}(Y)$ is (r,ℓ,t,C,α)-*good* for (X,Y) if there exist sets

$$E_1, \dots, E_\ell \subseteq E(X), \quad \text{with} \quad \overline{d}(E_j) \geqslant 2^{-\ell}\alpha C \quad \text{for each } 1 \leqslant j \leqslant \ell\,,$$

such that for every $e_1 \in E_1, \dots, e_\ell \in E_\ell$, we have $\mathbf{e}^\ell \to S$.

The pair (X,Y) is (C,α)-*dense* in copies of $Z_\ell^{r,t}$ if there exist at least $2^{-\ell}\alpha|Y|^s$ families $S \in \mathcal{F}(Y)$ which are (r,ℓ,t,C,α)-good for (X,Y).

To go with this definition we have the following lemma, whose proof is an inductive double counting argument, which converts richness into the more useable denseness.

Lemma A.6 (Lemma 24 of [3])**.** *If (X,Y) is (C,α)-rich in copies of $Z_\ell^{r,t}$, then (X,Y) is (C,α)-dense in copies of $Z_\ell^{r,t}$.*

Since we only obtain richness in copies of $Z_\ell^{3,t}$ from Proposition A.4, but the conclusion of Lemma 5.7 which we want to prove speaks of $Z_\ell^{r,t}$ for all $r \geqslant 3$, if $r \geqslant 4$ we need at some point to 'upgrade' the structure we have. The lemma which permits us to do this is the following. It is a small modification of Proposition 36 in [3].

Proposition A.7. *For every $r > 3$, $\ell, t \in \mathbb{N}$ and $d, \gamma > 0$ there exist $\ell^*, t^* \in \mathbb{N}$ such that for every $\alpha > 0$ and $C \in \mathbb{N}$, there exist $\varepsilon_1 > 0$ and $C^* \in \mathbb{N}$, such that for every $0 < \varepsilon < \varepsilon_1$ the following holds.*

Let G be a graph, and let X, Y and $Z_1, \dots, Z_{r-3}$ be disjoint subsets of $V(G)$, with $|Y| = |Z_j|$ for each $j \in [r-3]$. Let $Z := Z_1 \cup \cdots \cup Z_{r-3}$. Suppose that (Y, Z_j) and (Z_i, Z_j) are (ε, d)-regular for each $i \neq j$, and that for each $e \in G[X]$ and $j \in [r-3]$, the edge e has at least $\gamma|Z_j|$ common neighbours in Z_j.

If (X,Y) is (C^*,α)-dense in copies of $Z_{\ell^*}^{3,t^*}$, then there is some $S \in \mathcal{F}_\ell^{r,t}(Y \cup Z)$ such that S is (r,ℓ,t,C,α)-good for $(X, Y \cup Z)$.

The change here, as compared to [3], is that we insist that each edge $e \in G[X]$ has common neighbourhood $\gamma|Z_j|$ in each Z_j, as opposed to that each vertex of X has neighbourhood at least $\big(\frac{1}{2}+\gamma\big)|Z_j|$ in Z_j, which latter obviously implies the former. The change in the proof is similarly trivial: the only use made of the stronger condition in [3] is to imply the weaker condition. Nevertheless, we give the details.

For the proof, we combine an application of the Counting Lemma and two uses of the pigeonhole principle. As a preparation for these steps we need to show that there exists a family $S^* \in \mathcal{F}^{3,t^*}_{\ell^*}$ which is $(3,\ell^*,t^*,C^*,\alpha)$-good for (X,Y) and 'well-behaved' in the following sense. For each of the sets $S^*_I \subseteq Y$ given by S^*_I only a small positive fraction of the $(r-3)t$-element sets in Z has a common neighbourhood in S^*_I of less than t vertices. To this end we shall use the following lemma.

Recall that for a set T of vertices in a graph G, we write

$$N(T)\colon = \bigcap_{x\in T} N(x)\,.$$

Lemma A.8 (Lemma 37 of [3])**.** *For all $r,t \in \mathbb{N}$ and $\mu, d > 0$, there exist $t^* = t^*(r,t,\mu,d) \in \mathbb{N}$ and $\varepsilon_0 = \varepsilon_0(r,t,\mu,d) > 0$ such that for all $0 < \varepsilon < \varepsilon_0$ the following holds.*

Let G be a graph, and suppose that Y and $Z_1,\dots,Z_{r-3}$ are disjoint subsets of $V(G)$ such that (Y,Z_j) is (ε,d)-regular for each $j \in [r-3]$. Let $Z := Z_1 \cup \ldots \cup Z_{r-3}$, and define

$$\mathcal{B}(S) := \Big\{ T \in \binom{Z}{(r-3)t} \,:\, |N(T)\cap S| < t \Big\}$$

for each $S \subseteq Y$. Then we have

$$\mathcal{S} := \Big\{ S \in \binom{Y}{t^*} \,:\, |\mathcal{B}(S)| \geqslant \mu |Z|^{(r-3)t} \Big\} \;\leqslant\; \sqrt{\varepsilon}|Y|^{t^*}\,.$$

We shall now prove Proposition A.7.

Proof of Proposition A.7. We start by defining the constants. Given $r > 3$, $\ell, t \in \mathbb{N}$ and $\gamma, d > 0$, we set

$$\mu := \frac{\gamma^{(r-3)t}}{8\big((r-3)t\big)!(r-3)^{(r-3)t}}\Big(\frac{d}{2}\Big)^{\binom{r-3}{2}t^2} \quad\text{and}\quad \ell^* := \frac{\ell}{2\mu}\,. \tag{24}$$

Let t^* and ε_0 be given by Lemma A.8 with input $r,t,\mu' := 2^{-\ell^*}\mu, d$. Given $\alpha > 0$ and C, we choose

$$\varepsilon_1 := \min\Big(\frac{\alpha^2}{2^{4\ell^*+1}}, \frac{d\gamma}{4(\gamma+1)(r-3)t}, \varepsilon_0\Big) \quad\text{and}\quad C^* := \frac{2^{\ell^*}C}{\alpha\mu}\,. \tag{25}$$

Now let $0 < \varepsilon < \varepsilon_1$, let G be a graph, and let X, Y and $Z_1,\dots,Z_{r-3}$ be disjoint subsets of $V(G)$ as described in the statement, so in particular, (X,Y) is (C^*,α)-dense in copies of $Z^{3,t^*}_{\ell^*}$. The goal is to show that there exists $S \in \mathcal{F}^{r,t}_{\ell}(Y \cup Z)$ such that S is (r,ℓ,t,C,α)-good for $(X, Y \cup Z)$.

Our first step is to show that there is a 'well-behaved' function $S^* \in \mathcal{F}^{3,t^*}_{\ell^*}(Y)$.

Claim 3. *There is a function $S^* \in \mathcal{F}^{3,t^*}_{\ell^*}(Y)$ which is $(3, \ell^*, t^*, C^*, \alpha)$-good for (X, Y) and has the property that for every $I \subseteq [\ell^*]$, the set*

$$\mathcal{B}(S^*_I) = \left\{ T \in \binom{Z}{(r-3)t} : \left| N(T) \cap S^*_I \right| \leqslant t \right\}$$

in $\binom{Z}{(r-3)t}$ has size at most $2^{-\ell^*}\mu|Z|^{(r-3)t}$.

Proof of Claim 3. By Lemma A.8 (with input $r, t, \mu' = 2^{-\ell^*}\mu, d$), the total number of 'bad' t^*-subsets S' of Y, i.e., those for which $\mathcal{B}(S') \geqslant 2^{-\ell^*}\mu|Z|^{(r-3)t}$, is at most $\sqrt{\varepsilon}|Y|^{t^*}$. Let $\mathcal{S}$ be the set of functions S^* in $\mathcal{F}^{3,t^*}_{\ell^*}(Y)$ which do *not* have the property that for every $I \subseteq [\ell^*]$ we have $\mathcal{B}(S^*_I) < 2^{-\ell^*}\mu|Z|^{(r-3)t}$. We can obtain any function S^* in $\mathcal{S}$ by taking a set $I \subseteq [\ell^*]$ and one of the at most $\sqrt{\varepsilon}|Y|^{t^*}$ 'bad' t^*-sets to be S^*_I, and choosing the $2^{\ell^*} - 1$ remaining sets of S^* in any way from $\binom{Y}{t^*}$. It follows that

$$|\mathcal{S}| \leqslant 2^{\ell^*}\sqrt{\varepsilon}|Y|^{t^*}|Y|^{(2^{\ell^*}-1)t^*} = 2^{\ell^*}\sqrt{\varepsilon}|Y|^{2^{\ell^*}t^*}.$$

Since (X, Y) is (C^*, α)-dense in copies of $Z^{3,t^*}_{\ell^*}$, there are at least $2^{-\ell^*}\alpha|Y|^{2^{\ell^*}t^*}$ functions in $\mathcal{F}^{3,t^*}_{\ell^*}(Y)$ which are $(3, \ell^*, t^*, C^*, \alpha)$-good for (X, Y). Since by (25) we have $2^{-\ell^*}\alpha > 2^{\ell^*}\sqrt{\varepsilon}$, at least one of these functions is not in $\mathcal{S}$, as required. □

For the remainder of the proof, S^* will be a fixed function satisfying the conclusion of Claim 3. Since S^* is $(3, \ell^*, t^*, C^*, \alpha)$-good for (X, Y), there exist sets

$$E^*_1, \ldots, E^*_{\ell^*} \subseteq E(X), \quad \text{with} \quad \overline{d}(E^*_j) \geqslant 2^{-\ell^*}\alpha C^* \quad \text{for each} \quad 1 \leqslant j \leqslant \ell^*,$$

such that for every $e_1 \in E^*_1, \ldots, e_{\ell^*} \in E^*_{\ell^*}$, we have $\mathbf{e}^{\ell^*} \to S^*$.

Our next claim comprises two applications of the pigeonhole principle to find a copy of $K_{r-3}(t)$ in Z.

Claim 4. *There exists a copy T of $K_{r-3}(t)$ with t vertices in Z_j for each $j \in [r-3]$, and a set $L \subseteq [\ell^*]$ of size $|L| = \ell$ such that:*

(i) $|N(T) \cap S^*_I| \geqslant t$ for every $I \subseteq [\ell^*]$,
(ii) $N(T)$ contains at least $\mu|E^*_j|$ edges of E^*_j, for each $j \in L$.

Proof of Claim 4. By assumption, each edge $e \in E^*_1 \cup \ldots \cup E^*_{\ell^*}$ has at least $\gamma|Z_j|$ common neighbours in Z_j. By the Slicing Lemma, the common neighbours of e in Z_i and Z_j form an $(\varepsilon/\gamma, d)$-regular pair for each $1 \leqslant i < j \leqslant r-3$. By (25) we have $d - \varepsilon - (r-3)t\varepsilon/\gamma > d/2$. Hence, applying the Counting Lemma with ε replaced by ε/γ to the graph $H = K_{r-3}(t)$, it follows that there are at least

$$\frac{1}{\text{Aut}(H)}\left(d - \frac{\varepsilon}{\gamma}v(H)\right)^{e(H)}\left(\frac{\gamma|Z|}{r-3}\right)^{v(H)}$$
$$\geqslant \frac{1}{((r-3)t)!}\left(\frac{d}{2}\right)^{\binom{r-3}{2}t^2}\left(\frac{\gamma|Z|}{r-3}\right)^{(r-3)t} \overset{(24)}{\geqslant} 8\mu|Z|^{(r-3)t}$$

copies of $K_{r-3}(t)$ in $N(e) \cap Z$, each with t vertices in each Z_j.

There are therefore, for each $j \in [\ell^*]$, at least $8\mu|Z|^{(r-3)t}|E_j^*|$ pairs (e, T), where $e \in E_j^*$ and T is a copy of $K_{r-3}(t)$ as described, such that $T \subseteq N(e)$, or equivalently $e \subseteq N(T)$. Since we have

$$8\mu|Z|^{(r-3)t}|E_j^*| = 4\mu|Z|^{(r-3)t}|E_j^*| + 4\mu|E_j^*||Z|^{(r-3)t} ,$$

by the pigeonhole principle, it follows that there are at least $4\mu|Z|^{(r-3)t}$ copies of $K_{r-3}(t)$ in Z each of which has at least $4\mu|E_j^*|$ edges of E_j^* in its common neighbourhood. Let us denote by $\mathcal{T}_j$ the collection of such copies of $K_{r-3}(t)$. For a copy T of $K_{r-3}(t)$, let $L(T) = \{j : T \in \mathcal{T}_j\}$.

We claim that there is a set $\mathcal{T}$ containing at least $2\mu|Z|^{(r-3)t}$ copies T of $K_{r-3}(t)$ in Z, each with $|L(T)| \geqslant \ell$. Indeed, this follows once again by the pigeonhole principle, since there are at least

$$\ell^* \cdot 4\mu|Z|^{(r-3)t} \overset{(24)}{=} \ell|Z|^{(r-3)t} + \ell^* \cdot 2\mu|Z|^{(r-3)t}$$

pairs (T, j) with $T \in \mathcal{T}_j$.

Now, recall that S^* satisfies the conclusion of Claim 3, i.e., for each $I \subseteq [\ell^*]$, there are at most $2^{-\ell^*}\mu|Z|^{(r-3)t}$ sets $T \in \binom{Z}{(r-3)t}$ such that $|N(T) \cap S_I^*| \leqslant t$. Since $|\mathcal{T}| \geqslant 2\mu|Z|^{(r-3)t}$, there is a copy T of $K_{r-3}(t) \in \mathcal{T}$ such that for each $I \subseteq [\ell^*]$, we have $|N(T) \cap S_I^*| \geqslant t$. If we let L be any subset of $L(T)$ of size ℓ, then T and L satisfy the conclusions of the claim. □

Let T and L be as given by Claim 4 and for each $j \in L$ let $E_j \subseteq X$ be a set of $\mu|E_j^*|$ edges of E_j^* contained in $N(T)$ as promised by Claim 4(ii). We construct a function $S \in \mathcal{F}_\ell^{r,t}(Y)$ by choosing, for each $I \subseteq L$, a subset $S_I \subseteq S_I^*$ of size t in $N(T) \cap Y$ (which is possible by Claim 4(i)), and letting the sets S_i, $i \in [r-3]$, be the parts of T.

Claim 5. *S is (r, ℓ, t, C, α)-good for $(X, Y \cup Z)$.*

Proof of Claim 5. Recall that $|L| = \ell$, and assume without loss of generality that $L = \{1, \ldots, \ell\}$. By the choice of T and the definition of the sets S_I with $I \subseteq L$ and the sets S_i with $i \in [r-3]$, we have that S_i is completely adjacent to each $S_{i'}$ with $i \neq i'$, to each S_I, and to each edge $e \in \bigcup_{j \in L} E_j$. Since $\mathbf{e}^{\ell^*} \to S^*$ for each $\mathbf{e}^{\ell^*} \in E_1^* \times \ldots \times E_{\ell^*}^*$, it follows that $\mathbf{e}^\ell \to S$ for each $\mathbf{e}^\ell \in E_1 \times \ldots \times E_\ell$. Finally, for each $j \in L$, since $|E_j| \geqslant \mu|E_j^*|$, we have

$$\overline{d}(E_j) \geqslant \mu\overline{d}(E_j^*) \geqslant \mu 2^{-\ell^*}\alpha C^* \overset{(25)}{=} C ,$$

as required. □

Thus there exists a function $S \in \mathcal{F}_\ell^{r,t}(Y)$ which is (r, ℓ, t, C, α)-good for $(X, Y \cup Z)$, as required. □

Our final lemma states that the existence of a good function as provided by the previous proposition indeed implies the existence of the desired r-near-acyclic graph.

Lemma A.9 (Lemma 25 of [3])**.** *Let X and Y be disjoint vertex sets in G. Given $r, \ell, t \in \mathbb{N}$, $\alpha > 0$, and trees $T_1, \ldots, T_\ell$, if $C \geqslant 2^{\ell+3}\alpha^{-1}\sum_{i=1}^{\ell}|T_i|$ and $S \in \mathcal{F}_\ell^{r,t}(Y)$ is (r, ℓ, t, C, α)-good for (X, Y), then $Z_\ell^{r,t}(T_1, \ldots, T_\ell) \subseteq G$.*

We can now complete the proof of Lemma 5.7.

Proof of Lemma 5.7. Let H be an r-near-acyclic graph, with $r \geqslant 3$, and let $\gamma > 0$. We set $d = \gamma = \gamma$. Because H is r-near-acyclic, by Observation A.2 there exist trees $T_1, \dots, T_\ell$ and a number $t \in \mathbb{N}$ such that $H \subseteq Z_\ell^{r,t}(T_1, \dots, T_\ell)$. We now set constants as follows. First, we choose $d = \gamma$. Given r, ℓ, t, d and γ, Proposition A.7 returns integers ℓ^* and t^*. Now Proposition A.4, with input ℓ^*, t^* and d, returns $\alpha > 0$. Next, consistent with Lemma A.9 we set $C := 2^{\ell+3}\alpha^{-1}\sum_{i=1}^{\ell}|T_i|$. Feeding α and C into Proposition A.7 yields $\varepsilon > 0$ and C^*. Putting C^* into Proposition A.4 yields a constant $D = C'$.

Now suppose we are given a graph G containing pairwise disjoint vertex sets X, Y, $Z_1, \dots, Z_{r-2}$ such that $G[X]$ has chromatic number at least $D+1$, and such that $|Y| = |Z_1| = \cdots = |Z_{r-2}| = m$, such that each vertex of X has at least γm neighbours in Y and each edge of $G[X]$ has at least γm common neighbour in each $Z_1, \dots, Z_{r-2}$, and such that each pair from $Y, Z_1, \dots, Z_{r-2}$ forms an ε-regular pair in G of density at least γ. Our aim is to prove $H \subseteq G$.

We apply Proposition A.4, with input ℓ^*, t^*, d and C^*, to (X, Y). By assumption, we have $|N(x) \cap Y| \geqslant d|Y|$ for each $x \in X$. Since $\chi\big(G[X]\big) \geqslant D + 1$, so (X, Y) is (C^*, α)-rich in copies of $Z_{\ell^*}^{3,t^*}$.

By Lemma A.6 the pair (X, Y) is (C^*, α)-dense in copies of $Z_{\ell^*}^{3,t^*}$. We now apply Proposition A.7, with input r, ℓ, t, d, γ, α, C, and ε to $X, Y, Z_1, \dots, Z_{r-3}$. By assumption, each edge of $G[X]$ has at least $\gamma|Z_i|$ common neighbours in each Z_i, and any pair of $Y, Z_1, \dots, Z_{r-3}$ is (ε, d)-regular. By Proposition A.7, there exists a function $S \in \mathcal{F}_\ell^{r,t}(Y \cup Z_1 \cup \cdots \cup Z_{r-3})$ which is (r, ℓ, t, C, α)-good for $(X, Y \cup Z_1 \cup \cdots \cup Z_{r-3})$. Finally, we apply Lemma A.9, with input r, ℓ, t, α and $T_1, \dots, T_\ell$, to X and $Y \cup Z_1 \cup \cdots \cup Z_{r-3}$ with the function S. By the definition of C, this lemma gives that $Z_\ell^{r,t}(T_1, \dots, T_\ell)$ is contained in G, and so $H \subseteq G$ as desired. $\square$

References

[1] P. Allen, J. Böttcher, S. Griffiths, Y. Kohayakawa, and R. Morris, *Chromatic thresholds in dense random graphs*, Submitted (arXiv:1508.03870).

[2] ———, *Chromatic thresholds in sparse random graphs*, Submitted (arXiv:1508.03875).

[3] ———, *The chromatic thresholds of graphs*, Adv. Math. **235** (2013), 261–295.

[4] P. Allen, J. Böttcher, H. Hàn, Y. Kohayakawa, and Y. Person, *Blow-up lemmas for sparse graphs*, in preparation.

[5] P. Allen, J. Böttcher, Y. Kohayakawa, and B. Roberts, *Triangle-free subgraphs of random graphs*, Submitted (arXiv:1507.05226).

[6] J. Balogh, R. Morris, and W. Samotij, *Independent sets in hypergraphs*, J. Amer. Math. Soc. **28** (2015), no. 3, 669–709.

[7] J. Böttcher, Y. Kohayakawa, and A. Taraz, *Almost spanning subgraphs of random graphs after adversarial edge removal*, Combin. Probab. Comput. **22** (2013), no. 5, 639–683.

[8] D. Conlon and W. T. Gowers, *Combinatorial theorems in sparse random sets*, Submitted (arXiv:1011.4310).

[9] D. Conlon, W. T. Gowers, W. Samotij, and M. Schacht, *On the KLR conjecture in random graphs*, Israel J. Math. **203** (2014), no. 1, 535–580.

[10] B. DeMarco and J. Kahn, *Turán's Theorem for random graphs*, Submitted (arXiv:1501.01340).

[11] ———, *Mantel's Theorem for random graphs*, Random Structures Algorithms **47** (2015), no. 1, 59–72.

[12] P. Erdős and M. Simonovits, *Supersaturated graphs and hypergraphs*, Combinatorica **3** (1983), no. 2, 181–192.
[13] P. Erdős, *Graph theory and probability*, Canad. J. Math. **11** (1959), 34–38.
[14] ———, *On extremal problems of graphs and generalized graphs*, Israel J. Math. **2** (1964), 183–190.
[15] P. Erdős and M. Simonovits, *On a valence problem in extremal graph theory*, Discrete Math. **5** (1973), 323–334.
[16] P. Frankl and V. Rödl, *Large triangle-free subgraphs in graphs without K_4*, Graphs Combin. **2** (1986), no. 2, 135–144.
[17] S. Gerke, Y. Kohayakawa, V. Rödl, and A. Steger, *Small subsets inherit sparse ε-regularity*, J. Comb. Theory Ser. B **97** (2007), no. 1, 34–56.
[18] S. Janson, T. Łuczak, and A. Ruciński, *Random graphs*, Wiley-Interscience, New York, 2000.
[19] Y. Kohayakawa, T. Łuczak, and V. Rödl, *On K^4-free subgraphs of random graphs*, Combinatorica **17** (1997), no. 2, 173–213.
[20] J. Komlós and M. Simonovits, *Szemerédi's regularity lemma and its applications in graph theory*, Combinatorics, Paul Erdős is eighty, Vol. 2 (Keszthely, 1993), Bolyai Soc. Math. Stud., vol. 2, János Bolyai Math. Soc., Budapest, 1996, pp. 295–352.
[21] T. Łuczak and S. Thomassé, *Coloring dense graphs via VC-dimension*, Submitted (arXiv:1007.1670v1).
[22] J. Lyle, *On the chromatic number of H-free graphs of large minimum degree*, Graphs and Combinatorics (2010), 1–14.
[23] V. Rödl and M. Schacht, *Extremal results in random graphs*, Erdös centennial, Bolyai Soc. Math. Stud., vol. 25, János Bolyai Math. Soc., Budapest, 2013, pp. 535–583.
[24] W. Samotij, *Stability results for random discrete structures*, Random Structures Algorithms **44** (2014), no. 3, 269–289.
[25] D. Saxton and A. Thomason, *Hypergraph containers*, Invent. math. (2015), 1–68.
[26] M. Schacht, *Extremal results for random discrete structures*, Submitted.
[27] C. Thomassen, *On the chromatic number of triangle-free graphs of large minimum degree*, Combinatorica **22** (2002), no. 4, 591–596.
[28] ———, *On the chromatic number of pentagon-free graphs of large minimum degree*, Combinatorica **27** (2007), no. 2, 241–243.

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