Characterization of the dissipative mappings and their application to perturbations of dissipative-Hamiltonian systems

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Abstract

In this paper, we find necessary and sufficient conditions to identify pairs of matrices X and Y for which there exists $\Delta \in \mathbb{C}^{n,n}$ such that $\Delta + \Delta^*$ is positive semidefinite and $\Delta X = Y$. Such a Δ is called a dissipative mapping taking X to Y. We also provide two different characterizations for the set of all dissipative mappings, and use them to characterize the unique dissipative mapping with minimal Frobenius norm. The minimal-norm dissipative mapping is then used to determine the distance to asymptotic instability for dissipative-Hamiltonian systems under general structure-preserving perturbations. We illustrate our results over some numerical examples and compare them with those of Mehl, Mehrmann and Sharma (Stability Radii for Linear Hamiltonian Systems with Dissipation Under Structure-Preserving Perturbations, SIAM J. Mat. Anal. Appl. 37 (4): 1625-1654, 2016).

1   Introduction

In this paper, we consider the dissipative mapping problem. Given $X, Y \in \mathbb{C}^{n,m}$, the dissipative mapping problem can be divided in three subproblems:

1) *Existence*: find necessary and sufficient conditions on matrices X and Y for the existence of matrices $\Delta \in \mathbb{C}^{n,n}$ such that $\Delta + \Delta^*$ is positive semidefinite and $\Delta X = Y$.

2) *Characterization*: characterize all such dissipative matrices taking X to Y.

3) *Minimal norm*: characterize all solutions to the dissipative mapping problem that have minimal norm. In this paper, we focus on the Frobenius norm, which is the standard in the literature.

The dissipative mapping problem belongs to a wider class of mapping problems: for $X, Y \in \mathbb{C}^{n,m}$, the structured mapping problems require to find $\Delta \in \mathcal{S} \subseteq \mathbb{C}^{n,n}$ such that $\Delta X = Y$, where $\mathcal{S}$ stands for the structure of the mapping. The mapping problems have been extensively studied in [12] and [1, 2] for the structures that are associated with orthosymmetric scalar products. These include symmetric, skew-symmetric, Hermitian, skew-Hermitian, Hamiltonian, skew-Hamiltonian, Hermitian,

skew-Hermitian, persymmetric, and per-skew symmetric matrices. If $\mathcal{S} \subseteq \mathbb{R}^{n,n}$, then it is the real structured mapping problem. In particular, for given $X, Y \in \mathbb{C}^{n,m}$, the real dissipative mapping problem is to find $\Delta \in \mathbb{R}^{n,n}$ such that $\Delta + \Delta^T$ is positive semidefinite and $\Delta X = Y$.

The structured mapping problems occur and are useful in solving various distance problems related to structured matrices and matrix polynomials, see for example [5, 6, 16, 17, 18] and the references therein. A mapping problem closely related to the dissipative mapping is the positive semidefinite (PSD) mapping, when $\mathcal{S}$ is the cone of Hermitian positive semidefinite matrices. The PSD mapping problem has been recently solved and used in [16, 17] to derive formulas for structured distances to instability for linear time-invariant dissipative Hamiltonian (DH) systems. DH systems have the form

$$\dot{x}(t) = (J - R)Qx(t), \tag{1.1}$$

where $x(t)$ is the state vector at time t, $J^* = -J \in \mathbb{F}^{n,n}$, $R^* = R \in \mathbb{F}^{n,n}$ is positive semidefinite and is referred to as the dissipation matrix, and $Q = Q^* \in \mathbb{F}^{n,n}$ is positive definite and describes the energy of the system via the function $x \mapsto x^*Qx$. When the coefficient matrices in (1.1) are complex, that is,. $\mathbb{F} = \mathbb{C}$, then it is called a complex DH system, and when the coefficient matrices in (1.1) are real, that is, $\mathbb{F} = \mathbb{R}$, then it is called a real DH system. Such systems are special cases of port-Hamiltonian systems; see for example [8, 20, 21, 16, 17]. A linear time-invariant (LTI) control system $\dot{x} = Ax$ with $A \in \mathbb{C}^{n,n}$ is called *asymptotically stable* around its equilibrium point at the origin if (i) for any $\epsilon > 0$, there exists $\delta_1 > 0$ such that if $\|x(t_0)\| < \delta_1$, then $\|x(t)\| < \epsilon$, for all $t > t_0$, and (ii) there exists $\delta_2 > 0$ such that if $\|x(t_0)\| < \delta_2$, then $x(t) \to 0$ as $t \to \infty$. It is called *stable* if only (i) holds. An equivalent algebraic characterization of stability is given in terms of spectral conditions on A: the system $\dot{x} = Ax$ is called stable if all eigenvalues of the matrix A are in the closed left half of the complex plane and those on the imaginary axis are semisimple. It is asymptotically stable if all eigenvalues of A are in the open left half of the complex plane.

Any LTI stable system can be represented in the form (1.1) of a DH system [7]. DH systems are always stable and remain stable as long as the perturbations preserve the DH structure. Still, they may have eigenvalues on the imaginary axis, i.e., they are not necessarily asymptotically stable. Therefore it is useful to know the *distance to asymptotic instability* for DH systems [16, 17] defined as the smallest norm of the perturbation that makes system lose its asymptotic stability. Studying this question is an essential topic in many applications, like in power system and circuit simulation, see, e.g., [14, 15, 13], and multi-body systems, see, e.g., [9, 22]. This is also useful in some specific applications, e.g., the analysis of disk brake squeal [16, Example 1.1], of mass-spring-damper dynamical systems [17, Example 4.1], and of circuit simulation and power system modelling [16, Example 1.2], where the interest is in studying the stability or the instability under perturbation of the matrices J, R and Q. In the DH modelling of physical systems, the matrix J describes the energy flux among the system's energy storage elements, and R represents the damping effects in the system. Thus perturbing J and R together, or only one at a time, is of particular interest.

In [16], the authors derived various distances to asymptotic instability for DH systems (1.1) while focusing on perturbations that affect only one matrix from $\{J, R, Q\}$ at a time. Similarly in [17], the authors derived real distances to instability for real DH systems while perturbing only the dissipation matrix R. The framework suggested in [16] depends heavily on reformulating the instability radius problem in terms of an equivalent problem of minimizing the generalized Rayleigh quotient of two positive semidefinite matrices. This reformulation was achieved using minimal norm skew-Hermitian mappings in case only J is perturbed, and PSD mappings in case only R or Q is perturbed. However, we note that the framework suggested in [16] does not work if we allow perturbations in the DH system

that affect more than one matrix from $\{J, R, Q\}$ at a time. Analyzing the stability of the system (1.1) when more than one matrix from $\{J, R, Q\}$ is perturbed is one of the main motivations of our work.

In this paper, we focus on perturbations of DH systems that affect both J and R simultaneously. More precisely, we find (see Section 5) that the minimal-norm solution to the dissipative mapping problem can be a necessary tool in computing the structured distance to asymptotic instability for DH systems (1.1) when both J and R are simultaneously perturbed.

1.1 Contributions and outline of the paper

In Section 2, we present some preliminary results that will be needed to solve the dissipative mapping problem. For the solutions to the dissipative mapping problem, we present two different characterizations in terms of three matrix variables K, G, Z with symmetry and semidefinite structures. Both characterizations have an advantage over each other. The first characterization (see Section 3) results in a straightforward computation of the minimal-norm solutions to the dissipative mapping problem, but the matrix variables K, G, Z are highly constrained. We also derive necessary and sufficient conditions for solving the real dissipative mapping problem and compute solutions that are of minimal Frobenius norm. The second characterization (see Section 4) has the advantage of having a simple form in terms of matrix variables K, G, and Z. On the other hand, it is unclear how to find minimal-norm solutions via this second characterization. The minimal-norm dissipative mapping is used in Section 5 for studying the structured distance to instability for DH systems (1.1) for simultaneous perturbations of J and R. In Section 6, we present numerical examples comparing these distances to instability with those of [16] where perturbations affect only one of the matrices J, R, or Q.

Notation In the following, we denote the identity matrix of size $n \times n$ by I_n, the spectral norm of a matrix or a vector by $\| \cdot \|$, and the Frobenius norm by $\| \cdot \|_F$. The Moore-Penrose pseudoinverse of a matrix or a vector X is denoted by $X^\dagger$, and $\mathcal{P}_X = I_n - XX^\dagger$ denotes the orthogonal projection onto the null space of $n \times n$ matrix X^*. For a square matrix A, its Hermitian and skew-Hermitian parts are respectively denoted by $A_H = \frac{A+A^*}{2}$ and $A_S = \frac{A-A^*}{2}$. For $A = A^* \in \mathbb{F}^{n,n}$, where $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, we denote $A \succ 0$ $(A \prec 0)$ and $A \succeq 0$ $(A \preceq)$ if A is Hermitian positive definite (negative definite) and Hermitian positive semidefinite (negative semidefinite), respectively, and $\Lambda(A)$ denotes the set of all eigenvalues of the matrix A. For a given matrix $A \in \mathbb{C}^{n,m}$, we use the term SVD to denote the standard singular value decomposition of A, and reduced SVD for the decomposition $A = U_1 \Sigma_1 V_1^*$ in which Σ_1 is a square diagonal matrix of size equal to rank of A, r, with the nonzero singular values of A on its diagonal, and $U_1 \in \mathbb{C}^{n,r}$ and $V_1 \in \mathbb{C}^{m,r}$ have orthonormal columns.

2 Preliminaries

In this section, we discuss some basic results from the literature and derive some elementary lemmas that will be necessary to solve the dissipative mapping problem.

Let us start with two well-known lemmas for positive semidefinite matrices.

Lemma 2.1 [11] *Let $P \in \mathbb{C}^{n,n}$ and $X \in \mathbb{C}^{n,m}$. If $P \succeq 0$, then $X^*PX \succeq 0$.*

Lemma 2.2 [4] *Let the integer s be such that $0 < s < n$, and $R = R^* \in \mathbb{C}^{n,n}$ be partitioned as $R = \begin{bmatrix} B & C^* \\ C & D \end{bmatrix}$ with $B \in \mathbb{C}^{s,s}$, $C \in \mathbb{C}^{n-s,s}$ and $D \in \mathbb{C}^{n-s,n-s}$. Then $R \succeq 0$ if and only if*

1) $B \succeq 0$,

2) $\mathrm{null}(B) \subseteq \mathrm{null}(C)$, and

3) $D - CB^\dagger C^* \succeq 0$, where $B^\dagger$ denotes the Moore-Penrose pseudoinverse of B.

Next, we state a result [1, Theorem 2.2.3] for skew-Hermitian mappings in terms that allow a direct use in deriving the second characterization for dissipative mappings.

Theorem 2.3 *Let $X, Y \in \mathbb{C}^{n,k}$ and define $\mathcal{S} := \{\Delta \in \mathbb{C}^{n,n} : \Delta^* = -\Delta, \Delta X = Y\}$. Then $\mathcal{S} \neq \emptyset$ if and only if $(X^*Y)^* = -X^*Y$. Further, if $\mathcal{S} \neq \emptyset$, then*

$$\mathcal{S} = \left\{ YX^\dagger - (YX^\dagger)^* + (X^\dagger)^*X^*YX^\dagger + \mathcal{P}_X Z \mathcal{P}_X : Z \in \mathbb{C}^{n,n}, Z^* = -Z \right\}.$$

The next two lemmas will be used in deriving real dissipative matrices taking a complex $X \in \mathbb{C}^{n,m}$ to a complex $Y \in \mathbb{C}^{n,m}$.

Lemma 2.4 [3, Lemma 3.3] *Let $A, B \in \mathbb{C}^{n,p}$. Then $[A \ \overline{A}][B \ \overline{B}]^\dagger$ is a real matrix.*

Lemma 2.5 *Let $X, Y \in \mathbb{C}^{n,k}$ be such that $M = ([X \ \overline{X}]^*[Y \ \overline{Y}] + [Y \ \overline{Y}]^*[X \ \overline{X}])^{-1}$ exists. Then $[X \ \overline{X}]M[Y \ \overline{Y}]^*$ is a real matrix.*

Proof. Let $Q = \begin{bmatrix} O & I_k \\ I_k & O \end{bmatrix}$, where $O \in \mathbb{C}^{k,k}$ is the zero matrix. Then

$$\overline{[X \ \overline{X}]M[Y \ \overline{Y}]^*} = \overline{[X \ \overline{X}]Q^2 MQ^2 [Y \ \overline{Y}]^*} = \overline{([X \ \overline{X}]Q)(QMQ)(Q[Y \ \overline{Y}]^*)} = [X \ \overline{X}]\overline{QMQ}[Y \ \overline{Y}]^*,$$

because $[X \ \overline{X}]Q = [\overline{X} \ X]$ and $Q[Y \ \overline{Y}]^* = [\overline{Y} \ Y]^*$. Thus we prove the assertion by showing that $\overline{QMQ} = M$. This follows from the following derivations:

$$\begin{aligned}
\overline{QMQ} &= \overline{Q([X \ \overline{X}]^*[Y\overline{Y}] + [Y \ \overline{Y}]^*[X \ \overline{X}])^{-1}Q} \\
&= Q\overline{([X \ \overline{X}]^*[Y \ \overline{Y}] + [Y \ \overline{Y}]^*[X \ \overline{X}])^{-1}}Q \\
&= Q\overline{([X \ \overline{X}]^*[Y \ \overline{Y}] + [Y \ \overline{Y}]^*[X \ \overline{X}])}^{-1}Q \\
&= Q([\overline{X} \ X]^*[\overline{Y} \ Y] + [\overline{Y} \ Y]^*[\overline{X} \ X])^{-1}Q \\
&= \left(Q[\overline{X} \ X]^*[\overline{Y} \ Y]Q + Q[\overline{Y} \ Y]^*[\overline{X} \ X]Q\right)^{-1} \\
&= \left([X \ \overline{X}]^*[Y \ \overline{Y}] + [Y \ \overline{Y}]^*[X\overline{X}]\right)^{-1} \\
&= M.
\end{aligned}$$

□

We close the section by discussing some simple mapping results that will be necessary to compute the structured distances to instability in Section 5.

Lemma 2.6 *Let $x \in \mathbb{C}^q$ and $y \in \mathbb{C}^r$. Then*

$$\inf \left\{ \|\Delta_1\|_F^2 + \|\Delta_2\|_F^2 : \Delta_1, \Delta_2 \in \mathbb{C}^{r,q}, (\Delta_1 - \Delta_2)x = y \right\} = \inf \left\{ \frac{\|\Delta\|_F^2}{2} : \Delta \in \mathbb{C}^{r,q}, \Delta x = y \right\}. \quad (2.1)$$

Proof. First, let $\Delta \in \mathbb{C}^{r,q}$ such that $\Delta x = y$, and set $\Delta_1 = \Delta/2$ and $\Delta_2 = -\Delta/2$. Then $(\Delta_1 - \Delta_2)x = \Delta x = y$, and $\|\Delta_1\|_F^2 + \|\Delta_2\|_F^2 = \frac{\|\Delta\|_F^2}{4} + \frac{\|\Delta\|_F^2}{4} = \frac{\|\Delta\|_F^2}{2}$. This implies "$\leq$" in (2.1). Conversely, let $\Delta_1, \Delta_2 \in \mathbb{C}^{r,q}$ be such that $(\Delta_1 - \Delta_2)x = y$, and set $\Delta = \Delta_1 - \Delta_2$. Then $\Delta x = y$ and

$$
\begin{aligned}
\|\Delta\|_F^2 = \|\Delta_1 - \Delta_2\|_F^2 &\leq (\|\Delta_1\|_F + \|\Delta_2\|_F)^2 \\
&= \|\Delta_1\|_F^2 + \|\Delta_2\|_F^2 + 2\|\Delta_1\|_F\|\Delta_2\|_F \\
&\leq 2\left(\|\Delta_1\|_F^2 + \|\Delta_2\|_F^2\right).
\end{aligned}
$$

Thus $\frac{\|\Delta\|_F^2}{2} \leq \|\Delta_1\|_F^2 + \|\Delta_2\|_F^2$. This implies "$\geq$" in (2.1). $\square$

Lemma 2.7 *Let* $x, y \in \mathbb{C}^r$*. Then*

$$
\inf\left\{ \|\Delta_1\|_F^2 + \|\Delta_2\|_F^2 : \ \Delta_1, \Delta_2 \in \mathbb{C}^{r,r}, \Delta_1^* = -\Delta_1, \Delta_2 \preceq 0, (\Delta_1 - \Delta_2)x = y \right\}
$$

$$
= \inf\left\{ \|\Delta\|_F^2 : \ \Delta \in \mathbb{C}^{r,r}, \Delta + \Delta^* \succeq 0, \Delta x = y \right\}. \tag{2.2}
$$

Proof. The idea behind the proof is similar to Lemma 2.6. Let $\Delta \in \mathbb{C}^{r,r}$ be such that $\Delta + \Delta^* \succeq 0$ and $\Delta x = y$. Set $\Delta_1 = \Delta_S$ and $\Delta_2 = -\Delta_H$. Then clearly $\Delta_1^* = -\Delta_1$, $\Delta_2 \preceq 0$ since $\Delta + \Delta^* \succeq 0$, and $(\Delta_1 - \Delta_2)x = \Delta x = y$. Also $\|\Delta_1\|_F^2 + \|\Delta_2\|_F^2 = \|\Delta_S\|_F^2 + \|\Delta_H\|_F^2 = \|\Delta\|_F^2$. This implies "$\leq$" in (2.2). Conversely, let $\Delta_1, \Delta_2 \in \mathbb{C}^{r,r}$ be such that $\Delta_1^* = -\Delta_1$, $\Delta_2 \preceq 0$, and $(\Delta_1 - \Delta_2)x = y$. Now set $\Delta = \Delta_1 - \Delta_2$. Then clearly $\Delta x = y$ and $\Delta + \Delta^* = -2\Delta_2 \succeq 0$, since $\Delta_2 \preceq 0$. Further we have that $\|\Delta\|_F^2 = \|\Delta_S\|_F^2 + \|\Delta_H\|_F^2 = \|\Delta_1\|_F^2 + \|\Delta_2\|_F^2$. This proves "$\geq$" in (2.2). $\square$

Lemma 2.8 [16, Theorem 2.8] *Let* $B \in \mathbb{C}^{n,r}$ *with* $\operatorname{rank}(B) = r$, $x \in \mathbb{C}^r \setminus \{0\}$, $y \in \mathbb{C}^n \setminus \{0\}$*, and let* $\Delta \in \mathbb{C}^{r,r}$*. Then* $B\Delta x = y$ *if and only if* $\Delta x = B^\dagger y$ *and* $BB^\dagger y = y$.

3 First characterization of dissipative mappings

In this section, we derive the first characterization of dissipative mappings. This characterization allows us to find the minimal Frobenius norm solutions to the dissipative mapping problem which is shown to be unique. This minimal-norm solution will turn out to be a necessary tool in computing the structured stability radius in Section 5. For given $X \in \mathbb{C}^{n,m}$ and $Y \in \mathbb{C}^{n,m}$, we define the set of dissipative mappings from X to Y as follows

$$
\mathbb{S}(X, Y) := \{\Delta \in \mathbb{C}^{n,n} : \ \Delta + \Delta^* \succeq 0, \ \Delta X = Y\}.
$$

We will need the following lemma in characterizing the set of all solutions to the dissipative mapping problem.

Lemma 3.1 *Let* $X, Y \in \mathbb{C}^{n,m}$*. Suppose that* $\operatorname{rank}(X) = p$ *and consider the reduced singular value decomposition* $X = U_1 \Sigma_1 V_1^*$ *with* $U_1 \in \mathbb{C}^{n,p}$, $\Sigma_1 \in \mathbb{C}^{p,p}$ *and* $V_1 \in \mathbb{C}^{m,p}$*. If* $X^*Y + Y^*X \succeq 0$*, then* $U_1^*\left(YX^\dagger + (YX^\dagger)^*\right)U_1 \succeq 0$.

Proof. Since $X = U_1\Sigma_1 V_1^*$, we have $X^\dagger = V_1\Sigma_1^{-1}U_1^*$. Thus

$$
\begin{aligned}
X^*Y + Y^*X \succeq 0 \quad &\Longleftrightarrow \quad V_1\Sigma_1 U_1^*Y + Y^*U_1\Sigma_1 V_1^* \succeq 0 \\
&\Longrightarrow \quad V_1^*\left(V_1\Sigma_1 U_1^*Y + Y^*U_1\Sigma_1 V_1^*\right)V_1 \succeq 0 \quad (\because \text{ using Lemma 2.1}) \\
&\Longleftrightarrow \quad \Sigma_1 U_1^*YV_1 + V_1^*Y^*U_1\Sigma_1 \succeq 0 \qquad (\because V_1^*V_1 = I_p) \\
&\Longleftrightarrow \quad \Sigma_1^{-1}\left(\Sigma_1 U_1^*YV_1 + V_1^*Y^*U_1\Sigma_1\right)\Sigma_1^{-1} \succeq 0 \\
&\Longleftrightarrow \quad U_1^*YV_1\Sigma_1^{-1} + \Sigma_1^{-1}V_1^*Y^*U_1 \succeq 0 \\
&\Longleftrightarrow \quad U_1^*\left(YV_1\Sigma_1^{-1}U_1^* + U_1\Sigma_1^{-1}V_1^*Y^*\right)U_1 \succeq 0 \\
&\Longleftrightarrow \quad U_1^*\left(YX^\dagger + (YX^\dagger)^*\right)U_1 \succeq 0,
\end{aligned}
$$

which completes the proof. $\square$

Theorem 3.2 *Let $X, Y \in \mathbb{C}^{n,m}$, and suppose that $\mathrm{rank}(X) = p$. Then $\mathbb{S}(X,Y) \neq \emptyset$ if and only if $YX^\dagger X = Y$ and $X^*Y + Y^*X \succeq 0$. Moreover, if $\mathbb{S}(X,Y) \neq \emptyset$, then*

1) Characterization: *Let $X = U\Sigma V^*$ be the the singular value decomposition of X with $U = [U_1\ U_2]$, where $U_1 \in \mathbb{C}^{n,p}$. Then*

$$
\mathbb{S}(X,Y) = \Big\{ U \begin{bmatrix} U_1^*YX^\dagger U_1 & U_1^*(YX^\dagger)^*U_2 + U_1^*ZU_2 \\ U_2^*YX^\dagger U_1 & U_2^*(K+G)U_2 \end{bmatrix} U^* :
$$
$$
Z, K, G \in \mathbb{C}^{n,n} \text{ satisfying } (3.2)-(3.4) \Big\}, \tag{3.1}
$$

where

$$
G^* = -G, \quad K \succeq 0, \tag{3.2}
$$
$$
\mathrm{null}\left(U_1^*(YX^\dagger + (YX^\dagger)^*)U_1\right) \subseteq \mathrm{null}\left(U_2^*(2YX^\dagger + Z^*)U_1\right), \tag{3.3}
$$
$$
K - \frac{1}{8}(2YX^\dagger + Z^*)\left(XX^\dagger YX^\dagger + (YX^\dagger)^*XX^\dagger\right)^\dagger(2YX^\dagger + Z^*)^* \succeq 0. \tag{3.4}
$$

2) Minimal norm mapping:

$$
\inf_{\Delta \in \mathbb{S}(X,Y)} \|\Delta\|_F^2 \;=\; 2\|YX^\dagger\|_F^2 - \mathrm{trace}\left((YX^\dagger)^*XX^\dagger(YX^\dagger)\right), \tag{3.5}
$$

where the infimum is uniquely attained by the matrix $\mathcal{H} := YX^\dagger - (YX^\dagger)^*\mathcal{P}_X$, which is obtained by setting $K = 0$, $G = 0$, and $Z = -2U_1^*(YX^\dagger)^*U_2$ in (3.1).

Proof. First suppose that $\Delta \in \mathbb{S}(X,Y)$, i.e., $\Delta + \Delta^* \succeq 0$ and $\Delta X = Y$. Then $YX^\dagger X = \Delta XX^\dagger X = \Delta X = Y$. By Lemma 2.1

$$
X^*Y + Y^*X = X^*\Delta X + X^*\Delta^*X = X^*(\Delta + \Delta^*)X \succeq 0,
$$

since $\Delta + \Delta^* \succeq 0$. Conversely, suppose that X and Y satisfy $YX^\dagger X = Y$ and $X^*Y + Y^*X \succeq 0$. Then the matrix $\mathcal{H}$ satisfies $\mathcal{H}X = YX^\dagger X = Y$ and

$$
\begin{aligned}
\mathcal{H} + \mathcal{H}^* &= YX^\dagger - (YX^\dagger)^*\mathcal{P}_X + (YX^\dagger)^* - \mathcal{P}_X YX^\dagger \\
&= (YX^\dagger)^* XX^\dagger + XX^\dagger(YX^\dagger) \qquad (\because \mathcal{P}_X = I_n - XX^\dagger) \\
&= XX^\dagger \left((YX^\dagger)^* + YX^\dagger \right) XX^\dagger \qquad (\because X^\dagger XX^\dagger = X^\dagger, \ (XX^\dagger)^* = XX^\dagger) \\
&= U_1 U_1^* \left((YX^\dagger)^* + YX^\dagger \right) U_1 U_1^* \qquad (\because XX^\dagger = U_1 U_1^*) \\
&= \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} U_1^* \left((YX^\dagger)^* + YX^\dagger \right) U_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} U_1 & U_2 \end{bmatrix}^* \\
&\succeq 0,
\end{aligned}
$$

where the last identity follows by using Lemma 3.1, since $X^*Y + Y^*X \succeq 0$, and by Lemma 2.1.

Next, we prove (3.1). First suppose that $\Delta \in \mathbb{S}(X,Y)$, i.e., $\Delta + \Delta^* \succeq 0$ and $\Delta X = Y$. Let $\Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix}$ and $V = \begin{bmatrix} V_1 & V_2 \end{bmatrix}$, where $V_1 \in \mathbb{C}^{m,p}$, $V_2 \in \mathbb{C}^{m,m-p}$, and $\Sigma_1 \in \mathbb{C}^{p,p}$ such that $X = U_1 \Sigma_1 V_1^*$ is the reduced SVD of X. Then $X^* = V_1 \Sigma_1 U_1^*$ and $X^\dagger = V_1 \Sigma_1^{-1} U_1^*$. Consider $\Delta = UU^*\Delta UU^*$ and $\widetilde{\Delta} = U^*\Delta U = \widetilde{\Delta}_H + \widetilde{\Delta}_S$, where

$$
\widetilde{\Delta}_H = U^*\Delta_H U = \begin{bmatrix} H_{11} & H_{12} \\ H_{12}^* & H_{22} \end{bmatrix} \quad \text{and} \quad \widetilde{\Delta}_S = U^*\Delta_S U = \begin{bmatrix} S_{11} & S_{12} \\ -S_{12}^* & S_{22} \end{bmatrix}.
$$

Clearly, $\|\Delta\|_F = \|\widetilde{\Delta}\|_F$ and also $\Delta_H \succeq 0 \iff \widetilde{\Delta}_H \succeq 0$. As $\Delta X = Y$, we have

$$
\begin{aligned}
U^*\Delta UU^*X = U^*Y \implies & \widetilde{\Delta} \begin{bmatrix} U_1^* \\ U_2^* \end{bmatrix} X = \begin{bmatrix} U_1^*Y \\ U_2^*Y \end{bmatrix} \\
\implies & \begin{bmatrix} H_{11} + S_{11} & H_{12} + S_{12} \\ H_{12}^* - S_{12}^* & H_{22} + S_{22} \end{bmatrix} \begin{bmatrix} \Sigma_1 V_1^* \\ 0 \end{bmatrix} = \begin{bmatrix} U_1^*Y \\ U_2^*Y \end{bmatrix}.
\end{aligned}
$$

This implies that

$$
(H_{11} + S_{11}) \Sigma_1 V_1^* = U_1^*Y, \tag{3.6}
$$

and

$$
(H_{12}^* - S_{12}^*) \Sigma_1 V_1^* = U_2^*Y. \tag{3.7}
$$

Thus from (3.6), we have $H_{11} + S_{11} = U_1^*YV_1\Sigma_1^{-1} = U_1^*YX^\dagger U_1$, since $X^\dagger = V_1\Sigma_1^{-1}U_1^*$ and $X^\dagger U_1 = V_1\Sigma_1^{-1}$. This implies that

$$
H_{11} = U_1^* \left(\frac{(YX^\dagger) + (YX^\dagger)^*}{2} \right) U_1 \quad \text{and} \quad S_{11} = U_1^* \left(\frac{(YX^\dagger) - (YX^\dagger)^*}{2} \right) U_1. \tag{3.8}
$$

Note that since $X^*Y + Y^*X \succeq 0$, in view of Lemma 3.1, we have that $H_{11} \succeq 0$. Similarly, from (3.7) we have $H_{12}^* - S_{12}^* = U_2^*YV_1\Sigma_1^{-1} = U_2^*YX^\dagger U_1$. This implies that

$$
H_{12} = U_1^*(YX^\dagger)^*U_2 + S_{12}, \tag{3.9}
$$

where $S_{12} \in \mathbb{C}^{p,n-p}$ is a matrix variable. Thus from (3.8) and (3.9), $\widetilde{\Delta}$ has the form

$$
\widetilde{\Delta} = \begin{bmatrix} H_{11} + S_{11} & H_{12} + S_{12} \\ H_{12}^* - S_{12}^* & H_{22} + S_{22} \end{bmatrix} = \begin{bmatrix} U_1^*YX^\dagger U_1 & U_1^*(YX^\dagger)^*U_2 + 2S_{12} \\ U_2^*YX^\dagger U_1 & H_{22} + S_{22} \end{bmatrix}, \tag{3.10}
$$

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where $H_{22}, S_{22} \in \mathbb{C}^{n-p,n-p}$ and $S_{12} \in \mathbb{C}^{p,n-p}$ are such that $\widetilde{\Delta}_H \succeq 0$ and $\widetilde{\Delta}_S = -\widetilde{\Delta}_S^*$. That means, in view of Lemma 2.2, S_{22} satisfies that $S_{22}^* = -S_{22}$, and H_{22} and S_{12} satisfy the following constraints: $H_{22} \succeq 0$,

$$H_{22} - (U_2^* Y X^\dagger U_1 + S_{12}^*) \left(\frac{U_1^* (YX^\dagger + (YX^\dagger)^*) U_1}{2} \right)^\dagger (U_2^* Y X^\dagger U_1 + S_{12}^*)^* \succeq 0, \qquad (3.11)$$

and

$$\text{null} \left(\frac{U_1^* (YX^\dagger + (YX^\dagger)^*) U_1}{2} \right) \subseteq \text{null} \left(U_2^* Y X^\dagger U_1 + S_{12}^* \right). \qquad (3.12)$$

Thus from (3.10), we have

$$\Delta \;=\; U \widetilde{\Delta} U^* = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} U_1^* Y X^\dagger U_1 & U_1^* (YX^\dagger)^* U_2 + 2 S_{12} \\ U_2^* Y X^\dagger U_1 & H_{22} + S_{22} \end{bmatrix} \begin{bmatrix} U_1^* \\ U_2^* \end{bmatrix}. \qquad (3.13)$$

By setting $Z = 2 U_1 S_{12} U_2^*$, $K = U_2 H_{22} U_2^*$, $G = U_2 S_{22} U_2^*$, and by using the fact that $U_1 U_1^* + U_2 U_2^* = UU^* = I_n$, $U_1 U_1^* = XX^\dagger$, $U_1^* U_1 = I_p$, and $U_2^* U_2 = I_{n-p}$, we obtain that

$$\Delta = U \begin{bmatrix} U_1^* Y X^\dagger U_1 & U_1^* (YX^\dagger)^* U_2 + U_1^* Z U_2 \\ U_2^* Y X^\dagger U_1 & U_2^* (K + G) U_2 \end{bmatrix} U^*,$$

where G, Z, and K satisfy the conditions (3.2)–(3.4). This proves "$\subseteq$" in (3.1).

For the other inclusion in (3.1), let

$$A = U \begin{bmatrix} U_1^* Y X^\dagger U_1 & U_1^* (YX^\dagger)^* U_2 + U_1^* Z U_2 \\ U_2^* Y X^\dagger U_1 & U_2^* (K + G) U_2 \end{bmatrix} U^*,$$

where G, Z, and K satisfy the conditions (3.2)–(3.4), which can be written as

$$A = YX^\dagger + (YX^\dagger)^* \mathcal{P}_X + XX^\dagger Z \mathcal{P}_X + \mathcal{P}_X K \mathcal{P}_X + \mathcal{P}_X G \mathcal{P}_X. \qquad (3.14)$$

Clearly $AX = Y$ since $YX^\dagger X = Y$ and $\mathcal{P}_X X = 0$. Also $A + A^* \succeq 0$. Indeed,

$$U^* A U = \begin{bmatrix} U_1^* Y X^\dagger U_1 & U_1^* (YX^\dagger)^* U_2 + U_1^* Z U_2 \\ U_2^* Y X^\dagger U_1 & U_2^* (K + G) U_2 \end{bmatrix},$$

and thus

$$U^* (A + A^*) U = \begin{bmatrix} U_1^* \left(YX^\dagger + (YX^\dagger)^* \right) U_1 & 2 U_1^* (YX^\dagger)^* U_2 + U_1^* Z U_2 \\ 2 U_2^* Y X^\dagger U_1 + U_2^* Z^* U_1 & 2 U_2^* K U_2 \end{bmatrix} \succeq 0,$$

because of Lemma 2.2, since K, Z, G satisfy (3.2)–(3.4). This completes the proof of (3.1).

Suppose that $\mathbb{S}(X,Y) \neq \emptyset$ and let $\Delta \in \mathbb{S}(X,Y)$, then from (3.10) Δ satisfies

$$\|\Delta\|_F^2 = \|\widetilde{\Delta}\|_F^2 = \left\| \begin{bmatrix} U_1^* Y X^\dagger U_1 & U_1^* (YX^\dagger)^* U_2 + 2 S_{12} \\ U_2^* Y X^\dagger U_1 & H_{22} + S_{22} \end{bmatrix} \right\|_F^2,$$

where $S_{22} \in \mathbb{C}^{n-p,n-p}$ satisfies that $S_{22}^* = -S_{22}$, and $H_{22} \in \mathbb{C}^{n-p,n-p}$ and $S_{12} \in \mathbb{C}^{p,n-p}$ satisfy (3.11) and (3.12). This implies that

$$\begin{aligned} \|\Delta\|_F^2 &= \|U_1^* Y X^\dagger U_1\|_F^2 + \|U_2^* Y X^\dagger U_1\|_F^2 + \|U_1^* (YX^\dagger)^* U_2 + 2 S_{12}\|_F^2 + \|H_{22} + S_{22}\|_F^2 \\ &= \|Y X^\dagger U_1\|_F^2 + \|U_1^* (YX^\dagger)^* U_2 + 2 S_{12}\|_F^2 + \|H_{22}\|_F^2 + \|S_{22}\|_F^2, \end{aligned} \qquad (3.15)$$

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where the last equality follows as $\|U_1^* Y X^\dagger U_1\|_F^2 + \|U_2^* Y X^\dagger U_1\|_F^2 = \|U Y X^\dagger U_1\|_F^2 = \|Y X^\dagger U_1\|_F^2$ because $\|\cdot\|_F$ is unitary invariant, and by the fact that for any square matrix $A = A_H + A_S$ we have $\|A\|_F^2 = \|A_H\|_F^2 + \|A_S\|_F^2$. Further, from (3.15) we have that

$$\|\Delta\|_F^2 \geq \|Y X^\dagger U_1\|_F^2 + \|U_1^* (Y X^\dagger)^* U_2 + 2 S_{12}\|_F^2,$$

where the lower bound is attained by setting $H_{22} = 0$ and $S_{22} = 0$. Thus any $\Delta \in \mathbb{S}(X,Y)$ satisfies

$$\|\Delta\|_F^2 \geq \|Y X^\dagger U_1\|_F^2 + \|U_1^* (Y X^\dagger)^* U_2 + 2 S_{12}\|_F^2,$$

where S_{12} satisfies (3.11) and (3.12) with $H_{22} = 0$, but the only S_{12} that satisfies (3.11) with $H_{22} = 0$ is $S_{12} = -U_1^* (Y X^\dagger)^* U_2$ because $U_1^* ((Y X^\dagger) + (Y X^\dagger)^*) U_1 \succeq 0$. Hence by setting $H_{22} = 0$, $S_{22} = 0$, and $S_{12} = -U_1^* (Y X^\dagger)^* U_2$ in (3.10), we obtain the unique matrix

$$
\begin{aligned}
\Delta &= \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} U_1^* Y X^\dagger U_1 & -U_1^* (Y X^\dagger)^* U_2 \\ U_2^* Y X^\dagger U_1 & 0 \end{bmatrix} \begin{bmatrix} U_1^* \\ U_2^* \end{bmatrix} \\
&= (U_1 U_1^* + U_2 U_2^*) Y X^\dagger U_1 U_1^* - U_1 U_1^* (Y X^\dagger)^* U_2 U_2^* \\
&= Y X^\dagger X X^\dagger - X X^\dagger (Y X^\dagger)^* \mathcal{P}_X \\
&= Y X^\dagger - (Y X^\dagger)^* \mathcal{P}_X \\
&= \mathcal{H}
\end{aligned}
$$

which implies that (3.5) holds. $\square$

The following corollary of Theorem 3.2 for the vector case $(m = 1)$ of the minimal norm dissipative mapping is of particular interest for us as it will be used in Section 5 in finding the structured stability radius for DH systems when both J and R are perturbed.

Corollary 3.3 *Let $x, y \in \mathbb{C}^n \setminus \{0\}$. Then $\mathbb{S}(x,y) \neq \emptyset$ if and only if $\operatorname{Re}(x^* y) \geq 0$. Moreover, if $\mathbb{S}(x,y) \neq \emptyset$, then*

$$\inf_{\Delta \in \mathbb{S}(x,y)} \|\Delta\|_F^2 = 2 \frac{\|y\|^2}{\|x\|^2} - \frac{|x^* y|^2}{\|x\|^4}, \tag{3.16}$$

where the infimum is attained by the unique matrix $\mathcal{H} := \frac{y x^*}{\|x\|^2} - \frac{x y^*}{\|x\|^2} + (y^* x) \frac{x x^*}{\|x\|^4}$.

3.1 Real dissipative mappings

For $X, Y \in \mathbb{C}^{n,m}$, if we consider the real dissipative mapping problem, i.e., finding $\Delta \in \mathbb{R}^{n,n}$ such that $\Delta + \Delta^T \succeq 0$ and $\Delta X = Y$, then the minimal Frobenius norm solution can be easily obtained from Theorem 3.2. To see this, observe that for a real Δ, $\Delta X = Y$ if and only if $\Delta [X \ \overline{X}] = [Y \ \overline{Y}]$. In the following, we show that if there exists a complex dissipative mapping Δ satisfying $\Delta \mathcal{X} = \mathcal{Y}$ where $\mathcal{X} = [X \ \overline{X}]$ and $\mathcal{Y} = [Y \ \overline{Y}]$, then there also exists a real dissipative mapping. Moreover, from Theorem 3.2, we can easily find a minimal norm real dissipative mapping taking X to Y.

Theorem 3.4 *Let $X, Y \in \mathbb{C}^{n,m}$, and define $\mathbb{S}_\mathbb{R}(X,Y) := \{\Delta \in \mathbb{R}^{n,n} : \Delta + \Delta^T \succeq 0, \ \Delta X = Y\}$. Let $\mathcal{X} = [X \ \overline{X}]$ and $\mathcal{Y} = [Y \ \overline{Y}]$. Suppose that $\operatorname{rank}(\mathcal{X}) = p$. Then $\mathbb{S}_\mathbb{R}(X,Y) \neq \emptyset$ if and only if $\mathcal{Y} \mathcal{X}^\dagger \mathcal{X} = \mathcal{Y}$ and $\mathcal{X}^* \mathcal{Y} + \mathcal{Y}^* \mathcal{X} \succeq 0$. Moreover, if $\mathbb{S}_\mathbb{R}(X,Y) \neq \emptyset$ and if we consider the singular value decomposition $\mathcal{X} = U \Sigma V^T$ with $U = [U_1 \ U_2]$, where $U_1 \in \mathbb{C}^{n,p}$, then the set*

$$\left\{ U \begin{bmatrix} U_1^* \mathcal{Y}\mathcal{X}^\dagger U_1 & U_1^*(\mathcal{Y}\mathcal{X}^\dagger)^* U_2 + U_1^* Z U_2 \\ U_2^* \mathcal{Y}\mathcal{X}^\dagger U_1 & U_2^*(K+G)U_2 \end{bmatrix} U^* : Z, K, G \in \mathbb{R}^{n,n} \text{ satisfying } (3.17)-(3.19) \right\},$$

where

$$G^T = -G, \quad K \succeq 0, \tag{3.17}$$

$$\text{null}\left(U_1^*(\mathcal{Y}\mathcal{X}^\dagger + (\mathcal{Y}\mathcal{X}^\dagger)^T)U_1 \right) \subseteq \text{null}\left(U_2^*(2\mathcal{Y}\mathcal{X}^\dagger + Z^T)U_1 \right), \tag{3.18}$$

$$K - \frac{1}{8}(2\mathcal{Y}\mathcal{X}^\dagger + Z^T)\left(\mathcal{X}\mathcal{X}^\dagger \mathcal{Y}\mathcal{X}^\dagger + (\mathcal{Y}\mathcal{X}^\dagger)^T \mathcal{X}\mathcal{X}^\dagger \right)^\dagger (2\mathcal{Y}\mathcal{X}^\dagger + Z^T)^T \succeq 0, \tag{3.19}$$

is contained in $\mathbb{S}_\mathbb{R}(X,Y)$. Further,

$$\inf_{\Delta \in \mathbb{S}_\mathbb{R}(X,Y)} \|\Delta\|_F^2 = 2\|\mathcal{Y}\mathcal{X}^\dagger\|_F^2 - \text{trace}\left((\mathcal{Y}\mathcal{X}^\dagger)^* \mathcal{X}\mathcal{X}^\dagger(\mathcal{Y}\mathcal{X}^\dagger) \right),$$

where the infimum is attained by the matrix $\mathcal{H} := \mathcal{Y}\mathcal{X}^\dagger - (\mathcal{Y}\mathcal{X}^\dagger)^* \mathcal{P}_\mathcal{X}$.

Proof. In view of Lemma 2.4, the proof follows from Theorem 3.2. Indeed, from Theorem 3.2 there exists $\Delta \in \mathbb{S}(\mathcal{X}, \mathcal{Y})$ if and only if $\mathcal{Y}\mathcal{X}^\dagger \mathcal{X} = \mathcal{Y}$ and $\mathcal{X}^*\mathcal{Y} + \mathcal{Y}^*\mathcal{X} \succeq 0$. Assuming the latter two conditions hold true, and consider a family of mappings defined by

$$\Delta(Z, K, G) := U \begin{bmatrix} U_1^* \mathcal{Y}\mathcal{X}^\dagger U_1 & U_1^*(\mathcal{Y}\mathcal{X}^\dagger)^* U_2 + U_1^* Z U_2 \\ U_2^* \mathcal{Y}\mathcal{X}^\dagger U_1 & U_2^*(K+G)U_2 \end{bmatrix} U^*,$$

where $Z, K, G \in \mathbb{R}^{n,n}$ satisfy (3.17)-(3.19). Using Lemma 2.4, it is easy to check that $\Delta(Z, K, G)$ is real, which implies that $\Delta(Z, K, G) \in \mathbb{S}_\mathbb{R}(X,Y)$ since $\mathbb{S}_\mathbb{R}(\mathcal{X}, \mathcal{Y}) = \mathbb{S}_\mathbb{R}(X,Y)$. Further, we have that

$$\inf_{\Delta \in \mathbb{S}(\mathcal{X}, \mathcal{Y})} \|\Delta\|_F \leq \inf_{\Delta \in \mathbb{S}_\mathbb{R}(X,Y)} \|\Delta\|_F, \tag{3.20}$$

and the left hand side infimum in (3.20) is attained by the unique map $\mathcal{H} = \mathcal{Y}\mathcal{X}^\dagger - (\mathcal{Y}\mathcal{X}^\dagger)^* \mathcal{P}_\mathcal{X}$. Observe from Lemma 2.4 that the matrix $\mathcal{H}$ is real, that is, $\mathcal{H} \in \mathbb{S}_\mathbb{R}(X,Y)$. This implies that

$$\inf_{\Delta \in \mathbb{S}_\mathbb{R}(X,Y)} \|\Delta\|_F^2 = \|\mathcal{H}\|_F^2 = 2\|\mathcal{Y}\mathcal{X}^\dagger\|_F^2 - \text{trace}\left((\mathcal{Y}\mathcal{X}^\dagger)^* \mathcal{X}\mathcal{X}^\dagger(\mathcal{Y}\mathcal{X}^\dagger) \right), \tag{3.21}$$

which completes the proof. $\quad\square$

4 Second characterization of dissipative mappings

In this section, for given $X, Y \in \mathbb{C}^{n,m} \setminus \{0\}$ such that $X^*Y + Y^*X$ is of full rank, we provide a different characterization of dissipative mappings from X to Y. The main advantage of this characterization over Theorem 3.2 is that it is explicitly in terms of the matrix variables $K, Z, G \in \mathbb{C}^{n,n}$ such that $K \succeq 0$ and $G^* = -G$, and the matrices K, Z, G do not have to satisfy other constraints like (3.2)-(3.4). However, extracting the minimal Frobenius norm solutions from this characterization cannot be obtained easily.

Theorem 4.1 *Let $X, Y \in \mathbb{C}^{n,m} \setminus \{0\}$ and suppose that $M := (X^*Y + Y^*X)^{-1}$ exists. Then $\mathbb{S}(X, Y) \neq \emptyset$ if and only if $X^*Y + Y^*X \succ 0$ and $YX^\dagger X = Y$. Moreover,*

$$\mathbb{S}(X, Y) = \{H + \widetilde{H}(K, G, Z) \mid K, G, Z \in \mathbb{C}^{n,n}, K \succeq 0, \ G^* = -G\}, \tag{4.1}$$

where H and $\widetilde{H}$ are respectively defined by

$$H := \frac{1}{2}\left(YX^\dagger - X^{\dagger^*}Y^*\right) + \frac{1}{2}\Big\{YMY^* + YMX^*YX^\dagger + (YX^\dagger)^*XMY^* \\ + (YX^\dagger)^*XMX^*YX^\dagger\Big\} \tag{4.2}$$

and

$$\widetilde{H}(K, G, Z) := \frac{1}{2}\Big(\mathcal{P}_X K \mathcal{P}_X + \mathcal{P}_X G \mathcal{P}_X - \mathcal{P}_X Z^* XX^\dagger + X^{\dagger^*}X^\dagger Z\mathcal{P}_X + YMX^*Z\mathcal{P}_X$$

$$+ (YX^\dagger)^*XMX^*Z\mathcal{P}_X + \mathcal{P}_X Z^*XMY^* + \mathcal{P}_X Z^*XMX^*YX^\dagger + \mathcal{P}_X Z^*XMX^*Z\mathcal{P}_X\Big). \tag{4.3}$$

Proof. If there exists Δ such that $\Delta X = Y$ and $\Delta + \Delta^* \succeq 0$, then

$$X^*Y + Y^*X = X^*(Y + \Delta^*X) = X^*(\Delta + \Delta^*)X.$$

This implies that $X^*Y + Y^*X$ is positive definite as $\Delta + \Delta^* \succeq 0$ and $X^*Y + Y^*X$ is invertible. For the converse, let X, Y be such that $X^*Y + Y^*X \succ 0$. Then the matrix H in (4.2) is well defined and satisfies $HX = Y$. Also $H + H^*$ is a positive semidefinite matrix of the form BB^* for some $B \in \mathbb{C}^{n,n}$. Indeed, if we let $M = M_1 M_1^*$, where M_1 is the Cholesky factor of M, then we have

$$\begin{aligned} H + H^* &= YMY^* + YMX^*YX^\dagger + (YX^\dagger)^*XMY^* + (YX^\dagger)^*XMX^*YX^\dagger \\ &= \left(YM_1 + (YX^\dagger)^*XM_1\right)\left(YM_1 + (YX^\dagger)^*XM_1\right)^* \\ &= BB^*, \end{aligned}$$

where $B = YM_1 + (YX^\dagger)^*XM_1$.

Next we prove " $\subseteq$ " in (4.1). For this, let $\Delta \in \mathbb{S}$, that is, $\Delta X = Y$ and $\Delta + \Delta^* \succeq 0$. By [23, Lemma 1.3], there exists $Z \in \mathbb{C}^{n,n}$ such that

$$\Delta = YX^\dagger + Z\mathcal{P}_X. \tag{4.4}$$

Also, we can write Δ as

$$\Delta = \frac{\Delta + \Delta^*}{2} + \frac{\Delta - \Delta^*}{2}. \tag{4.5}$$

Now since $\Delta + \Delta^* \succeq 0$ we have $\Delta + \Delta^* = A^*A$ for some $A \in \mathbb{C}^{n,n}$. This implies that $(\Delta + \Delta^*)X = A^*AX$ and thus $\Delta X + \Delta^*X = A^*AX$. By setting $W := AX$, we obtain $Y + \Delta^*X = A^*W$ because $\Delta X = Y$. This yields the mappings $AX = W$ and $A^*W = Y + \Delta^*X$. The matrices X, Y and W satisfy

$$X^*(Y + \Delta^*X) = X^*Y + X^*\Delta^*X = X^*(\Delta + \Delta^*)X, = X^*A^*AX = W^*W, \tag{4.6}$$

$YX^\dagger X = Y$ and $WX^\dagger X = W$. Therefore, from [17, Theorem 2.1], A can be written as

$$A = WX^\dagger + \left((Y + \Delta^*X)W^\dagger\right)^* - \left((Y + \Delta^*X)W^\dagger\right)^*XX^\dagger + (I_n - WW^\dagger)R_1\mathcal{P}_X, \tag{4.7}$$

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for some $R_1 \in \mathbb{C}^{n,n}$. This can be further simplified as

$$A = ((Y + \Delta^* X)W^\dagger)^* + (I_n - WW^\dagger)R_1 \mathcal{P}_X, \tag{4.8}$$

using the fact that $((Y + \Delta^* X)W^\dagger))^* X X^\dagger = W X^\dagger$, since $(Y + \Delta^* X)^* X = W^* W$. Thus from (4.8), we have

$$
\begin{aligned}
A^* A &= \Big(((Y + \Delta^* X)W^\dagger)^* + (I_n - WW^\dagger)R_1 \mathcal{P}_X \Big)^* \Big(((Y + \Delta^* X)W^\dagger)^* + (I_n - WW^\dagger)R_1 \mathcal{P}_X \Big) \\
&= (Y + \Delta^* X)M(Y + \Delta^* X)^* + \mathcal{P}_X R_1^* (I_n - WW^\dagger)(I_n - WW^\dagger)R_1 \mathcal{P}_X.
\end{aligned} \tag{4.9}
$$

Define $K := R_1^* (I_n - WW^\dagger)(I_n - WW^\dagger)R_1$, then $K \succeq 0$, and from (4.9) we have

$$A^* A = (Y + \Delta^* X)M(Y + \Delta^* X)^* + \mathcal{P}_X K \mathcal{P}_X. \tag{4.10}$$

Now we consider the skew-Hermitian part $\Delta - \Delta^*$ of Δ. Let $S = \Delta - \Delta^*$. Then $SX = \Delta X - \Delta^* X = Y - \Delta^* X$. Again since

$$X^*(Y - \Delta^* X) = X^* \Delta X - X^* \Delta^* X = X^*(\Delta - \Delta^*)X = -(Y^* - X^* \Delta)X = -(Y - \Delta^* X)^* X,$$

by Theorem 2.3, we have

$$S = (Y - \Delta^* X)X^\dagger - ((Y - \Delta^* X)X^\dagger)^* \mathcal{P}_X + \mathcal{P}_X G \mathcal{P}_X, \tag{4.11}$$

for some skew-Hermitian matrix $G \in \mathbb{C}^{n,n}$. By inserting $\Delta = YX^\dagger + Z\mathcal{P}_X$ from (4.4) and $W^\dagger (W^\dagger)^* = ((Y + \Delta^* X)^* X)^{-1} = (X^* Y + Y^* X)^{-1}$ in (4.10) and (4.11), we respectively obtain

$$
\begin{aligned}
A^* A = &\; YMY^* + YMX^* YX^\dagger + YMX^* Z\mathcal{P}_X + (YX^\dagger)^* XMY^* + (YX^\dagger)^* XMX^* YX^\dagger \\
&+ (YX^\dagger)^* XMX^* Z\mathcal{P}_X + \mathcal{P}_X Z^* XMY^* + \mathcal{P}_X Z^* XMX^* YX^\dagger + \mathcal{P}_X Z^* XMX^* Z\mathcal{P}_X + \mathcal{P}_X K \mathcal{P}_X,
\end{aligned} \tag{4.12}
$$

and

$$S = YX^\dagger - X^{\dagger^*} Y^* XX^\dagger - \mathcal{P}_X Z^* XX^\dagger - X^{\dagger^*} Y^* \mathcal{P}_X + X^{\dagger^*} X^\dagger Z\mathcal{P}_X + \mathcal{P}_X G \mathcal{P}_X. \tag{4.13}$$

By inserting $\Delta + \Delta^* = A^* A$ from (4.12) and $\Delta - \Delta^* = S$ from (4.13) in (4.5), and by separating the terms with and without matrices G, K, Z, the matrix Δ can be written as

$$\Delta = H + \widetilde{H}(K, G, Z).$$

This proves "$\subseteq$" in (4.1).

Now, let us prove "$\supseteq$" in (4.1). Suppose, $\Delta = H + \widetilde{H}(K, G, Z)$, where H is defined in (4.2) and $\widetilde{H}(K, G, Z)$ is defined in (4.3) for some matrices $G, K, Z \in \mathbb{C}^{n,n}$ such that $K \succeq 0$ and $G^* = -G$. Then it is easy to check that $\Delta X = Y$ since H and $\widetilde{H}$ satisfy $HX = Y$ and $\widetilde{H}(K, G, Z)X = 0$. Also

$$\Delta + \Delta^* = \big(H + \widetilde{H}(K, G, Z)\big) + \big(H + \widetilde{H}(K, G, Z)\big)^* \succeq 0. \tag{4.14}$$

Indeed,

$$
\begin{aligned}
&\big(H + \widetilde{H}(K, G, Z)\big) + \big(H + \widetilde{H}(K, G, Z)\big)^* \\
&= (H + H^*) + (\widetilde{H}(K, G, Z) + \widetilde{H}(K, G, Z)^*) \\
&= YMY^* + YMX^* YX^\dagger + (YX^\dagger)^* XMY^* + (YX^\dagger)^* XMX^* YX^\dagger + YMX^* Z\mathcal{P}_X + \\
&\quad (YX^\dagger)^* XMX^* Z\mathcal{P}_X + \mathcal{P}_X Z^* XMY^* + \mathcal{P}_X Z^* XMX^* YX^\dagger + \mathcal{P}_X Z^* XMX^* Z\mathcal{P}_X + \mathcal{P}_X K \mathcal{P}_X \\
&= BB^* + \mathcal{P}_X K \mathcal{P}_X,
\end{aligned} \tag{4.15}
$$

where $B = YM_1 + (YX^\dagger)^*(XM_1) + \mathcal{P}_X Z^* X M_1$, and $M = M_1 M_1^*$ where M_1 is the Cholesky factor of M. The first term in (4.15) is PSD being a matrix of the form BB^* and the second term is PSD because $K \succeq 0$. Thus (4.15) is PSD being the sum of two PSD matrices. Hence $\Delta \in \mathbb{S}(X, Y)$. This shows "$\supseteq$" in (4.1) and hence completes the proof. $\square$

Remark 4.2 The assumption that $X^*Y + Y^*X$ is invertible is not necessary for the existence of a dissipative mapping which only requires $X^*Y + Y^*X$ to be positive semidefinite; see Theorem 3.2. This assumption is more of a technical need for the second characterization of dissipative mappings in Theorem 4.1. However, there are situations, such as computing the eigenpair backward error for DH matrices, where one has to find dissipative mappings taking $X \in \mathbb{C}^{n,r}$ to $Y \in \mathbb{C}^{n,r}$ such that $X^*Y + Y^*X$ is invertible. For example, consider a DH matrix $J - R$ with $Q = I_n$, $\lambda \in \mathbb{C}$ which is not an eigenvalue of $J - R$, and $\hat{X} \in \mathbb{C}^{n,r}$ with rank$(\hat{X}) = r$. Then the structured eigenpair backward error for the DH matrix $J - R$, for making λ an eigenvalue of multiplicity r and the columns of $\hat{X}$ as the corresponding eigenvectors, is the smallest norm of the perturbations $\Delta_J - \Delta_R$ such that $(J - \Delta_J) + (R - \Delta_R)$ is a DH matrix and $((J - \Delta_J) + (R - \Delta_R))\hat{X} = \lambda\hat{X}$. By Lemma 2.7, the latter condition is equivalent to finding a dissipative mapping Δ such that $\Delta\hat{X} = \hat{Y}$, where $\hat{Y} = (\lambda I_n - (J - R))\hat{X}$. Such a dissipative mapping exists if and only if $\hat{X}^*\hat{Y} + \hat{Y}^*\hat{X} = \hat{X}^*(\text{Re}(\lambda)I_n + R)\hat{X} \succeq 0$. Hence if $\text{Re}(\lambda) > 0$, then $\hat{X}^*\hat{Y} + \hat{Y}^*\hat{X} \succ 0$ since $R \succeq 0$.

A result analogous to Corollary 3.3 for the vector case (when $m = 1$), where the conditions on the vectors x and y are simpler than for Theorem 4.1, is stated below.

Corollary 4.3 *Let* $x, y \in \mathbb{C}^n \setminus \{0\}$. *Then* $\mathbb{S}(x, y) \neq \emptyset$ *if and only if* $\text{Re}(x^*y) \geq 0$. *Moreover, if* $\text{Re}(x^*y) > 0$, *then*

$$\mathbb{S}(x, y) = \{H + \widetilde{H}(K, G, Z) \mid K, G, Z \in \mathbb{C}^{n,n}, K \succeq 0, \ G^* = -G\},$$

where H and $\widetilde{H}$ are respectively defined as

$$H := \frac{1}{2}\left(\frac{yx^*}{\|x\|^2} - \frac{xy^*}{\|x\|^2}\right) + \frac{\alpha}{2}\left\{yy^* + (x^*y)\frac{yx^*}{\|x\|^2} + (y^*x)\frac{xy^*}{\|x\|^2} + (|x^*y|^2)\frac{xx^*}{\|x\|^4}\right\}$$

and

$$\widetilde{H}(K, G, Z) := \frac{1}{2}\Big(\mathcal{P}_x K \mathcal{P}_x + \mathcal{P}_x G \mathcal{P}_x + (\alpha(y^*x) + 1)xx^\dagger Z \mathcal{P}_x + (\alpha(x^*y) - 1)\mathcal{P}_x Z^* xx^\dagger$$
$$+ \alpha \mathcal{P}_x Z^* xy^* + \alpha yx^* Z \mathcal{P}_x + \alpha \mathcal{P}_x Z^* xx^* Z \mathcal{P}_x\Big),$$

where $\alpha = \frac{1}{2\,\text{Re}(x^*y)}$.

Remark 4.4 Along the lines of Section 3.1, we can also obtain a second family of real dissipative mappings from Theorem 4.1, where the restrictions on matrix variables K, G, Z are more simplified than Theorem 3.4. Indeed, let $X, Y \in \mathbb{C}^{n,m}$. Then for $\mathcal{X} = [X, \overline{X}]$ and $\mathcal{Y} = [Y, \overline{Y}]$, from Theorem 4.1 whenever there exists a complex dissipative mapping Δ satisfying $\Delta\mathcal{X} = \mathcal{Y}$, then there also exists a real dissipative mapping. In fact, in view of Lemmas 2.4 and 2.5, the matrix H in Theorem 4.1 is real. Moreover, $H + \widetilde{H}(K, G, Z)$ with real $K, G, Z \in \mathbb{R}^{n,n}$ such that $K \succeq 0$ and $G^T = -G$, gives a family of real dissipative mappings from X to Y.

5 Structured distance to asymptotic instability for DH systems

In this section, we study the distance to asymptotic instability for DH systems $\dot{x} = (J - R)Qx$ defined in (1.1). DH systems are always stable, but they are not necessarily asymptotically stable. In [16], the authors have obtained various structured distances to asymptotic instability for complex DH systems while perturbing J, R, or Q (only one matrix at a time). Similarly in [17], authors derived the real distances to asymptotic instability for real DH systems while perturbing only R. The main tool in the computation of stability radii in [16, 17] is minimal norm solutions to Hermitian, skew-Hermitian and semidefinite mapping problems. To compute such radii in the case of perturbing both J and R simultaneously, we will need minimal norm solutions to complex and real dissipative mappings that map one vector to another vector.

Therefore, in this section, we exploit the minimal norm solutions to the dissipative mapping problem from previous sections and derive the structured distance to asymptotic instability for DH systems while perturbing both J and R at a time. By following the terminology in [16], we define the unstructured and the structured distances to asymptotic instability for DH systems as follows:

Definition 5.1 Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. Consider a DH system of the form (1.1) and let $B \in \mathbb{F}^{n,r}$ and $C \in \mathbb{F}^{q,n}$ be given matrices. Then we define

1) the unstructured stability radius $r_\mathbb{F}(J, R; B, C)$ of system (1.1) with respect to general perturbations to J and R under the restrictions (B, C) by

$$r_\mathbb{F}(J, R; B, C) = \inf \left\{ \ \sqrt{\|\Delta_J\|_F^2 + \|\Delta_R\|_F^2} : \Delta_J, \Delta_R \in \mathbb{F}^{r,q}, \right.$$
$$\left. \Lambda\big((J - R)Q + B(\Delta_J - \Delta_R)CQ\big) \cap i\mathbb{R} \neq \emptyset \right\}; \qquad (5.1)$$

2) the structured stability radius $r_\mathbb{F}^{S_d}(J, R; B)$ of system (1.1) with respect to structure preserving skew-Hermitian perturbations to J and negative semidefinite perturbations to R from the set

$$S_d^\mathbb{F}(J, R; B) = \left\{ (\Delta_J, \Delta_R) \in (\mathbb{F}^{r,r})^2 : \ \Delta_J^* = \Delta_J, \Delta_R \preceq 0, R + B\Delta_R B^* \succeq 0 \right\}; \qquad (5.2)$$

    by

$$r_\mathbb{F}^{S_d}(J, R; B) = \inf \left\{ \ \sqrt{\|\Delta_J\|_F^2 + \|\Delta_R\|_F^2} : (\Delta_J, \Delta_R) \in S_d^\mathbb{F}(J, R; B), \right.$$
$$\left. \Lambda\big((J - R)Q + B(\Delta_J - \Delta_R)B^*Q\big) \cap i\mathbb{R} \neq \emptyset \right\}. \qquad (5.3)$$

In the complex case ($\mathbb{F} = \mathbb{C}$), $r_\mathbb{C}(J, R; B, C)$ and $r_\mathbb{C}^{S_d}(J, R; B)$ are respectively called the complex unstructured and complex structured stability radii. Similarly, in the real case ($\mathbb{F} = \mathbb{R}$), $r_\mathbb{R}(J, R; B, C)$ and $r_\mathbb{R}^{S_d}(J, R; B)$ are respectively called the real unstructured and real structured stability radii.

In order to obtain bounds for the structured stability radius $r_\mathbb{F}^{S_d}(J, R; B)$, we also define the structured eigenvalue backward error as follows

$$\eta_\mathbb{F}^{S_d}(J, R; B, \lambda) = \inf \left\{ \ \sqrt{\|\Delta_J\|_F^2 + \|\Delta_R\|_F^2} : (\Delta_J, \Delta_R) \in S_d^\mathbb{F}(J, R; B), \right.$$
$$\left. \det\big((J - R)Q + B(\Delta_J - \Delta_R)B^*Q - \lambda I_n\big) = 0 \right\}, \qquad (5.4)$$

where $\lambda \in \mathbb{C}$, $(J - R)Q$ is a DH matrix, and the perturbation set $S_d^\mathbb{F}(J, R; B)$ is as defined in (5.2). Consequently, we have the following result.

Theorem 5.2 *Consider an asymptotically stable DH system of the form* (1.1). *Then*

$$r_{\mathbb{F}}^{S_d}(J, R; B) = \inf_{w \in \mathbb{R}} \eta_{\mathbb{F}}^{S_d}(J, R; B, iw). \tag{5.5}$$

5.1 Complex stability radius

Consider a complex LTI DH system of the form

$$\dot{x}(t) = (J - R)Qx(t), \tag{5.6}$$

where $J, R, Q \in \mathbb{C}^{n,n}$ such that $J^* = -J$, $R^* = R \succeq 0$, and $Q^* = Q \succ 0$. Here, we study the complex stability radii $r_{\mathbb{C}}(J, R; B, C)$ and $r_{\mathbb{C}}^{S_d}(J, R; B)$.

Inspired by the proof of [10, Proposition 2.1], we obtain the following formula for the unstructured stability radius $r_{\mathbb{C}}(J, R; B, C)$.

Theorem 5.3 *Consider an asymptotically stable DH system of the form* (5.6). *Let $B \in \mathbb{C}^{n,r}$ and $C \in \mathbb{C}^{q,n}$ be given restriction matrices. Then $r_{\mathbb{C}}(J, R; B, C)$ is finite if and only if $G(w) = CQ\left(iwI_n - (J - R)Q\right)^{-1} B$ is not zero for some $w \in \mathbb{R}$. In the latter case, we have*

$$r_{\mathbb{C}}(J, R; B, C) = \frac{1}{\sqrt{2}} \inf_{w \in \mathbb{R}} \frac{1}{\|G(w)\|}.$$

Proof. In view of (5.1), note that for any $\Delta_J, \Delta_R \in \mathbb{C}^{r,q}$ $\Lambda\left((J - R)Q + B(\Delta_J - \Delta_R)CQ\right) \cap i\mathbb{R} \neq \emptyset$ if and only if $\det\left((J - R)Q + B(\Delta_J - \Delta_R)CQ - iwI_n\right) = 0$ for some $w \in \mathbb{R}$ if and only if $\det\left(I_n - (\Delta_J - \Delta_R)CQ(iwI_n - (J - R)Q)^{-1}B\right) = 0$ for some $w \in \mathbb{R}$. Using this in (5.1), we have

$$r_{\mathbb{C}}(J, R; B, C) = \inf\left\{ \sqrt{\|\Delta_J\|_F^2 + \|\Delta_R\|_F^2} : \Delta_J, \Delta_R \in \mathbb{C}^{r,q}, \ w \in \mathbb{R} \right.$$

$$\det\left(I_n - (\Delta_J - \Delta_R)CQ(iwI_n - (J - R)Q)^{-1}B\right) = 0 \Big\}$$

$$= \inf_{w \in \mathbb{R}} \inf\left\{ \sqrt{\|\Delta_J\|_F^2 + \|\Delta_R\|_F^2} : \Delta_J, \Delta_R \in \mathbb{C}^{r,q}, \ x \in \mathbb{C}^r \setminus \{0\} \right.$$

$$(\Delta_J - \Delta_R)CQ(iwI_n - (J - R)Q)^{-1}Bx = x \Big\}. \tag{5.7}$$

In view of Lemma 2.6, (5.7) becomes

$$r_{\mathbb{C}}(J, R; B, C)^2 = \inf_{w \in \mathbb{R}} \inf\left\{ \frac{\|\Delta\|_F^2}{2} : \Delta \in \mathbb{C}^{r,q}, \ x \in \mathbb{C}^r \setminus \{0\}, \Delta CQ(iwI_n - (J-R)Q)^{-1}Bx = x \right\}. \tag{5.8}$$

Note that $r_{\mathbb{C}}(J, R; B, C) \geq 0$. Therefore $r_{\mathbb{C}}(J, R; B, C)$ is finite if and only if the inner infimum in (5.8) is finite for some $w \in \mathbb{R}$, which is true if and only if $G(w) := CQ\left((J - R)Q - iwI_n\right)^{-1} B \neq 0$ for some $w \in \mathbb{R}$. Indeed, if $G(w) \neq 0$ for some $w \in \mathbb{R}$ then for any $x \in \mathbb{C}^r \setminus \{0\}$, there exists $\tilde{\Delta} \in \mathbb{C}^{r,q}$ such that $\tilde{\Delta}G(w)x = x$. This implies, using (5.8), that $r_{\mathbb{C}}(J, R; B, C) \leq \frac{\|\tilde{\Delta}\|_F^2}{2}$. Using the fact that $\|\Delta\|_F \geq \|\Delta\|$ for any Δ in (5.8), we have

$$r_{\mathbb{C}}(J, R; B, C) \geq \inf_{w \in \mathbb{R}} \inf\left\{ \frac{\|\Delta\|^2}{2} : \Delta \in \mathbb{C}^{r,q}, \ x \in \mathbb{C}^r \setminus \{0\}, \Delta CQ((J - R)Q - iwI_n)^{-1}Bx = x \right\} \tag{5.9}$$

$$= \frac{1}{2} \inf_{w \in \mathbb{R}} \frac{1}{\|G(w)\|^2},$$

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where the last equality follows by slightly modifying the proof of [10, Proposition 2.1]. In fact, we have the equality in (5.9) because for any $w \in \mathbb{R}$ the infimum in the right hand side of (5.9) is attained for a rank one matrix Δ for which we have $\|\Delta\| = \frac{1}{\|G(w)\|} = \|\Delta\|_F$ [10]. $\quad \square$

Next, using Theorem 5.2, we derive bounds for $r_\mathbb{C}^{S_d}(J, R; B)$ via bounds for the backward error $\eta_\mathbb{C}^{S_d}(J, R; B, \lambda)$.

Theorem 5.4 *Consider a DH matrix $(J - R)Q$ and let $w \in \mathbb{R}$ be such that $(J - R)Q - iwI_n$ is invertible. Let $B \in \mathbb{C}^{n,r}$ be of full column rank and define $\Omega = \text{null}\left((I_n - BB^\dagger)(iwI_n - (J - R)Q)\right)$. Suppose that $\Omega \neq \emptyset$. Then*

$$\eta_\mathbb{C}^{S_d}(J, R; B, iw)^2 \geq \inf_{x \in \Omega} \left\{ \frac{2\|B^\dagger(iwI_n - (J - R)Q)x\|^2}{\|B^*Qx\|^2} - \frac{|x^*Q(iwI_n - (J - R)Q)x|^2}{\|B^*Qx\|^4} \right\}. \quad (5.10)$$

Moreover, let the infimum in the right hand side of (5.10) be attained at $\hat{x} \in \Omega$ and define $\hat{\Delta}_R = -\hat{x}^*QRQ\hat{x}\frac{(B^*Q\hat{x})(B^*Q\hat{x})^*}{\|B^*Q\hat{x}\|^4}$. If $R + B\hat{\Delta}_RB^* \succeq 0$, then equality holds in (5.10). In this case, we have

$$\sqrt{2}\,\sigma_{\min}\left(B^\dagger(iwI_n - (J - R)Q)UW^{-*}\right) \geq \eta_\mathbb{C}^{S_d}(J, R; B, iw) \geq \sigma_{\min}\left(B^\dagger(iwI_n - (J - R)Q)UW^{-*}\right), \quad (5.11)$$

where the columns of U form an orthonormal basis for Ω and W is the Cholesky factor of U^*QBB^*QU.

Proof. By definition (5.4),

$$
\begin{aligned}
\eta_\mathbb{C}^{S_d}(J, R; B, iw) = {} & \inf \left\{ \sqrt{\|\Delta_J\|_F^2 + \|\Delta_R\|_F^2} : (\Delta_J, \Delta_R) \in S_d^\mathbb{C}(J, R; B), x \in \mathbb{C}^n \setminus \{0\}, \right. \\
& \left. \left((J - R)Q + B(\Delta_J - \Delta_R)B^*Q - iwI_n\right)x = 0 \right\}, \\
= {} & \inf \left\{ \sqrt{\|\Delta_J\|_F^2 + \|\Delta_R\|_F^2} : (\Delta_J, \Delta_R) \in S_d^\mathbb{C}(J, R; B), x \in \mathbb{C}^n \setminus \{0\}, \right. \\
& \left. B(\Delta_J - \Delta_R)B^*Qx = (iwI_n - (J - R)Q)x \right\}, \\
= {} & \inf \left\{ \sqrt{\|\Delta_J\|_F^2 + \|\Delta_R\|_F^2} : (\Delta_J, \Delta_R) \in S_d^\mathbb{C}(J, R; B), x \in \Omega \setminus \{0\}, \right. \\
& \left. (\Delta_J - \Delta_R)B^*Qx = B^\dagger(iwI_n - (J - R)Q)x \right\}, \quad (5.12) \\
= {} & \inf \left\{ \|\Delta\|_F : \Delta \in \mathbb{C}^{r,r}, \Delta_R = -(\Delta + \Delta^*) \preceq 0, R + B\Delta_RB^* \succeq 0, \right. \\
& \left. x \in \Omega \setminus \{0\}, \Delta B^*Qx = B^\dagger(iwI_n - (J - R)Q)x \right\}, \quad (5.13) \\
\geq {} & \inf \left\{ \|\Delta\|_F : \Delta \in \mathbb{C}^{r,r}, \Delta_R = -(\Delta + \Delta^*) \preceq 0, x \in \Omega \setminus \{0\}, \right. \\
& \left. \Delta B^*Qx = B^\dagger(iwI_n - (J - R)Q)x \right\}, \quad (5.14)
\end{aligned}
$$

where we have used Lemma 2.8 in (5.12) and Lemma 2.7 in (5.13). If $\Omega \neq \emptyset$, then infimum on the right hand side of (5.14) is finite. In fact, from Corollary 3.3 for any $x \in \Omega$ there exists $\Delta \in \mathbb{C}^{r,r}$ such that $\Delta + \Delta^* \succeq 0$ and $\Delta B^*Qx = B^\dagger(iwI_n - (J-R)Q)x$ if and only if $\text{Re}\left((B^*Qx)^*B^\dagger(iwI_n - (J - R)Q)x\right) \geq$

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0. Clearly for any $x \in \Omega$, we have

$$
\begin{aligned}
\mathrm{Re}\left((B^*Qx)^*B^\dagger(iwI_n - (J-R)Q)x\right) &= \mathrm{Re}\left(x^*QBB^\dagger(iwI_n - (J-R)Q)x\right) \\
&= \mathrm{Re}\left(x^*Q(iwI_n - (J-R)Q)x\right) \quad (\because x \in \Omega) \\
&= \mathrm{Re}\left(x^*((iwQ - QJQ) + QRQ)x\right) \\
&= \mathrm{Re}\left(x^*QRQx\right) \quad (\because iwQ - QJQ \text{ is skew-Hermitian}) \\
&= x^*QRQx \geq 0 \quad (\because R \succeq 0, \ Q \succ 0).
\end{aligned}
$$

Thus using the minimal norm mapping from Corollary 3.3 in (5.14), we obtain

$$
\eta_{\mathbb{C}}^{S_d}(J, R; B, iw)^2 \geq \inf_{x \in \Omega} \left\{ \frac{2\|B^\dagger(iwI_n - (J-R)Q)x\|^2}{\|B^*Qx\|^2} - \frac{|x^*Q(iwI_n - (J-R)Q)x|^2}{\|B^*Qx\|^4} \right\}. \tag{5.15}
$$

This proves (5.10). Now suppose infimum in the right hand side of (5.15) is attained at $\hat{x} \in \Omega$, then in view of Corollary 3.3 consider

$$
\hat{\Delta} = \frac{B^\dagger(iwI_n - (J-R)Q)\hat{x}\hat{x}^*QB}{\|B^*Q\hat{x}\|^2} + \frac{(B^\dagger(iwI_n - (J-R)Q)\hat{x})^*B^*Q\hat{x}}{\|B^*Q\hat{x}\|^4}(B^*Q\hat{x})(B^*Q\hat{x})^* \\
- \frac{B^*Q\hat{x}(B^\dagger(iwI_n - (J-R)Q)\hat{x})^*}{\|B^*Q\hat{x}\|^2}.
$$

Then $\hat{\Delta}B^*Q\hat{x} = B^\dagger(iwI_n - (J-R)Q)\hat{x}$ and $\hat{\Delta} + \hat{\Delta}^* \succeq 0$. Set $\hat{\Delta}_R = -\frac{\hat{\Delta}+\hat{\Delta}^*}{2}$ and $\hat{\Delta}_J = \frac{\hat{\Delta}-\hat{\Delta}^*}{2}$. Then $\hat{\Delta}_R \preceq 0$. Thus, if $R + B\hat{\Delta}_R B^* \succeq 0$, then clearly $(\hat{\Delta}_J, \hat{\Delta}_R) \in S_d^{\mathbb{C}}(J, R; B)$, and

$$
\|\hat{\Delta}_J\|_F^2 + \|\hat{\Delta}_R\|_F^2 = \|\hat{\Delta}\|_F^2 = \frac{2\|B^\dagger(iwI_n - (J-R)Q)\hat{x}\|^2}{\|B^*Q\hat{x}\|^2} - \frac{|\hat{x}^*Q(iwI_n - (J-R)Q)\hat{x}|^2}{\|B^*Q\hat{x}\|^4}.
$$

This show the equality in (5.15) and hence in (5.14), i.e.,

$$
\eta_{\mathbb{C}}^{S_d}(J, R; B, iw)^2 = \inf_{x \in \Omega} \left\{ \frac{2\|B^\dagger(iwI_n - (J-R)Q)x\|^2}{\|B^*Qx\|^2} - \frac{|x^*Q(iwI_n - (J-R)Q)x|^2}{\|B^*Qx\|^4} \right\}. \tag{5.16}
$$

Now suppose $k = \dim(\Omega)$ and let columns of $U \in \mathbb{C}^{n,k}$ forms an orthonormal basis for Ω. Then $x(\neq 0) \in \Omega$ if and only if $x = U\alpha$ for some $\alpha \in \mathbb{C}^k \setminus \{0\}$. Using this in (5.16), we obtain

$$
\eta_{\mathbb{C}}^{S_d}(J, R; B, iw)^2 = \inf_{\alpha \in \mathbb{C}^k \setminus \{0\}} \left\{ \frac{2\|B^\dagger(iwI_n - (J-R)Q)U\alpha\|^2}{\|B^*QU\alpha\|^2} - \frac{|\alpha^*U^*Q(iwI_n - (J-R)Q)U\alpha|^2}{\|B^*QU\alpha\|^4} \right\}. \tag{5.17}
$$

Note that B^*QU is a full rank matrix because if suppose $B^*QU\alpha = 0$ for some $\alpha \in \mathbb{C}^k \setminus \{0\}$. This implies that

$$
0 = B\hat{\Delta}B^*QU\alpha = (iwI_n - (J-R)Q)U\alpha,
$$

so that $iw \in \Lambda((J-R)Q)$ which is a contradiction. Thus the matrix U^*QBB^*QU is positive definite. Let $U^*QBB^*QU = WW^*$, where W is the unique Cholesky factor of U^*QBB^*QU, then by using $y = W^*\alpha$ in (5.17), we have

$$
\eta_{\mathbb{C}}^{S_d}(J, R; B, iw)^2 = \\
\inf_{y \in \mathbb{C}^k \setminus \{0\}} \left\{ \frac{2\|B^\dagger(iwI_n - (J-R)Q)UW^{-*}y\|^2}{\|y\|^2} - \frac{|y^*W^{-1}U^*Q(iwI_n - (J-R)Q)UW^{-*}y|^2}{\|y\|^4} \right\}. \tag{5.18}
$$

In view of the Cauchy-Schwarz inequality, note that for every $y \in \mathbb{C}^k \setminus \{0\}$ we have

$$
\begin{aligned}
\frac{2\|B^\dagger(iwI_n - (J-R)Q)UW^{-*}y\|^2}{\|y\|^2} \quad &\geq \quad \frac{2\|B^\dagger(iwI_n - (J-R)Q)UW^{-*}y\|^2}{\|y\|^2} \\
&\quad - \frac{|y^*W^{-1}U^*Q(iwI_n - (J-R)Q)UW^{-*}y|^2}{\|y\|^4} \\
&\geq \quad \frac{\|B^\dagger(iwI_n - (J-R)Q)UW^{-*}y\|^2}{\|y\|^2}.
\end{aligned}
$$

Using this in (5.18) yields

$$
\begin{aligned}
2 \inf_{y \in \mathbb{C}^k \setminus \{0\}} \frac{\|B^\dagger(iwI_n - (J-R)Q)UW^{-*}y\|^2}{\|y\|^2} \quad &\geq \quad \eta_{\mathbb{C}}^{S_d}(J,R;B,iw)^2 \\
&\geq \quad \inf_{y \in \mathbb{C}^k \setminus \{0\}} \frac{\|B^\dagger(iwI_n - (J-R)Q)UW^{-*}y\|^2}{\|y\|^2},
\end{aligned}
$$

and hence

$$
\sqrt{2}\,\sigma_{\min}\left(B^\dagger(iwI_n - (J-R)Q)UW^{-*}\right) \geq \eta_{\mathbb{C}}^{S_d}(J,R;B,iw) \geq \sigma_{\min}\left(B^\dagger(iwI_n - (J-R)Q)UW^{-*}\right).
$$

This completes the proof. ☐

By using Theorem 5.4 in Theorem 5.2, we obtain a lower bound for the structured stability radius $r_{\mathbb{C}}^{S_d}(J,R;B)$ as follows.

Theorem 5.5 *Consider an asymptotically stable DH system of the form* (5.6). *Let $B \in \mathbb{C}^{n,r}$ be of full column rank. For $w \in \mathbb{R}$, define $\Omega_w := \mathrm{null}\left((I_n - BB^\dagger)(iwI_n - (J-R)Q)\right)$. Suppose that $\Omega_w \neq \emptyset$ for some $w \in \mathbb{R}$. Then*

$$
r_{\mathbb{C}}^{S_d}(J,R;B)^2 \geq \inf_{w \in \mathbb{R}} \inf_{x \in \Omega_w} \left\{ \frac{2\|B^\dagger(iwI_n - (J-R)Q)x\|^2}{\|B^*Qx\|^2} - \frac{|x^*Q(iwI_n - (J-R)Q)x|^2}{\|B^*Qx\|^4} \right\}.
$$

Remark 5.6 We conclude the section with a few remarks about Theorem 5.4.

- Since $\hat{\Delta}_R$ in Theorem 5.4 is a rank-one matrix with only one negative eigenvalue, if $R \succ 0$ and $R + B\hat{\Delta}_R B^*$ is singular, then from [16, Lemma 4.4] $R + B\hat{\Delta}_R B^*$ is positive semidefinite, and this implies the equality in (5.10).

- As a particular case of Theorem 5.4, i.e., when $w = 0$, we obtain the distance to singularity $d^{S_d}(J,R;B) := \eta^{S_d}(J,R;B,0)$ with respect to structure-preserving perturbations to both J and R from the set $S_d(J,R;B)$ which is an analogous result to [16, Theorem 6.2]. Note that in [16], the structured distances to singularity for DH matrices were obtained for structure-preserving perturbations to Q only.

5.2 Real stability radius

Consider a real LTI DH system of the form

$$\dot{x}(t) = (J - R)Qx(t), \tag{5.19}$$

where $J, R, Q \in \mathbb{R}^{n,n}$ such that $J^T = -J$, $R^T = R \succeq 0$, and $Q^T = Q \succ 0$. In this section, we discuss the real unstructured distance $r_{\mathbb{R}}(J, R; B, C)$ defined by (5.1) and the real structured distance $r_{\mathbb{R}}^{S_d}(J, R; B)$ defined by (5.3) for real DH systems. For this, define, for a given $M \in \mathbb{C}^{p,m}$,

$$\mu_{\mathbb{R},q}(M) := \left(\inf \left\{ \|\Delta\|_q : \ \Delta \in \mathbb{R}^{m,p}, \ \det(I_m - \Delta M) = 0 \right\} \right)^{-1}, \tag{5.20}$$

where $q \in \{2, F\}$. Let us first state a result from [19] that will be useful in determining the unstructured distance $r_{\mathbb{R}}(J, R; B, C)$.

Theorem 5.7 ([19]) *We have*

$$\mu_{\mathbb{R},2}(M) = \inf_{\gamma \in (0,1]} \sigma_2 \left(\begin{bmatrix} \operatorname{Re} M & -\gamma \operatorname{Im} M \\ \gamma^{-1} \operatorname{Im} M & \operatorname{Re} M \end{bmatrix} \right),$$

where $\sigma_2(A)$ is the second largest singular value of a matrix A. Furthermore, an optimal Δ that attains the value of $\mu_{\mathbb{R},2}(M)$ can be chosen of rank at most two.

Let us now prove a result bounding $\mu_{\mathbb{R},F}(M)$ using $\mu_{\mathbb{R},2}(M)$.

Theorem 5.8 *We have*

$$\frac{1}{\sqrt{2}}\mu_{\mathbb{R},2}(M) \leq \mu_{\mathbb{R},F}(M) \leq \mu_{\mathbb{R},2}(M). \tag{5.21}$$

Proof. By definition,

$$
\begin{aligned}
\mu_{\mathbb{R},2}(M)^{-1} &= \ \inf \{\|\Delta\|_2 : \ \Delta \in \mathbb{R}^{m,p}, \ \det(I_m - \Delta M) = 0\} \\
&\leq \ \inf \{\|\Delta\|_F : \ \Delta \in \mathbb{R}^{m,p}, \ \det(I_m - \Delta M) = 0\} \\
&= \ \mu_{\mathbb{R},F}(M)^{-1},
\end{aligned} \tag{5.22}
$$

since for any $\Delta \in \mathbb{R}^{m,p}$, we have $\|\Delta\|_2 \leq \|\Delta\|_F$. Now suppose $\hat{\Delta} \in \mathbb{R}^{m,p}$ such that $\mu_{\mathbb{R},2}(M)^{-1} = \|\hat{\Delta}\|_2$. Then from the first part $\hat{\Delta}$ is of rank at most two. This implies that $\det(I_m - \hat{\Delta} M) = 0$ and $\|\hat{\Delta}\|_F \leq \sqrt{2}\|\hat{\Delta}\|_2$, which yields that

$$\mu_{\mathbb{R},F}(M)^{-1} \leq \sqrt{2}\|\hat{\Delta}\|_2 = \mu_{\mathbb{R},2}(M). \tag{5.23}$$

Hence from (5.22) and (5.23), we obtain (5.21). $\quad\square$

As an application of the above theorem we bound the real unstructured distance $r_{\mathbb{R}}(J, R; B, C)$.

Theorem 5.9 *Consider a real asymptotically stable DH system of the form* (5.19) *and let $B \in \mathbb{R}^{n,r}$ and $C \in \mathbb{R}^{q,n}$ be restriction matrices. Then*

$$\frac{1}{\sqrt{2}} \inf_{w \in \mathbb{R}} \mu_{\mathbb{R},2}(G(w))^{-1} \leq r_{\mathbb{R}}(J, R; B, C) \leq \inf_{w \in \mathbb{R}} \mu_{\mathbb{R},2}(G(w))^{-1},$$

where $\mu_{\mathbb{R},2}(G(w))^{-1} = \inf_{\gamma \in (0,1]} \sigma_2 \left(\begin{bmatrix} \operatorname{Re} G(w) & -\gamma \operatorname{Im} G(w) \\ \gamma^{-1} \operatorname{Im} G(w) & \operatorname{Re} G(w) \end{bmatrix} \right)$ and $G(w) = CQ\,(iwI_n - (J - R)Q)^{-1} B$.

Proof. By Definition (5.1), we have

$$r_\mathbb{R}(J, R; B, C)$$

$$= \inf\left\{\sqrt{\|\Delta_J\|_F^2 + \|\Delta_R\|_F^2} : \Delta_J, \Delta_R \in \mathbb{R}^{r,q}, \Lambda\big((J - R)Q + B(\Delta_J - \Delta_R)CQ\big) \cap i\mathbb{R} \neq \emptyset\right\}$$

$$= \inf\left\{\sqrt{\|\Delta_J\|_F^2 + \|\Delta_R\|_F^2} : \Delta_J, \Delta_R \in \mathbb{R}^{r,q}, \ w \in \mathbb{R} \right.$$
$$\left. \det\big((J - R)Q + B(\Delta_J - \Delta_R)CQ - iwI_n\big) = 0\right\}$$

$$= \inf_{w \in \mathbb{R}} \inf\left\{\sqrt{\|\Delta_J\|_F^2 + \|\Delta_R\|_F^2} : \Delta_J, \Delta_R \in \mathbb{R}^{r,q}, \right.$$
$$\left. \det\big(I_n - (\Delta_J - \Delta_R)CQ(iwI_n - (J - R)Q)^{-1}B\big) = 0\right\}$$

$$= \inf_{w \in \mathbb{R}} \inf\left\{\frac{\|\Delta\|_F}{\sqrt{2}} : \Delta \in \mathbb{R}^{r,q}, \det\big(I_n - \Delta CQ(iwI_n - (J - R)Q)^{-1}B\big) = 0\right\} \tag{5.24}$$

$$= \frac{1}{\sqrt{2}} \inf_{w \in \mathbb{R}} \big(\mu_{\mathbb{R},F} G(w)\big)^{-1}, \tag{5.25}$$

where $G(w) = CQ\,(iwI_n - (J - R)Q)^{-1}\,B$. The equality (5.24) follows from Lemma 2.6, and (5.25) follows by definition of $\mu_{\mathbb{R},F} G(w)$ in (5.20). Thus the result follows immediately using the inequality (5.23) in Theorem 5.8. $\quad\square$

We note that a result similar to Theorem 5.4 for the real backward error, $\eta_\mathbb{R}^{S_d}(J, R; B)$, defined in (5.4) may be obtained by using minimal-norm real dissipative mappings from Theorem 3.4. Thus in view of Theorem 5.2 we can obtain a lower bound for the real structured stability radius $r_\mathbb{R}^{S_d}(J, R; B)$. In the following we state this result for $\eta_\mathbb{R}^{S_d}(J, R; B)$ and skip its proof as it is similar to the proof of Theorem 5.4.

Theorem 5.10 *Consider a real DH system of the form (5.19). Let $w \in \mathbb{R}$ be such that $iwI_n - (J - R)Q$ is invertible. Let $B \in \mathbb{R}^{n,r}$ be of full column rank and let Ω be the subset of $\mathrm{null}\big((I_n - BB^\dagger)(iwI_n - (J - R)Q)\big)$ such that if $x \in \Omega$ then $x^* QRQx \geq |x^T(iwQ + QRQ)x|$. Suppose that $\Omega \neq \emptyset$. Then*

$$\eta_\mathbb{R}^{S_d}(J, R; B, iw)^2 \geq \inf_{x \in \Omega}\left(2\big\|\mathcal{Y}\mathcal{X}^\dagger\big\|_F^2 - trace\big((\mathcal{Y}\mathcal{X}^\dagger)^*(\mathcal{X}\mathcal{X}^\dagger)(\mathcal{Y}\mathcal{X}^\dagger)\big)\right), \tag{5.26}$$

where for $x \in \Omega$, $\mathcal{Y} = \left[\ B^\dagger(iwI_n - (J - R)Q)x \quad \overline{B^\dagger(iwI_n - (J - R)Q)x}\ \right]$ and $\mathcal{X} = \left[\ B^TQx \quad \overline{B^TQx}\ \right]$. Moreover, let the infimum in the right hand side of (5.26) be attained at $\tilde{x} \in \Omega$ and define $\tilde{\Delta}_R := \tilde{\mathcal{Y}}\tilde{\mathcal{X}}^\dagger - (\tilde{\mathcal{Y}}\tilde{\mathcal{X}}^\dagger)^*\mathcal{P}_{\tilde{\mathcal{X}}}$, where $\tilde{\mathcal{Y}} = \left[\ B^\dagger(iwI_n - (J - R)Q)\tilde{x} \quad \overline{B^\dagger(iwI_n - (J - R)Q)\tilde{x}}\ \right]$ and $\tilde{\mathcal{X}} = \left[\ B^TQ\tilde{x} \quad \overline{B^TQ\tilde{x}}\ \right]$. If $R + B\tilde{\Delta}_R B^T \succeq 0$, then equality holds in (5.26).

Proof. In view of Theorem 3.4, the proof is similar to Theorem 5.4. $\quad\square$

We close the section by noting that a result analogous to Theorem 5.5 can be obtain for real structured stability radius $r_\mathbb{R}^{S_d}(J, R; B)$ by using Theorem 5.10 in Theorem 5.2.

6 Numerical experiments

In this section, we illustrate the significance of our distances obtained in Theorems 5.3 and 5.5 and compare them with those of [16], where various structured stability radii have been obtained for DH

systems while perturbing only J or only R one at a time. In the following, l.b. stands for "lower bound", $r_{\mathbb{C}}(R; B, C)$ (resp. $r_{\mathbb{C}}(J; B, C)$) denotes the unstructured stability radius while perturbing only R (resp. J) with restriction matrices B and C [16, Theorem 3.3], $r_{\mathbb{C}}^{\mathcal{S}}(J; B)$ denotes the structured stability radius with respect to structure-preserving perturbations to J [16, Theorem 5.1], and $r_{\mathbb{C}}^{\mathcal{S}_d}(R; B)$ denotes the structured stability radius with respect to structure-preserving negative semidefinite perturbations to R [16, Theorem 4.2]. We note that $r_{\mathbb{C}}(R; B, C) = r_{\mathbb{C}}(J; B, C)$ [16, Theorem 3.3]. It is also clear that $r_{\mathbb{C}}(J, R; B, C) \leq r_{\mathbb{C}}(R; B, C)$ and the minimum of $r_{\mathbb{C}}^{\mathcal{S}}(J; B)$ and $r_{\mathbb{C}}^{\mathcal{S}_d}(R; B)$ gives an upper bound to $r_{\mathbb{C}}^{\mathcal{S}_d}(J, R; B)$.

All the experiments are performed in Matlab Version No. 9.1.0 (R2016b). To compute these values (or lower bounds), we proceed as follows.

- The formulas for the three radii $r_{\mathbb{C}}(R; B, B^*)$, $r_{\mathbb{C}}^{\mathcal{S}}(J; B)$, and $r_{\mathbb{C}}^{\mathcal{S}_d}(R; B)$ were obtained in [16] in terms of some non-convex optimization problems. In [16], the authors used the function *fminsearch* in Matlab to solve these optimization problems which only gives a local minimum starting from an initial guess. We instead used the *GlobalSearch* solver in Matlab, with its default parameters which attempts to locate the global solution. However, we can only guarantee that the computed solutions are lower bounds to the exact distances.

- $r_{\mathbb{C}}(J, R; B, B^*)$: a formula for $r_{\mathbb{C}}(J, R; B, B^*)$ is obtained in Theorem 5.3 which is $\frac{1}{\sqrt{2}}$ times the unstructured distance $r_{\mathbb{C}}(R; B, B^*)$.

- $r_{\mathbb{C}}^{\mathcal{S}_d}(J, R; B)$: a lower bound for $r_{\mathbb{C}}^{\mathcal{S}_d}(J, R; B)$ is obtained in Theorem 5.5. We again used the *GlobalSearch* solver in Matlab for this to get a good approximation for the lower bound. We note that solving the optimization problem involved in (5.10) is challenging, and beyond the scope of this this paper. This paper aims to solve the dissipative mapping problem and show it is useful in engineering applications such as DH systems. A possible future work would be to develop more sophisticated ways to solve (5.10).

Example 6.1 [16, Example 7.1] Consider a prototype example of disk brake squeal problem with the matrices G, M, K, D from [16]. We consider the DH system $\dot{x} = (J - R)Qx$ with

$$J = \begin{bmatrix} G & K + \frac{1}{2}N \\ -(K + \frac{1}{2}N^*) & 0 \end{bmatrix}, \quad R = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}, \quad \text{and} \quad Q = \begin{bmatrix} M & 0 \\ 0 & K \end{bmatrix}^{-1}.$$

We compute the various distances for the restriction matrices $B = I_4$ and $C = I_4$, which are given in the following table:

$r_{\mathbb{C}}(J, R; B, B^*)$	$r_{\mathbb{C}}(R; B, B^*)$	l.b. to $r_{\mathbb{C}}^{\mathcal{S}_d}(J, R; B)$	$r_{\mathbb{C}}^{\mathcal{S}}(J; B)$	$r_{\mathbb{C}}^{\mathcal{S}_d}(R; B)$
0.0218	0.0308	0.0310	2.4725	5.6149

The lower bound $0.0310 \leq r_{\mathbb{C}}^{\mathcal{S}_d}(J, R; B)$ is obtained by Theorem 5.5, which as expected shows that the unstructured radius, $r_{\mathbb{C}}(J, R; B, B^*) = 0.0218$, is smaller than the structured radius $r_{\mathbb{C}}^{\mathcal{S}_d}(J, R; B)$. If we replace the J matrix by $J_1 = \begin{bmatrix} G & K \\ -K & 0 \end{bmatrix}$, then the corresponding results are as follows:

$r_{\mathbb{C}}(J_1, R; B, B^*)$	$r_{\mathbb{C}}(R; B, B^*)$	l.b. to $r_{\mathbb{C}}^{\mathcal{S}_d}(J_1, R; B)$	$r_{\mathbb{C}}^{\mathcal{S}}(J_1; B)$	$r_{\mathbb{C}}^{\mathcal{S}_d}(R; B)$
0.0826	0.1169	0.1185	2.2340	5.7971

In this case, the difference between the unstructured radius $r_{\mathbb{C}}(J_1, R; B, B^*)$ and the structured radius $r_{\mathbb{C}}^{\mathcal{S}_d}(J, R; B)$ is more significant. This implies that if the magnitude of the structured perturbations to both J and R in Frobenius norm is smaller than 0.1185 than the system remains stable.

Example 6.2 To emphasize more on the stability radii $r_{\mathbb{C}}(J, R; B, B^*)$ and $r_{\mathbb{C}}^{\mathcal{S}_d}(J, R; B)$ obtained in this paper and to show that these general distances are indeed different than the ones in [16], we generate matrices J, R, Q, $B \in \mathbb{C}^{n,n}$ of different sizes ($n \leq 9$) randomly following a normal distribution with mean 0 and standard deviation 1 (randn(m, n) in Matlab) which we project on the feasible set, that is, $J = -J^*$ and $R, Q \succeq 0$, so that $\dot{x} = (J - R)Qx$ is a DH system, and all restricted stability radii were finite. The restriction matrices B and $C = B^*$ are chosen to be of full rank. We compute our distances $r_{\mathbb{C}}(J, R; B, B^*)$ and $r_{\mathbb{C}}^{\mathcal{S}_d}(J, R; B)$ and compare these results with those of [16] in Table 6.1. We see that (i) the various unstructured and structured distances are quite a different; (ii) as expected, the stability radii $r_{\mathbb{C}}(J, R; B, B^*)$ and $r^{\mathcal{S}_d}(J, R; B)$ are significantly smaller than their counterparts from [16] of perturbing only one of J and R; (iii) the results indicate that the lower bound to $r_{\mathbb{C}}^{\mathcal{S}_d}(J, R; B)$ in some cases ($n = 3, 8, 9$) is significantly larger than the unstructured distances (columns 2-3 in Table 6.1) and reasonably close to the upper bound (minimum of the last two columns in Table 6.1).

Table 6.1: Various disances to instability for DH matrices

size n	$r_{\mathbb{C}}(J, R; B, B^*)$	$r_{\mathbb{C}}(R; B)$	l.b. to $r_{\mathbb{C}}^{\mathcal{S}_d}(J, R; B)$	$r_{\mathbb{C}}^{\mathcal{S}}(J; B)$	$r_{\mathbb{C}}^{\mathcal{S}_d}(R; B)$
3	0.1162	0.1644	0.7353	7.8741	1.5038
4	0.0047	0.0067	0.5157	4.4828	620.6015
5	0.0550	0.0778	0.5108	1.2273	2.5896
6	0.0566	0.0801	0.2764	2.1462	1.1335
7	0.0165	0.0234	0.2421	0.8284	4.7189
8	0.0415	0.0587	0.9596	4.3467	1.9909
9	0.0632	0.0894	1.3540	1.7549	3.6928

7 Conclusion

In this paper, we have derived necessary and sufficient conditions for the existence of the dissipative mappings taking $X \in \mathbb{C}^{n,k}$ to $Y \in \mathbb{C}^{n,k}$, charactered the set of dissipative mappings, and found the minimal Frobenius norm solution. We have then applied these results to DH systems. In particular, we have used dissipative mappings to derive bounds for the structured distance to asymptotic instability for both complex and real DH systems (1.1) when both J and R are subject to perturbations.

The bounds computed in this paper involve solving a difficult non-convex optimization problem. Possible future works include more sophisticated methods to compute (5.10), and to study other applications of the dissipative mappings.

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