# List Coloring Triangle-Free Hypergraphs

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#### Abstract

A triangle in a hypergraph is a collection of distinct vertices u, v, w and distinct edges e, f, g with  $u, v \in e$ ,  $v, w \in f$ ,  $w, u \in g$  and  $\{u, v, w\} \cap e \cap f \cap g = \emptyset$ . Johansson [10] proved that every triangle-free graph with maximum degree  $\Delta_2$  has list chromatic number  $O(\Delta_2/\log \Delta_2)$ . Frieze and the second author [7] proved that every linear (meaning that every two edges share at most one vertex) triangle-free triple system with maximum degree  $\Delta_3$  has chromatic number  $O(\sqrt{\Delta_3/\log \Delta_3})$ . The restriction to linear triple systems was crucial to their proof.

We provide a generalization of these results. The *i*-degree of a vertex in a hypergraph is the number of edges of size *i* containing it. We prove that every triangle-free hypergraph of rank three (edges have size two or three) with maximum 3-degree  $\Delta_3$  and maximum 2-degree  $\Delta_2$  has list chromatic number at most

$$c \max \left\{ \frac{\Delta_2}{\log \Delta_2}, \left( \frac{\Delta_3}{\log \Delta_3} \right)^{\frac{1}{2}} \right\},$$

for some absolute positive constant c.

Thus our result removes the *linear* restriction from [7] and applies to the broader class of rank three hypergraphs, while reducing to the (best possible) result [10] for graphs. As an application, we prove that if  $C_3$  is the collection of 3-uniform triangles, then the Ramsey number  $R(C_3, K_t^3)$  satisfies

$$\frac{at^{3/2}}{(\log t)^{3/4}} \le R(\mathcal{C}_3, K_t^3) \le \frac{bt^{3/2}}{(\log t)^{1/2}}$$

for some positive constants a and b. The upper bound makes progress towards the recent conjecture of Kostochka, the second author, and Verstraëte [13] that  $R(C_3, K_t^3) = o(t^{3/2})$  where  $C_3$  is the linear triangle.

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## 1 Introduction

A hypergraph H = (V, E) is a tuple consisting of a set of vertices V and a set of edges E, which are subsets of V. The hypergraph has rank k if every edge contains at most k vertices and is called k-uniform if every edge contains exactly k vertices. A proper coloring of H is an assignment of colors to the vertices so that no edge is monochromatic. The chromatic number of H,  $\chi(H)$ , is the minimum number of colors needed in a proper coloring of H.

The chromatic number of graphs (2-uniform hypergraphs) has been studied extensively. A greedy coloring algorithm can be used to show that for any graph G with maximum degree  $\Delta$ ,  $\chi(G) \leq \Delta + 1$ ; this bound is tight for complete graphs and odd cycles. Brooks [4] extended this by showing that if G is not a complete graph or an odd cycle, then  $\chi(G) \leq \Delta$ .

A natural question to ask is what other structural properties can be put on a graph to decrease its chromatic number. One approach is to fix a graph K and consider the family of graphs which contain no copy of K. For example, if K is a tree on e edges and G contains no copy of K, then  $\chi(G) \leq e$ ; this follows from the fact that if G contains no copy of K, then G contains a vertex of degree at most e-1 (see [19], pg. 70).

When K is a cycle, the problem becomes more difficult. Kim [11] showed that if G contains no 4-cycles or 3-cycles, then  $\chi(G) \leq (1 + o(1))\Delta/\log \Delta$  as  $\Delta \to \infty$ , which is within a factor of 2 of the best possible bound. Shortly after, Johansson [10] showed that if G contains no 3-cycles, then  $\chi(G) \leq O(\Delta/\log \Delta)$ . Using Johansson's result, Alon, Krivelevich, and Sudakov [2] showed that if K is any graph containing a vertex K such that K - K is bipartite, then  $\chi(G) \leq O(\Delta/\log \Delta)$ .

Some analogous results for hypergraphs are known. Using the local lemma, one can show that  $\chi(H) \leq O(\Delta^{1/(k-1)})$  for any k-uniform hypergraph H. Bohman, Frieze, and the second author [3] showed that if K is a fixed k-uniform hypertree on e edges and H is a k-uniform hypergraph containing no copy of K, then  $\chi(H) \leq 2(k-1)(e-1)+1$ ; Loh [14] improved this to  $\chi(H) \leq e$ , matching the result for graphs.

A hypergraph is linear (or contains no 2-cycles) if any two of its edges intersect in at most one vertex. A triangle in a linear hypergraph is a set of three pairwise intersecting edges with no common point. In [7], Frieze and the second author showed that if H is a 3-uniform, linear, triangle-free hypergraph, then  $\chi(H) \leq O(\sqrt{\Delta}/\sqrt{\log \Delta})$ . They subsequently removed the triangle-free condition and generalized their result from 3 to k, showing that  $\chi(H) \leq O((\Delta/\log \Delta)^{1/(k-1)})$  for any k-uniform, linear hypergraph H. As shown in [3], these results are tight apart from the implied constants.

#### 1.1 Our Result

Our contribution is to remove the linear condition from [7]. However, in doing so, we also widen the definition of a triangle.

**Definition 1.** A triangle in a hypergraph H is a set of three distinct edges  $e, f, g \in H$  and three distinct vertices  $u, v, w \in V(H)$  such that  $u, v \in e, v, w \in f, w, u \in g$  and  $\{u, v, w\} \cap e \cap f \cap g$ .

For example, the three triangles in a 3-uniform hypergraph are the loose triangle  $C_3 = \{abc, cde, efa\}, F_5 = \{abc, bcd, aed\}, \text{ and } K_4^- = \{abc, bcd, abd\}.$ 

Given a set L(v) of colors for every vertex  $v \in V(H)$ , a proper list coloring of H is a proper coloring where every vertex v receives a color from L(v). The list chromatic number of H,  $\chi_l(H)$ , is the minimum l so that if  $|L(v)| \ge l$  for all v, then H has a proper list coloring. It is not hard to see that  $\chi(H) \le \chi_l(H)$ . As in [11] and [10], our main theorem can be stated in terms of list chromatic number. If H is a rank k hypergraph and  $i \le k$ , the i-degree of a vertex v is the number of size i edges containing v.

**Theorem 2.** Suppose H is a rank 3, triangle-free hypergraph with maximum 3-degree  $\Delta$  and maximum 2-degree  $\Delta_2$ . Then

$$\chi_l(H) \le c_1 \max\{\left(\frac{\Delta}{\log \Delta}\right)^{\frac{1}{2}}, \frac{\Delta_2}{\log \Delta_2}\},$$

for some constant  $c_1$ .

Theorem 2 generalizes the results of [10] and [7]. Additionally, it strengthens [7] by removing the linear hypothesis, which was a crucial ingredient in the proof. As mentioned above, for n-vertex 3-uniform hypergraphs H with maximum degree  $\Delta$ , one can easily show that the independence number of H is  $\Omega(n/\sqrt{\Delta})$  and  $\chi(H) = O(\sqrt{\Delta})$ ; however, adding a local restriction to the hypergraph in order to significantly improve either of these bounds appears to be a hard problem. There are two conjectures in this regard. De Caen [5] conjectured that if we add the hypothesis that every vertex subset S spans at most  $c|S|^2$  edges (for some fixed constant c), and  $\Delta = \Theta(n)$ , then the lower bound on the independence number can be improved by a factor that tends to infinity with  $\Delta$ . More recently, [7] conjectured that if there is a fixed hypergraph F with  $F \not\subset H$ , then  $\chi(H) < c_F \sqrt{\Delta/\log \Delta}$ . Guruswami and Sinop [8] showed that this conjecture implies certain hardness results in computer science.

We prove Theorem 2 by using a semi-random algorithm to properly color the hypergraph. Our algorithm is similar to the algorithm in [7], however, several new ideas are developed

to deal with the non-linear case. At each iteration, we randomly color a few of the vertices. When a vertex in a 3-edge is colored c, we add a c-colored 2-edge between the remaining two vertices to record the fact that those two vertices cannot both be colored c in the future. [7] assumed the hypergraph was linear, which implied that at most one such 2-edge could be added between two vertices. Here we maintain a 2-graph for every color and allow two vertices to share an edge in multiple graphs. This allows us to extend our algorithm to rank 3 hypergraphs: for each 2-edge in the original hypergraph, we simply add a copy of that 2-edge to every color graph. After several iterations, we color the remaining vertices with the asymmetric version of the local lemma. This prevents the 3-edges from becoming monochromatic, while also enforcing the constraints from the 2-graphs.

### 1.2 Application to Hypergraph Ramsey Numbers

Let  $C_3^r$  be the collection of r-uniform hypergraph triangles. Notice that for graphs,  $C_3^2$  consists of only the 3-vertex cycle, and for triple systems,  $C_3^3 = \{C_3, F_5, K_4^-\}$ . The hypergraph Ramsey number  $R(C_3^r, K_t^r)$  is the smallest n so that in every red-blue coloring of the edges of the complete r-uniform hypergraph  $K_n^r$ , there exists a red triangle or a blue  $K_t^r$ . Ajtai-Komlós-Szemerédi [1] and Kim [12] proved that  $R(C_3^2, K_t^2) = \Theta(t^2/\log t)$ .

In [13], Kostochka, the second author, and Verstraëte proved a version of this result for r = 3. In this setting,  $R(C_3, K_t^3)$  is the smallest n so that in every red-blue coloring of the edges of the complete 3-uniform hypergraph  $K_n^3$ , there exists a red  $C_3$  or a blue  $K_t^3$ . [13] showed that there exist constants a, b such that

$$\frac{at^{3/2}}{(\log t)^{3/4}} \le R(C_3, K_t^3) \le bt^{3/2},$$

and they conjectured that the upper bound could be reduced to  $o(t^{3/2})$ . We prove a weaker form of this conjecture, namely that  $R(\mathcal{C}_3^3, K_t^3) = O(t^{3/2}/\sqrt{\log t})$ . Since the  $C_3$ -free construction given in [13] is also  $F_5$  and  $K_4^-$  free, this implies that for some constants a and b,

$$\frac{at^{3/2}}{(\log t)^{3/4}} \le R(\mathcal{C}_3^3, K_t^3) \le b \frac{t^{3/2}}{(\log t)^{1/2}}.$$

## 1.3 Organization

In Section 2, we present the probabilistic tools we will need to analyze our algorithm. In Section 3, we describe our algorithm. The presentation is similar to Vu's description in [18] of Johansson's algorithm. Section 4 contains an analysis of our algorithm. This

analysis does not use triangle-free anywhere, but is instead based on parameters which can be given to the algorithm. In Section 5, we show how triangle-free can be used to set these parameters in a way that implies Theorem 2.

### 2 Tools

#### 2.1 Local Lemma

**Asymmetric Local Lemma** ([17]). Consider a set  $\mathcal{E} = \{A_1, \ldots, A_n\}$  of (typically bad) events that such each  $A_i$  is mutually independent of  $\mathcal{E} - (\mathcal{D}_i \cup A_i)$ , for some  $\mathcal{D}_i \subset \mathcal{E}$ . If for each  $1 \leq i \leq n$ 

- $Pr[A_i] \leq 1/4$ , and
- $\sum_{A_j \in \mathcal{D}_i} \Pr[A_j] \le 1/4$ ,

then with positive probability, none of the events in  $\mathcal{E}$  occur.

### 2.2 Concentration Theorems

The first result is due to Hoeffding [9].

**Theorem 3.** Suppose that  $X = X_1 + \cdots + X_m$ , where the  $X_i$  are independent random variables satisfying  $|X_i| \leq a_i$  for all i. Then for any t > 0,

$$\Pr[X \ge \mathbf{E}[X] + t] \le e^{-\frac{2t^2}{\sum_{i=1}^{m} a_i^2}},$$

and

$$\Pr[X \le \mathbf{E}[X] - t] \le e^{-\frac{2t^2}{\sum_{i=1}^{m} a_i^2}}.$$

We will also use the following theorem, which is Theorem 2.7 from [16].

**Theorem 4.** Suppose that  $X = X_1 + \cdots + X_m$ , where the  $X_i$  are independent random variables satisfying  $X_i \leq \mathbf{E}[X_i] + b$  for all i. Then for any t > 0,

$$\Pr[X \ge \mathbf{E}[X] + t] \le e^{-\frac{t^2}{2\operatorname{Var}[X] + bt}}.$$

McDiarmid [15] proved the following generalization of Theorem 3.

**Theorem 5.** Let  $Z_1, \ldots, Z_n$  be independent random variables, with  $Z_i$  taking values in a set  $A_i$  for each i. Suppose that the (measurable) function  $g: \prod A_k \to \mathbb{R}$  satisfies  $|g(x) - g(x')| \le d_i$  whenever the vectors x and x' differ only in the i<sup>th</sup> coordinate. Let W be the random variable  $g(Z_1, \ldots, Z_n)$ . Then for any t > 0,

$$\Pr[W > \mathbf{E}[W] + t) \le e^{-2t^2 \sum_{i=1}^n d_i^2}$$

Note that in the above theorem, we may view  $\prod A_k$  as a probability space induced by the random variables  $Z_1, \ldots, Z_n$ . We will use the following corollary, which resembles Theorem 7.2 from [6].

Corollary 6. Let  $X_1, \ldots, X_n$  be independent random variables, with  $X_i$  taking values in a set  $\mathcal{B}_i$  for each i. Let  $\mathcal{A}_1, \ldots, \mathcal{A}_n$  be events, where each  $\mathcal{A}_i \subset \mathcal{B}_i$ . Set  $\mathcal{A} = \prod_{i=1}^n \mathcal{A}_i$ . Suppose that the (measurable) function  $f: \prod \mathcal{B}_k \to \mathbb{R}$  is non-negative and satisfies  $|f(x) - f(x')| \leq d_i$  for any two vectors  $x, x' \in \mathcal{A}$  differing only in the  $i^{th}$  coordinate. Let Y be the random variable  $f(X_1, \ldots, X_n)$ . Then

$$\Pr[Y > \mathbf{E}[Y]/\Pr[\mathcal{A}] + t] \le e^{-2t^2/\sum_{i=1}^n d_i^2} + \Pr[\bar{\mathcal{A}}].$$

Proof. Define  $g: \mathcal{A} \to \mathbb{R}$  by g(x) := f(x) (in other words,  $g = f|\mathcal{A}$ ). For each i, let  $Z_i: X_i^{-1}(\mathcal{A}_i) \to \mathcal{A}_i$  be the random variable with  $Z_i(s) = X_i(s)$  for all  $s \in X_i^{-1}(\mathcal{A}_i)$ . Let W be the random variable  $g(Z_1, \ldots, Z_n)$ . Since the  $X_i$  are independent, the  $Z_i$  are also independent, so we will be able to apply Theorem 5 to bound  $\Pr[W > \mathbf{E}[W] + t]$ .

By total probability and the non-negativity of f,

$$\mathbf{E}[Y] = \mathbf{E}[Y|\mathcal{A}] \Pr[\mathcal{A}] + \mathbf{E}[Y|\bar{\mathcal{A}}] \Pr[\bar{\mathcal{A}}] \ge \mathbf{E}[Y|\mathcal{A}] \Pr[\mathcal{A}]$$

SO

$$\mathbf{E}[W] = \mathbf{E}[Y|\mathcal{A}] \le \mathbf{E}[Y]/\Pr[\mathcal{A}].$$

Combining this with Theorem 5 implies

$$\Pr[Y > \frac{\mathbf{E}[Y]}{\Pr[\mathcal{A}]} + t] = \Pr[Y > \frac{\mathbf{E}[Y]}{\Pr[\mathcal{A}]} + t|\mathcal{A}] \Pr[\mathcal{A}] + \Pr[Y > \frac{\mathbf{E}[Y]}{\Pr[\mathcal{A}]} + t|\bar{\mathcal{A}}] \Pr[\bar{\mathcal{A}}]$$

$$\leq \Pr[Y > \frac{\mathbf{E}[Y]}{\Pr[\mathcal{A}]} + t|\mathcal{A}] + \Pr[\bar{\mathcal{A}}]$$

$$\leq \Pr[Y > \mathbf{E}[Y|\mathcal{A}] + t|\mathcal{A}] + \Pr[\bar{\mathcal{A}}]$$

$$= \Pr[W > \mathbf{E}[W] + t] + \Pr[\bar{\mathcal{A}}]$$

$$\leq e^{-2t^2/\sum_{i=1}^n d_i^2} + \Pr[\bar{\mathcal{A}}].$$

## 3 Coloring Algorithm

The input to our algorithm is a rank 3 hypergraph with maximum 3-degree  $\Delta$  and maximum 2-degree  $\Delta_2$ . Let H denote the input hypergraph restricted to its size 3 edges, and let G denote the input hypergraph restricted to its size 2 edges. At the beginning, each vertex u has a list C(u) of acceptable colors. We assume |C(u)| = C for all vertices u. For each vertex u and color c, we set

$$p_u^0(c) = \begin{cases} 1/C, & \text{if } c \in C(u) \\ 0, & \text{if } c \notin C(u). \end{cases}$$

We define a parameter  $\hat{p}$ , which will serve as an upper bound on the weights  $p_u^i(c)$ . Set  $W^0(u) = \{p_u^0(c) : c \in \bigcup_v C(v)\}$ . We start with the hypergraph  $H^0 = H$  and the collection  $\{W^0(u)\}_u$ . For each color c, we also construct a graph  $G_c^0$ , which is initially a copy of the 2-graph G. Finally, we assign to each vertex an empty set  $B^0(u)$ .

At the  $(i+1)^{th}$  step,  $i=0,1,\ldots,T-1$ , our input to the algorithm is a quadruple,  $(H^i, \{G_c^i\}_c, \{W_u^i\}_u, \{B^i(u)\}_u)$ . We generate a small random set of colors at each vertex u as follows: For each color c, we choose c with probability  $\theta p_u^i(c)$ . Let

$$\gamma_u^i(c) = \begin{cases} 1, & \text{if } c \text{ is chosen at } u, \\ 0, & \text{otherwise.} \end{cases}$$

Note that the  $\gamma_u^i(c)$  are independent random variables.

Consider a vertex u. We define the set of colors lost at u as

$$L(u) = \{c : \exists e \in E(H^i) \cup E(G_c^i) \text{ such that } u \in e \text{ and } \gamma_v^i(c) = 1 \ \forall v \in e - u\}.$$

We say a color c survives at u if  $c \notin B^i(u)$  and  $c \notin L(u)$ . For  $c \notin B^i(u)$ , we define

$$q_u^i(c) := \Pr[c \text{ survives at } u] = \Pr[\bigcap_{uvw \in H^i} (\gamma_v^i(c) = 0 \cup \gamma_w^i(c) = 0) \bigcap_{uv \in G_c^i} \gamma_v^i(c) = 0].$$
 (3.1)

In other words, if  $c \notin B^i(u)$ , then  $q_u^i(c) = \Pr[c \notin L(u)]$ . Note that at the  $(i+1)^{th}$  step,  $q_u^i(c)$  is a fixed number, which can be computed given  $H^i$ ,  $G_c^i$ , and all of the  $p_v^i(c)$ ; it does not depend on the random variables  $\gamma_u^i(c)$ . In the analysis below, we will use the bound

$$q_u^i(c) = 1 - \Pr\left[\bigcup_{uvw \in H^i} (\gamma_v^i(c) = 1 \cap \gamma_w^i(c) = 1) \bigcup_{uv \in G_c^i} \gamma_v(c) = 1\right]$$

$$\geq 1 - \sum_{uvw \in H^i} \theta^2 p_v^i(c) p_w^i(c) - \sum_{uv \in G_c^i} \theta p_v^i(c). \tag{3.2}$$

Let I[X] denote the 0,1 indicator variable for the event X. Define  $p_u^{i+1}(c)$  as:

• If  $p_u^i(c)/q_u^i(c) < \hat{p}$  and  $c \notin B^i(u)$ , then

$$p_u^{i+1}(c) = p_u^i(c) \frac{\mathbf{I}[c \text{ survives at } u]}{\Pr[c \text{ survives at } u]} = \begin{cases} p_u^i(c)/q_u^i(c), & \text{if } c \text{ is survives at } u, \\ 0, & \text{else.} \end{cases}$$
(3.3)

• If  $p_u^i(c)/q_u^i(c) \geq \hat{p}$  or  $c \in B^i(u)$ , then we toss a biased coin with  $\Pr[Head] = p_u^i(c)/\hat{p}$ . We then set

$$\eta_u^i(c) = \mathbf{I}[Head],$$

and

$$p_u^{i+1}(c) = p_u^i(c) \frac{\mathbf{I}[Head]}{\Pr[Head]} = \begin{cases} \hat{p}, & \text{if } \eta_u^i(c) = 1\\ 0, & \text{else.} \end{cases}$$
(3.4)

Crucially, (3.3) and (3.4) imply

$$\mathbf{E}[p_u^{i+1}(c)] = p_u^i(c). \tag{3.5}$$

Color u with c if c survives at u and  $\gamma_u^i(c)=1$  (if there are multiple such c, pick one arbitrarily). Let  $U^{i+1}$  denote the set of uncolored vertices in H after the iteration i. Let  $H^{i+1}$  be the hypergraph induced from H by  $U^{i+1}$ , let  $B^{i+1}(u)=\{c:p_u^{i+1}(c)=\hat{p}\}$ , and let  $W_u^{i+1}=\{p_u^{i+1}(c)\}$ . To form  $G_c^{i+1}$ , start with  $G_c^i$ , and for each triple  $u,v,w\in U^i$  with  $u,v\in U^{i+1}$ ,  $uv\notin G_c^i$ , and w colored c, add an edge uv to  $G_c^{i+1}$ . Then delete any vertex from  $G_c^{i+1}$  that is not in  $U^{i+1}$ .

Observe that if uvw is an edge in  $H^i$  and u and v are both colored c in the current round, then  $p_w^{i+1}(c) \in \{0, \hat{p}\}$ ; in particular, c is never considered for w in a future round. Similarly, if  $vw \in G_c^i$  and v is colored with c in the current round, then c is never considered for w in the future. Thus the algorithm always maintains a proper partial coloring of H.

After T iterations, some vertices will remain uncolored. We color these in one final step, which is described in Section 4.5.

#### 3.1 Parameters and Notation

We summarize all of the variables used in the algorithm and its analysis in the two tables below. The first table contains descriptions of the independent variables in our algorithm. We set them for one family of hypergraphs in Section 5, when we prove that our algorithm works for triangle-free hypergraphs. The values of the remaining parameters are defined in the second table.

Our algorithm requires that the parameter  $\omega_0$  satisfy the following properties:

• For any edge uvw in  $H^i$  and any color c,

$$\Pr[c \notin L(u) \cup L(v) \cup L(w)] \le q_u^i(c)q_v^i(c)q_w^i(c)(1+1/\omega_0). \tag{3.6}$$

• For any color c and any pair u, v with  $uvw \in H^i$  for some w,

$$\Pr[c \notin L(u) \cup L(v)] \le q_u^i(c)q_v^i(c)(1 + 1/\omega_0). \tag{3.7}$$

• For any color c and any edge uv in  $G_c^i$ ,

$$\Pr[c \notin L(u) \cup L(v)] \le q_u^i(c)q_v^i(c)(1 + 1/\omega_0). \tag{3.8}$$

The parameters  $\omega_1$  through  $\omega_6$  are error terms used in the analysis of the algorithm.

#### Description

$\Delta$	Maximum degree of 3-graph
$\Delta_2$	Maximum degree of 2-graph
$\delta$	Maximum codegree
$\omega$	Color bound, tending to $\infty$ with $\Delta$

- $\epsilon$  Small constant
- $\omega_0$  Error term depending on H
- $\hat{p}$  Threshold probability

	Value	Description
$\overline{C}$	$\sqrt{\Delta}/\sqrt{\omega}$	Number of colors
T	$(5\omega/\epsilon)\log\omega$	Number of iterations
$\theta$	$\epsilon/\omega$	Activation probability
m	21	Used to control codegrees
$\omega_1$	$T \log C$	Error term
$\omega_2$	$\omega_0/16\omega$	Error term
$\omega_3$	$\omega^2$	Error term
	$\omega^2$	Error term
$\omega_5$	$\Delta^{19/20}$	Error term
$\omega_6$	$\Delta^{1/4}$	Error term

We will use the following notation:

$$\begin{split} N_H^i(u) &= \{v \in V(H^i) - u : \exists e \in H^i \text{ with } u, v \in e\} \\ N_H^i(u,v) &= \{w \in V(H^i) - \{u,v\} : \{u,v,w\} \in H^i\} \\ N_c^i(u) &= \{v \in V(G_c^i) - u : \exists e \in G_c^i \text{ with } u, v \in e\} \\ N^i(u) &= N_H^i(u) \cup \cup_c N_c^i(u) \\ N_G^0(u) &= \{v \in V(G) : uv \in E(G)\} \\ d_H^i(u) &= |\{e \in H^i : u \in e\}| \\ d_H^i(u,v) &= |\{e \in H^i : u, v \in e\}| \\ d_{G_c}^i(u) &= |\{v \in G_c^i : uv \in G_c\}|. \end{split}$$

At the beginning of iteration i of the algorithm, we also define the following parameters:

$$\begin{split} w(p_u^i) &= \sum_c p_u^i(c) \\ f_u^i(c) &= \sum_{uv \in G_c^i} p_v^i(c) \\ f_u^i &= \sum_c \sum_{uv \in G_c^i} p_u^i(c) p_v^i(c) \\ e_{uvw}^i &= \sum_c p_u^i(c) p_v^i(c) p_w^i(c) \\ e_u^i &= \sum_{uvw \in H^i} e_{uvw}^i \\ e_u^i(c) &= \sum_{uvw \in H^i} p_v^i(c) p_w^i(c) \\ h_u^i &= -\sum_c p_u^i(c) \log p_u^i(c), \text{ where } x \log x := 0 \text{ if } x = 0 \ . \end{split}$$

Our analysis assumes that the parameters of the algorithm satisfy the following relations. All asymptotic notation assumes  $\Delta \to \infty$ .

(R1) 
$$\theta \log(\hat{p}C) \ge 85$$

(R2) 
$$1/\omega_0 = o(\theta)$$

(R3) 
$$2/\omega_1^2 C\hat{p}^2 > 6\log\Delta$$

(R4) 
$$T/\omega_1 = o(1)$$

(R5) 
$$(T \log C)/\omega_1 < \epsilon/\theta$$

(R6) 
$$2/(4\Delta^2\omega_2^2C\hat{p}^6) > 6\log\Delta$$

(R7) 
$$\theta T/\omega_2 = o(1)$$

(R8) 
$$\omega\omega_2 + T < \omega_0/2$$

(R9) 
$$1/\omega_2 \le (1 - \theta/4)^T \omega$$

(R10) 
$$1/(4\omega_3^2(6\omega_6T\theta\hat{p}^5\Delta^2 + 4m\hat{p}^5\Delta^{2+1/2m} + Cm^2\hat{p}^6\Delta^{2+1/m})) \ge 7\log\Delta$$

(R11) 
$$2/(4\omega_3^2 C(m\Delta^{1+1/2m}\hat{p}^3 + \delta\Delta^{1/2+1/2m}\hat{p}^3)^2) \ge 7\log\Delta$$

(R12) 
$$2/(\omega_4^2 C(-\hat{p}\log\hat{p})^2) > 6\log\Delta$$

(R13) 
$$1/\omega_4 \le \epsilon (1 - \theta/4)^T$$

(R14) 
$$2\omega_5^2/(C(m\Delta^{1+1/2m}\hat{p} + \Delta^{1/2+1/2m}\hat{p}\delta)^2) \ge 7\log\Delta$$

(R15) 
$$\omega_5 < (\theta/6)(1 - \theta/3)^T \Delta$$

(R16) 
$$\omega_6 \Delta \theta \hat{p}/(5\delta) \ge 6 \log \Delta$$

(R17) 
$$\theta\omega(1-\theta/4)^T \ge \theta T/\omega_2 + 1/\omega_3$$

(R18) 
$$1 - 10\epsilon \ge 3/4$$

(R19) 
$$\Delta_2 \leq \omega_6 \theta \Delta \hat{p}$$

(R20) 
$$\Delta_2 \leq \sqrt{\Delta}\sqrt{\omega}$$

(R21) 
$$\hat{p} \ge \Delta^{-1/2}$$
.

The analysis in Section 4 only requires that (3.6), (3.7), (3.8), and (R1)-(R21) hold; the parameters  $\omega$ ,  $\epsilon$ ,  $\hat{p}$ , and  $\omega_0$  depend on the structure of the hypergraph. For instance, we will use the following bounds when applying the analysis to triangle-free hypergraphs.

Claim 7. The following inequalities are consistent, and if they hold, then (R1)-(R21) also hold:

$$\epsilon \leq 1/40 \qquad \Delta_2 \leq \sqrt{\Delta}\sqrt{\omega}$$

$$\omega < (1/26)(\epsilon/86)\log \Delta \qquad \delta \leq \Delta^{6/10}$$

$$\omega_0 > 10\omega^3 \log \omega \qquad \hat{p} > e^{86\omega/\epsilon}\sqrt{\omega}/\sqrt{\Delta}$$

$$\hat{p} \leq \Delta^{-11/24}.$$

*Proof.* The bounds on  $\omega$  and  $\epsilon$  imply

$$\frac{e^{86\omega/\epsilon}\sqrt{\omega}}{\sqrt{\Delta}} < \Delta^{1/26-1/2}\sqrt{\omega} = \Delta^{-6/13}\sqrt{\omega} \le \Delta^{-11/24},$$

so the inequalities are consistent. Checking that they satisfy (R1)-(R21) (for  $\Delta$  sufficiently large) is straightforward.

## 4 Analysis of Algorithm

**Theorem 8.** If (3.6), (3.7), (3.8), and (R1)-(R21) hold and  $|C(u)| \leq C$  for all vertices u, then the algorithm produces a proper list coloring of  $H \cup G$ .

*Proof.* By Lemma 9, our algorithm proceeds for T iterations, coloring most of the vertices. Since Lemmas 9, 10 and 12 hold after iteration T, we may color the remaining vertices as described in Section 4.5.

**Lemma 9** (Main Lemma). If (3.6), (3.7), (3.8), and (R1)-(R21) hold, then for each i = 0, 1, ..., T, the following properties hold:

$$(P1) |1 - w(p_u^i)| \le i/\omega_1.$$

$$(P2) e_u^i \le (1 - \theta/3)^i \omega + i/\omega_2$$

(P3) 
$$f_u^i \le 8(1 - \theta/4)^i \omega$$

$$(P4) \ h_u^i \ge h_u^0 - 21\epsilon \sum_{j=0}^{i-1} (1 - \theta/4)^j$$

$$(P5) \ d_H^i(u) \le (1 - \theta/3)^i \Delta$$

$$(P6) \ d_{G_c}^i(u) \le 3\omega_6 i\theta \Delta \hat{p}.$$

The proof of the Main Lemma relies on the next three lemmas.

**Lemma 10.** For any i = 0, 1, ..., T - 1, if (3.6), (3.7), (3.8), and (R1)-(R21) hold and  $|B^i(u)| \le \epsilon/\hat{p}$  for all  $u \in U^i$ , then there is an assignment of colors to the vertices in  $U^i$  so that the following properties hold:

$$(Q1) |w(p_u^{i+1}) - w(p_u^i)| \le 1/\omega_1$$

$$(Q2) e_{uvw}^{i+1} \le e_{uvw}^{i} + 1/(\Delta\omega_2)$$

(Q3) 
$$f_u^{i+1} \le f_u^i (1 - \theta/2) + \theta e_u^i + 1/\omega_3$$

$$(Q4) h_u^i - h_u^{i+1} \le 2\theta (f_u^i + e_u^i) + 1/\omega_4$$

$$(Q5) \ d_H^{i+1}(u) \le (1 - \theta/2) d_H^i(u) + \omega_5$$

$$(Q6) \ d_{G_c}^{i+1}(u) \le d_{G_c}^{i}(u) + 2\omega_6 \theta \Delta \hat{p}.$$

**Lemma 11.** If (Q1)-(Q6) hold for i and (P1)-(P6) hold for i, then (P1)-(P6) hold for i+1.

**Lemma 12.** If (P1)-(P6) hold for i + 1 and (R1) and (R5) hold, then  $|B^{i+1}(u)| \le \epsilon/\hat{p}$ .

#### 4.1 Proof of Main Lemma

The proof relies on Lemmas 10, 11 and 12. Assuming these lemmas, we proceed inductively as follows: properties (P1)-(P6) hold for i = 0 ((P3) holds by (R20)). Assume (P1)-(P6) hold for i. By Lemma 12,  $|B^i(u)| \le \epsilon/\hat{p}$ , so by Lemma 10, (Q1)-(Q6) hold for i. Thus Lemma 11 implies (P1)-(P6) hold for i + 1.

#### 4.2 Proof of Lemma 11

**Proof of (P1)**. By (P1) (for i) and (Q1),

$$|1 - w(p_u^{i+1})| = |1 - w(p_u^i) + w(p_u^i) - w(p_u^{i+1})|$$

$$\leq |1 - w(p_u^i)| + |w(p_u^{i+1}) - w(p_u^i)|$$

$$\leq (i+1)/\omega_1.$$

**Proof of (P5)**. Using (P5) (for i),

$$\begin{split} d_H^{i+1}(u) &\overset{\text{\tiny (Q5)}}{\leq} (1 - \theta/2) d_H^i(u) + \omega_5 \overset{\text{\tiny (P5)}}{\leq} (1 - \theta/2) (1 - \theta/3)^i \Delta + \omega_5 \\ &= (1 - \theta/3)^{i+1} \Delta - \frac{\theta}{6} (1 - \theta/3)^i \Delta + \omega_5 \\ &\leq (1 - \theta/3)^{i+1} \Delta - \frac{\theta}{6} (1 - \theta/3)^T \Delta + \omega_5 \\ &\overset{\text{\tiny (R15)}}{\leq} (1 - \theta/3)^{i+1} \Delta. \end{split}$$

Proof of (P2). By (Q2),

$$e_{uvw}^{i+1} \le e_{uvw}^0 + (i+1)/\Delta\omega_2 \le C(1/C^3) + (i+1)/\Delta\omega_2 = \omega/\Delta + (i+1)/\Delta\omega_2.$$

So by (P5) (for i + 1),

$$e_u^{i+1} = \sum_{uvw} e_{uvw}^{i+1} \le (1 - \theta/3)^{i+1} \Delta(\omega/\Delta + (i+1)/\Delta\omega_2) \le (1 - \theta/3)^{i+1} \omega + (i+1)/\omega_2.$$

**Proof of (P3)**. By (P3) and (P2) (for i),

$$\begin{split} f_u^{i+1} &\overset{\text{(Q3)}}{\leq} f_u^i (1 - \theta/2) + \theta e_u^i + 1/\omega_3 \\ &\overset{\text{(P3)}}{\leq} 8(1 - \theta/4)^i \omega (1 - \theta/2) + \theta e_u^i + 1/\omega_3 \\ &\overset{\text{(P2)}}{\leq} 8(1 - \theta/4)^i \omega (1 - \theta/2) + \theta \omega (1 - \theta/3)^i + \theta T/\omega_2 + 1/\omega_3 \\ &= 8(1 - \theta/4)^i \omega (1 - \theta/4 - \theta/4) + \theta \omega (1 - \theta/3)^i + \theta T/\omega_2 + 1/\omega_3 \\ &= 8(1 - \theta/4)^{i+1} \omega - 2\theta \omega (1 - \theta/4)^i + \theta \omega (1 - \theta/3)^i + \theta T/\omega_2 + 1/\omega_3 \\ &< 8(1 - \theta/4)^{i+1} \omega - \theta \omega (1 - \theta/4)^i + \theta T/\omega_2 + 1/\omega_3 \\ &\overset{\text{(R17)}}{\leq} 8(1 - \theta/4)^{i+1} \omega. \end{split}$$

Proof of (P4). We have

$$T/\omega_2 \stackrel{\text{(R17)}}{<} \omega (1 - \theta/4)^T \le \omega (1 - \theta/4)^i.$$
 (4.1)

Therefore, using  $\epsilon = \omega \theta$  and (P4) (for i),

$$\begin{split} h_u^{i+1} &\overset{\text{(Q4)}}{\geq} h_u^i - 2\theta (f_u^i + e_u^i) - 1/\omega_4 \\ &\overset{\text{(P3)}}{\geq} h_u^i - 2\theta (8(1 - \theta/4)^i \omega + e_u^i) - 1/\omega_4 \\ &\overset{\text{(P2)}}{\geq} h_u^i - 2\theta (8(1 - \theta/4)^i \omega + (1 - \theta/3)^i \omega + T/\omega_2) - 1/\omega_4 \\ & \geq h_u^i - 2\theta (9(1 - \theta/4)^i \omega + T/\omega_2) - 1/\omega_4 \\ &\overset{\text{(4.1)}}{\geq} h_u^i - 2\theta (10(1 - \theta/4)^i \omega) - 1/\omega_4 \\ &= h_u^i - 20\epsilon (1 - \theta/4)^i - 1/\omega_4 \\ &\overset{\text{(R13)}}{\geq} h_u^i - 21\epsilon (1 - \theta/4)^i \\ &\overset{\text{(P4)}}{\geq} h_u^0 - 21\epsilon \sum_{j=0}^{i-1} (1 - \theta/4)^j - 21\epsilon (1 - \theta/4)^i \\ &= h_u^0 - 21\epsilon \sum_{j=0}^{i} (1 - \theta/4)^j. \end{split}$$

**Proof of (P6)**. By (Q6) and (R19),

$$d_{G_c}^{i+1}(u) \stackrel{\text{(Q6)}}{<} \Delta_2 + 2\omega_6(i+1)\theta\Delta\hat{p} \stackrel{\text{(R19)}}{<} 3\omega_6(i+1)\theta\Delta\hat{p}.$$

#### 4.3 Proof of Lemma 12

First,

$$|B^{i+1}(u)|\hat{p}\log(\hat{p}C) = \sum_{c \in B^{i+1}(u)} \hat{p}\log(\hat{p}C) = \sum_{c \in B^{i+1}(u)} p_u^{i+1}(c)\log(p_u^{i+1}(c)C)$$

$$\leq \sum_{c \in C(u)} p_u^{i+1}(c)\log(p_u^{i+1}(c)C)$$

$$= \sum_{c \in C(u)} p_u^{i+1}(c)\log p_u^{i+1}(c) + \sum_{c \in C(u)} p_u^{i+1}(c)\log C$$

$$= -h_u^{i+1} + \log C \sum_{c \in C(u)} p_u^{i+1}(c). \tag{4.2}$$

Using  $p_u^0(c) = 1/C$  for all  $c \in C(u)$ ,

$$\begin{split} h_u^0 &= -\sum_{c \in C(u)} p_u^0(c) \log p_u^0(c) \\ &= \log C \sum_{c \in C(u)} p_u^0(c) \\ &= \log C \sum_{c \in C(u)} (p_u^0(c) - p_u^{i+1}(c)) + \log C \sum_{c \in C(u)} p_u^{i+1}(c) \\ &= \log C (1 - w(p_u^{i+1})) + \log C \sum_{c \in C(u)} p_u^{i+1}(c) \\ &\stackrel{\text{(P1)}}{\geq} - (T \log C) / \omega_1 + \log C \sum_{c \in C(u)} p_u^{i+1}(c) \\ &\stackrel{\text{(R5)}}{>} - \epsilon / \theta + \log C \sum_{c \in C(u)} p_u^{i+1}(c). \end{split}$$

Using  $\sum_{j=0}^{i} (1 - \theta/4)^j \le 4/\theta$ , the above inequality, and inequality (4.2),

$$h_u^{i+1} \stackrel{\text{(P4)}}{\geq} h_u^0 - 21\epsilon \sum_{j=0}^i (1 - \theta/4)^j \ge h_u^0 - 84\epsilon/\theta \ge \log C \sum_{c \in C(u)} p_u^{i+1}(c) - 85\epsilon/\theta$$

$$\stackrel{\text{(4.2)}}{\geq} h_u^{i+1} + |B^{i+1}(u)| \hat{p} \log(\hat{p}C) - 85\epsilon/\theta.$$

So

$$|B^{i+1}(u)| \le \frac{85\epsilon}{\theta \hat{p} \log(\hat{p}C)} \stackrel{\text{\tiny (R1)}}{\le} \epsilon/\hat{p}.$$

#### 4.4 Proof of Lemma 10

We are going to apply the Local Lemma. Our probability space is determined by coin flips at each vertex which determine the random variables  $\gamma_u(c)$  and  $\eta_u(c)$ . The random

variable  $p_u(c)$  is determined by the coin flips in N(u). The events "(Q1) fails to hold for u" and "(Q4) fails to hold for u" are therefore determined by these coin flips. The events "(Q3) fails to hold for u" and "(Q5) fails to hold for u" are determined by the coin flips in N(N(u)). The event "(Q2) fails to hold for edge uvw" is determined by the coin flips in N(N(u)) + N(N(v)) + N(N(w)). The event "(Q6) fails to hold for u and c" is determined by the coin flips in N(N(u)). Each event is therefore mutually independent of at most  $5(\Delta + \Delta_2)^4$  (Q1), (Q3), (Q4), (Q5), or (Q6) events and at most  $\Delta(3\Delta + 3\Delta_2)^4$  (Q3) events. By (R20),  $\Delta_2 < \Delta$ , so each event is mutually independent of at most  $7^4\Delta^5$  other events.

It therefore suffices to show that the probability that (Qi) fails is less than  $4(7^4)\Delta^{-5}$ . We prove this for (Q1), (Q2), (Q4), and (Q6) first, and then move on to (Q3) and (Q5). Throughout the proof, we drop the notation i+1 and i, and use, for instance,  $p'_u(c)$  and  $p_u(c)$  to denote values in iterations i+1 and i, respectively.

**Proof of (Q1)**. By (3.5),  $\mathbf{E}[p'_u(c)] = p_u(c)$  for each color c. By linearity of expectation,

$$\mathbf{E}[w(p_u')] = w(p_u).$$

Since  $w(p'_u)$  is the sum of C independent non-negative random variables, each bounded by  $\hat{p}$ , Theorem 3 and (R3) imply

$$\Pr[|w(p_u') - w(p_u)| \ge 1/\omega_1] \le 2e^{-2/(C\hat{p}^2\omega_1^2)} < 2e^{-6\log\Delta}.$$

**Proof of (Q2)**. Suppose  $uvw \in H$ . We first prove

$$\mathbf{E}[p_u'(c)p_v'(c)p_w'(c)] \le p_u(c)p_v(c)p_w(c)(1+1/\omega_0). \tag{4.3}$$

Assume that  $p'_u(c)$ ,  $p'_v(c)$ , and  $p'_w(c)$  are determined by (3.3). If  $c \in L(u) \cup L(v) \cup L(w)$ , then  $p'_u(c)p'_v(c)p'_w(c) = 0$ , so by (3.6),

$$\mathbf{E}[p'_{u}(c)p'_{v}(c)p'_{w}(c)] \leq \frac{p_{u}(c)}{q_{u}(c)} \frac{p_{v}(c)}{q_{v}(c)} \frac{p_{w}(c)}{q_{w}(c)} \Pr[c \notin L(u) \cup L(v) \cup L(w)]$$
  
$$\leq p_{u}(c)p_{v}(c)p_{w}(c)(1 + 1/\omega_{0}).$$

Suppose  $p'_u(c)$  and  $p'_v(c)$  are determined by (3.3), and  $p'_w(c)$  is determined by (3.4). Then  $p'_w(c)$  is independent of  $p'_u(c)$  and  $p'_v(c)$ , so by (3.7),

$$\mathbf{E}[p'_u(c)p'_v(c)p'_w(c)] = \mathbf{E}[p'_u(c)p'_v(c)] \mathbf{E}[p'_w(c)]$$

$$\leq \frac{p_u(c)}{q_u(c)} \frac{p_v(c)}{q_v(c)} \Pr[c \notin L(u) \cup L(v)] p_w(c)$$

$$\leq p_u(c)p_v(c)p_w(c)(1 + 1/\omega_0).$$

If at least two of  $p'_u(c)$ ,  $p'_v(c)$ , and  $p'_w(c)$  are determined by (3.4), then all three are independent of each other, and

$$\mathbf{E}[p'_{u}(c)p'_{v}(c)p'_{w}(c)] = p_{u}(c)p_{v}(c)p_{w}(c),$$

finishing the proof of (4.3).

By definition,  $e_{uvw}^0 \leq C/C^3 = \omega/\Delta$ . So by (Q2) (for i) and (R8),

$$e_{uvw}/\omega_0 \stackrel{\text{\tiny (Q2)}}{\leq} (e^0_{uvw} + \frac{i}{\Delta\omega_2}) \frac{1}{\omega_0} \leq (\frac{\omega}{\Delta} + \frac{T}{\Delta\omega_2}) \frac{1}{\omega_0} = \frac{\omega\omega_2 + T}{\omega_0} \frac{1}{\Delta\omega_2} \stackrel{\text{\tiny (R8)}}{<} 1/(2\Delta\omega_2).$$

So by (4.3),

$$\mathbf{E}[e'_{uvw}] = \sum_{c} \mathbf{E}[p'_{u}(c)p'_{v}(c)p'_{w}(c)] \le \sum_{c} p_{u}(c)p_{v}(c)p_{w}(c)(1+1/\omega_{0})$$

$$= e_{uvw}(1+1/\omega_{0})$$

$$< e_{uvw} + 1/(2\Delta\omega_{2}).$$

Now  $e'_{uvw}$  is the sum of C independent random variables, each bounded by  $\hat{p}^3$ . Thus Theorem 3 and (R6) yield

$$\Pr[e'_{uvw} \ge e_{uww} + 1/(\Delta\omega_2)] \le \Pr[e'_{uvw} \ge e_{uvw} + 1/(2\Delta\omega_2) + 1/(2\Delta\omega_2)]$$

$$\le \Pr[e'_{uvw} \ge \mathbf{E}[e'_{uvw}] + 1/(2\Delta\omega_2)]$$

$$< e^{-2/(4\Delta^2\omega_2^2C\hat{p}^6)}$$

$$< e^{-6\log\Delta}.$$

**Proof of (Q4)**. By (3.3) and (3.4),  $p'_u(c) = p_u(c) \mathbf{I}[A] / \Pr[A]$  for some event A. Thus, using  $x \log x = 0$  for  $x \in \{0, 1\}$ ,

$$\begin{split} \mathbf{E}[p_u'(c)\log p_u'(c)] &= \mathbf{E}[p_u(c)\,\mathbf{I}[A]/\Pr[A]\log(p_u(c)\,\mathbf{I}[A]/\Pr[A])] \\ &= \mathbf{E}[p_u(c)\,\mathbf{I}[A]/\Pr[A]\log p_u(c) + p_u(c)\,\mathbf{I}[A]/\Pr[A]\log \left(\mathbf{I}[A]/\Pr[A]\right)] \\ &= \frac{p_u(c)\log p_u(c)}{\Pr[A]}\,\mathbf{E}[\mathbf{I}[A]] + \frac{p_u(c)}{\Pr[A]}\,\mathbf{E}[\mathbf{I}[A]\log \left(\mathbf{I}[A]/\Pr[A]\right)] \\ &= p_u(c)\log p_u(c) + \frac{p_u(c)}{\Pr[A]}\,\mathbf{E}[\mathbf{I}[A]\log \mathbf{I}[A]] - \frac{p_u(c)}{\Pr[A]}\,\mathbf{E}[\mathbf{I}[A]\log \Pr[A]] \\ &= p_u(c)\log p_u(c) + \frac{p_u(c)}{\Pr[A]}\,\mathbf{E}[0] - p_u(c)\log \Pr[A] \\ &= p_u(c)\log p_u(c) - p_u(c)\log \Pr[A]. \end{split}$$

Recall that

$$q_u(c) = \Pr\left[\bigcap_{uvw \in H} (\gamma_v(c) = 0 \cup \gamma_w(c) = 0) \bigcap_{uv \in G_c} \gamma_v(c) = 0\right].$$

Also,  $1 - rx \ge (1 - x)^r$  for  $r, x \in (0, 1)$ . Finally, the event  $\gamma_v(c) = 0$  is monotone decreasing, so by the FKG inequality,

$$q_{u}(c) \stackrel{\text{FKG}}{\geq} \prod_{uvw \in H} \Pr[\gamma_{v}(c) = 0 \cup \gamma_{w}(c) = 0] \prod_{uv \in G_{c}} \Pr[\gamma_{v}(c) = 0]$$

$$= \prod_{uvw \in H} (1 - \theta^{2} p_{v}(c) p_{w}(c)) \prod_{uv \in G_{c}} (1 - \theta p_{v}(c))$$

$$\geq \prod_{uvw \in H} (1 - \theta)^{\theta p_{v}(c) p_{w}(c)} \prod_{uv \in G_{c}} (1 - \theta)^{p_{v}(c)}.$$

By the algorithm,  $\Pr[A] \ge q_u(c)$ . Also,  $\log(1-x) \ge -x - x^2$  for  $x \in [0, 1/3]$ . Combining these inequalities with the previous inequality, we obtain

$$\log \Pr[A] \ge \log q_u(c) \ge \log \left( \prod_{uvw \in H} (1 - \theta)^{\theta p_v(c) p_w(c)} \prod_{uv \in G_c} (1 - \theta)^{p_v(c)} \right)$$

$$= \sum_{uvw \in H} \theta p_v(c) p_w(c) \log(1 - \theta) + \sum_{uv \in G_c} p_v(c) \log(1 - \theta)$$

$$\ge \sum_{uvw \in H} \theta p_v(c) p_w(c) (-\theta - \theta^2) + \sum_{uv \in G_c} p_v(c) (-\theta - \theta^2)$$

$$= (-\theta^2 - \theta^3) \sum_{uvw \in H} p_v(c) p_w(c) + (-\theta - \theta^2) \sum_{uv \in G_c} p_v(c)$$

$$= -(\theta^2 + \theta^3) e_u(c) - (\theta + \theta^2) f_u(c).$$

Therefore, using the definition of  $h_u$  and  $\theta < 1/2$ ,

$$\mathbf{E}[h_u - h'_u] = h_u + \sum_c \mathbf{E}[p'_u(c)\log p'_u(c)]$$

$$= h_u + \sum_c p_u(c)\log p_u(c) - \sum_c p_u(c)\log \Pr[A]$$

$$= -\sum_c p_u(c)\log \Pr[A]$$

$$\leq \sum_c p_u(c)((\theta + \theta^2)f_u(c) + (\theta^2 + \theta^3)e_u(c))$$

$$= (\theta + \theta^2)f_u + (\theta^2 + \theta^3)e_u$$

$$< 2\theta(f_u + e_u).$$

The terms in  $\sum_{c} -p'_{u}(c) \log p'_{u}(c)$  are independent and, since  $-x \log x$  is increasing for  $0 < x \le \hat{p}$ , bounded by  $-\hat{p} \log \hat{p}$ . Thus, by Theorem 3 and (R12),

$$\Pr[h_u - h'_u \ge 2\theta(f_u + e_u) + 1/\omega_4] < e^{-2/(\omega_4^2 C(-\hat{p}\log\hat{p})^2)} < e^{-6\log\Delta}.$$

**Proof of (Q6)**. Fix  $c \in C(u)$ . For each  $v \in N_H(u)$ , set

$$X_v = d_H(u, v)\gamma_v(c),$$

and set

$$X = \sum_{v \in N_H(u)} X_v.$$

Then

$$\mathbf{E}[X] = \sum_{v \in N_H(u)} d_H(u, v) p_v(c) \theta \le \hat{p}\theta \sum_{v \in N_H(u)} d_H(u, v) \le 2\Delta \hat{p}\theta.$$

Since the  $X_v$  are independent from each other (because the  $\gamma_v(c)$  are independent), and x(1-x) is increasing for x < 1/2,

$$\begin{aligned} \mathbf{Var}[X] &= \sum_{v \in N_H(u)} \mathbf{Var}[X_v] = \sum_{v \in N_H(u)} (\mathbf{E}[X_v^2] - \mathbf{E}[X_v]^2) \\ &= \sum_{v \in N_H(u)} (d_H(u, v)^2 p_v(c) \theta - d_H(u, v)^2 p_v(c)^2 \theta^2) \\ &\leq \sum_{v \in N_H(u)} d_H(u, v)^2 \hat{p} \theta (1 - \hat{p} \theta) \\ &= \hat{p} \theta (1 - \hat{p} \theta) \sum_{v \in N_H(u)} d_H(u, v)^2 \\ &\leq \hat{p} \theta (1 - \hat{p} \theta) \delta \sum_{v \in N_H(u)} d_H(u, v) \\ &= \hat{p} \theta (1 - \hat{p} \theta) 2\Delta \delta \\ &< \hat{p} \theta 2\Delta \delta. \end{aligned}$$

If  $uv \notin G_c$  and  $uv \in G'_c$ , then there exists an edge  $uvw \in H$  such that  $\gamma_w(c) = 1$ . Hence

$$d'_{G_c}(u) - d_{G_c}(u) \le \sum_{uvw \in H} (\gamma_v(c) + \gamma_w(c)) = \sum_{v \in N_H(u)} d_H(u, v) \gamma_v(c) = X.$$

Applying Theorem 4 (with  $b = \delta$ ) and (R16),

$$\Pr[d'_{G_c}(u) - d_{G_c}(u) \ge 2\omega_6 \Delta \hat{p}\theta] \le \Pr[X \ge \omega_6 \Delta \hat{p}\theta + \omega_6 \Delta \hat{p}\theta]$$

$$\le \Pr[X \ge \mathbf{E}[X] + \omega_6 \Delta \hat{p}\theta]$$

$$\le e^{-\omega_6^2 \Delta^2 \hat{p}^2 \theta^2 / (4\hat{p}\theta \Delta \delta + \delta \omega_6 \Delta \hat{p}\theta)}$$

$$\le e^{-\omega_6^2 \Delta^2 \hat{p}^2 \theta^2 / 5\delta \omega_6 \Delta \hat{p}\theta}$$

$$< e^{-\omega_6 \Delta \hat{p}\theta / 5\delta}$$

$$\stackrel{\text{(R16)}}{\le} e^{-6 \log \Delta}.$$

We now prove (Q3) and (Q5). The following two claims will be used in both proofs.

Claim 13. For any  $v \in U$  and  $c \in C(v)$ ,

$$\Pr[v \notin U' | c \notin L(v)] \ge \Pr[v \notin U'] \ge 3\theta/4,$$

and if  $uv \in G_c$ , then

$$\Pr[v \notin U' | c \notin L(u)] \ge \Pr[v \notin U'] - \theta \hat{p} \ge 5\theta/8,$$

$$\Pr[v \notin U' | c \notin L(u) \cup L(v)] \ge \Pr[v \notin U'] - \theta \hat{p} \ge 5\theta/8.$$

Proof of claim. The vertex v is colored (i.e.,  $v \notin U'$ ) if and only if for some color  $d \notin B(v)$ ,  $\gamma_v(d) = 1$  and  $d \notin L(v)$ . Let  $R_d$  denote the event that  $\gamma_v(d) = 1$  and  $d \notin L(v)$ . If  $c \in B(v)$ , then v cannot be colored c, so the event  $v \notin U'$  is independent of the events  $c \notin L(v)$  and  $c \notin L(u)$ ; hence

 $\Pr[v \notin U'] = \Pr[v \notin U' | c \notin L(v)] = \Pr[v \notin U' | c \notin L(u)] = \Pr[v \notin U' | c \notin L(u) \cup L(v)].$  Otherwise,

$$\Pr[v \notin U' | c \notin L(v)] = \frac{\Pr[v \notin U', c \notin L(v)]}{\Pr[c \notin L(v)]}$$

$$= \frac{\Pr[\bigcup_{d \notin B(u)} R_d, c \notin L(v)]}{\Pr[c \notin L(v)]}$$

$$= \frac{\Pr[(\bigcup_{d \notin B(u) + c} R_d \cup R_c), c \notin L(v)]}{\Pr[c \notin L(v)]}$$

$$= \frac{\Pr[(\bigcup_{d \notin B(u) + c} R_d \cup \gamma_v(c) = 1), c \notin L(v)]}{\Pr[c \notin L(v)]}$$

$$= \frac{\Pr[(\bigcup_{d \notin B(u) + c} R_d \cup \gamma_v(c) = 1)] \Pr[c \notin L(v)]}{\Pr[c \notin L(v)]}$$

$$= \Pr[(\bigcup_{d \notin B(v) + c} R_d) \cup (\gamma_v(c) = 1)]$$

$$\geq \Pr[(\bigcup_{d \notin B(v) + c} R_d) \cup R_c]$$

$$= \Pr[v \notin U'].$$

Suppose  $uv \in G_c$ . If  $c \notin L(u)$ , then  $\gamma_w(c) = 0$  for all  $w \in N_{G_c}(u)$ , so in particular,  $\gamma_v(c) = 0$ . Consequently,

$$\Pr[R_c|c \notin L(u) \cup L(v)] = \Pr[\gamma_v(c) = 1 \cap c \notin L(v)|c \notin L(u) \cup L(v)] = 0.$$

So by the independence of colors and the inequality

$$\Pr[\bigcup_{d \in C(v) - B(v)} R_d] \le \Pr[\bigcup_{d \in C(v) - B(v) - c} R_d] + \Pr[R_c],$$

we obtain

$$\Pr[v \notin U' | c \notin L(u) \cup L(v)] = \Pr[\bigcup_{d \notin B(v)} R_d | c \notin L(u) \cup L(v)]$$

$$= \Pr[\bigcup_{d \notin B(v) + c} R_d]$$

$$= \Pr[\bigcup_{d \in C(v) - B(v) - c} R_d]$$

$$\geq \Pr[\bigcup_{d \in C(v) - B(v)} R_d] - \Pr[R_c]$$

$$\geq \Pr[v \notin U'] - \theta \hat{p}.$$

Since we only used the condition  $c \notin L(u)$ , this also implies

$$\Pr[v \notin U' | c \notin L(u)] \ge \Pr[v \notin U'] - \theta \hat{p}.$$

To finish the proof of the claim, we now show  $\Pr[v \notin U'] \geq 3\theta/4$ . First,

$$\Pr[v \notin U'] = \Pr[\bigcup_{d \notin B(v)} R_d] 
\geq \sum_{d \notin B(v)} \Pr[R_d] - \sum_{d,d' \notin B(v)} \Pr[R_d] \Pr[R_{d'}] 
= \sum_{d \notin B(v)} \theta p_v(d) q_v(d) - \sum_{d,d' \notin B(v)} \theta^2 p_v(d) p_v(d') q_v(d) q_v(d') 
\geq \theta \sum_{d \in C(v)} p_v(d) q_v(d) - \theta \sum_{d \in B(v)} p_v(d) q_v(d) - \theta^2 \sum_{d,d' \notin B(v)} p_v(d) p_v(d') 
\geq \theta \sum_{d \in C(v)} p_v(d) q_v(d) - \theta |B(v)| \hat{p} - \theta^2 \sum_{d,d' \notin B(v)} p_v(d) p_v(d').$$

By (3.2),

$$q_{v}(d) \ge 1 - \sum_{uvw \in H} \theta^{2} p_{u}(d) p_{w}(d) - \sum_{uv \in G_{d}} \theta p_{u}(d)$$

$$= 1 - \theta^{2} \sum_{uvw \in H} p_{u}(d) p_{w}(d) - \theta \sum_{uv \in G_{d}} p_{u}(d)$$

$$= 1 - \theta^{2} e_{v}(d) - \theta f_{v}(d).$$

Since  $\sum_{d \in C(v)} p_v(c) \le \sqrt{2}$  (by (P1) and (R4)),

$$\theta^2 \sum_{d,d' \notin B(v)} p_v(d) p_v(d') \le \frac{1}{2} \theta^2 \sum_{d \in C(v)} \sum_{d' \in C(v) - d} p_v(d) p_v(d') \le \frac{1}{2} \theta^2 (\sum_{d \in C} p_v(d))^2 \le \theta^2.$$

By our lemma's assumption,  $|B(v)| \leq \epsilon/\hat{p}$ . By (P3),  $f_v < 8\omega$ , so  $\theta f_v < 8\epsilon$ . By (P2),  $e_v \leq \omega + T/\omega_2$ , so (R7) implies  $\theta^2 e_v < \epsilon/3$ . Using these three inequalities,  $\sum_{d \in C(v)} p_v(c) \geq (1 - \epsilon/3)$ , and (R18), we finally obtain

$$\Pr[v \notin U'] \ge \theta \sum_{d \in C(v)} p_v(d)(1 - \theta^2 e_v(d) - \theta f_v(d)) - \theta |B(v)| \hat{p} - \theta^2$$

$$= \theta \sum_{d \in C(v)} p_v(d) - \theta^3 \sum_{d \in C(v)} p_v(d) e_v(d) - \theta^2 \sum_{d \in C(v)} p_v(d) f_v(d) - \theta |B(v)| \hat{p} - \theta^2$$

$$\ge \theta \sum_{d \in C(v)} p_v(d) - \theta^3 \sum_{d \in C(v)} p_v(d) e_v(d) - \theta^2 \sum_{d \in C(v)} p_v(d) f_v(d) - \theta \epsilon - \theta^2$$

$$= \theta \sum_{d \in C(v)} p_v(d) - \theta^3 e_v - \theta^2 f_v - \theta \epsilon - \theta^2$$

$$\ge \theta (1 - \epsilon/3) - \theta \epsilon/3 - 8\theta \epsilon - \theta \epsilon - \theta \epsilon/3$$

$$= \theta (1 - 10\epsilon)$$

$$> 3\theta/4.$$

Recall that m is a fixed constant.

**Claim 14.** For each l = 0, ..., m - 2, let

$$N^0(u,l) = \{ v \in N_H^0(u) - N_G^0(u) : \Delta^{l/2m} < d_H^0(u,v) \le \Delta^{(l+1)/2m} \},$$

and for l = m - 1, let

$$N^{0}(u, l) = \{ v \in N_{H}^{0}(u) : d_{H}^{0}(u, v) > \Delta^{l/2m} \} \cup N_{G}^{0}(u).$$

For each l and color c, let  $\mathcal{A}_{c,l}$  be the event that  $\gamma_v(c) = 1$  for at most  $\Delta^{1-l/2m}\hat{p}$  vertices  $v \in N^0(u, l)$ . Let  $\mathcal{A}$  denote the event that  $\mathcal{A}_{c,l}$  holds for all l and c. Then

$$\Pr[\bar{\mathcal{A}}] \le e^{-10\log \Delta}.$$

Proof of claim. Suppose l < m-1. Since each  $v \in N^0(u, l)$  contributes at least  $\Delta^{l/2m}$  edges to  $d_H^0(u)$ , and each edge is counted at most twice,

$$|N^{0}(u,l)| \le 2\Delta/\Delta^{l/2m} = 2\Delta^{1-l/2m}.$$

If l = m - 1,

$$|N^{0}(u,l)| \le 2\Delta/\Delta^{l/2m} + \Delta_{2} = 2\Delta^{1-l/2m} + \Delta_{2} \stackrel{\text{(R20)}}{<} 3\Delta^{1-l/2m}$$

Thus  $|N^0(u,l)| < 3\Delta^{1-l/2m}$  for each l.

Since  $\Pr[\gamma_v(c) = 1] \le \hat{p}\theta$  and  $3e\theta < 1/e$ ,

$$\Pr[\bar{\mathcal{A}}_{c,l}] \leq \binom{|N^{0}(u,l)|}{\Delta^{1-l/2m}\hat{p}} (\hat{p}\theta)^{\Delta^{1-l/2m}\hat{p}} \leq \binom{3\Delta^{1-l/2m}}{\Delta^{1-l/2m}\hat{p}} (\hat{p}\theta)^{\Delta^{1-l/2m}\hat{p}} \\ \leq (\frac{3e}{\hat{p}})^{\Delta^{1-l/2m}\hat{p}} (\hat{p}\theta)^{\Delta^{1-l/2m}\hat{p}} \\ = (3e\theta)^{\Delta^{1-l/2m}\hat{p}} \\ < e^{-\Delta^{1-l/2m}\hat{p}} \\ < e^{-\Delta^{1-l/2m}\hat{p}} \\ = e^{-\Delta^{1/2m}}.$$

So by the union bound,

$$\Pr[\bar{\mathcal{A}}] \le Cme^{-\Delta^{1/2m}} \le e^{-10\log \Delta}.$$

**Proof of (Q3)**. Observe that

$$\begin{split} f'_u &= \sum_{c} \sum_{uv \in G'_c} p'_u(c) p'_v(c) \\ &= \sum_{c} \sum_{uv \in G_c} p'_u(c) p'_v(c) \, \mathbf{I}[uv \in G'_c] + \sum_{c} \sum_{\substack{uv \notin G_c \\ uv \in G'_c}} p'_u(c) p'_v(c) \\ &\leq \sum_{c} \sum_{uv \in G_c} p'_u(c) p'_v(c) \, \mathbf{I}[v \in U'] \\ &+ \sum_{c} \sum_{uvw \in H} (p'_u(c) p'_v(c) \, \mathbf{I}[\gamma_w(c) = 1] + p'_u(c) p'_w(c) \, \mathbf{I}[\gamma_v(c) = 1]) \\ &= D_1 + D_2, \end{split}$$

where

$$D_1 = \sum_{c} \sum_{uv \in G_c} p'_u(c) p'_v(c) \mathbf{I}[v \in U'],$$

and

$$D_2 = \sum_{c} \sum_{uvw \in H} (p'_u(c)p'_v(c) \mathbf{I}[\gamma_w(c) = 1] + p'_u(c)p'_w(c) \mathbf{I}[\gamma_v(c) = 1]).$$

To bound  $D_1$ , we first prove that for  $uv \in G_c$ ,

$$\mathbf{E}[p_u'(c)p_v'(c)\,\mathbf{I}[v\in U']] \le p_u(c)p_v(c)(1-9\theta/16). \tag{4.4}$$

First assume that  $p'_u(c)$  and  $p'_v(c)$  are determined by (3.3). If  $c \in L(u) \cup L(v)$ , then  $p'_u(c)p'_v(c) = 0$ , so using (3.8), Claim 13, and then (R2),

$$\begin{split} \mathbf{E}[p'_{u}(c)p'_{v}(c)\,\mathbf{I}[v\in U']] &= \mathbf{E}[p'_{u}(c)p'_{v}(c)|v\in U']\,\mathrm{Pr}[v\in U'] \\ &\leq \frac{p_{u}(c)}{q_{u}(c)}\frac{p_{v}(c)}{q_{v}(c)}\,\mathrm{Pr}[c\notin L(u)\cup L(v)|v\in U']\,\mathrm{Pr}[v\in U'] \\ &= \frac{p_{u}(c)}{q_{u}(c)}\frac{p_{v}(c)}{q_{v}(c)}\,\mathrm{Pr}[v\in U'|c\notin L(u)\cup L(v)]\,\mathrm{Pr}[c\notin L(u)\cup L(v)] \\ &\stackrel{(3.8)}{\leq} p_{u}(c)p_{v}(c)(1+1/\omega_{0})\,\mathrm{Pr}[v\in U'|c\notin L(u)\cup L(v)] \\ &\stackrel{(3.8)}{\leq} p_{u}(c)p_{v}(c)(1+1/\omega_{0})(1-5\theta/8) \\ &\stackrel{(\mathrm{R2})}{\leq} p_{u}(c)p_{v}(c)p_{v}(c)(1-9\theta/16). \end{split}$$

Suppose  $p'_u(c)$  is determined by (3.3) and  $p'_v(c)$  is determined by (3.4). Then  $p'_u(c)$  and

 $p'_v(c)$  are independent of each other, and  $p'_v(c)$  is independent of the event  $v \in U'$ , so

$$\begin{split} \mathbf{E}[p'_{u}(c)p'_{v}(c)\,\mathbf{I}[v\in U']] &= \mathbf{E}[p'_{u}(c)p'_{v}(c)|v\in U']\,\mathrm{Pr}[v\in U'] \\ &= \mathbf{E}[p'_{u}(c)|v\in U']\,\mathbf{E}[p'_{v}(c)]\,\mathrm{Pr}[v\in U'] \\ &\stackrel{(3.5)}{\leq} \,\mathbf{E}[p'_{u}(c)|v\in U']p_{v}(c)\,\mathrm{Pr}[v\in U'] \\ &\leq \frac{p_{u}(c)}{q_{u}(c)}\,\mathrm{Pr}[c\notin L(u)|v\in U']\,\mathrm{Pr}[v\in U']p_{v}(c) \\ &= \frac{p_{u}(c)}{q_{u}(c)}\,\mathrm{Pr}[v\in U'|c\notin L(u)]\,\mathrm{Pr}[c\notin L(u)]p_{v}(c) \\ &= p_{u}(c)p_{v}(c)\,\mathrm{Pr}[v\in U'|c\notin L(u)] \\ &\stackrel{(3.5)}{\leq} p_{u}(c)p_{v}(c)(1+1/\omega_{0})(1-5\theta/8) \\ &\stackrel{(3.5)}{\leq} p_{u}(c)p_{v}(c)(1-9\theta/16). \end{split}$$

Similarly, if  $p'_u(c)$  is determined by (3.4) and  $p'_v(c)$  is determined by (3.3),

$$\mathbf{E}[p'_{u}(c)p'_{v}(c)\,\mathbf{I}[v\in U']] \leq p_{u}(c)p_{v}(c)\,\Pr[v\in U'|c\notin L(v)]$$

$$\stackrel{\text{C.13}}{\leq} p_{u}(c)p_{v}(c)(1+1/\omega_{0})(1-5\theta/8)$$

$$\stackrel{\text{(R2)}}{\leq} p_{u}(c)p_{v}(c)(1-9\theta/16).$$

If  $p'_u(c)$  and  $p'_v(c)$  are both determined by (3.4),

$$\mathbf{E}[p'_u(c)p'_v(c)\,\mathbf{I}[v\in U']] = \mathbf{E}[p'_u(c)p'_v(c)]\,\mathrm{Pr}[v\in U']$$

$$= \mathbf{E}[p'_u(c)]\,\mathbf{E}[p'_v(c)]\,\mathrm{Pr}[v\in U']$$

$$\stackrel{\text{(C.13)}}{\leq} p_u(c)p_v(c)(1-3\theta/4)$$

$$< p_u(c)p_v(c)(1-9\theta/16),$$

concluding the proof of (4.4).

By (4.4),

$$\mathbf{E}[D_1] = \sum_{c} \sum_{uv \in G_c} \mathbf{E}[p'_u(c)p'_v(c) \mathbf{I}[v \in U']]$$

$$\leq \sum_{c} \sum_{uv \in G_c} p_u(c)p_v(c)(1 - 9\theta/16)$$

$$= f_u(1 - 9\theta/16).$$

For  $c \in C(u)$ , let

$$T_c = \{ \gamma_v(c) : v \in N(N(u)) \} \cup \{ \eta_v(c) : v \in N(N(u)) \}.$$

Then each  $T_c$  is a (vector valued) random variable, and the set of random variables  $\{T_c : c \in C(u)\}$  are mutually independent and determine the variable  $D_1$ . We will now apply Corollary 6 with parameters:

- Independent random variables  $T_c: \{c\} \to \{0,1\}^{2|N(N(u))|}$ , for each  $c \in C(u)$
- Events  $\mathcal{A}_c = \bigcap_{l=1}^m \mathcal{A}_{c,l}$ , for each  $c \in C(u)$  (where  $\mathcal{A}_{c,l}$  is from Claim 14)
- $\mathcal{A} = \prod_{c \in C(u)} \mathcal{A}_c$ , for each  $c \in C(u)$  (this is the same  $\mathcal{A}$  as in Claim 14)
- $D_1$  (which is non-negative) in the role of Y
- $d_{G_c(u)}\hat{p}^2 + m\hat{p}^3\Delta^{1+1/2m}$  in the role of  $d_c$ .

Our goal is thus to bound the effect of  $T_c$  on  $D_1$  given that  $\mathcal{A}$  holds. Note first that

$$D_1 = \sum_{uv \in G_c} p'_u(c) p'_v(c) \mathbf{I}[v \in U'] + \sum_{l=0}^{m-1} \sum_{v \in N^0(u,l)} \mathbf{I}[v \in U'] \sum_{\substack{d \neq c: \\ uv \in G_d}} p'_u(d) p'_v(d).$$

The total effect of  $T_c$  on the left hand sum is at most  $d_{G_c}(u)\hat{p}^2$ , so consider the right hand sum. The  $p'_u(d)p'_v(d)$  terms are always independent of  $T_c$ . Observe that if  $\gamma_v(c) = 0$ , then  $\mathbf{I}[v \in U']$  is also independent of  $T_c$ ; this is because if  $\gamma_v(c) = 0$ , then v can not be colored c in the current round, so  $T_c$  has no impact on whether or not  $v \in U'$ . Thus  $T_c$  only affects the term

$$\mathbf{I}[v \in U'] \sum_{\substack{d \neq c \\ uv \in G_d}} p'_u(d)p'_v(d)$$

if  $\gamma_v(c) = 1$ . So given the event  $\mathcal{A}_{c,l}$  from Claim 14,  $T_c$  affects at most  $\Delta^{1-l/2m}\hat{p}$  such terms for each l. If  $v \in N^0(u,l)$ , where  $l \leq m-2$ , the effect is at most  $d_H^0(u,v)\hat{p}^2 \leq \Delta^{(l+1)/2m}\hat{p}^2$ . If l=m-1, the effect is at most  $C\hat{p}^2 < \Delta^{1/2}\hat{p}^2$ . Therefore, given  $\mathcal{A}$ , the effect of  $T_c$  on the right hand sum is at most

$$\sum_{l=0}^{m-2} (\Delta^{1-l/2m} \hat{p}) \Delta^{(l+1)/2m} \hat{p}^2 + (\Delta^{1-(m-1)/2m} \hat{p}) \Delta^{1/2} \hat{p}^2 = m \hat{p}^3 \Delta^{1+1/2m}.$$

Given  $\mathcal{A}$ ,  $T_c$  thus affects  $D_1$  by at most

$$d_{G_c}(u)\hat{p}^2 + m\hat{p}^3\Delta^{1+1/2m}$$
.

Since  $\sum_c d_{G_c}(u) \le \Delta + \Delta_2 < 2\Delta$  and, by (P6),  $d_{G_c}(u) \le 3\omega_6 T \theta \Delta \hat{p}$ ,

$$\begin{split} & \sum_{c} (d_{G_c}(u)\hat{p}^2 + m\hat{p}^3\Delta^{1+1/2m})^2 \\ & \leq \hat{p}^4 \sum_{c} d_{G_c}(u)^2 + 4m\hat{p}^5\Delta^{1+1/2m}\Delta + Cm^2\hat{p}^6\Delta^{2+1/m} \\ & \leq 3\hat{p}^5\omega_6 T\theta\Delta \sum_{c} d_{G_c}(u) + 4m\hat{p}^5\Delta^{1+1/2m}\Delta + Cm^2\hat{p}^6\Delta^{2+1/m} \\ & \leq 6\omega_6 T\theta\hat{p}^5\Delta^2 + 4m\hat{p}^5\Delta^{2+1/2m} + Cm^2\hat{p}^6\Delta^{2+1/m}. \end{split}$$

Together with Claim 14 and (R10), Corollary 6 now implies

$$\Pr[D_{1} > f_{u}(1 - \theta/2) + 1/2\omega_{3}] \leq \Pr[D_{1} > f_{u}(1 - 9\theta/16) / \Pr[\mathcal{A}] + 1/2\omega_{3}]$$

$$\leq \Pr[D_{1} > \mathbf{E}[D_{1}] / \Pr[\mathcal{A}] + 1/2\omega_{3}]$$

$$\stackrel{C.6}{\leq} e^{-1/4\omega_{3}^{2}(6\omega_{6}T\theta\hat{p}^{5}\Delta^{2} + 4m\hat{p}^{5}\Delta^{2+1/2m} + Cm^{2}\hat{p}^{6}\Delta^{2+1/m})} + \Pr[\bar{\mathcal{A}}]$$

$$\stackrel{(R10)}{\leq} e^{-7\log\Delta} + \Pr[\bar{\mathcal{A}}]$$

$$\stackrel{C.14}{\leq} e^{-7\log\Delta} + e^{-10\log\Delta}$$

$$\leq e^{-6\log\Delta}.$$

We now bound  $D_2$ . We first prove that for any edge uvw,

$$\mathbf{E}[p_u'(c)p_v'(c)|\gamma_w(c) = 1] \le p_u(c)p_v(c)(1 + 1/\omega_0). \tag{4.5}$$

Assume that both  $p'_u(c)$  and  $p'_v(c)$  are determined by (3.3). If  $c \in L(u)$  or  $c \in L(v)$ , then  $p'_u(c)p'_v(c) = 0$ , so by (3.8),

$$\mathbf{E}[p'_{u}(c)p'_{v}(c)|\gamma_{w}(c) = 1] \leq \frac{p_{u}(c)}{q_{u}(c)} \frac{p_{v}(c)}{q_{v}(c)} \Pr[c \notin L(u) \cup L(v)|\gamma_{w}(c) = 1]$$

$$\leq \frac{p_{u}(c)}{q_{u}(c)} \frac{p_{v}(c)}{q_{v}(c)} \Pr[c \notin L(u) \cup L(v)]$$

$$\stackrel{(3.8)}{\leq} p_{u}(c) p_{v}(c) (1 + 1/\omega_{0}).$$

Suppose  $p'_u(c)$  is determined by (3.3) and  $p'_v(c)$  is determined by (3.4). Then  $p'_u(c)$  and  $p'_v(c)$  are independent of each other, and  $p'_v(c)$  is independent of the event  $\gamma_w(c) = 1$ , so

$$\mathbf{E}[p'_{u}(c)p'_{v}(c)|\gamma_{w}(c) = 1] = \mathbf{E}[p'_{u}(c)|\gamma_{w}(c) = 1] \mathbf{E}[p'_{v}(c)]$$

$$\stackrel{(3.5)}{=} \mathbf{E}[p'_{u}(c)|\gamma_{w}(c) = 1]p_{v}(c)$$

$$\leq \frac{p_{u}(c)}{q_{u}(c)} \Pr[c \notin L(u)|\gamma_{w}(c) = 1]p_{v}(c)$$

$$\leq \frac{p_{u}(c)}{q_{u}(c)} \Pr[c \notin L(u)]p_{v}(c)$$

$$= p_{u}(c)p_{v}(c)$$

$$< p_{u}(c)p_{v}(c)(1 + 1/\omega_{0}).$$

If  $p'_u(c)$  and  $p'_v(c)$  are both determined by (3.4), then

$$\mathbf{E}[p_u'(c)p_v'(c)|\gamma_w(c) = 1] = \mathbf{E}[p_u'(c)p_v'(c)] = \mathbf{E}[p_u'(c)] \mathbf{E}[p_v'(c)] \stackrel{\text{(3.5)}}{=} p_u(c)p_v(c),$$

which establishes (4.5).

Now, by (4.5),

$$\mathbf{E}[D_{2}] = \sum_{c} \sum_{uvw} (\mathbf{E}[p'_{u}(c)p'_{v}(c) \mathbf{I}[\gamma_{w}(c) = 1]] + \mathbf{E}[p'_{u}(c)p'_{w}(c) \mathbf{I}[\gamma_{v}(c) = 1]])$$

$$= \sum_{c} \sum_{uvw} \mathbf{E}[p'_{u}(c)p'_{v}(c)|\gamma_{w}(c) = 1] \Pr[\gamma_{w}(c) = 1]$$

$$+ \sum_{c} \sum_{uvw} \mathbf{E}[p'_{u}(c)p'_{w}(c)|\gamma_{v}(c) = 1] \Pr[\gamma_{v}(c) = 1]$$

$$\leq (1 + 1/\omega_{0}) \sum_{c} \sum_{uvw} (p_{u}(c)p_{v}(c) \Pr[\gamma_{w}(c) = 1] + p_{u}(c)p_{w}(c) \Pr[\gamma_{v}(c) = 1])$$

$$= (1 + 1/\omega_{0}) \sum_{c} \sum_{uvw} (p_{u}(c)p_{v}(c)\theta p_{w}(c) + p_{u}(c)p_{w}(c)\theta p_{v}(c))$$

$$= (1 + 1/\omega_{0})2\theta e_{u}.$$

Again, let

$$T_c = \{ \gamma_v(c) : v \in N(N(u)) \} \cup \{ \eta_v(c) : v \in N(N(u)) \}.$$

Then  $D_2$  is determined by the set of random variables  $\{T_c:c\in C(u)\}$ . Observe that

$$D_2 = \sum_{c} \sum_{l=0}^{m-1} \sum_{v \in N_H(u) \cap N^0(u,l)} \mathbf{I}[\gamma_v(c) = 1] \sum_{w \in N_H(u,v)} p'_u(c) p'_w(c).$$

The random variable  $T_c$  does not affect terms of the form  $\mathbf{I}[\gamma_v(d) = 1] \sum_{w \in N(u,v)} p'_u(d) p'_w(d)$ , where  $d \neq c$ .  $T_c$  affects the term  $\mathbf{I}[\gamma_v(c) = 1] \sum_{w \in N(u,v)} p'_u(c) p'_w(c)$  only if  $\gamma_v(c) = 1$ ; in this case, the effect is at most  $d_H(u,v)\hat{p}^2$ . Thus, given the event  $\mathcal{A}$  from Claim 14, the total effect of  $T_c$  on  $D_2$  is bounded by

$$\sum_{l=0}^{m-2} \Delta^{1-l/2m} \hat{p} \Delta^{(l+1)/2m} \hat{p}^2 + \Delta^{1-(m-1)/2m} \hat{p} \delta \hat{p}^2 < m \Delta^{1+1/2m} \hat{p}^3 + \delta \Delta^{1/2+1/2m} \hat{p}^3.$$

By Corollary 6, (R11), and Claim 14,

$$\Pr[D_{2} > 3\theta e_{u} + 1/2\omega_{3}] \leq \Pr[D_{2} > (1 + 1/\omega_{0})2\theta e_{u}/\Pr[\mathcal{A}] + 1/2\omega_{3}]$$

$$\stackrel{\text{C.6}}{\leq} e^{-2/(4\omega_{3}^{2}C(m\Delta^{1+1/2m}\hat{p}^{3} + \delta\Delta^{1/2+1/2m}\hat{p}^{3})^{2})} + \Pr[\bar{\mathcal{A}}]$$

$$\stackrel{\text{(R11)}}{\leq} e^{-7\log\Delta} + \Pr[\bar{\mathcal{A}}]$$

$$\stackrel{\text{C.14}}{\leq} e^{-7\log\Delta} + e^{-10\log\Delta}$$

$$\leq e^{-6\log\Delta}.$$

Therefore, with probability at least  $1 - 2\Delta^{-5}$ ,

$$f'_u \le f_u(1 - \theta/2) + 1/2\omega_3 + 3\theta e_u + 1/2\omega_3$$
  
$$\le f_u(1 - \theta/2) + 3\theta e_u + 1/\omega_3.$$

Proof of (Q5). Since

$$d'_{H}(u) = \frac{1}{2} \sum_{v \in N_{H}(u)} \sum_{w \in N_{H}(u,v)} \mathbf{I}[v, w \in U'] \le \frac{1}{2} \sum_{v \in N_{H}(u)} d_{H}(u,v) \mathbf{I}[v \in U'],$$

Claim 13 implies

$$\mathbf{E}[d'_H(u)] \le \frac{1 - 3\theta/4}{2} \sum_{v \in N_H(u)} d_H(u, v) = (1 - 3\theta/4) d_H(u).$$

We prove concentration in the same way as in the proof of (Q3). Let

$$T_c = \{ \gamma_v(c) : v \in N(N(u)) \} \cup \{ \eta_v(c) : v \in N(N(u)) \}.$$

The random variable  $d'_H(u)$  is determined by the set of random variables  $\{T_c : c \in C(u)\}$ . For  $v \in N(u)$ ,  $T_c$  affects the term  $d_H(u,v) \mathbf{I}[v \in V']$  only if  $\gamma_v(c) = 1$ , and in this case, the effect is at most  $d_H(u,v)$ . Thus, given the event  $\mathcal{A}$  from Claim 14,  $T_c$  affects  $d'_H(u)$  by at most

$$\sum_{l=0}^{m-2} \Delta^{1-l/2m} \hat{p} \Delta^{(l+1)/2m} + \Delta^{1-(m-1)/2m} \hat{p} \delta < m \Delta^{1+1/2m} \hat{p} + \Delta^{1/2+1/2m} \hat{p} \delta.$$

By Corollary 6, (R14), and Claim 14,

$$\Pr[d'_{H}(u) > (1 - \theta/2)d_{H}(u) + \omega_{5}] \leq \Pr[d'_{H}(u) > (1 - 3\theta/4)d_{H}(u) / \Pr[\mathcal{A}] + \omega_{5}]$$

$$\stackrel{\text{C.6}}{\leq} e^{-2\omega_{5}^{2}/C(m\Delta^{1+1/2m}\hat{p}+\Delta^{1/2+1/2m}\hat{p}\delta)^{2}} + \Pr[\bar{\mathcal{A}}]$$

$$\stackrel{\text{(R14)}}{\leq} e^{-7\log\Delta} + \Pr[\bar{\mathcal{A}}]$$

$$\stackrel{\text{C.14}}{\leq} e^{-7\log\Delta} + e^{-10\log\Delta}$$

$$\leq e^{-6\log\Delta}.$$

## 4.5 Final Step

After the iterative portion of the algorithm, some vertices will still be uncolored. Assuming (R1)-(R21) and Lemmas 9, 10, and 12 hold, we color them using the Asymmetric Local Lemma as follows. Suppose u has not been colored. By (P1), (R4), Lemma 12, and (R18),

$$\sum_{c \in C(u) - B^{T}(u)} p_{u}^{T}(c) = \sum_{c \in C(u)} p_{u}^{T}(c) - \sum_{c \in B^{T}(u)} p_{u}^{T}(c) \stackrel{\text{(P1)}}{\geq} 1 - T/\omega_{1} - |B^{T}(u)| \hat{p}$$

$$\stackrel{\text{(R4)}}{\geq} 1 - o(1) - |B^{T}(u)| \hat{p}$$

$$\stackrel{\text{L.12}}{\geq} 1 - o(1) - \epsilon$$

$$\stackrel{\text{(R18)}}{\geq} 1/2.$$

For each  $c \notin B^T(u)$ , define

$$p_u^*(c) := \frac{p_u^T(c)}{\sum_{c \in C(u) - B^T(u)} p_u^T(c)} \le 2p_u^T(c).$$

For each uncolored vertex u, randomly assign u one color from the distribution given by  $p_u^*$ . For an edge  $e = uvw \in H^T$ , let  $A_{uvw}$  denote the event that u, v, and w receive the same color. By (R7) and definition of  $\theta$ ,  $T/\omega_2 = o(\omega/\epsilon)$ ; in particular,  $T/\omega_2 = o(\omega)$ . So by (Q2),

$$e_{unv}^T \le e_{unv}^0 + T/\Delta\omega_2 = 1/C^2 + o(\omega/\Delta) = \omega/\Delta + o(\omega/\Delta).$$

Therefore

$$\Pr[A_{uvw}] = \sum_{c} p_u^*(c) p_v^*(c) p_w^*(c) \le 8 \sum_{c} p_u^T(c) p_v^T(c) p_w^T(c) = 8e_{uvw}^T \le 9\omega/\Delta.$$

For each c and each pair  $uv \in G_c^T$ , let  $B_{uv,c}$  denote the event that u and v both receive color c. By (P3), for each u,

$$\sum_{c \in C(u)} \sum_{ux \in G_c^T} \Pr[B_{ux,c}] \le 4 \sum_{c \in C(u)} \sum_{ux \in G_c^T} p_u^T(c) p_x^T(c) = 4 f_u^T \le 32 (1 - \theta/4)^T \omega.$$

The event  $A_{uvw}$  depends on any event  $A_e$  or  $B_{f,d}$ , where u, v, or w is in the edge e or the edge f. Using (P5),

$$\sum_{e \in H^T: u \in e} \Pr[A_e] + \sum_{e \in H^T: v \in e} \Pr[A_e] + \sum_{e \in H^T: w \in e} \Pr[A_e]$$

$$+ \sum_{c \in C(u)} \sum_{ux \in G_c^T} \Pr[B_{ux,c}] + \sum_{c \in C(v)} \sum_{vx \in G_c^T} \Pr[B_{vx,c}] \sum_{c \in C(w)} \sum_{wx \in G_c^T} \Pr[B_{wx,c}]$$

$$\leq 3(9\omega/\Delta)(1 - \theta/3)^T \Delta + 3(32)(1 - \theta/4)^T \omega$$

$$\leq 123(1 - \theta/4)^T \omega$$

$$\leq 123e^{-\theta T/4} \omega$$

$$= 123e^{-5\log \omega/4} \omega$$

$$= 123(\frac{1}{\omega})^{5/4} \omega$$

$$< 1/4.$$

The event  $B_{uv,c}$  depends on any event  $A_e$  or  $B_{f,d}$ , where u or v is in e or f. Since

$$\sum_{e \in H^T: u \in e} \Pr[A_e] + \sum_{e \in H^T: v \in e} \Pr[A_e] + \sum_{c \in C(u)} \sum_{ux \in G_c^T} \Pr[B_{ux,c}] + \sum_{c \in C(v)} \sum_{vx \in G_c^T} \Pr[B_{vx,c}] \\
\leq 18(1 - \theta/3)^T \omega + 64(1 - \theta/4)^T \omega \\
\leq 1/4,$$

the Asymmetric Local Lemma implies that there exists a coloring where none of the events  $A_{uvw}$  or  $B_{uv,c}$  occur. Since no color in  $B^T(u)$  and no color with  $p_c^T(u) = 0$  was assigned to u, this coloring, combined with the partial coloring from the algorithm, is a proper list coloring of  $H \cup G$ .

## 5 Triangle-free hypergraphs

We will derive Theorem 2 as a corollary of the following theorem:

**Theorem 15.** Set  $c_0 = 1/86,000$ . Suppose H is a rank 3, triangle-free hypergraph with maximum 3-degree at most  $\Delta$ , maximum 2-degree at most  $(c_0 \Delta \log \Delta)^{1/2}$ , and maximum codegree at most  $\Delta^{6/10}$ . Then

$$\chi_l(H) \le \left(\frac{\Delta}{c_0 \log \Delta}\right)^{1/2}.$$

To prove this using Theorem 8, we need to find values for the parameters  $\omega$ ,  $\epsilon$ ,  $\omega_0$ , and  $\hat{p}$  which satisfy (R1)-(R21), (3.6), (3.7), and (3.8), and  $\omega = c_0 \log \Delta$ . We will show that the following values satisfy these criteria:

$$\epsilon = 1/40$$
  $\omega = (1/25)(\epsilon/86) \log \Delta$   $\hat{p} = \Delta^{-11/24}$   $\omega_0 = 1/19\theta \hat{p}$ .

By Claim 7, these parameters satisfy (R1)-(R21), so all that remains is to show that inequalities (3.6), (3.7), and (3.8) hold. Fix a color c. In Claim 16, we first show that that hypergraph  $H \cup G_c$  remains triangle-free throughout the algorithm. The next three claims then show that if the hypergraph remains triangle-free, we will have enough independence to derive (3.6), (3.7), and (3.8). Throughout the rest of this section, we will be taking intersections and unions over edges; when we do this, we use the notation e in place of  $e \in E(H) \cup E(G_c)$ .

Claim 16. For iteration i, if  $H^i \cup G^i_c$  is triangle-free, then  $H^{i+1} \cup G^{i+1}_c$  is triangle-free.

Proof. It suffices to show that when the algorithm creates  $G_c^{i+1}$  from  $G_c^i$  by adding an edge uv to  $G_c^i$ , no triangle is created. Toward a contradiction, suppose that a triangle is created with distinct edges  $uv, e, f \in H^{i+1} \cup G_c^{i+1}$  and distinct vertices u, v, w such that  $u \in e, v \in f, w \in e \cap f$ , and  $u \notin f, v \notin e$ . Note that  $u, v, w \in V(H^i \cup G_c^i)$  and  $e, f \in H^i \cup G_c^i$ . Since  $w \in V(H^i \cup G_c^i)$ , w has not been colored. Thus there exists a vertex  $x \in V(H^i) - w$  and an edge  $uvx \in H^i$  which gave rise to the edge uv. The edges uvx, e, and f form a triangle with vertices u, v, and w in  $H^i + G_c^i$ , a contradiction.  $\square$ 

In the rest of this section, we define

$$d(u, v) = |\{e \in H \cup G_c : u, v \in e\}|.$$

In addition, we drop the superscript from  $H^i$  and  $G_c^i$ .

Claim 17. Suppose  $uvw \in H$ ,  $d(u,v) \ge 2$ , and  $d(w,v) \ge 2$ . Then d(u,w) = 1.

Proof. Since  $d(u, v) \geq 2$  and  $d(w, v) \geq 2$ , there exist distinct edges  $e, f \neq uvw$  such that  $u, v \in e$  and  $w, v \in f$ . If there exists  $x \neq v$  such that  $uwx \in H$ , then e, f, and uxw form a triangle with corresponding vertices u, v, and w. If  $uw \in G_c$ , then e, f, and uw form a triangle with vertices u, v, and w.

Claim 18. If uvw is an edge and d(u, w) = 1, then

$$\left(\bigcup_{e:u\in e;v\notin e}e-u\right)\cap\left(\bigcup_{e:w\in e;v\notin e}e-w\right)=\emptyset,\tag{5.1}$$

$$\left(\bigcup_{e:u\in e;v\notin e}e-u\right)\cap\left(\bigcup_{e:v\in e;u\notin e}e-v\right)=\emptyset,\tag{5.2}$$

and

$$\left(\bigcup_{e:w\in e:v\notin e}e-u\right)\cap\left(\bigcup_{e:v\in e:w\notin e}e-v\right)=\emptyset. \tag{5.3}$$

*Proof.* Let  $x \in U$ , and let e be an edge such that  $u \in e$ ,  $v \notin e$ , and  $x \in e - u$ . Then  $e \neq uvw$ , and since d(u, w) = 1,  $x \notin \{u, v, w\}$ .

Suppose f is an edge such that  $w \in f$ ,  $v \notin f$ , and  $x \in f - w$ . Then, since  $x \in f$ ,  $f \neq uvw$ . Using d(u, w) = 1,  $u \in e$ ,  $w \in f$  and e,  $f \neq uvw$ , we get  $e \neq f$ ,  $u \notin f$ , and  $w \notin e$ . Since  $x \notin uvw$ , we obtain a triangle with edges e, f, and uvw and vertices u, w, and x.

Now suppose that  $v, x \in f$  and  $u \notin f$ . Again,  $f \neq uvw$ . Because  $u \in e$  and  $u \notin f$ ,  $e \neq f$ . Since  $u \notin f$ ,  $v \notin e$ , and  $v \notin f$ . By symmetry, this also gives (5.3).

Claim 19. If  $uv \in G_c$ , then

$$\left(\bigcup_{e:u\in e;v\notin e}e-u\right)\cap\left(\bigcup_{e:v\in e;u\notin e}e-v\right)=\emptyset.$$
(5.4)

Proof. If there exist edges e and f and a vertex x such that  $u \in e$ ,  $v \notin e$ ,  $v \in f$ ,  $u \notin f$ , and  $x \in e - u \cap f - v$ , then e, f, and uv form a triangle with vertices u, v, and x in  $H \cup G_c$ .

For a set of vertices S, let  $\gamma_S(c) = 1$  denote the event that  $\gamma_v(c) = 1$  for all  $v \in S$ , and let  $\gamma_S(c) \neq 1$  denote the event that  $\gamma_v(c) = 0$  for some  $v \in S$ .

Claim 20. For any three vertices x, y, and z,

$$\Pr[\bigcap_{e:x\in e;y\notin e}\gamma_{e-x}(c)\neq 1]\leq \Pr[\bigcap_{e:x\in e;y,z\notin e}\gamma_{e-x}(c)\neq 1]\leq q_x(c)(1+3\theta\hat{p}).$$

*Proof.* Note first that

$$\Pr\left[\bigcap_{e:x\in e;y\in e}\gamma_{e-x}(c)\neq 1\right] \geq \Pr\left[\gamma_y(c)=0\right] \geq 1-\theta\hat{p}.$$

Similarly,

$$\Pr[\bigcap_{e:x\in e:z\in e}\gamma_{e-x}(c)\geq 1-\theta\hat{p}.$$

Since the events  $\bigcap_{x \in e; y \notin e} \gamma_{e-x}(c) \neq 1$  and  $\bigcap_{x \in e; y \in e} \gamma_{e-x}(c) \neq 1$  are monotone decreasing, the FKG inequality and then the previous two inequalities yield

$$q_{x}(c) = \Pr\left[\bigcap_{e:x \in e; y, z \notin e} \gamma_{e-x}(c) \neq 1 \bigcap_{e:x,y \in e} \gamma_{e-x}(c) \neq 1 \bigcap_{e:x,z \in e} \gamma_{e-x}(c) \neq 1\right]$$

$$\geq \Pr\left[\bigcap_{e:x \in e; y, z \notin e} \gamma_{e-x}(c) \neq 1\right] \Pr\left[\bigcap_{e:x \in e, y \in e} \gamma_{e-x}(c) \neq 1\right] \Pr\left[\bigcap_{e:x \in e, z \in e} \gamma_{e-x}(c) \neq 1\right]$$

$$\geq \Pr\left[\bigcap_{e:x \in e; y,z \notin e} \gamma_{e-x}(c) \neq 1\right] (1 - \theta \hat{p})^{2}$$

$$\geq \Pr\left[\bigcap_{e:x \in e; y,z \notin e} \gamma_{e-x}(c) \neq 1\right] (1 - 2\theta \hat{p}).$$

Thus

$$\Pr\left[\bigcap_{e: x \in e; y, z \notin e} \gamma_{e-x}(c) \neq 1\right] \le q_x(c)/(1 - 2\theta\hat{p}) \le q_x(c)(1 + 3\theta\hat{p}).$$

We can now prove (3.6), (3.7), and (3.8). Suppose uvw is an edge. By Claim 17, we may assume d(u, w) = 1. The events  $\bigcap_{u \in e; v \notin e} \gamma_{e-u}(c) \neq 1$ ,  $\bigcap_{w \in e; v \notin e} \gamma_{e-w}(c) \neq 1$ , and  $\bigcap_{v \in e; u, w \notin e} \gamma_{e-v}(c) \neq 1$  depend only on the sets of random variables

$$\{\gamma_x(c): x \in \bigcup_{e: u \in e; v \notin e} e - u\},$$

$$\{\gamma_x(c): x \in \bigcup_{e:w \in e; v \notin e} e - w\},$$

and

$$\{\gamma_x(c): x \in \bigcup_{e:v \in e; u, w \notin e} e - v\},$$

respectively. By (5.1), (5.2), and (5.3), these sets are pairwise disjoint, so the three events are independent of each other. Therefore, applying Claim 20,

$$\begin{aligned} &\Pr[c \notin L(u) \cup L(v) \cup L(w)] \\ &= \Pr[\bigcap_{e:u \in e} \gamma_{e-u}(c) \neq 1 \bigcap_{e:v \in e} \gamma_{e-v}(c) \neq 1 \bigcap_{e:w \in e} \gamma_{e-w}(c) \neq 1] \\ &\leq \Pr[\bigcap_{e:u \in e;v \notin e} \gamma_{e-u}(c) \neq 1 \bigcap_{e:v \in e;u,w \notin e} \gamma_{e-v}(c) \neq 1 \bigcap_{e:w \in e;v \notin e} \gamma_{e-w}(c) \neq 1] \\ &= \Pr[\bigcap_{e:u \in e;v \notin e} \gamma_{e-u}(c) \neq 1] \Pr[\bigcap_{e:v \in e;u,w \notin e} \gamma_{e-v}(c) \neq 1] \Pr[\bigcap_{e:w \in e;v \notin e} \gamma_{e-w}(c) \neq 1] \\ &\leq q_u(c)q_v(c)q_w(c)(1 + 3\theta\hat{p})^3 \\ &< q_u(c)q_v(c)q_w(c)(1 + 19\theta\hat{p}) \\ &= q_u(c)q_v(c)q_w(c)(1 + 1/\omega_0). \end{aligned}$$

This proves (3.6). The proof of (3.7) is the same, except we start with any two vertices in uvw instead of all three.

Suppose now that  $uv \in G_c$  for some color c. By (5.4) and Claim 20,

$$\Pr[c \notin L(u) \cup L(v)] = \Pr\left[\bigcap_{e:u \in e} \gamma_{e-u}(c) \neq 1 \bigcap_{e:v \in e} \gamma_{e-v}(c) \neq 1\right]$$

$$\leq \Pr\left[\bigcap_{e:u \in e; v \notin e} \gamma_{e-u}(c) \neq 1 \bigcap_{e:v \in e; u \notin e} \gamma_{e-v}(c) \neq 1\right]$$

$$\stackrel{(5.4)}{=} \Pr\left[\bigcap_{e:u \in e; v \notin e} \gamma_{e-u}(c) \neq 1\right] \Pr\left[\bigcap_{e:v \in e; u \notin e} \gamma_{e-v}(c) \neq 1\right]$$

$$\stackrel{(5.20)}{\leq} q_u(c)q_v(c)(1 + 3\theta\hat{p})^2$$

$$< q_u(c)q_v(c)(1 + 7\theta\hat{p})$$

$$< q_u(c)q_v(c)(1 + 1/\omega_0),$$

completing the proof of (3.8) and Theorem 15.

**Proof of Theorem 2**: Recall that  $c_0 = 1/86,000$ . Let H be a rank 3, triangle-free hypergraph with maximum 3-degree  $\Delta$  and maximum 2-degree  $\Delta_2$ . The original hypergraph H may have some pairs of vertices with codegree too large to apply Theorem 15, so we will work on a modified hypergraph instead. Let

$$K(u) = \{ v \in N(u) : d(u, v) \ge \Delta^{6/10} \}.$$

Define a new hypergraph H' with V(H') = V(H) and

$$E(H') = E(H) - (\bigcup_{u \in V(H)} \bigcup_{v \in K(u)} \{e : u, v \in e\}) + (\bigcup_{u \in V(H)} \bigcup_{v \in K(u)} \{u, v\})$$

Let  $\Delta'$ ,  $\Delta'_2$ , and  $\delta'$  denote the maximum 3-degree, maximum 2-degree, and maximum codegree of H', respectively. Note that H' is still triangle-free,  $\chi_l(H) \leq \chi_l(H')$ ,  $\delta' \leq \Delta^{6/10}$ , and  $\Delta' \leq \Delta$ .

Suppose  $\Delta_2' \leq \sqrt{\Delta} \sqrt{c_0 \log \Delta}$ . Since  $\Delta' \leq \Delta$  and  $\delta' \leq \Delta^{6/10}$ , Theorem 15 implies

$$\chi_l(H) \le \chi_l(H') \le \left(\frac{\Delta}{c_0 \log \Delta}\right)^{1/2}.$$

On the other hand, suppose  $\Delta_2' > \sqrt{\Delta} \sqrt{c_0 \log \Delta}$ . Then, since

$$\Delta \ge d_H(u) \ge \frac{1}{2} \sum_{v \in N_H(u)} d_H(u, v) \ge \frac{1}{2} \sum_{\substack{v \in N_H(u) \\ d_H(u, v) > \Delta^{6/10}}} d_H(u, v) \ge |K(u)| \Delta^{6/10}/2,$$

we have

$$\Delta_2' \le \Delta_2 + 2\Delta^{4/10} < \Delta_2 + \Delta_2'/2.$$

Choose  $\Delta''$  so that  $\Delta'_2 = \sqrt{\Delta''}\sqrt{c_0\log\Delta''}$ . Since  $\Delta'_2 > \sqrt{\Delta}\sqrt{c_0\log\Delta}$ ,  $\Delta'' > \Delta$ . Then the maximum 3-degree of H' is at most  $\Delta < \Delta''$ , the maximum 2-degree of H' is at most  $\Delta'_2 \leq \sqrt{\Delta''}\sqrt{c_0\log\Delta''}$ , and the maximum codegree of H' is at most  $\Delta^{6/10} < \Delta''^{6/10}$ , so Theorem 15 implies

$$\chi_l(H) \le \chi_l(H') \le \left(\frac{\Delta''}{c_0 \log \Delta''}\right)^{1/2} = \frac{\Delta'_2}{c_0 \log \Delta''} < \frac{\Delta'_2}{c_0 \log \Delta'_2} < \frac{2\Delta_2}{c_0 \log 2\Delta_2}.$$

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