

The Minimum Expectation Selection Problem

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Abstract

We define the *min-min expectation selection problem* (resp. *max-min expectation selection problem*) to be that of selecting k out of n given discrete probability distributions, to minimize (resp. maximize) the expectation of the minimum value resulting when independent random variables are drawn from the selected distributions. We assume each distribution has finitely many atoms. Let d be the number of distinct values in the support of the distributions. We show that if d is a constant greater than 2, the min-min expectation problem is NP-complete but admits a fully polynomial time approximation scheme. For d an arbitrary integer, it is NP-hard to approximate the min-min expectation problem with any constant approximation factor. The max-min expectation problem is polynomially solvable for constant d ; we leave open its complexity for variable d . We also show similar results for *binary selection problems* in which we must choose one distribution from each of n pairs of distributions.

1 Introduction

Suppose we are given an integer n and the distributions of n independent random variables Y_i , for $1 \leq i \leq n$. We will let $l_0 < l_1 < \dots < l_{d-1}$ denote the possible values assumed by each of the random variables, and assume these are specified in the input. We assume that each Y_i is then specified in the input by giving, for each $j \in 1, 2, \dots, d-1$, the value of $\Pr\{Y_i \geq l_j\}$. (Note that $\Pr\{Y_i \geq l_0\}$ need not be specified since it will always be 1.) For

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complexity purposes, we assume that all integer values in the input are given in binary, and all other values are specified as ratios of integers. Let N be the total number of bits required to specify the input.

Suppose we wish to choose k of the random variables so as to minimize the expected value of the minimum of the selected variables. More formally, if we wish to choose a subset $S \subseteq \{1, 2, \dots, n\}$ with $|S| = k$ so as to minimize

$$\mathbb{E}\left[\min_{i \in S} Y_i\right], \quad (1)$$

we call this the *min-min expectation subset selection problem*. We call the variation in which we wish to maximize (1) the *max-min expectation subset selection problem*.

When we consider approximation results, we will assume

$$l_0 \geq 0. \quad (2)$$

(Note that without this assumption we could use any approximation ratio bound to solve the decision problem, by translating all of the l_i appropriately.)

We also consider the problem of maximizing (resp. minimizing) the expected value of the maximum of the selected variables, which we call the *max-max* and *min-max* problems. By a simple change of sign of all of the l_i , we see that the exact optimization forms of the max-max and min-min problems, and of the min-max and max-min problems, are equivalent in difficulty. (Note that this equivalence does not carry over to the approximation forms of the problems, since the sign change would cause a violation of (2).)

To motivate the min-min problem, suppose you are a user of a peer-to-peer file sharing service such as Gnutella. After performing a search, you have located several servers hosting copies of a file that you urgently need to download. If bandwidth at your end of the network is not a limiting factor, you may be able to speed your download by requesting downloads from more than one server, and stopping when the first of these downloads reaches completion. Suppose you have the capacity for k simultaneous requests, and you have information from each server such as its connection type and echo time from which you can estimate a distribution on its download times. Which k servers do you choose in order to minimize the expected time until you have downloaded a complete copy of your file?

To motivate the min-max problem, consider an editor who wishes to select k referees out of n qualified candidates so as to process an article as quickly as possible. Assume that the editor can estimate the time used by each referee as a random variable, and assume that the times used by each referee are independent. Then, assuming the editor will wait until all referee reports are received, s/he wishes to choose a set of referees that minimizes the maximum of their times.

Call the variations in which we are given $2n$ independent random variables $Y_{i,s}$, for $1 \leq i \leq n$ and $0 \leq s \leq 1$, and asked to choose a function $\chi : \{1, 2, \dots, n\} \rightarrow \{0, 1\}$ so as

to optimize

$$\mathbf{E}\left[\min_{i=1}^n Y_{i,\mathcal{X}(i)}\right], \quad (3)$$

the min-min expectation *binary* selection problem or the max-min expectation *binary* selection problem.

We consider the complexity and approximability of these problems both for the case in which d is fixed and the case in which d is a parameter given in the input. Our results hold for either the binary selection or subset selection form of the problem. In Section 2 we show that the variation of the min-min problem in which d is fixed is NP-complete (assuming $d \geq 3$), but admits a fully polynomial time approximation scheme. (See [4] for definitions.) Curiously, the max-min problem with fixed d can be solved in polynomial time. In Section 3 we show that when d is variable, the min-min problem cannot be approximated to within any fixed ratio in polynomial time unless $P = NP$; we leave open the complexity of the max-min problem with variable d .

The following formula, which follows from summation by parts, will be useful. Suppose that a random variable X assumes only the values l_0, l_1, \dots, l_k . Then

$$\mathbf{E}[X] = l_0 + \sum_{j=1}^{d-1} (l_j - l_{j-1}) \Pr\{X \geq l_j\}. \quad (4)$$

2 Fixed d

In this section we consider the case in which d is fixed. It will be convenient to consider scaled negative logarithms of the probabilities: we describe a given random variable X by a vector $\vec{L} = (L_1, L_2, \dots, L_{d-1})$ where $L_j = -\gamma^{-1} \ln \Pr\{X \geq l_j\}$, so that

$$\Pr\{X \geq l_j\} = e^{-\gamma L_j}. \quad (5)$$

(Since $\Pr\{X \geq l_0\}$ will always be one, we need not specify the value of L_0 .) Here γ is a positive value to be chosen later. Note that the expectation of such a random variable is given by the function

$$f(\vec{L}) = l_0 + \sum_{j=1}^{d-1} (l_j - l_{j-1}) e^{-\gamma L_j}, \quad (6)$$

and that this function f is convex, i.e., for $\alpha \in [0, 1]$ and arbitrary \vec{L} and \vec{L}' , we have

$$f(\alpha \vec{L} + (1 - \alpha) \vec{L}') \leq \alpha f(\vec{L}) + (1 - \alpha) f(\vec{L}').$$

Note also that if \vec{L} and \vec{L}' specify the distributions of two random variables, then the distribution of their minimum is specified by $\vec{L} + \vec{L}'$.

Assume we are dealing with the binary selection problem and let $P_{i,s}$ be the vector of scaled negative logarithms corresponding to the random variable $Y_{i,s}$, i.e, the j th component L_j of $P_{i,s}$ satisfies

$$\Pr \{Y_{i,s} \geq l_j\} = e^{-\gamma L_j}.$$

Then given a selection $\chi : \{0, 1, \dots, n-1\} \rightarrow \{0, 1\}$, we have

$$\mathbb{E} \left[\min_{i=1}^n Y_{i,\chi(i)} \right] = f \left(\sum_{i=1}^n P_{i,\chi(i)} \right). \quad (7)$$

2.1 NP-completeness of the min-min problem

The min-min problem is hard even when the all random variables must assume values chosen from a set of size three.

Theorem 1 *The min-min expectation binary selection problem is NP-complete for any fixed $d \geq 3$.*

Proof. Membership in NP is apparent. To prove completeness we perform a polynomial transformation from the subset-sum problem. Suppose we are given a set $\{z_1, \dots, z_n\}$ of nonnegative integers and asked whether the sum of some subset is equal to a given integer T . Let M be the maximum of the z_i ; we will assume that $T \leq nM$ since otherwise the problem is trivial. We show how to transform this problem into the min-min expectation binary selection problem with $d = 3$. (The result for larger d follows trivially by a padding argument.)

We choose $l_0 = 0$, $l_1 = 1$, and

$$l_2 - l_1 = e^{2\gamma(nM-T)}, \quad (8)$$

where γ is a positive number to be specified later. For $1 \leq i \leq n$ set

$$P_{i,0} = (0, 2M)$$

and

$$P_{i,1} = (z_i, 2M - z_i).$$

Note that each of these gives a valid set of probabilities; in particular, since $0 \leq z_i \leq M$ we have

$$1 \geq e^{-\gamma z_i} \geq e^{-\gamma(2M-z_i)} \geq 0.$$

We ask whether we can choose χ so that the expected minimum of the selected variables is at most $2e^{-\gamma T}$.

Now suppose that a selection function χ has been specified and let $S = \{i \mid \chi(i) = 1\}$. Then using (6) and (7), and letting $\sigma = \sum_{i \in S} z_i$, we have

$$\begin{aligned}
\mathbb{E}\left[\min_{i=1}^n Y_{i,\chi(i)}\right] &= f\left(\sum_{i=1}^n P_{i,\chi(i)}\right) \\
&= l_0 + (l_1 - l_0) \exp\left(-\gamma \sum_{i \in S} z_i\right) + (l_2 - l_1) \exp\left(-\gamma\left(2nM - \sum_{i \in S} z_i\right)\right) \\
&= \exp\left(-\gamma \sum_{i \in S} z_i\right) + e^{2\gamma(nM-T)} \exp\left(-\gamma\left(2nM - \sum_{i \in S} z_i\right)\right) \\
&\hspace{15em} \text{(where we have used (8))} \\
&= e^{-\gamma\sigma} + e^{2\gamma(nM-T)} e^{-\gamma(2nM-\sigma)} \\
&= e^{-\gamma\sigma} + e^{\gamma(\sigma-2T)} \\
&= e^{-\gamma T} \left(e^{\gamma(T-\sigma)} + e^{\gamma(\sigma-T)}\right).
\end{aligned}$$

Since $e^x + e^{-x}$ is minimized at $x = 0$, it is clear that we can achieve an expected minimum of $2e^{-\gamma T}$ if and only if we can choose S to make $\sigma = T$, i.e., if and only if the answer to the subset sum problem is yes.

Two technical points need to be addressed. First, since the magnitudes of z_i , T , and M can be exponentially large in the length of the input, one might fear that this transformation would produce an image of exponential length. Second, of course, we cannot output arbitrary reals in the constructed problem so we must use finite precision and consider rounding problems. To resolve these problems, set $\gamma = 1/(2nM)$; then it is easy to verify that no number output is larger than $e + 1$. Also note that to resolve the constructed decision problem it is sufficient to be able to distinguish $e^{-\gamma T}(e^\gamma + e^{-\gamma})$ from $2e^{-\gamma T}$. It is easy to verify that we need only give a number of bits that is polynomial in the input size to achieve this. \blacksquare

This same construction shows that the min-min expectation subset selection problem is also NP-complete: Since all n of the variables $Y_{i,0}$ constructed have the same distribution, picking any n out of the $2n$ variables is equivalent to picking one from each pair.

Although the problem is NP-complete, the optimum solution can be easily approximated for fixed d , assuming that the l_i are nonnegative.

Theorem 2 *With nonnegative l_i and a fixed value of d , the min-min expectation binary selection problem admits a fully polynomial time approximation scheme.*

Proof. We use a standard rounding and dynamic programming approach. Assume we are given some positive

$$\epsilon \leq 1. \tag{9}$$

Let the scale factor in (5) be set at

$$\gamma = \frac{\epsilon}{6n}. \quad (10)$$

Note that the components of each $P_{i,s}$ are positive real numbers (or $+\infty$ when the corresponding probability is 0). Let $\hat{P}_{i,s}$ be a vector in which each real component x of $P_{i,s}$ is rounded down to any integer in the range $[x - 2, x]$. (The entries that are ∞ will not be rounded. We allow some flexibility in the rounding, rather than rounding to $\lfloor x \rfloor$, to avoid having to do extremely precise calculations when x is very close to an integer.) We will build a table A indexed by d -tuples of integers, specifying which vectors are achievable as sums $\sum_{i=1}^n P_{i,\chi(i)}$ for some selection function χ .

A minor technical problem is that some components of $\sum_{i=1}^n P_{i,\chi(i)}$ may be infinite. To deal with this we note that the maximum value of any finite component of $\sum_{i=1}^n \hat{P}_{i,\chi(i)}$ is bounded by $\gamma^{-1} \ln 1/p^*$, where p^* is the product of all of the nonzero probabilities given in the input. Let T be an integer between $\gamma^{-1} \ln 1/p^*$ and $\gamma^{-1} \ln 1/p^* + 2$. Then we need not consider any finite indices of elements of A which exceed T . Recalling that N is the number of bits required to specify the input, we see that $\ln 1/p^*$ is $O(N)$, so

$$T = O(\gamma^{-1}N). \quad (11)$$

Formally, we now define the value of $A(s, L_1, L_2, \dots, L_{d-1})$, for $0 \leq s \leq n$ and $L_i \in \{0, 1, \dots, T, T+1\}$, to be a boolean which is **true** if there exists a selection function χ such that for each j , the j th component of $\sum_{i=1}^s \hat{P}_{i,\chi(i)}$ is equal to

$$\begin{cases} L_j & \text{if } L_j \leq T, \text{ and} \\ \infty & \text{if } L_j = T + 1. \end{cases}$$

This table has $O(nT^{d-1})$ entries, and successive entries can be computed in constant time by a standard dynamic programming approach. Hence in view of (11) and then (10) the time required to build the table is $O(nT^{d-1}) = O(n(nN/\epsilon)^{d-1})$, which is polynomial in the input size and ϵ^{-1} .

Once the table has been constructed, we simply evaluate the function f given in (6) at the tuples that the table tells us are achievable. More formally, let $g(\vec{L})$ be a function which maps components of \vec{L} that are equal to $T+1$ back to the infinity they represent, i.e., the j th component of $g(L_1, L_2, \dots, L_{d-1})$ is

$$\begin{cases} L_j & \text{if } L_j \leq T, \text{ and} \\ \infty & \text{if } L_j = T + 1. \end{cases}$$

Then our estimate of the minimum expectation is

$$E_{\min} = \min_{\vec{L}: A(n,\vec{L})=\text{true}} f(g(\vec{L})).$$

Let \vec{L}_{\min} be the value of \vec{L} at which the minimum is achieved, and let χ_{\min} be a selection that achieves this minimum, so

$$\sum_{i=1}^n \hat{P}_{i, \chi_{\min}(i)} = \vec{L}_{\min}.$$

Since \hat{P} was computed from P by rounding down, and f is a decreasing function, we know that E_{\min} is greater than or equal to the true optimum. Moreover, since each component of \hat{P} was rounded down by at most 2, and we have considered all possibilities for χ when constructing A , from inspection of f we know that E_{\min} exceeds the true minimum by a ratio of at most

$$e^{2n\gamma} = e^{2n\epsilon/(6n)} = e^{\epsilon/3} \leq 1 + \epsilon/2,$$

where first step used (10) and the last step used (9).

This achieves a ratio of $1 + \epsilon/2$ assuming that the computations are exact; it is not hard to verify that the arithmetic can be done to only polynomially many places and achieve a ratio of $1 + \epsilon$. ■

Again, a similar algorithm gives an approximation scheme for the min-min subset expectation selection problem.

2.2 The max-min problem is in P

We were surprised to find that, in contrast to the difficulty of the min-min problem, the max-min expectation binary selection problem with fixed d can be solved in polynomial time.

Some background is useful. Let L_i , for $1 \leq i \leq n$, be line segments (considered as point sets). The *Minkowski sum* of these line segments is

$$\left\{ \sum_{i=1}^n v_i \mid v_i \in L_i \right\}.$$

The Minkowski sum of a set of line segments is called a *zonotope*. For example, see Figure 1, where, as suggested by Edelsbrunner [2], the construction of a zonotope is illustrated inductively. If the n line segments correspond to the unit vectors for each of n dimensions, the Minkowski sum is a hypercube in n dimensions, with 2^n vertices. With fixed dimension $d-1$, the number of vertices in the Minkowski sum of n segments grows much more slowly: there are $O(n^{d-2})$ vertices, and they can be listed in $O(n^{d-2} + n \log n)$ time [3].

Going back to our problem, let

$$S = \left\{ \sum_{i=1}^n \alpha_i P_{i,0} + (1 - \alpha_i) P_{i,1} \mid \alpha_i \in \{0, 1\} \right\}.$$

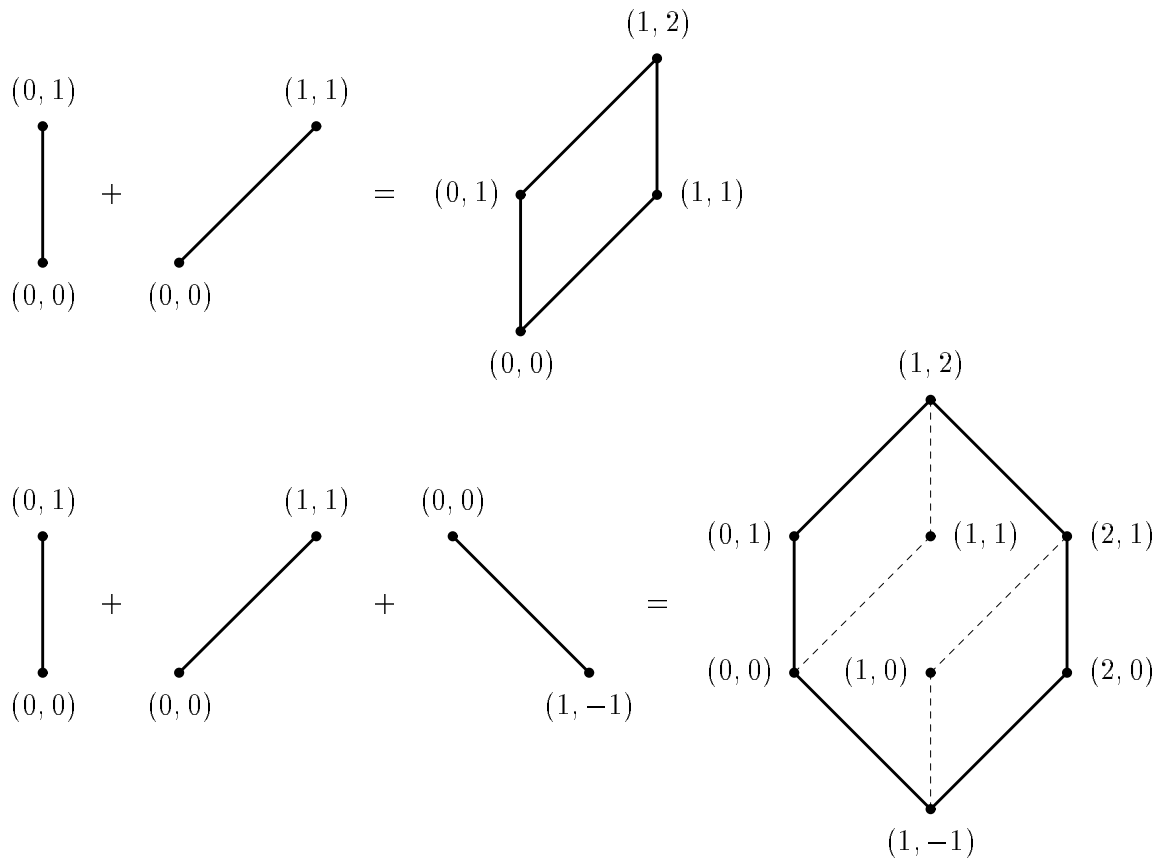


Figure 1: Illustration of the construction of a zonotope. We start with the interval from $(0,0)$ to $(0,1)$. We sweep this by the second interval, from $(0,0)$ to $(1,1)$, to form a rhombus. We then sweep this rhombus by the third interval, from $(0,0)$ to $(1,-1)$, to form a hexagon. Note that $(1,0)$ and $(1,1)$ are not vertices of this hexagon.

We wish to find the maximum value of f at any point in S . Let K be the convex hull of these points, so K is the zonotope

$$K = \left\{ \sum_{i=1}^n \alpha_i P_{i,0} + (1 - \alpha_i) P_{i,1} \mid \alpha_i \in [0, 1] \right\}.$$

All vertices of K are in S , but not all points in S are necessarily vertices of K . Since f is convex, we know that the maximum value of f over S will be achieved at a vertex of K . Thus we can find the maximum in polynomial time by an exhaustive search of the vertices of K .

Solving the max-min expectation subset selection problem (picking k out of n variables) is done similarly, except that we need to define

$$S = \left\{ \sum_{i=1}^n \alpha_i P_i \mid \alpha_i \in \{0, 1\} \text{ and } \sum_{i=1}^n \alpha_i = k \right\}$$

and

$$K = \left\{ \sum_{i=1}^n \alpha_i P_i \mid \alpha_i \in [0, 1] \text{ and } \sum_{i=1}^n \alpha_i = k \right\}.$$

This K is not in general a zonotope, but rather the intersection of a zonotope with the hyperplane given by

$$\sum_{i=1}^n \alpha_i = k.$$

By results of [1], we can still list the vertices in polynomial time, and hence produce a polynomial time algorithm by exhaustive search.

A technical point arises here: we have been assuming that we do arithmetic with real numbers. To show that the problem is in P , we sketch here a proof that we need only carry the computations to $O(N)$ places of accuracy in order to determine the point at which the exact optimum occurs. First let q be the product of all of the denominators of fractions appearing in the input, and note that $\log q = O(N)$. It is then easy to see that the expectation of the minimum of any subset of the random variables can be expressed as i/q for some integer i . Thus to see where the maximum occurs, we only need to be able to perform the computations accurately enough to be able to correctly compare quantities that differ by at least $1/q$. Since the correct answer, even without the assumption that the l_i are positive, is bounded by $\max(|l_i|)$, it follows that computations need only be done to $O(N)$ places of accuracy.

3 Variable d

Theorem 3 *The problem of approximating the min-min expectation binary selection with unrestricted d to within any constant factor r is NP-hard.*

Proof. We will transform CNF-sat to min-min expectation binary selection in a way which gives a large gap between the optimum solutions for satisfiable expressions and unsatisfiable expressions.

Suppose we are given a boolean formula F on variables X_1, \dots, X_n , with clauses C_0, C_1, \dots, C_{c-1} . We assume without loss of generality that $n \geq 1$ and $c \geq 1$. For any literal L and clause C , let $I(L, C)$ be 1 if L is present in C , and 0 otherwise. Let $Atom(p, x)$ denote an atom of weight p and location x . Choose

$$p = \frac{1}{r(c+1)} \quad (12)$$

and

$$v = r^n(c+1)^n = p^{-n}. \quad (13)$$

Construct a min-min expectation binary selection problem M with, for $1 \leq i \leq n$, variables

$$Y_{i,1} \text{ with distribution } Atom(p^c, v^c) + \sum_{s=0}^{c-1} Atom\left((1-p)p^s, v^{s-I(X_i, C_s)}\right) \quad (14)$$

and

$$Y_{i,0} \text{ with distribution } Atom(p^c, v^c) + \sum_{s=0}^{c-1} Atom\left((1-p)p^s, v^{s-I(\bar{X}_i, C_s)}\right). \quad (15)$$

We show that if F is satisfiable then the optimum solution for M is at most $1/r$, but if F is unsatisfiable the optimum solution is at least 1.

To begin, fix integers i and s , with $0 \leq s < c$, and consider the probability that a single variable $Y_{i,1}$ is at least v^s . If the literal X_i is not in clause C_s , then from (14) the total weight of all atoms in $Y_{i,1}$ with locations at or beyond v^s is $p^c + \sum_{l=s}^{c-1} (1-p)p^l = p^s$. If, however, the literal X_i is in clause C_s , then a term $(1-p)p^s$ disappears from this sum so $Y_{i,1}$ is at least v^s with probability only p^{s+1} . In summary,

$$\Pr\{Y_{i,1} \geq v^s\} = \begin{cases} p^{s+1} & \text{if } X_i \text{ is in } C_s \\ p^s & \text{if } X_i \text{ is not in } C_s. \end{cases} \quad (16)$$

Similarly from (15)

$$\Pr\{Y_{i,0} \geq v^s\} = \begin{cases} p^{s+1} & \text{if } \bar{X}_i \text{ is in } C_s \\ p^s & \text{if } \bar{X}_i \text{ is not in } C_s. \end{cases} \quad (17)$$

Next consider the probability that the minimum of the selected variables is at least v^s , i.e.,

$$\Pr\left\{\min_{i=1}^n Y_{i,\chi(i)} \geq v^s\right\} = \prod_{i=1}^n \Pr\{Y_{i,\chi(i)} \geq v^s\}.$$

Using (16) and (17), if the assignment $X_i = \chi(i)$ does not satisfy clause C_s , this product is just p^{sn} . If, on the other hand, the assignment does satisfy clause C_s , then at least one of the terms in the product will be bounded by p^{s+1} , so the product is at most p^{sn+1} . Summarizing,

$$\Pr \left\{ \min_{i=1}^n Y_{i,\chi(i)} \geq v^s \right\} \quad \text{is} \quad \begin{cases} = p^{sn} & \text{if } \chi \text{ does not satisfy } C_s \\ \leq p^{sn+1} & \text{if } \chi \text{ satisfies } C_s. \end{cases} \quad (18)$$

Finally, consider the expectation of the minimum of the selected variables. Note that the possible values for the random variables are $v^{-1}, 1, v, v^2, \dots, v^{c-1}$, so using (4) we have

$$\mathbb{E} \left[\min_{i=1}^n Y_{i,\chi(i)} \right] = v^{-1} + \sum_{s=0}^{c-1} \Pr \left\{ \min_{i=1}^n Y_{i,\chi(i)} \geq v^s \right\} (v^s - v^{s-1}).$$

Using (18), we see that if χ satisfies all of the clauses, this becomes

$$\begin{aligned} \mathbb{E} \left[\min_{i=1}^n Y_{i,\chi(i)} \right] &\leq v^{-1} + \sum_{s=0}^{c-1} p^{sn+1} (v^s - v^{s-1}) \\ &= v^{-1} + (1 - v^{-1}) \sum_{s=0}^{c-1} p^{sn+1} v^s \\ &= p^n + (1 - p^n) \sum_{s=0}^{c-1} p \\ &\quad \text{(by using (13))} \\ &= p^n + (1 - p^n)cp. \end{aligned} \quad (19)$$

Since $n \geq 1$ this is less than

$$p + cp = p(c + 1) = \frac{1}{r}.$$

If, on the other hand, χ violates some clause, say C_s , again using (18) we have

$$\begin{aligned} \mathbb{E} \left[\min_{i=1}^n Y_{i,\chi(i)} \right] &\geq v^{-1} + p^{sn} (v^s - v^{s-1}) \\ &= v^{-1} + p^{sn} (1 - v^{-1}) v^s \\ &= p^n + (1 - p^n) = 1. \end{aligned} \quad \text{(by using (13))} \quad (20)$$

Thus approximating the solution to within a factor of r would enable us to distinguish between satisfiable and unsatisfiable expressions. ■

Theorem 4 *The problem of approximating the min-min expectation subset selection with unrestricted d to within any constant factor r is NP-hard.*

Proof sketch. The proof is similar to that of the previous theorem, so we only briefly describe the necessary changes. We set

$$p = \frac{1}{r(n+c+1)}. \quad (21)$$

Now, for $1 \leq i \leq n$, we construct variables $Y_{i,1}$ with distribution

$$Atom(p^{c+n}, v^{c+n}) + \sum_{s=0}^{c-1} Atom\left((1-p)p^s, v^{s-I(X_i, C_s)}\right) + \sum_{s=0}^{n-1} Atom\left((1-p)p^{c+s}, v^{c+s-\delta_{i,s}}\right) \quad (22)$$

and variables $Y_{i,0}$ with distribution

$$Atom(p^{c+n}, v^{c+n}) + \sum_{s=0}^{c-1} Atom\left((1-p)p^s, v^{s-I(\bar{X}_i, C_s)}\right) + \sum_{s=0}^{n-1} Atom\left((1-p)p^{c+s}, v^{c+s-\delta_{i,s}}\right), \quad (23)$$

where as usual we define

$$\delta_{i,s} = \begin{cases} 1 & \text{if } i = s \\ 0 & \text{otherwise.} \end{cases}$$

The problem we construct is to choose n of these $2n$ variables so as to minimize the expected value of the minimum of the selected variables; let Z be distributed as the minimum of the selected variables. If for some i neither of $Y_{i,0}$ and $Y_{i,1}$ is selected, arguing much as before we have $\mathbb{E}[Z] \geq 1$. Thus in order to get an expectation below 1 we must pick at least one of each pair. But since there are n pairs and we must pick exactly n variables, this means we can also pick at most one, and hence must pick exactly one, from each pair. Again as before, if the selected variables do not correspond to a satisfying assignment, we will have $\mathbb{E}[Z] \geq 1$. On the other hand, if they do correspond to a satisfying assignment, we will have

$$\mathbb{E}[Z] \leq p^n + (1-p^n)(c+n)p < p + (c+n)p = \frac{1}{r}.$$

Thus approximating the solution to within a factor of r would again enable us to distinguish between satisfiable and unsatisfiable expressions. ■

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