A Complete Solution to the Cvetković-Rowlinson Conjecture

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Abstract

In 1990, Cvetković and Rowlinson [The largest eigenvalue of a graph: a survey, Linear Multilinear Algebra 28(1-2) (1990), 3–33] conjectured that among all outerplanar graphs on n vertices, $K_1 \vee P_{n-1}$ attains the maximum spectral radius. In 2017, Tait and Tobin [Three conjectures in extremal spectral graph theory, J. Combin. Theory, Ser. B 126 (2017) 137-161] confirmed the conjecture for sufficiently large values of n. In this article, we show the conjecture is true for all $n \geq 2$ except for $n = 6$.

Keywords: Spectral radius; Planar graphs; Outerplanar graphs; Minor

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There is a long tradition of studying planar graphs. In particular, the study of spectral radius of planar graphs is a fruitful topic in spectral graph theory and can be traced back at least to Schwenk and Wilson [13] who asked "what can be said about the eigenvalues of a planar graph?". In 1988, Hong [10] proved the first non-trivial result that $\lambda(\Gamma) \leq \sqrt{5n-11}$, where $\lambda(\Gamma)$ is the spectral radius of a planar graph Γ on $n \geq 3$ vertices. Hong's bound was improved to $4+\sqrt{3n-9}$ by Cao and Vince [3], and to $2\sqrt{2}+\sqrt{3n-\frac{15}{2}}$ by Hong [11] himself, and finally to $2+\sqrt{2n-6}$ by Ellingham and Zha [7]. On the other hand, Boots and Royle [2], and independently, Cao and Vince [3], conjectured that $P_2 \vee P_{n-2}$ attains the maximum spectral radius among all planar graphs on $n \geq 9$ vertices. Only recently, Tait and Tobin [16] published a proof of the conjecture for sufficiently large graphs.

A graph G is outerplanar if it has a planar embedding $\widetilde{G}$ in which all vertices lie on the boundary of its outer face. In fact, earlier than the Boots-Royle-Cao-Vince Conjecture, Cvetković and Rowlinson [6] proposed the following conjecture on outerplanar graphs in 1990. In what follows, K_1 denotes a single vertex, P_{n-1} denotes the path on $n-1$ vertices, and "$\vee$" is the join operation.

Conjecture 1 (Cvetković, Rowlinson [6])**.** Among all outerplanar graphs on n vertices, $K_1 \vee P_{n-1}$ attains the maximum spectral radius.

Cvetković and Rowlinson [6] considered the above conjecture as study on indices of Hamiltonian graphs. Rowlinson [12] proved Conjecture 1 for outerplanar graphs without internal triangles, where an internal triangle of an outerplanar graph is a 3-cycle which has no edges in common with the unique Hamiltonian cycle of the graph. For upper bounds

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of the spectral radius $\lambda(G)$ of an outerplanar graph G, Cao and Vince [3] showed that $\lambda(G) \leq 1 + \sqrt{2+\sqrt{2}} + \sqrt{n-5}$. This was improved by Shu and Hong [14] to $\lambda(G) \leq \frac{3}{2} + \sqrt{n - \frac{7}{4}}$. In 2017, Tait and Tobin[16] confirmed Conjecture 1 for sufficiently large n.

Theorem 1 (Tait, Tobin [16]). *The Cvetković-Rowlinson Conjecture is true for all sufficiently large n.*

Some variant of the Cvetković-Rowlinson Conjecture was considered by Yu, Kang, Liu and Shan [18]. For related topics on spectral properties of planar graphs, we refer to the introduction part of [16] and references therein.

The humble goal of this article is to give a solution to the Cvetković-Rowlinson Conjecture for all n. The complete proof consists of two parts. We first prove the conjecture for $n \geq 17$, and then prove the case that $2 \leq n \leq 16$ where $n \neq 6$, with the aid of a computer. We disprove the conjecture for the case of $n = 6$.

Theorem 2. *Among all outerplanar graphs on $n \geq 17$ vertices, $K_1 \vee P_{n-1}$ attains the maximum spectral radius.*

Before our proof of Theorem 2, let us introduce some necessary notations and terminology. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$ and $S \subseteq V(G)$. We denote by $G[S]$ the subgraph of G induced by S and $G - S$ the subgraph $G[V(G)\backslash S]$. For any $v \in V(G)$, $N_G(v)$ denotes the set of neighbors of v in G, $d_G(v)$ is defined as $|N_G(v)|$, and $d_S(v) := |N_G(v) \cap S|$. Let $A, B \subset V(G)$ be two disjoint sets. We denote by $N_A(B) := \bigcup_{v \in B} N_A(v)$, by $d_A(B) := |N_A(B)|$ and by $e_G(A, B)$ the number of edges with one end-vertex in A and the other one in B. If there is no danger of ambiguity, we use $e(A, B)$ instead of $e_G(A, B)$. Let G_1 and G_2 be two disjoint graphs. The *join* of G_1 and G_2, denoted by $G_1 \vee G_2$, is defined as a graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2) \cup \{xy : x \in V(G_1), y \in V(G_2)\}$. Let $A(G)$ be the adjacency matrix of G and $\lambda(G)$ be the spectral radius of $A(G)$.

A graph H is a *minor* of a graph G if H can be obtained from G by a sequence of vertex and edge deletions and edge contractions. A complete characterization of outerplanar graphs states that a graph is outerplanar if and only if it is $K_{2,3}$-minor free and K_4-minor free. It is clear that a subgraph of an outerplanar graph is also outerplanar. An outerplanar graph is *edge-maximal* (or in short, maximal), if no edge can be added to the graph without violating outerplanarity. It is well-known that every outerplanar graph on n vertices has at most $2n - 3$ edges if $n \geq 2$. These properties will be used frequently in our proof. For some nice article on minors in spectral graph theory, we refer to [15].

Our proof of Theorem 2 also needs a well-known fact and an upper bound of the spectral radius of an outerplanar graph as following:

Lemma 1 ([1, Exercise 11.2.7]). *Let G be an edge-maximal outerplanar graph of order $n \geq 3$. Then G has a planar embedding whose outer face is a Hamilton cycle, all other faces being triangles.*

Lemma 2 (Shu, Hong [14]). *Let G be a connected outerplanar graph on $n \geq 3$ vertices. Then $\lambda(G) \leq \frac{3}{2} + \sqrt{n - \frac{7}{4}}$.*

Now we present a proof of Theorem 2.

Proof of Theorem 2. For any integer $n \geq 17$, let G_n be an outerplanar graph which attains the maximum spectral radius among all outerplanar graphs of order n, and let $\lambda := \lambda(G_n)$ be its spectral radius. In the rest, we use G instead of G_n for convenience. Obviously, G is connected and maximal. By the Perron-Frobenius Theorem, G has the Perron vector such that each component is positive. Let X be a normalized one such that maximum entry is 1. For any vertex $v \in V(G)$, we write x_v for the eigenvector entry which corresponds to v. Let $u \in V(G)$ such that $x_u = 1$, $A = N_G(u)$ and $B = V(G) - (\{u\} \cup A)$.

The first claim gives us a nearly tight lower bound of λ.

Claim 1. $\lambda \geq \sqrt{n} + 1 - \frac{1}{n-\sqrt{n}}$.

Proof. Let $\Gamma = K_1 \vee C_{n-1}$, where C_{n-1} denotes a cycle on $n-1$ vertices. Suppose that $Y = (y_1, y_2, \ldots, y_n)^t$ is the Perron vector of Γ, where y_1 corresponds to the vertex of degree $n-1$. By symmetry, $y_2 = y_3 = \cdots = y_n$. Then $\lambda(\Gamma)y_1 = (n-1)y_2$, $\lambda(\Gamma)y_2 = y_1 + 2y_2$ and $y_1^2 + (n-1)y_2^2 = 1$. It follows that $\lambda(\Gamma) = 1 + \sqrt{n}$ and $y_2^2 = \frac{1}{2(n-\sqrt{n})}$. Let $e \in E(C_{n-1})$ and $\Gamma' = \Gamma - e$. Then by Rayleigh principle, $\lambda(\Gamma') \geq Y^t A(\Gamma') Y = Y^t A(\Gamma) Y - 2y_2^2 = \sqrt{n} + 1 - \frac{1}{n-\sqrt{n}}$. Obviously, Γ' is outerplanar, and $\lambda(G) \geq \lambda(\Gamma') \geq \sqrt{n} + 1 - \frac{1}{n-\sqrt{n}}$, as required. □

As a warm up, we quickly determine the structure of $G[A]$ approximately.

Claim 2. $G[A]$ is a union of disjoint induced paths or an induced path. (In particular, we also view an isolated vertex in $G[A]$ as an induced path.)

Proof. We first claim that $G[A]$ contains no vertex of degree at least 3 in A. If not, then there is a $K_{2,3}$ in $G[A \cup \{u\}]$, a contradiction.

We then claim that there is no cycle in $G[A]$. Suppose to the contrary that there is a cycle in $G[A]$. Then we can contract the cycle into a triangle, and there is a K_4 in the resulting graph. That is, there is a K_4-minor in G, a contradiction.

From the two claims mentioned above, we conclude that $G[A]$ is the union of some induced paths or an induced paths, in which we view each isolated vertex as an induced path. □

Let

$$S = \{v : v \in A, d_{G[A]}(v) = 1\}.$$

For two vertices $x, y \in V(G)$, we write $x \sim y$ if x is adjacent to y. By Claim 1, we have $d(u) := d_u \geq \lambda \geq \sqrt{n} + 1 - \frac{1}{n-\sqrt{n}} > 5$.

We want to show that d_u is very close to $n-1$. As a first step, we must associate d_u with λ by the following.

Claim 3.

$$\lambda^2 \leq d_u + 2\lambda - \frac{2}{\sqrt{n - \frac{7}{4}} + \frac{3}{2}} + \sum_{v \in B} d_A(v) x_v. \tag{1}$$

Proof. Note that for any $v \in S$, we have $\lambda x_v > x_u = 1$. By Lemma 2, we obtain $x_v > \frac{1}{\lambda} \geq \frac{1}{\frac{3}{2} + \sqrt{n - \frac{7}{4}}}$. The first equality below was used by Tait and Tobin (see the proof of Lemma 4 in [16]), which also appeared in several references, see [9] for example:

$$\lambda^2 = \lambda^2 x_u = d_u + \sum_{y \sim u} \sum_{z \in N(y) \cap A} x_z + \sum_{y \sim u} \sum_{z \in N(y) \cap B} x_z = d_u + \sum_{v \in A} d_A(v) x_v + \sum_{v \in B} d_A(v) x_v.$$

If $G[A]$ consists of isolated vertices, i.e., without any edge, then $\sum_{v\in A} d_A(v)x_v = 0$. Thus, we have

$$\lambda^2 = d_u + \sum_{v\in B} d_A(v)x_v.$$

Otherwise, $G[A]$ contains at least one edge, and it follows $|S| \geq 2$. We have

$$\begin{aligned}\lambda^2 &= d_u + \sum_{v\in A} d_A(v)x_v + \sum_{v\in B} d_A(v)x_v \\ &= d_u + \sum_{v\in S} x_v + \sum_{v\in\{v\in A: d_A(v)=2\}} 2x_v + \sum_{v\in B} d_A(v)x_v \\ &\leq d_u + 2\lambda - \sum_{v\in S} x_v + \sum_{v\in B} d_A(v)x_v \\ &\leq d_u + 2\lambda - \frac{2}{\frac{3}{2}+\sqrt{n-\frac{7}{4}}} + \sum_{v\in B} d_A(v)x_v.\end{aligned}$$

Since $2\lambda - \frac{2}{\frac{3}{2}+\sqrt{n-\frac{7}{4}}} > 0$ for $n \geq 16$, we have proved the claim. □

Our goal of the most of the rest is to show that $|B| \leq 1$ firstly, and then show $B = \emptyset$. We prove this fact by contradiction. Suppose to the contrary that

$$|B| \geq 2. \tag{2}$$

Since G is outerplanar, $G[B]$ is also outerplanar, and so $e(G[B]) \leq 2|B| - 3$ by (2). In the rest, let $B_1, B_2, \ldots, B_t$ be the vertex sets of all components of $G[B]$, respectively. The coming claim gives a tight upper bound of the sum of all degrees of vertices of B in G, which plays a central role in our proof. Since adding a new edge can increase the value of the spectral radius (recall G is connected), G is a maximal outerplanar graph. Therefore, Lemma 1 can be used below.

Claim 4. (i) For each $i \in [1,t]$, $d_A(B_i) = 2$. (ii) If $|B_i| \geq 2$, then $2e(G[B_i]) + e(A, B_i) \leq 4|B_i| - 3$. In particular, $2e(G[B]) + e(A,B) \leq 4|B| - 3$ (recall $|B| \geq 2$).

Proof. (i) Since G is $K_{2,3}$-minor free, B_i has at most 2 neighbors in A for any $i \in [1,t]$. Indeed, if not, we contract all vertices of B_i into a single vertex, and would find a $K_{2,3}$ in the resulting graph. Thus, $d_A(B_i) \leq 2$. Recall that there is a Hamilton cycle in G. Thus, $d_A(B_i) = 2$. This proves Claim 4 (i).

(ii) By Claim 4 (i), we can assume that $N_A(B_i) = \{x, x'\}$ for any $i \in [1,t]$. By Lemma 1, there is a planar embedding of G, say $\widetilde{G}$, such that its outer-face is a Hamilton cycle. Let $P := xp_1p_2\cdots p_sx'$ be the (x,x')-path on the Hamilton cycle passing through all vertices in B_i. That is, $B_i = \{p_1, \ldots, p_s\}$. In the rest of the proof, when there is no danger of ambiguity, we do not distinguish G and $\widetilde{G}$.

Suppose that $|B_i| \geq 2$. We first claim that there are no subscripts j,k such that $1 \leq j < k \leq s$ and $xp_k, x'p_j \in E(G)$. Suppose not. Then we first contract three paths $p_1\ldots p_j$, $p_k\ldots p_s$ and xux' into vertices w_1, w_2 and an edge xx', respectively, and then contract the path $w_1p_{j+1}\ldots p_{k-1}w_2$ into an edge w_1w_2, resulting in a K_4. In this way, we can find a K_4-minor in G, a contradiction. In the following, set $l_1 := \max\{q : p_qx \in E(G)\}$ and $l_2 := \min\{q : p_qx' \in E(G)\}$. Therefore $l_1 \leq l_2$. Also, $G_1 := G[\{x, p_1, \ldots, p_{l_1}\}]$ is outerplanar, and hence $e(G_1) \le 2(l_1+1)-3 = 2l_1-1$. Note that $G_2 := G[p_{l_1},\dots,p_{l_2},x,x'] - xx'$ is outerplanar. Thus, if $l_2 \ge l_1+1$, then $e(G_2) \le e(G[\{p_{l_1},\dots,p_{l_2}\}]) + 2 \le 2(l_2-l_1+1) - 3 + 2 = 2(l_2-l_1)+1$; if $l_1 = l_2$ then $e(G_2) = 2$. Let $G_3 := G[\{p_{l_2},\dots,p_s,x'\}]$. Then $e(G_3) \le 2(s-l_2+1+1) - 3 = 2(s-l_2)+1$.

Observe that for any $i \in [1,l_1]$ and $j \in [l_2,s]$ such that $j \ge i+2$, we have $p_ip_j \notin E(G)$, since otherwise we can find a K_4-minor in G similarly as above. Hence $e(G[B_i \cup \{x,x'\}] - xx') = e(G_1)+e(G_2)+e(G_3)-2$, where the term "-2" comes from the fact that the edges $xp_{l_1}, x'p_{l_2}$ are counting twice when we compute the value of $e(G_1)+e(G_2)+e(G_3)$.

If $l_2 \ge l_1+1$, then $e(G[B_i\cup\{x,x'\}]-xx') = e(G_1)+e(G_2)+e(G_3)-2 \le (2l_1-1) + (2(l_2-l_1)+1) + (2(s-l_2)+1) - 2 = 2s-1$, Thus, $2e(G[B_i]) + e(A,B_i) \le 2e(G[B_i \cup \{x,x'\}] - xx') - e(A,B_i) \le 2(2s-1)-3 = 4s-5$, where $e(A,B_i) \ge 3$ since $|B_i| \ge 2$ and each face inside $\widetilde{G}[B_i \cup \{x,x'\}]$ is a triangle.

If $l_2 = l_1$, then $e(G_2) = 2$. In this case, $e(G[B_i \cup \{x,x'\}] - xx') = e(G_1)+e(G_2)+e(G_3)-2 \le (2l_1-1)+2+(2(s-l_2)+1)-2 = 2s$. Then $2e(G[B_i]) + e(A,B_i) \le 2e(G[B_i\cup\{x,x'\}]-xx') - e(A,B_i) \le 2\cdot(2s)-3 = 4s-3$, where $e(A,B_i) \ge 3$ since $|B_i| \ge 2$.

Thus, for any $i \in [1,t]$ with $|B_i| \ge 2$, we have $2e(G[B_i]) + e(A,B_i) \le 4s-3$. If $|B_i| = 1$ then $2e(G[B_i]) + e(A,B_i) \le 2$. Summing over all indices i, we have $e(B,A) + 2e(G[B]) \le 4|B|-3$. This proves Claim 4 (ii). □

By using Claim 4 (ii), we can estimate the upper bound of $\sum_{v\in B} d_A(v)x_v$ as follows.

Claim 5.

$$\sum_{v\in B} d_A(v)x_v \le \frac{5n-5d_u-7}{\sqrt{n}+1-\frac{1}{n-\sqrt{n}}}. \tag{3}$$

Proof. Recall that $B_1, B_2, \dots, B_t$ are all components of $G[B]$. For any $i \in [1,t]$, by Claim 4 (i), B_i has two neighbors in A. Since G contains no $K_{2,3}$, there is at most one vertex in B_i with two neighbors in A. Set $x_i' := \max\{x_v : v \in B_i\}$. Thus, if $|B_i| \ge 2$ then

$$\begin{aligned}\sum_{v\in B_i} d_A(v)x_v &\le \sum_{v\in B_i} x_v + x_i' = \frac{1}{\lambda}(\sum_{v\in B_i}\lambda x_v + \lambda x_i')\\ &\le \frac{1}{\lambda}(\sum_{v\in B_i} d_G(v) + (|B_i|-1+2))\\ &= \frac{1}{\lambda}\left(e(A,B_i) + 2e(G[B_i]) + |B_i| + 1\right)\\ &= \frac{1}{\lambda}(5|B_i|-2).\end{aligned}$$

If $|B_i| = 1$ then $\sum_{v\in B_i} d_A(v)x_v \le \frac{2}{\lambda}\sum_{w\in N_A(B_i)} x_w \le \frac{4}{\lambda} = \frac{1}{\lambda}(5|B_i|-1)$. Observe that if $|B_i| = 1$ for every i, then $t \ge 2$ since $|B| \ge 2$. Summing over all $i \in [1,t]$, we have

$$\sum_{v\in B} d_A(v)x_v \le \frac{5|B|-2}{\sqrt{n}+1-\frac{1}{n-\sqrt{n}}} = \frac{5n-5d_u-7}{\sqrt{n}+1-\frac{1}{n-\sqrt{n}}}.$$

This proves the claim. □

In what follows, we aim to show that

$$\left(1-\frac{5}{\sqrt{n}+1-\frac{1}{n-\sqrt{n}}}\right)d_u > \max\left\{(n-1)\cdot\left(1-\frac{5}{\sqrt{n}+1-\frac{1}{n-\sqrt{n}}}\right),0\right\}. \tag{4}$$

holds for $n \geq 17$. This finally results in $d(u) > n-1$, and implies that $|B| \geq 2$ does not hold.

By (1), (2) and (3), we infer

$$\begin{aligned}
&\left(1-\frac{5}{\sqrt{n}+1-\frac{1}{n-\sqrt{n}}}\right)d_u \\
&\geq \lambda^2 - 2\lambda + \frac{2}{\frac{3}{2}+\sqrt{n-\frac{7}{4}}} - \frac{5n-7}{\sqrt{n}+1-\frac{1}{n-\sqrt{n}}} \\
&\geq n-1-\frac{2}{\sqrt{n}-1}+\frac{1}{n(\sqrt{n}-1)^2}+\frac{2}{\frac{3}{2}+\sqrt{n-\frac{7}{4}}}-\frac{5n-7}{\sqrt{n}+1-\frac{1}{n-\sqrt{n}}} \\
&> (n-1)\cdot\left(1-\frac{5}{\sqrt{n}+1-\frac{1}{n-\sqrt{n}}}\right)-\frac{2}{\sqrt{n}-1}+\frac{2}{\frac{3}{2}+\sqrt{n-\frac{7}{4}}}+\frac{2}{\sqrt{n}+1} \\
&> (n-1)\cdot\left(1-\frac{5}{\sqrt{n}+1-\frac{1}{n-\sqrt{n}}}\right)+\frac{2\sqrt{n-\frac{7}{4}}-7}{n-1} \\
&\geq (n-1)\cdot\left(1-\frac{5}{\sqrt{n}+1-\frac{1}{n-\sqrt{n}}}\right) > 0
\end{aligned}$$

for $n \geq 17$. (Note that $(1-\frac{5}{\sqrt{n}+1-\frac{1}{n-\sqrt{n}}}) < 0$ when $n = 16$.)

Therefore, we have $|B| \leq 1$. Suppose that $|B| = 1$. At this point, we can know more information on $G[A]$ than Claim 2.

Claim 6. $G[A]$ is an induced path.

Proof. By Claim 2, $G[A]$ is a union of disjoint induced paths or an induced path. Since G is a maximal outerplanar graph, by Lemma 1, G has a planar embedding, say $\widetilde{G}$, whose outer face is a Hamilton cycle, all other faces being triangles. If $G[A]$ is not an induced path, then the fact $|B| = 1$ implies there is an inner face in $\widetilde{G}$ which is not a triangle, a contradiction. This proves the claim. □

Finally, we show that, indeed, B is an empty set.

Claim 7. $B = \emptyset$.

Proof. Suppose that $|B| = 1$. Let $B = \{v\}$. Since G is $K_{2,3}$-free, we have $d(v) = 2$. Set $N(v) = \{v_i, v_j\}$. Recall that $G[A]$ is an induced path. Let $G[A] = v_1v_2\ldots v_{n-2}$. If $|i-j| \neq 1$, then G contains a $K_{2,3}$-minor, a contradiction. Thus, $|i-j| = 1$. Without loss of generality, set $j = i+1$. Since $d_u > 5$, at least one vertex of $\{v_i, v_{i+1}\}$ has degree two in $G[A]$. Moreover, $vv_i, vv_{i+1} \in E(G)$. Let $X = (x_u, x_1, \ldots, x_{n-2}, x_v)^t$ be the eigenvector corresponding to $\lambda(G)$, where $x_u = 1$, x_v corresponds to v, and x_k corresponds to v_k for $k = 1, \ldots, n-2$. Set $x_s := \max\{x_k : k = 1, \ldots, n-2\}$. By Claim 1, $\lambda \geq \sqrt{n}+1-\frac{1}{n-\sqrt{n}} \geq 5$.

Then $\lambda x_v = x_i + x_{i+1} \leq 2x_s$, which implies that $x_v < x_s$. Since $\lambda x_s \leq x_u + \sum_{v_k \sim v_s} x_k + x_v < 1 + 3x_s$, it follows that $x_s < \frac{1}{\lambda-3}$. Also, since $\lambda x_1 > x_u$, we have $x_1 > \frac{1}{\lambda}$.

Now let $G' := G - vv_i - vv_{i+1} + vu + vv_1$. Note that G' is also outerplanar and $\lambda(G') - \lambda(G) \geq 2X^t(A(G') - A(G))X = 2x_v(x_u + x_1 - x_i - x_{i+1}) > 2x_v(1 + \frac{1}{\lambda} - \frac{2}{\lambda-3})$. By simple algebra, $1 + \frac{1}{\lambda} - \frac{2}{\lambda-3} = \frac{\lambda^2-4\lambda-3}{\lambda(\lambda-3)}$. In order to prove the inequality $\frac{\lambda^2-4\lambda-3}{\lambda(\lambda-3)} > 0$, it suffices to show $\lambda > 2+\sqrt{7}$. By Claim 1,$\lambda > 2+\sqrt{7}$ when $n \geq 16$. Therefore $\lambda(G') > \lambda(G)$, a contradiction. This proves the claim. □

It follows that $G = K_1 \vee P_{n-1}$, completing the proof. ■

The case of $2 \leq n \leq 16$.

Throughout this part, we use the notation NIM-outerplanar graphs instead of non-isomorphic maximal outerplanar graphs.

Let G be a maximal outerplanar graph with order n and vertex set $V(G) = \{v_1, \ldots, v_n\}$. By Lemma 1, G has a planar embedding, say $\widetilde{G}$, such that the outer-face is a Hamilton cycle. One can easily find such a Hamilton cycle is unique, since otherwise there is a K_4-minor in G. In the following, we do not distinguish a maximal outerplanar graph and its planar embedding when there is no ambiguity. Let $v_1 \ldots v_n v_1$ be the Hamilton cycle mentioned above.

A property of a maximal outerplanar graph on $n \geq 3$ vertices states that every such graph has a vertex of degree 2 and has a subgraph which is also maximal outerplanar by deleting the vertex. Let G' be a maximal outerplanar graph obtained from G by adding a new vertex v_{n+1}, which is adjacent to some vertex(vertices) of G. Since $e(G') = 2(n+1) - 3 = e(G) + 2$, we have $d_{G'}(v_{n+1}) = 2$. Furthermore, $N_{G'}(v_{n+1}) = \{v_i, v_{i+1}\}$ for some $i \in [1, n]$. Let $S(n)$ denote the number of NIM-outerplanar graphs with order n. It follows that $S(n+1) \leq nS(n)$. In fact, by using matlab, the number of NIM-outerplanar graphs with order at most 13 can be computed as shown in Table 1. In particular, $S(14) \leq 29666$, $S(15) \leq 415324$ and $S(16) \leq 6229860$.

n	6	7	8	9	10	11	12	13	14	15	16
$S(n)$	3	4	12	27	82	228	733	2282	≤ 29666	≤ 415324	≤ 6229860

Table 1: The numbers of NIM-outerplanar graphs with order at least 6 and at most 16.

For the case of $2 \leq n \leq 5$, one can check by hand calculation. For the case of $6 \leq n \leq 13$, the problem can be completely solved by a computer. When $14 \leq n \leq 16$, we first develop a program to determine all NIM-outerplanar graphs of order 13, and then compare the spectral radius of each graph in the family of no more than 29666 (415324, 6229860) graphs and of $K_1 \vee P_{n-1}$, respectively (see https://github.com/HuiqiuLin-83/Outerplanar-graph/ for the code). Furthermore, we find out $\lambda(G_1) = 3.2361 > \lambda(K_1 \vee P_5) = 3.2227$. In summary, we get the following result (together with Theorem 2).

Theorem 3. *Among all outerplanar graphs on n vertices, $K_1 \vee P_{n-1}$ attains the maximum spectral radius, with the only exceptional case of $n = 6$, in which G_1 attains the maximum spectral radius. (see Figure 1).*

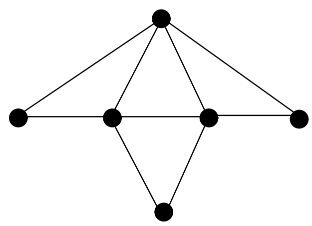

Figure 1: The graph G_1.

Concluding remarks

We would like to compare our proof with the methods applied by Tait and Tobin in [16]. Generality speaking, the Tait-Tobin Method is somewhat motivated by Regularity Lemma. For any $\varepsilon > 0$, they partitioned the vertex set $V(G)$ into two parts $V_L = \{v \in V(G) : x_v > \varepsilon\}$ and $V_S = V(G)\backslash V_L$. "This enables them to determine the structure on large (linear sized) pieces of the graph to understand approximately what the extremal graph looks like."[1] Our proof heavily relies on the complete characterization of outerplanar graphs, and the concept of "minor" plays an important role in our proof. More importantly for us, we need a Hamilton cycle in a maximal outerplanar graph to label the vertices in order. We need much more details (see Claims 4 and 5) to control the bound of eigenvector entries, besides using Shu-Hong's inequality.

On the other hand, the Tait-Tobin Method seems to be powerful for many problems (on large graphs) in spectral graph theory. Till now, it has been successfully used to make progress on Cvetković-Rowlinson Conjecture [6], Boots-Royle-Cao-Vince Conjecture [2, 3], and Cioabă-Gregory Conjecture [5], etc. We would like to refer to the very recent spectral version [4] of extremal numbers of friendship graphs [8].

In closing, we shall mention the following conjecture again, which is still open for small n. (For example, let us confirm this conjecture for all $n \geq 100$.) We note that Tait and Tobin [16] verified it for sufficiently large n.

Conjecture 2 (Boots-Royle [2], and independently by Cao-Vince [3])**.** The planar graph on $n \geq 9$ vertices of maximum spectral radius is $P_2 \vee P_{n-2}$.

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References

[1] J.A. Bondy, U.S.R. Murty, Graph theory. Graduate Texts in Mathematics, 244. Springer, New York, 2008. xii+651 pp. ISBN: 978-1-84628-969-9.

[2] B.N. Boots, G.F. Royle, A conjecture on the maximum value of the principal eigenvalue of a planar graph, *Geogr. Anal.* **23** (3) (1991), 276–282.

[1] This sentence is essentially borrowed from an email from M. Tait to the authors, who explained the ideas on the Tait-Tobin Method to them.

[3] D. Cao, A. Vince, The spectral radius of a planar graph, *Linear Algebra Appl.* **187** (1993), 251–257.

[4] S.M. Cioabă, L. Feng, M. Tait, X. Zhang, The maximum spectral radius of graphs without friendship subgraphs, *Electron. J. Combin.* **27** (2020) no. 4, P4.22.

[5] S.M. Cioabă, D.A. Gregory, Principal eigenvectors of irregular graphs, *Electron. J. Linear Algebra* **16** (2007), 366–379.

[6] D. Cvetković, P. Rowlinson, The largest eigenvalue of a graph: a survey, *Linear Multilinear Algebra* 28(1-2) (1990), 3–33.

[7] M N. Ellingham, X. Zha, The spectral radius of graphs on surfaces, *J. Combin. Theory, Ser. B* 78 (1) (2000), 45–56.

[8] P. Erdős, Z. Füredi, R.J. Gould, D.S. Gunderson, Extremal graphs for intersecting triangles, *J. Combin. Theory, Ser. B* **64** (1995), 89–100.

[9] O. Favaron, M. Mahéo, J-F. Saclé, Some eigenvalue properties in graphs (conjectures of Graffiti II), *Discrete Math.* **111** (1) (1993), 197–220.

[10] Y. Hong, A bound on the spectral radius of graphs, *Linear Algebra Appl.* **108** (1988), 135–139.

[11] Y. Hong, On the spectral radius and the genus of graphs, *J. Combin. Theory, Ser. B* **65** (2) (1995), 262–268.

[12] P. Rowlinson, On the index of certain outerplanar graphs, *Ars Combin.* **29** (1990), 221–225.

[13] A. J. Schwenk, R. J. Wilson, On the eigenvalues of a graph, in: Lowell W. Beineke, Robin J. Wilson (Eds.), Selected Topics in Graph Theory, Academic Press, London, 1978, pp. 307–336, chapter 11.

[14] J. Shu, Y. Hong, Upper bounds for the spectral radii of outerplanar graphs and Halin graphs (in Chinese), *Chinese Ann. Math. Ser. A* **21** (2000), no. 6, 677–682.

[15] M. Tait, The Colin de Verdière parameter, excluded minors, and the spectral radius, *J. Combin. Theory, Ser. A* **166** (2019), 42–58.

[16] M. Tait, J. Tobin, Three conjectures in extremal spectral graph theory, *J. Combin. Theory, Ser. B* **126** (2017), 137–161.

[17] M. Tait, J. Tobin, Characterizing graphs of maximum principal ratio, *Electron. J. Linear Algebra* **34** (2018), 61–70.

[18] Z. Yu, L. Kang, L. Liu, E. Shan, The extremal α-index of outerplanar and planar graphs, *Appl. Math. Comput.* **343** (2019), 90–99.