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STRONGLY EVEN-CYCLE DECOMPOSABLE GRAPHS

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Abstract. A graph is *strongly even-cycle decomposable* if the edge set of every subdivision with an even number of edges can be partitioned into cycles of even length. We prove that several fundamental composition operations that preserve the property of being Eulerian also yield strongly even-cycle decomposable graphs. As an easy application of our theorems, we give an exact characterization of the set of strongly even-cycle decomposable cographs.

1. Introduction

A graph G is *even-cycle decomposable* if the edge set of G can be partitioned into cycles of even length. Even-cycle decomposable graphs have been the subject of substantial attention. For a summary of relevant results we refer the reader to the surveys of Jackson [4] or Fleischner [2]. In addition, Huynh, Oum and Verdian-Rizi [3] recently investigated even-cycle decomposable graphs with respect to odd minors.

In this paper we instead focus on a stronger decomposition property. Namely, we define a graph G to be *strongly even-cycle decomposable* if every subdivision of G with an even number of edges is even-cycle decomposable. Note that a strongly even-cycle decomposable graph G with $|V(G)| \geq 3$ and no isolated vertices is necessarily Eulerian, loopless and 2-connected. For us, an *Eulerian graph* will always mean a (not necessarily connected) graph in which all vertex degrees are even. An *anti-Eulerian graph* is a graph in which every vertex has odd degree.

One motivation for introducing strongly even-cycle decomposable graphs is that inductive arguments tend to work more smoothly for strongly even-cycle decomposable graphs as opposed to even-cycle decomposable graphs. For this reason, sometimes the easiest way to prove that a graph is even-cycle decomposable is to prove that it is strongly even-cycle decomposable. In addition, there are indeed interesting classes of graphs that are strongly even-cycle decomposable. For example, Seymour [6] proved that planar graphs that satisfy the obvious necessary conditions are strongly even-cycle decomposable.

Theorem 1.1 (Seymour [6])**.** *Every loopless 2-connected Eulerian planar graph is strongly even-cycle decomposable.*

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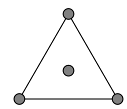

FIGURE 1. A co-claw.

Theorem 1.1 was generalized by Zhang [7] to graphs having no K_5-minor.

Theorem 1.2 (Zhang [7]). *Every loopless 2-connected Eulerian K_5-minor-free graph is strongly even-cycle decomposable.*

We note that K_5 is an example where the obvious necessary conditions are not sufficient. To see this, observe that K_5 has 10 edges but every even-length cycle of K_5 is a 4-cycle. Thus, K_5 is not even-cycle decomposable (and hence not strongly even-cycle decomposable). Markström [5] recently gave a construction for an infinite class of 4-regular 2-connected graphs that are not even-cycle decomposable; the construction is based on a gadget that places K_5 in an edge.

In this paper we prove that several fundamental composition operations that preserve the property of being Eulerian also yield strongly even-cycle decomposable graphs. Our main composition operations (from which the others can be derived) are the following. Definitions are deferred until later.

Theorem 5.1. *Let G be a simple strongly even-cycle decomposable graph, let v be a non-isolated vertex of G, and let H be a simple Eulerian graph with an odd number of vertices. Then the substitution of v by H in G is strongly even-cycle decomposable, provided that H is not K_3 or $\deg_G(v) \geq 4$.*

Theorem 6.1. *Let G be a simple strongly even-cycle decomposable graph and let u and v be non-isolated, non-adjacent twin vertices of G. If H is a simple Eulerian graph with an even number of vertices, then the twin substitution of $\{u, v\}$ by H in G is also strongly even-cycle decomposable, provided that H is not a co-claw or $\deg_G(v) \geq 4$.*

Theorem 6.2. *Let G be a simple strongly even-cycle decomposable graph and let u and v be adjacent twin vertices of G having degree at least 4. If H is a simple anti-Eulerian graph, then the twin substitution of $\{u, v\}$ by H in G is also strongly even-cycle decomposable.*

Theorem 7.3. *Let G be a simple Eulerian graph that is a join of two graphs each having at least two vertices. Then G is strongly even-cycle decomposable if and only if G is neither K_5 nor K_5 with an edge subdivided.*

See Figure 1 for a co-claw and Figure 2 for K_5 with an edge subdivided. These theorems suggest that K_5 is essentially the unique obstruction to the strongly even-cycle decomposable property. In particular, we obtain the following corollary as an easy application of our composition theorems. A *cograph* is a simple graph with no induced path of length 3. Here is an exact characterization of the set of strongly even-cycle decomposable cographs.

Corollary 8.1. *Let G be a cograph with no isolated vertices. Then G is strongly even-cycle decomposable if and only if G is 2-connected Eulerian and G is neither K_5 nor K_5 with an edge subdivided.*

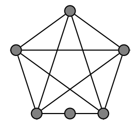

FIGURE 2. K_5 with an edge subdivided.

The rest of the paper is organized as follows. In Section 2, we define *signed graphs* and recast our problem in their language. In Section 3 we prove that Eulerian complete bipartite graphs are strongly even-cycle decomposable. In Section 4, we present several classes of strongly even-cycle decomposable graphs to be used in other sections as base cases. In Sections 5, 6, and 7 we introduce our composition operations and prove that they yield strongly even-cycle decomposable graphs. Finally, in Section 8 we give a quick derivation of Corollary 8.1.

2. Signed graphs

A *signed graph* is a pair (G, Σ) consisting of a graph G together with a *signature* $\Sigma \subseteq E(G)$. The edges in Σ are *odd* and the other edges are *even*. We extend this terminology and define a cycle (or path) to be *even* or *odd* according as it contains an even or odd number of odd edges. To avoid confusion, we will always use the term *even-length cycle* if we need to refer to a cycle with an even number of edges. For a graph G, a subset of $E(G)$ is called a *signing* of $E(G)$; a signing Σ of $E(G)$ induces a signed graph (G, Σ).

A signed graph (G, Σ) is *even-cycle decomposable*, if $E(G)$ can be partitioned into even cycles. The relationship between even-cycle decompositions of signed graphs and graphs is as follows. Let G be a graph and let H be a subdivision of G with $|E(H)|$ even. Consider the signed graph $\mathcal{G}_H := (G, \Sigma)$, where $e \in \Sigma$ if and only if the subdivided path in H corresponding to e has an odd number of edges. Observe that H is even-cycle decomposable if and only if $\mathcal{G}_H$ is even-cycle decomposable. Thus, we have the following equivalent definition of strongly even-cycle decomposable graphs.

Definition 2.1. A graph G is strongly even-cycle decomposable if and only if for each signing Σ of $E(G)$ with $|\Sigma|$ even, the signed graph (G, Σ) is even-cycle decomposable.

For $X \subseteq V(G)$, we let $\delta_G(X)$ be the set of non-loop edges with exactly one end in X. We say that $\delta_G(X)$ is the *cut induced by* X. Two signatures $\Sigma_1, \Sigma_2 \subseteq E(G)$ are *equivalent* if their symmetric difference is a cut. Note that signature equivalence is an equivalence relation. The operation of changing to an equivalent signature is called *re-signing*. The main reason for working with signed graphs is that if $\Sigma_1, \Sigma_2 \subseteq E(G)$ are equivalent signatures, then (G, Σ_1) and (G, Σ_2) have exactly the same set of even cycles. Thus, for equivalent signatures Σ_1 and Σ_2, (G, Σ_1) is even-cycle decomposable if and only if (G, Σ_2) is even-cycle decomposable. The *parity* of a vertex v in a signed graph (G, Σ) is the parity of the number of odd non-loop edges incident with v.

We will frequently use the following well-known lemma without explicit reference.

Lemma 2.2. *Let (G, Σ) be a signed graph. For any $F \subseteq E(G)$ which does not contain a cycle, there is a signature Σ_F which is equivalent to Σ such that $\Sigma_F \cap F = \emptyset$.*

Observe that Lemma 2.2 implies that (G, Σ_1) and (G, Σ_2) have the same set of even cycles if and only if Σ_1 and Σ_2 are equivalent.

An *almost even-cycle decomposition* of a signed graph (G, Σ) is a partition of $E(G)$ into cycles, at most one of which is odd.

Lemma 2.3. *Let G be a strongly even-cycle decomposable graph and let Σ be a signing of $E(G)$. For each edge e of G, there exists an almost even-cycle decomposition $\{C\} \cup \mathcal{C}$ of (G, Σ) such that $e \in E(C)$ and all cycles in $\mathcal{C}$ are even.*

Proof. We may assume that $|\Sigma|$ is odd. Let $\Sigma' = \Sigma \cup \{e\}$ if $e \notin \Sigma$ and $\Sigma' = \Sigma \setminus \{e\}$ otherwise. Since G is strongly even-cycle decomposable, (G, Σ') has an even-cycle decomposition. Such an even-cycle decomposition corresponds to an almost even-cycle decomposition of (G, Σ) whose odd cycle contains e. □

3. Complete bipartite graphs

In this section, we prove that Eulerian complete bipartite graphs are strongly even-cycle decomposable. In fact, we will need to prove something slightly stronger. Namely, after removing the edges of a 4-cycle from an Eulerian complete bipartite graph, the resulting graph is strongly even-cycle decomposable.

Let us write $K_{n,m} - C_4$ to denote a subgraph of $K_{n,m}$ obtained by deleting the edges of a fixed 4-cycle of $K_{n,m}$. We proceed via a sequence of lemmas.

Lemma 3.1. *$K_{2,n}$ is strongly even-cycle decomposable for all positive even integers n.*

Proof. Consider a signing Σ of $E(K_{2,n})$ with $|\Sigma|$ even. Let $\{u, v\}$ and $\{x_1, \ldots, x_n\}$ be the bipartition of $K_{2,n}$. By Lemma 2.2, we may assume that all edges incident with u are even. By re-indexing vertices if necessary, we may assume that there exists an even index i such that the edges $vx_1, \ldots, vx_i$ are all odd and all the other edges incident with v are even. But now,

$$\{ux_1vx_2, ux_3vx_4, ux_5vx_6, \ldots, ux_{n-1}vx_n\}$$

is an even-cycle decomposition of $(K_{2,n}, \Sigma)$. □

Lemma 3.2. *$K_{4,4} - C_4$ is strongly even-cycle decomposable.*

Proof. Let $G := K_{4,4} - C_4$ with bipartition $\{a, b, c, d\}$ and $\{w, x, y, z\}$ such that $C_1 := aybz$, $C_2 := cydz$, and $C_3 := wcxd$ are edge-disjoint 4-cycles and the deleted C_4 is $awbx$. Let Σ be a signing of $E(G)$ with $|\Sigma|$ even. If either C_1 or C_3 is even then (G, Σ) is even-cycle decomposable by Lemma 3.1. So we may assume C_1 and C_3 are both odd and therefore C_2 is even.

Note that $E(C_1)$, $E(C_2)$, and $E(C_3)$ are disjoint edge cuts. We may therefore assume by re-signing that each of C_1 and C_3 contains exactly one odd edge and that C_2 has either zero or two odd edges. Assume without loss of generality that the edges ay and cw are odd. There are three cases to consider: C_2 has no odd edges, C_2 has two odd edges forming a matching (which we can choose, by re-signing), or C_2 has two odd edges not forming a matching (which can be chosen to meet at y or c by re-signing). The even-cycle decompositions for each of these cases are shown in Figure 3. □

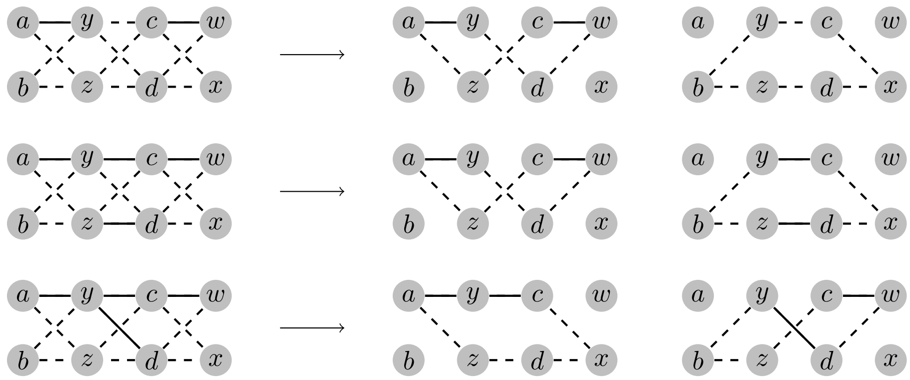

FIGURE 3. Three possible signatures for $K_{4,4} - C_4$ in the proof of Lemma 3.2, and corresponding even-cycle decompositions. Odd edges are solid and even edges are dashed.

Lemma 3.3. *$K_{n,m} - C_4$ is strongly even-cycle decomposable for all even integers $n, m \geq 2$.*

Proof. We proceed by induction on $n+m$. Note that $K_{2,j} - C_4$ is $K_{2,j-2}$ (together with two isolated vertices). By Lemma 3.1 we may assume that $n, m \geq 4$. Next, by Lemma 3.2, we may assume that $m \geq 6$. Now let $X := \{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_m\}$ be a bipartition of $K_{n,m} - C_4$, and let Σ be a signing of its edges with $|\Sigma|$ even. By re-indexing, we may assume that the vertices of the missing C_4 are $\{x_1, x_2, y_1, y_2\}$. Since $m \geq 6$, two vertices in $\{y_3, y_4, \ldots, y_m\}$ (say, y_{m-1} and y_m) must have the same parity. Thus, the subgraphs induced by $X \cup \{y_{m-1}, y_m\}$ and $X \cup \{y_1, \ldots, y_{m-2}\}$ both contain an even number of odd edges. But now we are finished since the first subgraph is isomorphic to $K_{2,n}$ (and hence is strongly even-cycle decomposable by Lemma 3.1), while the second is isomorphic to $K_{n,m-2} - C_4$ (and is strongly even-cycle decomposable by the induction hypothesis). □

Lemma 3.4. *Let $n, m \geq 2$ be even integers. For every signing Σ of $E(K_{n,m})$, there exists a 4-cycle C in $K_{n,m}$ such that $|\Sigma \cap E(C)| \equiv |\Sigma| \pmod 2$.*

Proof. Let (A, B) be the bipartition of $K_{n,m}$. By re-signing, we may assume that there exists a vertex $a_1 \in A$ such that all edges incident with a_1 are even. Now, since $|A|$ and the parity of a_1 are even, it follows that there is a vertex $a_2 \in A$ such that a_2 has the same parity as $|\Sigma|$ and $a_2 \neq a_1$. Therefore, there exist $b_1, b_2 \in B$ such that $C := a_1 b_1 a_2 b_2$ is a 4-cycle with the same parity as $|\Sigma|$. □

Lemma 3.5. *For all even integers $n, m \geq 2$, $K_{n,m}$ is strongly even-cycle decomposable.*

Proof. Let Σ be a signing of $E(K_{n,m})$ with $|\Sigma|$ even. By Lemma 3.4, there exists an even 4-cycle C of $K_{n,m}$. We are done by Lemma 3.3. □

4. Small graphs

We have previously noted that K_5 is not strongly even-cycle decomposable. On the other hand, up to re-signing, there is only one even-size signature Σ of $E(K_5)$ such that (K_5, Σ) is not even-cycle decomposable.

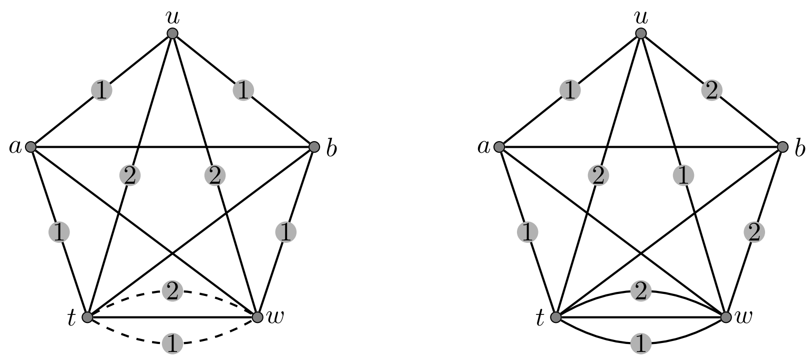

FIGURE 4. Even-cycle decompositions of signed graphs in the proof of Lemma 4.3. Solid edges are odd and dashed edges are even. The bent edges represent e_1 and e_2. The labelled edges illustrate the union of two even cycles whose removal leaves an even 4-cycle.

Lemma 4.1. *Let Σ be a signing of $E(K_5)$ with $|\Sigma|$ even. If (K_5, Σ) is not even-cycle decomposable, then Σ and $E(K_5)$ are equivalent.*

Proof. Since the complement of a 5-cycle in K_5 is another 5-cycle, every 5-cycle of (K_5, Σ) must be odd, else (K_5, Σ) is even-cycle decomposable. Every 4-cycle in (K_5, Σ) is the symmetric difference of two 5-cycles, and thus is even. Every 3-cycle in (K_5, Σ) is the symmetric difference of a 4-cycle and a 5-cycle, and thus is odd. Thus, (K_5, Σ) and $(K_5, E(K_5))$ have the same set of even cycles, and so Σ and $E(K_5)$ are equivalent. $\square$

For a graph G, $e \in E(G)$ and a non-negative integer m, we define $G + me$ to be the graph obtained from G by adding m additional edges in parallel with e.

Lemma 4.2. *If G is a strongly even-cycle decomposable graph, then $G + me$ is strongly even-cycle decomposable for all $e \in E(G)$ and all even m.*

Proof. Consider a signing Σ of $E(G + me)$ with $|\Sigma|$ even. Let t and w be the ends of e. Note that there is a set of $m/2$ edge-disjoint even 2-cycles between t and w in $(G + me, \Sigma)$. Thus, we can remove this set of even 2-cycles and then use the fact that G is strongly even-cycle decomposable. $\square$

The next lemma shows that the converse of Lemma 4.2 fails for $G = K_5$.

Lemma 4.3. *$K_5 + me$ is strongly even-cycle decomposable for all even $m \geq 2$ and $e \in E(K_5)$.*

Proof. Let $e = tw$ and Σ be a signing of $E(K_5 + me)$ with $|\Sigma|$ even. By repeatedly removing even 2-cycles between t and w in $(K_5 + me, \Sigma)$, we may assume $m = 2$. Let e_1 and e_2 be edges between t and w with the same sign. If $((K_5 + 2e) \setminus e_1 \setminus e_2, \Sigma \setminus \{e_1, e_2\})$ is not equivalent to $((K_5 + 2e) \setminus e_1 \setminus e_2, E((K_5 + 2e) \setminus e_1 \setminus e_2))$, then we are done by Lemma 4.1. Thus, by re-signing, we may assume that all edges in $E(K_5 + 2e) \setminus \{e_1, e_2\}$ are odd. Either e_1 and e_2 are both odd, or e_1 and e_2 are both even. Even-cycle decompositions for both possibilities are shown in Figure 4. $\square$

Let $\overline{K_n}$ denote the graph with n vertices and no edges. Let G and H be graphs. The *join of* G *and* H is the graph obtained from the disjoint union of G and H by adding an edge uv for all $u \in V(G)$ and $v \in V(H)$. Two distinct vertices u and v of a graph G are *twins* if no vertex in $V(G) \setminus \{u, v\}$ is adjacent to exactly one of u and v.

Lemma 4.4. *Let G be a graph having pairwise non-adjacent twins x, y, z such that no loops or parallel edges are incident with x, y or z. If $G - x - y$ is strongly even-cycle decomposable, then G is strongly even-cycle decomposble.*

Proof. Let Σ be a signing of $E(G)$ with $|\Sigma|$ even. By the pigeonhole principle, there are distinct vertices $v_1, v_2 \in \{x, y, z\}$ such that the parity of v_1 is equal to that of v_2. Note that the subgraph of G induced by all edges between $G - x - y - z$ and $\{v_1, v_2\}$ is strongly even-cycle decomposable by Lemma 3.1 and $G - v_1 - v_2$ is isomorphic to $G - x - y$, which is strongly even-cycle decomposable. □

Lemma 4.5. *Let G be a simple Eulerian graph with at most one cycle with $|V(G)|$ even. Let n be a positive integer. Then the join of G with $\overline{K_{2n}}$ is strongly even-cycle decomposable if and only if $n > 1$ or G is not a co-claw.*

Proof. The forward implication is trivial because the join of a co-claw and $\overline{K_2}$ is a subdivision of K_5.

For the other direction, we proceed by induction on $|E(G)| + n$. Let G' be the join of G and $\overline{K_{2n}}$ and let Σ be a signing of $E(G')$ with $|\Sigma|$ even.

If $|\Sigma \cap E(G)|$ is even, then we are done by Lemma 3.5. Thus, we may assume $|\Sigma \cap E(G)|$ is odd, and so $E(G) = E(C)$, where C is an odd cycle.

Let $X = V(\overline{K_{2n}})$ and $K_{C,X}$ be the subgraph of G' consisting of all edges between $V(C)$ and X. Suppose that $K_{C,X}$ contains an odd 4-cycle D. Observe that $C \cup D$ can be decomposed into two even cycles. The remaining edges are even-cycle decomposable by Lemma 3.3. Thus, we may assume $K_{C,X}$ does not contain an odd 4-cycle. Since the 4-cycles of $K_{C,X}$ span its cycle space, all cycles of $K_{C,X}$ are even. By Lemma 2.2, we may assume that all edges of $K_{C,X}$ are even. Since $|\Sigma|$ is even, but C is odd, there must be a vertex $u \in V(G) \setminus V(C)$ of odd parity in (G', Σ). We handle two separate cases depending on the size of X.

Case 1. $|X| = 2$. In this case, by hypothesis, we may assume that G is not a co-claw. Let a and b be the two vertices in X. Since u has odd parity, the edges ua and ub have different signs. We first suppose that C has length at least 4. We choose two vertices x, y in C such that one of the x-y paths in C, say P_1, is even with at most two edges. Since C has length at least 4, the other x-y path P_2 has an internal vertex z. But then $D_1 = \{xb, bz, za, ay\} \cup E(P_1)$ and $D_2 = \{xa, au, ub, by\} \cup E(P_2)$ are even cycles. By Lemma 3.5, the remaining edges are even-cycle decomposable.

We may therefore assume that C is a triangle. Since G is not a co-claw and $|V(G)|$ is even, the number m of isolated vertices in G is at least 3 and odd. But now G' is a subdivision of $K_5 + (m - 1)e$, and so we are done by Lemma 4.3.

Case 2. $|X| \geq 4$. In this case, we can apply Lemma 4.4 and the induction hypothesis, unless $|X| = 4$ and G is a co-claw. Let $X = \{a, b, c, d\}$. Again recall that C is odd, and we have re-signed so that all edges between C and X are even. Let xy be an odd edge in C. Since one of $P_1 := aub$ and $P_2 := cud$ is odd and the other is even, $C \cup P_1 \cup \{xa, yb, xc, yd\} \cup P_2$ can be partitioned into two even cycles.

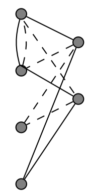

FIGURE 5. A decomposition of a graph into two 5-cycles in the proof of Lemma 4.7.

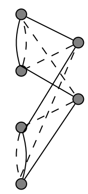

FIGURE 6. A decomposition of a graph into two 6-cycles in the proof of Lemma 4.7.

We are done since the signed graph of remaining edges of G' is Eulerian with no odd edge, and hence is even-cycle decomposable. □

Lemma 4.6. *Let G be a simple Eulerian graph with at most one cycle with $|V(G)|$ odd. Then the join of G and K_2 is strongly even-cycle decomposable if and only if $G \neq K_3$.*

Proof. The forward implication is trivial because the join of K_3 and K_2 is K_5.

Let G' be the join of G and K_2. Let G'' be the join of $G \cup \{u\}$ and the $\overline{K_2}$, where u is a new isolated vertex added to G. Since G'' is a subdivision of G', it follows that G' is strongly even-cycle decomposable if and only if G'' is strongly even-cycle decomposable. Thus, provided $G \neq K_3$, by Lemma 4.5, G' is strongly even-cycle decomposable. □

Lemma 4.7. *Let G_1 be the disjoint union of one or two 2-cycles together with an even number of isolated vertices. Then, the join of G_1 and $\overline{K_{2n}}$ is strongly even-cycle decomposable for all $n \geq 1$.*

Proof. Let $G_2 = \overline{K_{2n}}$, G be the join of G_1 and G_2, and Σ be a signing of $E(G)$ with $|\Sigma|$ even. By Lemma 4.4, we may assume that $|V(G_2)| = 2$ and that G_1 has 0 or 2 isolated vertices.

We first handle the case that G_1 has only one 2-cycle C. We may assume that C is odd, else we can remove $E(C)$ and apply Lemma 3.5. If G_1 has no isolated vertices, then $E(G)$ can be decomposed into two even triangles. So, we may assume that G_1 has exactly two isolated vertices. Figure 5 shows a decomposition of $E(G)$ into two 5-cycles. Since the 2-cycle C is odd, we can ensure that both these 5-cycles are even, as required.

We finish by considering the case that G_1 has exactly two 2-cycles C_1 and C_2. By the previous case, we may assume that both C_1 and C_2 are odd. We first handle the subcase that G_1 has no isolated vertices. In this case, Figure 6 shows a decomposition of $E(G)$ into two 6-cycles. Again, because of the odd 2-cycle C_1, we

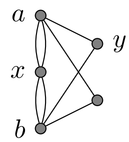

FIGURE 7. $K_{2,3}$ with added two parallel edges incident with a degree-2 vertex.

can ensure that both 6-cycles are even. So, we may assume that G_1 has exactly two isolated vertices x and y. For $i \in \{1,2\}$, define H_i to be C_i together with all edges between $V(C_i)$ and $V(G_2)$. Define H_3 to be the 4-cycle induced by $\{x,y\} \cup V(G_2)$. Note that H_1 and H_2 are strongly even-cycle decomposable by the previous case, and H_3 is obviously strongly even-cycle decomposable. By the pigeonhole principle, there exist $i \neq j$ such that $|E(H_i) \cap \Sigma| \equiv |E(H_j) \cap \Sigma| \pmod 2$. But now we are done since $E(H_1) \cup E(H_2) \cup E(H_3)$ is a partition of $E(G)$. □

Lemma 4.8. *The join of a* 4*-cycle and a* 2*-cycle is strongly even-cycle decomposable.*

Proof. Let G be the join of a 4-cycle C and a 2-cycle D. Let Σ be a signing of $E(G)$ with $|\Sigma|$ even. If the 2-cycle D is even, then (G,Σ) is even-cycle decomposable because $G \setminus E(D)$ is strongly even-cycle decomposable by Lemma 4.5. Thus, we may assume that D is odd. Similarly, we may assume that C is odd by Lemma 4.7. By Lemma 3.4, there is an even 4-cycle F in $G \setminus (E(C) \cup E(D))$. Observe that $C \cup D \cup F$ can be decomposed into two even cycles and the remaining edges are even-cycle decomposable by Lemma 3.3. □

Lemma 4.9. *The join of two co-claws is strongly even-cycle decomposable.*

Proof. Let G be the join of two co-claws. Let C_1 and C_2 be the triangles of the two co-claws. Let Σ be a signing of $E(G)$ with $|\Sigma|$ even. We may assume that C_1 and C_2 are both odd, else we can apply Lemma 4.5. Let x_1 and y_1 be distinct vertices in C_1. By an easy parity argument, there are vertices x_2 and y_2 in C_2 such that $C := x_1x_2y_1y_2$ is an even 4-cycle. Then $C \cup C_1 \cup C_2$ can be decomposed into two even cycles. The remaining edges of G can be decomposed into even cycles by Lemma 3.3. □

Lemma 4.10. *The graph in Figure 7 is strongly even-cycle decomposable.*

Proof. Let G be the graph in Figure 7. Let Σ be a signing of $E(G)$ with $|\Sigma|$ even. If both 2-cycles in G are even, then trivially we can decompose $E(G)$ into three even cycles. Thus by symmetry, we may assume that the cycle ax is odd. Thus we may choose an even cycle $C = axby$ by selecting one of two edges joining ax of the correct parity. Then both C and $G \setminus E(C)$ are even cycles. □

Lemma 4.11. *The graph $K_7 - C_4 - C_3$ in Figure 8 is strongly even-cycle decomposable.*

Proof. See Figure 8 for the labels of vertices of $G = K_7 - C_4 - C_3$. Let Σ be a signing of $E(G)$ with $|\Sigma|$ even. Since the symmetric difference of the three 4-cycles $xcya$, $ycza$, and $xcza$ is empty, we may assume that $xcya$ is even.

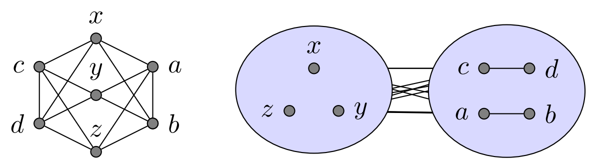

FIGURE 8. Two drawings of $K_7 - C_4 - C_3$.

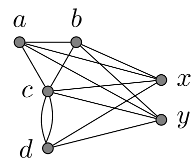

FIGURE 9. The join of $\overline{K_2}$ with a graph having one 2-cycle and one triangle sharing a vertex.

If the 4-cycle $xdyb$ is odd, then either $xdzb$ or $ydzb$ is even. By symmetry between x and y, we may assume that the 4-cycle $xdzb$ is even. Then $\{xcya, xdzb, ydczab\}$ is an even-cycle decomposition. Thus, we may assume that $xdyb$ is even. If the 3-cycle cdz is even, then the other 3-cycle abz is even and so $\{xcya, xdyb, cdz, abz\}$ is an even-cycle decomposition. Thus, we may assume that cdz is an odd 3-cycle. Then either $C = dzby$ or $C' = dczby$ is an even cycle. If C is even, then $\{xcya, dzby, dczabx\}$ is an even-cycle decomposition. If C' is even, then $\{xcya, dczby, dzabx\}$ is an even-cycle decomposition. □

Lemma 4.12. *Let G_1 be two 2-cycles or two triangles meeting at a vertex u and let $G_2 = \overline{K_{2n}}$ for some $n \geq 1$. Let G be the graph obtained from the disjoint union of G_1 and G_2 by joining each vertex of $V(G_1) \setminus \{u\}$ to each vertex of $V(G_2)$. Then G is strongly even-cycle decomposable.*

Proof. Let Σ be a signing of $E(G)$ with $|\Sigma|$ even. By Lemma 4.4, we may assume $|V(G_2)| = 2$. If G_1 is two 2-cycles meeting at u, then G is strongly even-cycle decomposable by Lemma 4.10. So, we may assume that G_1 is two triangles meeting at u. Then G is isomorphic to $K_7 - C_4 - C_3$ in Figure 8 and the conclusion follows from Lemma 4.11. □

Lemma 4.13. *Let G be a graph with one 2-cycle and one triangle meeting at a vertex c. Then the join of G and $\overline{K_{2n}}$ is strongly even-cycle decomposable for all $n \geq 1$.*

Proof. Let G' be the join of G and $\overline{K_{2n}}$. Let Σ be a signing of $E(G')$ with $|\Sigma'|$ even. By Lemma 4.4, we may assume that $n = 1$. We use the labels in Figure 9 for the vertices of G.

Let C_1 be the 3-cycle abc, C_2 be the 4-cycle $bcdx$, and C_3 be the 5-cycle $abcdx$. Since $G \setminus E(C_1)$ is strongly even-cycle decomposable by Lemma 4.7, we may assume that C_1 is odd. Since $G \setminus E(C_2)$ is a subdivision of $K_3 + e_1 + e_2 + e_3$ where $E(K_3) = \{e_1, e_2, e_3\}$, which is strongly even-cycle decomposable, we may assume that C_2 is odd.

Since $E(C_3) = E(C_1)\Delta E(C_2)$, C_3 is even. Furthermore $G\backslash E(C_3)$ is a subdivision of $C_4 + 2e$ for an edge e in C_4, which is strongly even-cycle decomposable by Lemma 4.2. □

5. Substituting a vertex

Let G and H be simple graphs and let $v \in V(G)$. The *substitution of v by H in G* is the graph obtained from the disjoint union of $G - v$ and H by adding an edge xy for all $x \in V(H)$ and all neighbors y of v.

Theorem 5.1. *Let G be a simple strongly even-cycle decomposable graph, let v be a non-isolated vertex of G, and let H be a simple Eulerian graph with an odd number of vertices. Then the substitution of v by H in G is strongly even-cycle decomposable, provided that H is not K_3 or* $\deg_G(v) \geq 4$.

Proof. Let G' be the substitution of v by H in G, and Σ be a signing of $E(G')$ with $|\Sigma|$ even. Let N be the set of neighbors of v. We proceed by induction on $|E(H)| + |V(H)|$.

For us, a *block* of H is a maximal subgraph B of H such that $|V(B)| \geq 2$ and $B - x$ is connected for all $x \in V(B)$. If $(H, E(H) \cap \Sigma)$ contains an even cycle C, then we can apply the induction hypothesis to $H \setminus E(C)$. Thus, we may assume each block of $(H, E(H) \cap \Sigma)$ is an odd cycle or a K_2. Since H is Eulerian, no block can be a K_2. A *leaf block* of H is a block of H that contains at most 1 cut vertex of H.

For each leaf block B of H, we define W_B to be the set of non-cut vertices of H contained in $V(B)$. For distinct vertices $x, y \in W_B$, we define $G'(x, y)$ to be the union of B together with all edges between $\{x, y\}$ and N. We let $\overline{G'}(x, y) = G' \setminus E(G'(x, y)) - x - y$ and $\overline{H}(x, y) = H \setminus E(B) - x - y$. Observe that $G'(x, y)$ is a subdivision of the join of a 2-cycle and $\overline{K_{|N|}}$. By Lemma 4.7, $G'(x, y)$ is strongly even-cycle decomposable. Furthermore $\overline{G'}(x, y)$ is obtained from G by substituting v with $\overline{H}(x, y)$.

For each $t \in V(H)$, define $G_t := G' - V(H - t)$ and observe that G_t is isomorphic to G'. We also define $p(t)$ to be the number of edges of Σ between t and N.

Suppose $|E(G'(x, y)) \cap \Sigma|$ is even for some leaf block B of H and $x, y \in W_B$. Then we are done by induction, unless $|N| = 2$ and $\overline{H}(x, y) = K_3$. That is, H is the union of two triangles meeting at a vertex c. If $|E(G_t) \cap \Sigma|$ is even for some $t \in V(H) \setminus \{c\}$, then we are done since G_t is strongly-even cycle decomposable, and the remaining edges are even-cycle decomposable by Lemma 4.13. Thus, we may assume $p(u) \equiv p(w) \pmod 2$ for all $u, w \in V(H) \setminus \{c\}$. Define H^+ to be H together with all edges between $V(H) \setminus \{c\}$ and N. Observe that $|E(H^+) \cap \Sigma|$ is even, and that H^+ is strongly even-cycle decomposable by Lemma 4.12. Since $G' \setminus E(H^+)$ is isomorphic to G' together with 4 isolated vertices, we are done.

We may hence assume $|E(G'(x, y)) \cap \Sigma|$ is odd for all leaf blocks B of H and $x, y \in W_B$. Suppose that H contains distinct leaf blocks B_1 and B_2. Let a, b, c, d be distinct vertices with $a, b \in W_{B_1}$ and $c, d \in W_{B_2}$. Let $G'(a, b, c, d) := G'(a, b) \cup G'(c, d)$ and let $\overline{G'}(a, b, c, d) = G' \setminus E(G'(a, b, c, d)) - a - b - c - d$. Since $|E(G'(a, b)) \cap \Sigma|$ and $|E(G'(c, d)) \cap \Sigma|$ are both odd, $|E(G'(a, b, c, d)) \cap \Sigma|$ is even. Note that $G'(a, b, c, d)$ is a subdivision of a graph that is strongly even-cycle decomposable by either Lemma 4.7 or Lemma 4.12, and so we are done by the induction hypothesis

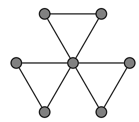

FIGURE 10. The windmill graph.

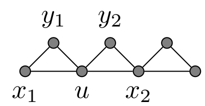

FIGURE 11. A path of three triangles.

unless $|N| = 2$ and H is the windmill graph (see Figure 10) or $|N| = 2$ and H is a path of three triangles (see Figure 11).

Suppose H is the windmill graph. By the pigeonhole principle, there are distinct blocks B_1 and B_2 of H and $x_1 \in W_{B_1}$ and $x_2 \in W_{B_2}$ such that $p(x_1) \equiv p(x_2) \pmod 2$. Let G'' be the subgraph of G' consisting of $B_1 \cup B_2$ together with all edges between $\{x_1, x_2\}$ and N. Note that G'' contains and even number of odd edges . We are done since G'' is strongly even-cycle decomposable by Lemma 4.12, while the remaining edges are even-cycle decomposable by the induction hypothesis.

Suppose H is a path of three triangles. Let $B_1 := x_1y_1u$ be a leaf block and $B_0 := ux_2y_2$ be a triangle of H sharing a vertex u with B_1 (See Figure 11). Let H' be the subgraph of H consisting of $E(B_1) \cup E(B_0)$ together with all edges between $\{x_1, y_1, x_2, y_2\}$ and N. If $p(x_2) \equiv p(y_2) \pmod 2$, then H' contains an even number of odd edges. Moreover, H' is strongly even-cycle decomposable by Lemma 4.12, and the remaining edges are even-cycle decomposable by the induction hypothesis. Thus, we may assume $p(x_2) \not\equiv p(y_2) \pmod 2$. It follows that we may pick $z \in \{x_2, y_2\}$ such that $p(x_1) \equiv p(z) \pmod 2$. Let G'' be the subgraph of G' consisting of $E(B_1) \cup E(B_0)$ together with all edges between $\{x_1, z\}$ and N. Note that G'' contains an even number of odd edges. We are thus finished since G'' is strongly even-cycle decomposable by Lemma 4.12, and the remaining edges are even-cycle decomposable by the induction hypothesis.

We have therefore reduced to the case that H has at most one leaf block. By Lemma 4.4, we may assume that H has at most 2 isolated vertices. So, we may assume that H is a cycle C together with a set S of isolated vertices with $|S| \le 2$.

If $|V(H)| \ge 7$, then then there exist two vertices $x, y \in V(C)$ such that $p(x) \equiv p(y) \pmod 2$. Since $K_{2,|N|}$ is strongly even-cycle decomposable by Lemma 3.5, we may shrink H by suppressing x and y and apply the induction hypothesis (as $|V(H)| \ge 7$).

We may hence assume that $|V(H)| \le 5$, and so H is a 5-cycle, a 4-cycle with one isolated vertex, a triangle with two isolated vertices, or a triangle. We finish by handling each of these cases separately.

Suppose H is a 5-cycle. Let x and y be distinct vertices of H such that $p(x) \equiv p(y) \pmod 2$. Then we may shrink H by suppressing x and y and apply the induction hypothesis, unless $|N| = 2$. So, we may assume $|N| = 2$. Let u be an arbitrary vertex of H. By Lemma 2.3, there is an almost even-cycle decomposition

$\{C'\} \cup \mathcal{C}$ of $(G_u, \Sigma \cap E(G_u))$, where all cycles in $\mathcal{C}$ are even and $u \in V(C')$. Let G'' be the graph induced by the edges in $E(G') \setminus (E(G_u) \setminus E(C'))$. Observe that G'' is isomorphic to a subdivision of the join of a 5-cycle and K_2. By Lemma 4.6, G'' is strongly even-cycle decomposable.

Suppose H is a 4-cycle with an isolated vertex s. If $|E(G_s) \cap \Sigma|$ is even, then we are done since the graph induced by the remaining edges is the join of C_4 and $\overline{K_{|N|}}$, which is strongly even-cycle decomposable by Lemma 4.5. Therefore we may assume that $|E(G_s) \cap \Sigma|$ is odd.

Suppose $|N| \geq 4$. If $p(s) \equiv p(x) \pmod 2$ for some $x \in V(C)$, then the set F of edges from $\{s, x\}$ to N is even-cycle decomposable. Since $|N| \geq 4$, the remaining edges are even-cycle decomposable by the induction hypothesis. Therefore we may assume $|E(G_x) \cap \Sigma|$ is even for all $x \in V(C)$. Then for every vertex x of C, the graph obtained from $G' \setminus E(G_x)$ by suppressing x is strongly even-cycle decomposable by Lemma 4.5.

It remains to consider the case when $|N| = 2$. Let $\{C'\} \cup \mathcal{C}$ be an almost even-cycle decomposition of $(G_s, \Sigma \cap E(G_s))$, where all cycles in $\mathcal{C}$ are even and $s \in V(C')$. Let G'' be the graph induced by $E(G') \setminus (E(G_s) \setminus E(C'))$. Note that G'' is a subdivision of the join of C_4 and a 2-cycle, which is strongly even-cycle decomposable by Lemma 4.8.

Suppose H is a triangle with two isolated vertices. Let $S = \{x, y\}$. Suppose $|N| \geq 4$. We first handle the subcase that $p(x) \equiv p(y) \pmod 2$. Here, we shrink H by suppressing x and y and apply the induction hypothesis to deduce that (G', Σ) is even-cycle decomposable since $|N| \geq 4$. So, we may assume that $p(x) \not\equiv p(y)$ (mod 2). By symmetry, we may assume that G_x contains an even number of odd edges. Let G'' be the graph induced by $E(G') \setminus E(G_x)$. Observe that G'' is the join of a co-claw and $\overline{K_{|N|}}$, and we are done by Lemma 4.5.

We may hence assume that $|N| = 2$. Let $\{C'\} \cup \mathcal{C}$ be an almost even-cycle decomposition of $(G_x, \Sigma \cap E(G_x))$, where all cycles in $\mathcal{C}$ are even and $x \in V(C')$. Let G'' be the graph induced by $E(G') \setminus (E(G_x) \setminus E(C'))$. Observe that G'' is a subdivision of $K_5 + 2e$, which is strongly even-cycle decomposable by Lemma 4.3.

Suppose H is a triangle abc. By hypthosis, $|N| \geq 4$. Let $\{C'\} \cup \mathcal{C}$ be an almost even-cycle decomposition of $(G_a, \Sigma \cap E(G_a))$, where all cycles in $\mathcal{C}$ are even and $a \in V(C')$. Let G'' be the graph obtained from $G' \setminus (E(G_a) \setminus E(C'))$ by deleting isolated vertices and suppressing degree-2 vertices not in N. It is enough to prove that G'' is strongly even-cycle decomposable. Notice that $G'' - b - c$ is a simple graph with exactly one cycle and $|V(G'' - b - c)| \geq 5$. Since G'' is the join of K_2 and $G'' - b - c$, G'' is strongly even-cycle decomposable by Lemma 4.6. □

6. Substituting twin vertices

If u and v are twins of G, and H is a simple graph, then the *twin substitution of* $\{u, v\}$ *by* H *in* G is the graph obtained from the disjoint union of $G - u - v$ and H by adding an edge xy for all $x \in V(H)$ and all neighbors y of v.

Theorem 6.1. *Let G be a simple strongly even-cycle decomposable graph and let u and v be non-isolated, non-adjacent twin vertices of G. If H is a simple Eulerian graph with an even number of vertices, then the twin substitution of $\{u, v\}$ by H in G is also strongly even-cycle decomposable, provided that H is not a co-claw or $\deg_G(v) \geq 4$.*

Proof. We proceed by induction on $|E(H)|$. If H has an isolated vertex, then it follows easily from Theorem 5.1. Thus, we may assume that H has no isolated vertices.

Let G' be the twin substitution of $\{u,v\}$ by H in G. We may assume that G has no isolated vertices and $|V(H)| \geq 4$. Let N be the set of neighbors of v in G. Let Σ be a signing of $E(G')$ with $|\Sigma|$ even. If $(H, E(H)\cap\Sigma)$ has an even cycle C, then we apply the induction hypothesis with $H' = H \setminus E(C)$. Thus we may assume that $(H, E(H)\cap\Sigma)$ has no even cycles. Since H is Eulerian, every block of H is an odd cycle.

For each leaf block B of H, we define W_B to be the set of non-cut vertices of H contained in $V(B)$. For distinct vertices $x, y \in W_B$, we define $G'(x,y)$ to be a subgraph of G' that is the union of B together with all edges between $\{x,y\}$ and N. We let $\overline{G'}(x,y) = G' \setminus E(G'(x,y)) - x - y$. Observe $G'(x,y)$ is a subdivision of the join of a 2-cycle and $\overline{K_{|N|}}$. By Lemma 4.7, $G'(x,y)$ is strongly even-cycle decomposable. Thus we may assume for all such choices x, y of B, $|E(G'(x,y))\cap\Sigma|$ is odd.

If H has at least two odd cycles, then let B_1, B_2 be distinct leaf blocks of H and let a, b, c, d be distinct vertices with $a,b \in W_{B_1}$ and $c,d \in W_{B_2}$. Let $G'(a,b,c,d) = G'(a,b) \cup G'(c,d)$ and let $\overline{G'}(a,b,c,d) = G \setminus E(G'(a,b,c,d)) - a - b - c - d$. Since $|E(G'(a,b))\cap\Sigma|$ and $|E(G'(c,d))\cap\Sigma|$ are both odd, $|E(G'(a,b,c,d))\cap\Sigma|$ is even. Note that $G'(a,b,c,d)$ is a subdivision of a graph that is strongly even-cycle decomposable by Lemma 4.7 or Lemma 4.12. Furthermore, $\overline{G'}(a,b,c,d)$ is strongly even-cycle decomposable by the induction hypothesis. Thus, we may assume $(H, \Sigma\cap E(H))$ is an odd cycle of even length.

Let x and y be two non-adjacent vertices in H. Let $G_1 = G' - (V(H) \setminus \{x,y\})$ and $G_2 = G' \setminus E(G_1) \setminus E(H)$. Note that G_1 is isomorphic to G, and $E(G') = E(G_1) \cup E(G_2) \cup E(H)$.

Suppose $|E(G_1)\cap\Sigma|$ is even and hence $|E(G_2)\cap\Sigma|$ is odd. By Lemma 3.4, there exists an odd 4-cycle C in $(G_2, E(G_2)\cap\Sigma)$. Then $C \cup H$ can be decomposed into two even cycles, and by Lemma 3.3, $G_2 \setminus E(C)$ is strongly even-cycle decomposable.

Hence $|E(G_1)\cap\Sigma|$ is odd. Thus there is an almost even-cycle decomposition $\mathcal{C}$ of G_1 such that x belongs to an odd cycle $C_1 \in \mathcal{C}$.

If $|V(H)| \geq 6$, then let F be the set of edges from $\{x,y\}$ to N. As $|G'(x,y)\cap\Sigma|$ is odd, $|F\cap\Sigma|$ is even and so F is even-cycle decomposable by Lemma 3.1. The remaining edges are even-cycle decomposable by the induction hypothesis applied to the graph obtained from H by suppressing x and y. Therefore we may assume $|V(H)| = 4$ and H is a 4-cycle.

Suppose $|N| \geq 4$. Let G_1' be the spanning subgraph of G_1 on $V(G_1)$ whose edge set is precisely $E(C_1)$. Let G_1'' be the graph obtained from G_1' by deleting all isolated vertices and suppressing all degree-2 vertices not in $N \cup \{x,y\}$ so that $V(G_1'') = N \cup \{x,y\}$. Let G'' be the graph induced by $E(G') \setminus (E(G_1) \setminus E(C_1))$. Since G_1'' is a simple graph with exactly one cycle and $|V(G_1'')| \geq 6$, G'' is strongly even-cycle decomposable by Lemma 4.5 because G'' is a subdivision of the join of $\overline{K_2}$ and G_1''.

Thus we may assume that $|N| = 2$. Let C_2 be the cycle in $\mathcal{C}$ containing y (possibly $C_1 = C_2$). Then $C_1 \cup C_2$ can be decomposed into two xy-paths, and thus $C_1 \cup C_2 \cup H$ can be decomposed into two even cycles. □

Theorem 6.2. *Let G be a simple strongly even-cycle decomposable graph and let u and v be adjacent twin vertices of G having degree at least 4. If H is a simple anti-Eulerian graph, then the twin substitution of $\{u,v\}$ by H in G is also strongly even-cycle decomposable.*

Proof. Let G' be the twin substitution of $\{u,v\}$ by H in G. We proceed by induction on $|E(H)|$. Let Σ be a signing of $E(G')$ with $|\Sigma|$ even. Let $N = N_G(v) \setminus \{u\}$. By assumption, $|N| \geq 3$ and $|N|$ is odd.

If $(H, E(H) \cap \Sigma)$ has an even cycle C, then we apply the induction hypothesis to G and $H \setminus E(C)$. Thus, we may assume that $(H, E(H) \cap \Sigma)$ has no even cycle. Therefore every block of H is either an odd cycle or a K_2.

If B is a leaf block of H, then $B = K_2$ because H is anti-Eulerian. Since B has at most one cut vertex, each leaf block has at least one vertex of degree 1. Conversely, every vertex of degree 1 in H is in a leaf block.

If H has an odd cycle C, then there exist vertices x, y of degree 1 in H such that there are two paths P_1 and P_2 from x to y having different parity. For $i = 1, 2$, let G_i be the subgraph of G induced by all edges between $\{x, y\}$ and N and all edges in P_i. Then there exists $i \in \{1, 2\}$ such that $|E(G_i) \cap \Sigma|$ is even. Since G_i is a subdivision of the join of $\overline{K_{|N|}}$ with K_2, G_i is strongly even-cycle decomposable by Lemma 4.6. It remains to show that $G' \setminus E(G_i)$ is strongly even-cycle decomposable, which is implied by the induction hypothesis with G and $H \setminus E(P_i) - x - y$.

Therefore, we may assume that H has no cycles. Suppose that H has a component with three leaves v_1, v_2, v_3. For each $1 \leq i < j \leq 3$, let P_{ij} be the unique path from v_i to v_j in H. Let G_{ij} be the subgraph of G induced by all edges between $\{v_i, v_j\}$ and N and all edges in P_{ij}. Since $E(G_{12}) \Delta E(G_{23}) \Delta E(G_{31}) = \emptyset$, there exist $1 \leq i < j \leq 3$ such that $|E(G_{ij}) \cap \Sigma|$ is even. As before, G_{ij} is strongly even-cycle decomposable by Lemma 4.6 and $G' \setminus E(G_{ij})$ is strongly even-cycle decomposable by the induction hypothesis.

Thus, we may assume that each component of H is isomorphic to K_2. Let $H_1, \ldots, H_m$ be the components of H. If $m = 1$, then G' is isomorphic to G. So, we may assume $m \geq 2$. For each $1 \leq i \leq m$, let $G'_i = G' - (V(H) \setminus V(H_i))$ and let G''_i be the subgraph of G' induced by H_i and all edges incident with a vertex in H_i. Observe that G''_i is isomorphic to the join of K_2 and $\overline{K_{|N|}}$, which is strongly even-cycle decomposable by Lemma 4.6. If $|E(G''_i) \cap \Sigma|$ is even, then we are done since $G' \setminus E(G''_i)$ is strongly even-cycle decomposable by the induction hypothesis.

Thus, we may assume that $|E(G''_i) \cap \Sigma|$ is odd for all $1 \leq i \leq m$. Then $|E(G''_1 \cup G''_2) \cap \Sigma|$ is even. Since $K_7 - C_4 - C_3$ is strongly even-cycle decomposable by Lemma 4.11 and $K_7 - C_4 - C_3$ is the join of $\overline{K_3}$ with $2K_2$, $G''_1 \cup G''_2$ is strongly even-cycle decomposable by Lemma 4.4. If $m > 2$, then $G' \setminus E(G''_1 \cup G''_2)$ is strongly even-cycle decomposable by the induction hypothesis and therefore (G', Σ) is even-cycle decomposable.

Hence, we may assume that $m = 2$. Then $E(G'_1) \cup E(G''_2) = E(G')$ and $E(G'_1) \cap E(G''_2) = \emptyset$ and so $|E(G'_1) \cap \Sigma|$ is odd.

Claim 1. *If $(G'_1, E(G'_1) \cap \Sigma)$ has an almost even-cycle decomposition $\mathcal{C}$ such that the unique odd cycle $C \in \mathcal{C}$ uses at least two vertices of N, then (G', Σ) is even-cycle decomposable.*

Subproof. Let $\mathcal{C}$ be an almost even-cycle decomposition of $(G'_1, E(G'_1) \cap \Sigma)$ such that the unique odd cycle $C \in \mathcal{C}$ uses at least two vertices of N. Let $G^*_1 = (V(G'_1), E(C))$

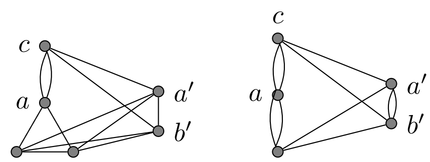

FIGURE 12. Two remaining possibilities in the proof of Theorem 6.2.

be the spanning subgraph of G'_1 whose edge set is $E(C)$. It is enough to show that $G^*_1 \cup G''_2$ is strongly even-cycle decomposable.

Let G^{**}_1 be the graph obtained from G^*_1 by removing isolated vertices not in N and suppressing all vertices of degree 2 not in N. Observe G^{**}_1 has exactly one cycle, $V(G^{**}_1) = N$, $V(G''_2) = N \cup V(H_2)$, and $G^*_1 \cup G''_2$ is strongly even-cycle decomposable if $G^{**}_1 \cup G''_2$ is strongly even-cycle decomposable. Also note $G^{**}_1 \cup G''_2$ is the join of G^{**}_1 and K_2.

If G^{**}_1 does not have a 2-cycle, then $G^{**}_1 \cup G''_2$ is strongly-even cycle decomposable by Lemma 4.6. Thus, we may assume that G^{**}_1 has a 2-cycle. By subdividing the edge in H_2, we conclude that $G^{**}_1 \cup G''_2$ is strongly even-cycle decomposable by Lemma 4.7. ∎

Let $e = ab$ be the unique edge in H_1 and let $\mathcal{C}$ be an almost even-cycle decomposition of $(G'_1, E(G'_1) \cap \Sigma)$ such that e belongs to the unique odd cycle $C_1 \in \mathcal{C}$. If C_1 uses at least two vertices of N, then we are done by Claim 1. So, we may assume that $C_1 = abc$, with $c \in N$. Let $C_2 \neq C_1$ be a cycle in $\mathcal{C}$ such that $a \in V(C_2)$. Note that $|E(C_1 \cup C_2) \cap \Sigma|$ is odd, since C_2 is even.

If $c \in V(C_2)$, then $C_1 \cup C_2$ can be decomposed into two cycles C'_1 and C'_2 such that C'_1 and C'_2 both use at least two vertices of N. One of C'_1 and C'_2 is odd and the other is even, so we are done by Claim 1.

Suppose $c \notin V(C_2)$ and $b \in V(C_2)$. If C_2 is not a 4-cycle, then $C_1 \cup C_2$ can be decomposed into two cycles C'_1 and C'_2 such that C'_1 and C'_2 both use at least two vertices of N, so we are again done by Claim 1. We may thus assume that C_2 is a 4-cycle. Let $f = a'b'$ be the unique edge in H_2. Consider $G^*_2 = G''_2 \cup E(C_1) \cup E(C_2)$. It suffices to show that G^*_2 is strongly even-cycle decomposable. Let G^{**}_2 be the graph obtained from G^*_2 by removing isolated vertices and suppressing degree-2 vertices. Note that there are $|N| - 2$ parallel edges between a' and b' in G^{**}_2. Let G^{***}_2 be the graph obtained from G^{**}_2 by deleting $|N| - 3$ edges between a' and b'. By Lemma 4.2 it suffices to show that G^{***}_2 is strongly even-cycle decomposable. But $G^{***}_2 = K_7 - C_4 - C_3$, so we are done by Lemma 4.11.

The remaining case is $b \notin V(C_2)$ and $c \notin V(C_2)$. Again let $G^*_2 = G''_2 \cup E(C_1) \cup E(C_2)$ and $f = a'b'$ be the unique edge in H_2. Let G^{**}_2 be the graph obtained from G^*_2 by removing isolated vertices and suppressing degree-2 vertices. It suffices to show that G^{**}_2 is strongly even-cycle decomposable. Let N^{**} be the vertices of N that have not been suppressed. Let Σ^{**} be a signing of $E(G^{**}_2)$ with $|\Sigma^{**}|$ even. By removing even 2-cycles between a' and b', we may assume that there are k edges between a' and b', where $k \in \{1, 2\}$. By removing even 4-cycles between $\{a', b'\}$ and $N^{**} \setminus \{c\}$, and then suppressing degree-2 vertices, we may assume that $|N^{**}| \in \{2, 3\}$. Since G^{**}_2 is Eulerian, the only possibilities are $k = 1$ and $|N^{**}| = 3$ or $k = 2$, and $|N^{**}| = 2$, see Figure 12.

Suppose $k = 2$, and $|N^{**}| = 2$. If the 2-cycle $a'b'a'$ is even, then we can remove it and apply Lemma 4.12. So, we may assume that the 2-cycle $a'b'a'$ is odd. Observe that $E(G^{**})$ decomposes into two 5-cycles. We are thus done, since we can ensure that both these 5-cycles are even because of the odd 2-cycle $a'b'a'$.

Suppose $k = 1$ and $|N^{**}| = 3$. If the 2-cycle aca is even, then by removing it we get a graph which is a subdivision of the join of two 2-cycles. Clearly, the join of two 2-cycles is strongly even-cycle decomposable, so we may assume that aca is odd. Observe that $E(G^{**})$ decomposes into two 6-cycles. We are thus done, since we can ensure that both these 6-cycles are even because of the odd 2-cycle aca. □

7. Joins

The main result of this section is that if G is a simple Eulerian graph that is a join of two graphs G_1 and G_2 with $|V(G_1)|, |V(G_2)| \geq 2$, then G is strongly even-cycle decomposable if and only if G is neither K_5 nor K_5 with an edge subdivided.

Lemma 7.1. *Let G_1 and G_2 be simple Eulerian graphs with an even number of vertices. Then the join of G_1 and G_2 is strongly even-cycle decomposable if and only if it is not K_5 with an edge subdivided.*

Proof. The forward direction is obvious. For the converse, let G be the join of G_1 and G_2 and assume G is not K_5 with an edge subdivided. If both G_1 and G_2 are either $\overline{K_2}$ or a co-claw, then G is $K_{2,2}$ or the join of two co-claws, both of which are strongly even-cycle decomposable (the latter by Lemma 4.9). Thus we may assume that G_1 is neither $\overline{K_2}$ nor a co-claw. In particular, $|V(G_1)| \geq 4$. Let $X = \{a, b\}$ and $Y = \{c, d\}$ be a bipartition of $K_{2,2}$. Let H be the twin substitution of $\{a, b\}$ by G_1 in $K_{2,2}$. Since G_1 is not a co-claw, H is strongly even-cycle decomposable by Theorem 6.1. Then G is isomorphic to the twin substitution of $\{c, d\}$ by G_2 in H, which is strongly even-cycle decomposable by Theorem 6.1 because $\deg_H(c) \geq 4$. □

We also prove a variant of our previous lemma for anti-Eulerian graphs.

Lemma 7.2. *Let G_1 be a simple Eulerian graph with $|V(G_1)| \geq 3$ and $|V(G_1)|$ odd. Let G_2 be a simple anti-Eulerian graph with $|V(G_2)| \geq 2$. Then the join of G_1 and G_2 is strongly even-cycle decomposable if and only if $G_1 \neq K_3$ or $G_2 \neq K_2$.*

Proof. The forward direction is obvious since the join of K_3 and K_2 is K_5. For the converse, let G be the join of G_1 and G_2, where $G_1 \neq K_3$ or $G_2 \neq K_2$. First suppose that $G_1 \neq K_3$. Let G_3 be the join of G_1 and K_2. Let $V(K_3) = \{a, b, c\}$. Since G_3 is isomorphic to the substitution of a by G_1 in K_3, G_3 is strongly even-cycle decomposable by Theorem 5.1. Since G is isomorphic to the twin substitution of $\{b, c\}$ by G_2 in G_3, G is strongly even-cycle decomposable by Theorem 6.2.

Hence we may assume that $G_1 = K_3$ and $G_2 \neq K_2$. In particular, $|V(G_2)| \geq 4$. Let $v \in V(G_1)$, $G_1' = G - (V(K_3) \setminus \{v\})$, and $G_2' = G_1 - v$. Then G is isomorphic to the join of G_1' and G_2', where $G_1' \neq K_3$ since $|V(G_1')| \geq 5$. By the previous argument with (G_1, G_2) replaced by (G_1', G_2'), we conclude that G is strongly even-cycle decomposable. □

Combining the previous two lemmas, we obtain the main result of this section.

Theorem 7.3. *Let G be a simple Eulerian graph that is a join of two graphs each having at least two vertices. Then G is strongly even-cycle decomposable if and only if G is neither K_5 nor K_5 with an edge subdivided.*

Proof. The forward direction is trivial. For the converse, suppose that G is the join of two graphs G_1 and G_2 with $|V(G_1)|, |V(G_2)| \geq 2$. If both G_1 and G_2 have an even number of vertices, then both are Eulerian and we apply Lemma 7.1. If $|V(G_1)|$ is even and $|V(G_2)|$ is odd, then G_1 is anti-Eulerian and G_2 is Eulerian and so we apply Lemma 7.2. If both G_1 and G_2 have an odd number of vertices, then G_1 is anti-Eulerian, contradicting the fact that every anti-Eulerian graph has an even number of vertices. □

8. Cographs

As promised, we finish our paper with the following characterization of strongly even-cycle decomposable cographs.

Corollary 8.1. *Let G be a cograph with no isolated vertices. Then G is strongly even-cycle decomposable if and only if G is 2-connected Eulerian and G is neither K_5 nor K_5 with an edge subdivided.*

Proof. The forward direction is trivial. Let us prove the converse. It is well known that if G is a cograph with at least two vertices, then either G or its complement is disconnected [1]. We may assume that G has at least 4 vertices, as K_3 is trivially strongly even-cycle decomposable. Since G is connected, there exists a partition (L, R) of $V(G)$ such that all edges are present between L and R. Note that $|L|$ and $|R|$ cannot both be odd. We choose $|R| \geq 2$ and $|L|$ maximum. If $|L| \geq 2$, then we are done by Theorem 7.3. Suppose $|L| = 1$. Then because G is 2-connected, $G[R]$ is a connected cograph and so $G[R]$ is the join of two graphs G_1 and G_2 with $|V(G_1)| \leq |V(G_2)|$. But now $(L \cup V(G_1), V(G_2))$ contradicts the choice of (L, R). □

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