On the maximum number of maximum independent sets in connected graphs

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Abstract

We characterize the connected graphs of given order n and given independence number α that maximize the number of maximum independent sets. For $3 \leq \alpha \leq n/2$, there is a unique such graph that arises from the disjoint union of α cliques of orders $\lceil \frac{n}{\alpha} \rceil$ and $\lfloor \frac{n}{\alpha} \rfloor$, by selecting a vertex x in a largest clique and adding an edge between x and a vertex in each of the remaining $\alpha - 1$ cliques. Our result confirms a conjecture of Derikvand and Oboudi [On the number of maximum independent sets of graphs, Transactions on Combinatorics 3 (2014) 29-36].

1 Introduction

Moon and Moser's [5] classical result on the number of maximal cliques immediately yields a characterization of the graphs of a given order that have the maximum number of maximum independent sets. Similarly, the characterization of the connected graphs of a given order with that property follows from a result of Griggs, Grinstead, and Guichard [2]; see [3]. Using a result of Zykov [9] allows to characterize the graphs of a given order and a given independence number that have the maximum number of maximum independent sets; see Theorem 1 below. Our contribution in the present paper is the connected version of this result; that is, we characterize the connected graphs of a given order and a given independence number that have the maximum number of maximum independent sets. Our results confirm a recent conjecture of Derikvand and Oboudi [1].

We consider only finite, simple, and undirected graphs, and use standard terminology and notation. An *independent set* in a graph G is a set of pairwise non-adjacent vertices of G. The *independence number* $\alpha(G)$ of G is the maximum cardinality of an independent set in G. An independent set in G is *maximum* if it has cardinality $\alpha(G)$. For a graph G, let $\sharp\alpha(G)$ be the number of maximum independent sets in G. For a vertex u of G, let $\sharp\alpha(G, u)$ be the number of maximum independent sets in G that contains u.

Let n and α be positive integers with $\alpha < n$.

Let the graph $G(n,\alpha)$ be the disjoint union of one clique C_0 of order $\lceil\frac{n}{\alpha}\rceil$, and $\alpha-1$ further cliques $C_1,\ldots,C_{\alpha-1}$ of orders $\lceil\frac{n}{\alpha}\rceil$ and $\lfloor\frac{n}{\alpha}\rfloor$, that is, the graph $G(n,\alpha)$ is the complement of the *Turán graph* of order n and clique number α. Let the graph $F(n,\alpha)$ arise from $G(n,\alpha)$ by adding the edges $x_0x_1,\ldots,x_0x_{\alpha-1}$, where x_i is a vertex in C_i for every i in $\{0,\ldots,\alpha-1\}$. We will call the vertex x_0 the *special cutvertex* of $F(n,\alpha)$. Note that x_0 may not be unique if $\alpha\leq 2$, and n is a multiple of α.

For $\frac{n}{\alpha}\geq 2$, let

$$\mathcal{F}(n,\alpha)=\begin{cases}\big\{F(n,\alpha),C_5\big\} & \text{, if } (n,\alpha)=(5,2), \text{ and}\\ \big\{F(n,\alpha)\big\} & \text{, otherwise,}\end{cases}$$

where C_5 denotes the cycle of order 5, and for $\frac{n}{\alpha}<2$, let $\mathcal{F}(n,\alpha)$ be the set of all connected graphs G that have a vertex x_0 such that $G-x_0$ is isomorphic to $G(n-1,\alpha)$. It is easy to see that every graph in $\mathcal{F}(n,\alpha)$ for $\frac{n}{\alpha}<2$ is isomorphic to a graph that arises from $F(n,\alpha)$ by possibly adding further edges incident with the special cutvertex x_0 of $F(n,\alpha)$.

See Figure 1 for an illustration.

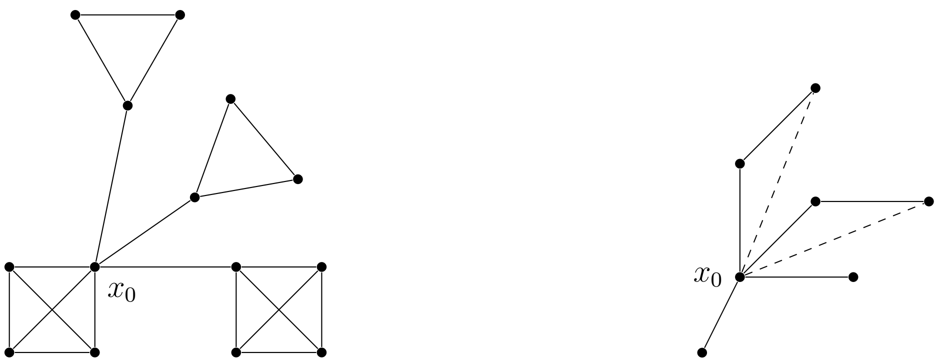

Figure 1: The graph $F(14,4)$ on the left and a member of $\mathcal{F}(7,4)$ on the right, where dashed lines are potential edges.

The graph $G(n,\alpha)$ has exactly $\alpha-n \bmod \alpha$ components of order $\lfloor\frac{n}{\alpha}\rfloor$, which implies

$$\sharp\alpha(G(n,\alpha))=g(n,\alpha):=\left\lfloor\frac{n}{\alpha}\right\rfloor^{\alpha-(n \bmod \alpha)}\left\lceil\frac{n}{\alpha}\right\rceil^{n \bmod \alpha}.$$

For $\frac{n}{\alpha}\geq 2$, we have that $F(n,\alpha)-x_0$ is isomorphic to $G(n-1,\alpha)$, which implies

$$\sharp\alpha(F(n,\alpha)) \;=\; f(n,\alpha):=g(n-1,\alpha)+\left(\left\lfloor\frac{n}{\alpha}\right\rfloor-1\right)^{\alpha-n \bmod \alpha}\left(\left\lceil\frac{n}{\alpha}\right\rceil-1\right)^{n \bmod \alpha-1},$$

where the term added to $g(n-1,\alpha)$ counts the maximum independent sets in $F(n,\alpha)$ that contain x_0. For $\frac{n}{\alpha}<2$, the added term evaluates to 0, that is, $f(n,\alpha)$ equals $g(n-1,\alpha)$.

Furthermore, we obtain $\alpha \leq n-1 \leq 2\alpha - 2$, which implies that $G(n-1,\alpha)$ has

$$\alpha - (n-1) \bmod \alpha = \alpha - (n-1-\alpha) = 2\alpha - n + 1 \geq 2$$

isolated vertices. This implies that the vertex x_0 whose removal from a graph G in $\mathcal{F}(n,\alpha)$ yields $G(n-1,\alpha)$ does not belong to any maximum independent set in G for $\frac{n}{\alpha} < 2$. Hence, also in this case, we obtain

$$\sharp\alpha(G) = f(n,\alpha) = g(n-1,\alpha)$$

for every graph G in $\mathcal{F}(n,\alpha)$.

Note that

$$\sharp\alpha(C_5) = \sharp\alpha(F(5,2)) = f(5,2) = 5.$$

The following result is an immediate consequence of Zykov's generalization [9] of Turán's theorem [7]; see [4] for a simple proof.

Theorem 1. *If G is a graph of order n and independence number α with $\alpha < n$, then $\sharp\alpha(G) \leq g(n,\alpha)$ with equality if and only if G is isomorphic to $G(n,\alpha)$.*

Our contribution in the present paper is the following connected version of Theorem 1, which was recently conjectured by Derikvand and Oboudi [1].

Theorem 2. *If G is a connected graph of order n and independence number α with $\alpha < n$, then $\sharp\alpha(G) \leq f(n,\alpha)$ with equality if and only if G is isomorphic to a graph in $\mathcal{F}(n,\alpha)$.*

In [1], Derikvand and Oboudi verify Theorem 2 for $\alpha \in \{1, 2, n-3, n-2, n-1\}$, that is, for very small and very large values of the independence number. The maximum number of maximum/maximal independent sets has been studied in some further classes of graphs, and we refer the reader to [3, 4, 6, 8].

The rest of the paper is devoted to the proof of our main result.

2 Proof of Theorem 2

We begin with two preparatory lemmas.

Lemma 3. *Let G be a connected graph of order n and independence number α with $\alpha < n$. If some vertex u of G is contained in no maximum independent set in G, then $\sharp\alpha(G) \leq f(n,\alpha)$ with equality if and only if $G \in \mathcal{F}(n,\alpha)$.*

Proof. By the hypothesis and Theorem 1, we obtain $\sharp\alpha(G) = \sharp\alpha(G-u) \leq g(n-1,\alpha)$ with equality if and only if $G-u$ is isomorphic to $G(n-1,\alpha)$. It follows that $\sharp\alpha(G) \leq g(n-1,\alpha) \leq f(n,\alpha)$, and that $\sharp\alpha(G) = f(n,\alpha)$ holds if and only if $G-u$ is isomorphic to $G(n-1,\alpha)$, and $g(n-1,\alpha) = f(n,\alpha)$. Since $g(n-1,\alpha) = f(n,\alpha)$ implies $\frac{n}{\alpha} < 2$, the definition of $\mathcal{F}(n,\alpha)$ for $\frac{n}{\alpha} < 2$ implies that $\sharp\alpha(G) = f(n,\alpha)$ holds if and only if $G \in \mathcal{F}(n,\alpha)$. $\square$

The second lemma concerns graphs whose structure is similar to the structure of the graphs in $\mathcal{F}(n,\alpha)$.

Lemma 4. *Let n and α be positive integers with $\alpha < n$.*

(i) Let G be a connected graph of order n and independence number α, whose vertex set is the disjoint union of the vertex sets of α cliques $C_0,\ldots,C_{\alpha-1}$. Let all edges of G that do not lie in one of these cliques be incident with a vertex x_0 in C_0, and let x_0 have exactly one neighbor in each of the cliques $C_1,\ldots,C_{\alpha-1}$.

Under these assumptions $\sharp\alpha(G)\leq f(n,\alpha)$ with equality if and only if G is isomorphic to $F(n,\alpha)$.

(ii) Let the graph G' arise from $F(n,\alpha)$ by adding an edge uv between two non-adjacent vertices of $F(n,\alpha)$.

If $\frac{n}{\alpha}\geq 2$, then $\alpha(G')=\alpha$ and $\sharp\alpha(G')<f(n,\alpha)$, and,

if $\frac{n}{\alpha}<2$, and u and v are distinct from the special cutvertex x_0 of $F(n,\alpha)$, then

either $\alpha(G')<\alpha$

or $\alpha(G')=\alpha$ and $\sharp\alpha(G')<f(n,\alpha)$.

Proof. (i) If C_i has order n_i for i in $\{0,\ldots,\alpha-1\}$, then

$$\sharp\alpha(G) = (n_0-1)\prod_{k=1}^{\alpha-1}n_k+\prod_{k=1}^{\alpha-1}(n_k-1)=\prod_{k=0}^{\alpha-1}n_k-\prod_{k=1}^{\alpha-1}n_k+\prod_{k=1}^{\alpha-1}(n_k-1).$$

In view of the desired statement, we may assume that the n_i are such that $\sharp\alpha(G)$ is as large as possible. By symmetry, we may assume $n_1\geq\ldots\geq n_{\alpha-1}$. In order to complete the proof, it suffices to show that $n_0\geq n_1$ and $n_{\alpha-1}\geq n_0-1$.

If $n_i=1$ for some $i\in\{0,\alpha-1\}$, then every maximum independent set in G contains the unique vertex, say u, in C_i. It follows that some neighbor, say v, of u belongs to no maximum independent set in G, and Lemma 3 implies the desired statement. Hence, we may assume $n_0,n_{\alpha-1}\geq 2$.

First, we suppose that that $n_0+1\leq n_1$. Moving one vertex from C_1 to C_0 results in a graph G' of order n and independence number α with

$$\begin{aligned}\sharp\alpha(G') &= n_0\left(\frac{n_1-1}{n_1}\right)\prod_{k=1}^{\alpha-1}n_k+\left(\frac{n_1-2}{n_1-1}\right)\prod_{k=1}^{\alpha-1}(n_k-1)\\ &= \prod_{k=0}^{\alpha-1}n_k-n_0\prod_{k=2}^{\alpha-1}n_k+\prod_{k=1}^{\alpha-1}(n_k-1)-\prod_{k=2}^{\alpha-1}(n_k-1).\end{aligned}$$

Since

$$\sharp\alpha(G')-\sharp\alpha(G) = \prod_{k=1}^{\alpha-1}n_k-n_0\prod_{k=2}^{\alpha-1}n_k-\prod_{k=2}^{\alpha-1}(n_k-1)$$

$$
\begin{aligned}
&= (n_1 - n_0)\prod_{k=2}^{\alpha-1} n_k - \prod_{k=2}^{\alpha-1}(n_k - 1) \\
&> 0,
\end{aligned}
$$

we obtain a contradiction to the choice of the n_i.

Next, we suppose that $n_{\alpha-1} \leq n_0 - 2$. Moving a vertex from C_0 to $C_{\alpha-1}$ results in a graph G' of order n and independence number α with

$$
\sharp\alpha(G') = (n_0 - 1)\frac{(n_0 - 2)(n_{\alpha-1} + 1)}{(n_0 - 1)n_{\alpha-1}}\prod_{k=1}^{\alpha-1} n_k + \left(\frac{n_{\alpha-1}}{n_{\alpha-1} - 1}\right)\prod_{k=1}^{\alpha-1}(n_k - 1).
$$

Since $\frac{(n_0-2)(n_{\alpha-1}+1)}{(n_0-1)n_{\alpha-1}} > 1$ and $\frac{n_{\alpha-1}}{n_{\alpha-1}-1} > 1$, we obtain $\sharp\alpha(G') > \sharp\alpha(G)$, which is a contradiction to the choice of the n_i, and completes the proof of (i).

(ii) We leave the simple proof of this to the reader. □

We proceed to the proof of our main result.

Proof of Theorem 2. Suppose, for a contradiction, that the theorem fails, and that n is the smallest order of a counterexample G_0, which has independence number α. Since the result is easily verified for $n \leq 5$ or $\alpha = 1$, we may assume that $n \geq 6$ and $\alpha \geq 2$. Furthermore, we may assume that the connected graph G_0 maximizes $\sharp\alpha(G_0)$ among all connected graphs of order n and independence number α. Since G_0 is a counterexample, we have

- either $\sharp\alpha(G_0) > f(n, \alpha)$
- or $\sharp\alpha(G_0) = f(n, \alpha)$ but $G_0 \notin \mathcal{F}(n, \alpha)$.

For the rest of the proof, let the vertex x of G_0 maximize $\sharp\alpha(G_0, x)$, that is, x is contained in the maximum number of maximum independent sets in G_0. Let the set N be the closed neighborhood $N_{G_0}[x]$ of x in G_0.

Applying the so-called *Moon-Moser operation*, we recursively construct a finite sequence of graphs

$$
G_0, \ldots, G_k
$$

such that, for every $i \in \{0, 1, \ldots, k\}$,

- G_i is a connected graph with vertex set $V(G_0)$,
- $N_{G_i}[x] = N$,
- G_i has independence number α,
- $\sharp\alpha(G_i) = \sharp\alpha(G_0)$, and
- $\sharp\alpha(G_0, x) = \sharp\alpha(G_i, x) \geq \sharp\alpha(G_i, u)$ for every vertex $u \in N$.

Trivially, G_0 has all these properties.

Now, suppose that G_{i-1} has been constructed for some positive integer i, and that N contains a vertex y_i such that y_i is not a cutvertex of G_{i-1}, and $N_{G_{i-1}}[y_i] \neq N$. In this case, we construct a further graph G_i in the sequence by removing all edges incident with y_i in G_{i-1}, and adding new edges between y_i and all vertices of $N \setminus \{y_i\}$, that is, we turn y_i into a *true twin* of x. If no such vertex exists, the sequence terminates with G_{i-1}.

Since G_{i-1} is connected, and y_i is not a cutvertex of G_{i-1}, the graph G_i is connected. By construction, $N_{G_i}[x] = N_{G_{i-1}}[x] = N$. Since a maximum independent set in G_{i-1} that contains x is also an independent set in G_i, we have $\alpha(G_i) \geq \alpha$. If some independent set I in G_i contains more than α vertices, then I necessarily contains y_i, and no other vertex from $N = N_{G_i}[y_i] = N_{G_{i-1}}[x]$, which implies the contradiction that $(I \setminus \{y_i\}) \cup \{x\}$ is an independent set in G_{i-1} with more than α elements. Hence, G_i has independence number α. By construction,

$$\sharp\alpha(G_i) = \sharp\alpha(G_{i-1}) - \sharp\alpha(G_{i-1}, y_i) + \sharp\alpha(G_{i-1}, x) \geq \sharp\alpha(G_{i-1}) = \sharp\alpha(G_0),$$

and the choice of G_0 implies $\sharp\alpha(G_i) = \sharp\alpha(G_0)$. Similarly, by construction,

$$\sharp\alpha(G_i, x) = \sharp\alpha(G_{i-1}, x) = \sharp\alpha(G_0, x) \quad \text{and} \quad \sharp\alpha(G_i, y_i) = \sharp\alpha(G_i, x).$$

Now, let $u \in N \setminus \{x, y_i\}$. Since every independent set in G_i that contains u does not contain y_i, it is also an independent set in G_{i-1}, which implies

$$\sharp\alpha(G_i, u) \leq \sharp\alpha(G_{i-1}, u) \leq \sharp\alpha(G_{i-1}, x) = \sharp\alpha(G_0, x) = \sharp\alpha(G_i, x).$$

Altogether, we established the desired properties for G_i.

The final graph in the sequence G_k has the additional property that $N_{G_k}[y] = N$ for every vertex y in N that is not a cutvertex of G_k. Let the graph G arise from G_k by removing iteratively as long as possible one by one edges between N and $V(G_0) \setminus N$ such that the resulting graph remains connected, and still has independence number α. Since the independence number does not change, we obtain $\sharp\alpha(G) \geq \sharp\alpha(G_k) = \sharp\alpha(G_0)$, and the choice of G_0 implies

$$\sharp\alpha(G) = \sharp\alpha(G_0),$$

that is, the removal of the edges in $E(G_0) \setminus E(G)$ does not lead to any new maximum independent set.

Claim 1. *G is isomorphic to a graph in $\mathcal{F}(n, \alpha)$.*

Proof of Claim 1. If some vertex of G is contained in no maximum independent set in G, then, by Lemma 3, $f(n, \alpha) \geq \sharp\alpha(G) = \sharp\alpha(G_0) \geq f(n, \alpha)$, which implies $\sharp\alpha(G) = f(n, \alpha)$. Again by Lemma 3, we obtain $G \in \mathcal{F}(n, \alpha)$. Hence, we may assume that

every vertex of G belongs to some maximum independent set in G.

Let B be the set of cutvertices of G in N. Note that the set $N \setminus B$ contains x, and that all vertices in $N \setminus B$ are true twins of x. Since G is connected, x is contained in some maximum independent set in G, and $\alpha \geq 2$, the set B is not empty. A component C of $G - N$ for which only one vertex y in B has neighbors in $V(C)$ is a *private component of* y. Since every vertex in B is a cutvertex, every such vertex has at least one private component.

In order to complete the proof of Claim 1, we insert two further claims.

Claim 2. *There is a vertex y in B, and a private component C of y such that C has order at least 2, and y has exactly one neighbor in $V(C)$.*

Proof of Claim 2. First, we assume that there is a vertex y in B as well as a private component C of y such that C has order at least 2. In view of the desired statement, we may assume that y has more than one neighbor in $V(C)$. Let z be a neighbor of y in $V(C)$. Since yz is not a bridge in G, the construction of G implies that $G - yz$ has an independent set I of order $\alpha + 1$. Clearly, the set I contains y and z. If y is the only vertex of B in I, than $(I \setminus \{y\}) \cup \{x\}$ is an independent set in G of order $\alpha + 1$, which is a contradiction. Hence, I contains more than one vertex from B. If, for every vertex y' in $(I \cap B) \setminus \{y\}$, there is some private component C' of y' such that $|I \cap C'| < \alpha(C')$, then the union of $\{x\}$ and maximum independent sets in the components of $G - N$ is an independent set in G that is at least as large as I, which is a contradiction. Hence, there is some vertex y' in $(I \cap B) \setminus \{y\}$ such that $|I \cap C'| = \alpha(C')$ for every private component C' of y'. Let C' be a private component of y'. Since $y' \in I$ and I intersects $V(C')$, the component C' has order at least 2. Since $y' \in I$, and $|I \cap C'| = \alpha(C')$, the removal of an edge between y' and a vertex in C' does not increase the independence number. Therefore, by the construction of G, the vertex y' has exactly one neighbor in C', and the desired statement follows for y' and C'.

Next, we assume that all private components have order exactly 1. Since every vertex of G belongs to some maximum independent set in G, there is a maximum independent set I in G that intersects B. Now, if I contains a vertex y from B, then I contains no vertex from any private component of y. Therefore, removing from I all vertices from B, and adding x as well as all vertices of all private components yields an independent set in G that is larger than I, which is a contradiction. This completes the proof of Claim 2. $\square$

For the rest of the proof, let $y \in B$, and a private component C of y be as in Claim 2.

Let z be the unique neighbor of y in C.

Claim 3. *The graph G has a cutvertex y' such that*

- *$G - y'$ has exactly two components C' and C'',*
- *C' is a clique,*
- *y' is adjacent to every vertex of C', and*
- *y' has exactly one neighbor in C''.*

Proof of Claim 3. If $\alpha(C)=1$, then $y'=z$ has the desired properties. Hence, we may assume that $\alpha(C)\geq 2$.

First, we assume that $\alpha(C)+\alpha(G-V(C))>\alpha$, which implies that every maximum independent set in G contains either y or z, but, trivially, not both. Since y and z both have degree at least 2, and $g(n,\alpha)$ is increasing in n, we obtain

$$\sharp\alpha(G) \;=\; \sharp\alpha(G-N_G[y])+\sharp\alpha(G-N_G[z])\leq 2g(n-3,\alpha-1).$$

Let the connected graph G' of order n and independence number α arise from $G(n-3,\alpha-1)$ by adding a clique K of order 3, and edges between one vertex in K and one vertex in each component of $G(n-3,\alpha-1)$. If $\frac{n-3}{\alpha-1}\geq 2$, then every component of $G(n-3,\alpha-1)$ has order at least 2, which implies that G' has strictly more than $2g(n-3,\alpha-1)$ maximum independent sets. In this case, Lemma 4 implies the contradiction

$$\sharp\alpha(G)\leq 2g(n-3,\alpha-1)<\sharp\alpha(G')\leq f(n,\alpha).$$

If $\frac{n-3}{\alpha-1}<2$, then $\sharp\alpha(G')=2g(n-3,\alpha-1)$, because one component of $G(n-3,\alpha-1)$ has order 1. Since K has order 3, Lemma 4 implies $\sharp\alpha(G')<f(n,\alpha)$, that is, also in this case we obtain the contradiction

$$\sharp\alpha(G)<f(n,\alpha).$$

Hence, we may assume that $\alpha(C)+\alpha(G-V(C))=\alpha$.

Let I_y and I_z be maximum independent sets in G that contain y and z, respectively. Clearly,

$$\begin{aligned}
|I_y\cap V(C)| &\leq \alpha(C),\\
|I_z\cap V(C)| &\leq \alpha(C),\\
|I_y\cap (V(G)\setminus V(C))| &\leq \alpha(G-V(C)), \text{ and}\\
|I_z\cap (V(G)\setminus V(C))| &\leq \alpha(G-V(C)).
\end{aligned}$$

Since $|I_y|=|I_z|=\alpha=\alpha(C)+\alpha(G-V(C))$, these four inequalities all hold with equality, that is, C has a maximum independent set containing z and another one not containing z, and $G-V(C)$ has a maximum independent set containing y and another one not containing y.

By Theorem 1, and the choice of n, we obtain

$$\begin{aligned}
\alpha(C-z) &= \alpha(C)\\
\alpha\big(G-(V(C)\cup\{y\})\big) &= \alpha(G-V(C))\\
1\leq\sharp\alpha(C-z) &< \sharp\alpha(C), && (1)\\
\sharp\alpha\big(G-(V(C)\cup\{y\})\big) &\leq g(n-n(C)-1,\alpha-\alpha(C)), \text{ and} && (2)\\
\sharp\alpha(G-V(C)) &= \sharp\alpha\big(G-(V(C)\cup\{y\})\big)+\sharp\alpha(G-V(C),y)\\
&\leq f(n-n(C),\alpha-\alpha(C)). && (3)
\end{aligned}$$

By (1), the linear program

$$\begin{array}{lrcl} \max & \sharp\alpha(C-z)\cdot r+\sharp\alpha(C)\cdot s & & \\ \text{such that} & s & \leq & g(n-n(C)-1,\alpha-\alpha(C)) \\ & r+s & \leq & f(n-n(C),\alpha-\alpha(C)) \\ & r,s & \geq & 0 \end{array}$$

has the unique optimal solution

$$\begin{aligned} r &= f(n-n(C),\alpha-\alpha(C))-g(n-n(C)-1,\alpha-\alpha(C)) \text{ and} \\ s &= g(n-n(C)-1,\alpha-\alpha(C))\big). \end{aligned}$$

Since x belongs to some maximum independent set in G, we have $\alpha(G-y)=\alpha(G)$, and using (2) and (3) as well as the unique optimal solution of the above linear program, we obtain

$$\begin{aligned} \sharp\alpha(G) &= \sharp\alpha(G,y)+\sharp\alpha(G-y) \\ &= \sharp\alpha(C-z)\cdot\sharp\alpha(G-V(C),y)+\sharp\alpha(C)\cdot\sharp\alpha\big(G-(V(C)\cup\{y\})\big) \\ &\leq \sharp\alpha(C-z)\cdot\Big(f(n-n(C),\alpha-\alpha(C))-g(n-n(C)-1,\alpha-\alpha(C))\Big) \qquad (4) \\ &\quad +\sharp\alpha(C)\cdot g(n-n(C)-1,\alpha-\alpha(C)), \end{aligned}$$

with equality in (4) if and only if (2) and (3) hold with equality. By Theorem 1, and the choice of n, this implies that (4) holds with equality if and only if

(i) $G-(V(C)\cup\{y\})$ is isomorphic to $G(n-n(C)-1,\alpha-\alpha(C))$, and

(ii) $G-V(C)$ is isomorphic to a graph in $\mathcal{F}(n-n(C),\alpha-\alpha(C))$.

If (i) or (ii) fails, then (4) is a strict inequality. In this case, replacing $G-V(C)$ within G by $F(n-n(C),\alpha-\alpha(C))$, and adding a bridge between the special cutvertex x_0 of $F(n-n(C),\alpha-\alpha(C))$ and the vertex z of C, yields a connected graph G' of order n and independence number α such that $\sharp\alpha(G')$ equals the right hand side of (4). Now, $\sharp\alpha(G_0)=\sharp\alpha(G)<\sharp\alpha(G')$, which contradicts the choice of G_0. Altogether, we obtain that (i) and (ii) hold.

If $G-V(C)$ is isomorphic to C_5, then the neighbor of x distinct from y is neither a cutvertex of G nor a true twin of x, which is a contradiction. Hence, $G-V(C)$ is not isomorphic to C_5. If $\alpha-\alpha(C)=1$, then $G-V(C)$ is a clique of order at least 2, and $y'=y$ has the desired properties. Hence, we may assume that $\alpha-\alpha(C)\geq 2$. By (i) and (ii), the vertex y is the special cutvertex x_0 of $G-V(C)$. If $\frac{n-n(C)}{\alpha-\alpha(C)}<2$, then no maximum independent set of $G-V(C)$ contains y, which implies the contradiction that no maximum independent set of G contains y. Hence, we may assume that $\frac{n-n(C)}{\alpha-\alpha(C)}\geq 2$. Now, (ii) implies the existence of a bridge yy' in $G-V(C)$ such that y' has the desired properties. This completes the proof of Claim 3. □

We are now in a position to complete the proof of Claim 1.

For the rest of the proof, let y' and C' be as in Claim 3.

Let $n' = n(C')$, and let x' be the unique neighbor of y' outside of C'.

Since y' and each vertex in C' belongs to some maximum independent set in G, we obtain

$$\begin{aligned}\alpha(G - y') &= \alpha,\\ \alpha\big(G - (V(C') \cup \{y'\})\big) &= \alpha - 1, \text{ and}\\ \alpha(G - N_G[y']) &= \alpha - 1.\end{aligned}$$

Now, Theorem 1 and the choice of n imply

$$\begin{aligned}\sharp\alpha(G) &= \sharp\alpha(G, y') + \sharp\alpha(G - y')\\ &= \sharp\alpha(G - N_G[y']) + n(C') \cdot \sharp\alpha\big(G - (V(C') \cup \{y'\})\big)\\ &\leq g(n - n' - 2, \alpha - 1) + n' \cdot f(n - n' - 1, \alpha - 1). \end{aligned} \tag{5}$$

By Theorem 1 and Lemma 4, the right hand side of (5) is an upper bound on the number of maximum independent sets of a suitable connected graph of order n and independence number α whose structure is as in Lemma 4(i). By Lemma 4, this implies

$$g(n - n' - 2, \alpha - 1) + n' \cdot f(n - n' - 1, \alpha - 1) \leq f(n, \alpha). \tag{6}$$

Since $\sharp\alpha(G) = \sharp\alpha(G_0) \geq f(n, \alpha)$, it follows that $\sharp\alpha(G) = f(n, \alpha)$, and (5) and (6) hold with equality. We obtain

$$\begin{aligned}\sharp\alpha(G - N_G[y']) &= g(n - n' - 2, \alpha - 1) \text{ and}\\ \sharp\alpha\big(G - (V(C') \cup \{y'\})\big) &= f(n - n' - 1, \alpha - 1),\end{aligned}$$

which, by Theorem 1 and the choice of n, imply that

(i) $G - N_G[y']$ is isomorphic to $G(n - n' - 2, \alpha - 1)$ and

(ii) $G - (V(C') \cup \{y'\})$ is isomorphic to a graph in $\mathcal{F}(n - n' - 1, \alpha - 1)$.

By (i), the graph $G - (V(C') \cup \{y'\})$ can not be isomorphic to C_5.

If $\alpha = 2$, then G arises by adding a bridge between two disjoint cliques, and Lemma 4 implies that G is isomorphic to a graph in $\mathcal{F}(n, \alpha)$.

If $\alpha \geq 3$, then (i) and (ii) together imply that x' is the special cutvertex x_0 of $G - (V(C') \cup \{y'\})$. Now, the construction of G from G_k, and Lemma 4 imply that G is isomorphic to a graph in $\mathcal{F}(n, \alpha)$. This complete the proof of Claim 1. $\square$

If $\frac{n}{\alpha} \geq 2$, then no edge can be added to G without reducing $\alpha(G)$ or $\sharp\alpha(G)$, which implies that $G_k = G$ in this case. If $\frac{n}{\alpha} < 2$, then the only edges that can be added to G without reducing $\alpha(G)$ or $\sharp\alpha(G)$, are incident with the special cutvertex x_0 of G. Altogether, it follows in both cases that G_k is isomorphic to a graph in $\mathcal{F}(n, \alpha)$.

Since G_0 is a counterexample, we have $k \geq 1$.

First, we assume that $\frac{n}{\alpha} \geq 2$. This implies that G_k is isomorphic to $F(n, \alpha)$. Let $C_0, \ldots, C_{\alpha-1}$ and $x_0, \ldots, x_{\alpha-1}$ be as in the definition of $F(n, \alpha)$. Note that x and y_k are true twins and no cutvertices of G_k, and, hence, belong to the same clique, say C_i. If $C_i \subseteq N_{G_{k-1}}[y_k]$, then G_{k-1} arises from G_k by adding edges incident with y_k, which implies the contradiction $\sharp\alpha(G_{k-1}) < \sharp\alpha(G_k)$. If $C_j \subseteq N_{G_{k-1}}[y_k]$ for some $j \in \{0, \ldots, \alpha-1\} \setminus \{i\}$, that is, G_{k-1} is a supergraph of a graph as in Lemma 4, then Lemma 4 implies that G_{k-1} is isomorphic to $F(n, \alpha)$, which implies the contradiction that y_k is not adjacent to x in G_{k-1}. Since $\alpha(G_k) = \alpha(G_{k-1})$, the structure of $F(n, \alpha)$ easily implies that

$$\begin{aligned} N_{G_{k-1}}[y_k] \cap C_0 &= C_0 \setminus \{x_0\} \text{ and} \\ N_{G_{k-1}}[y_k] \cap C_j &= C_j \setminus \{x_j\} \text{ for some } j \in \{1, \ldots, \alpha-1\} \text{ such that } i \in \{0, j\}. \end{aligned}$$

Similarly as in Lemma 4, we have

$$\sharp\alpha(G_k) = (n_0 - 1) \prod_{k=1}^{\alpha-1} n_k + \prod_{k=1}^{\alpha-1} (n_k - 1),$$

where n_k is the order of C_k for $k \in \{0, \ldots, \alpha-1\}$.

If $i = 0$, then, considering the maximum independent sets of G_{k-1} that contain neither x_0 nor y_k, those that contain x_0 but not y_k, those that contain y_k but not x_0, and that contain x_0 and y_k, we obtain

$$\begin{aligned} \sharp\alpha(G_{k-1}) &\leq (n_0 - 2) \prod_{k=1}^{\alpha-1} n_k + \prod_{k=1}^{\alpha-1} (n_k - 1) + \frac{1}{n_j} \prod_{k=1}^{\alpha-1} n_k + \frac{1}{n_j - 1} \prod_{k=1}^{\alpha-1} (n_k - 1) \\ &= \sharp\alpha(G_k) + \frac{1}{n_j} \prod_{k=1}^{\alpha-1} n_k + \frac{1}{n_j - 1} \prod_{k=1}^{\alpha-1} (n_k - 1) - \prod_{k=1}^{\alpha-1} n_k. \end{aligned}$$

Since either $\alpha \geq 3$ and $n_k \geq 2$ for every $k \in \{0, \ldots, \alpha-1\}$, or $\alpha = 2$ and $n_1 \geq 3$, this implies the contradiction $\sharp\alpha(G_{k-1}) < \sharp\alpha(G_k)$.

If $i = j$, then we obtain

$$\begin{aligned} \sharp\alpha(G_{k-1}) &\leq (n_0 - 1) \frac{(n_j - 1)}{n_j} \prod_{k=1}^{\alpha-1} n_k + \frac{(n_j - 2)}{(n_j - 1)} \prod_{k=1}^{\alpha-1} (n_k - 1) + \frac{1}{n_j} \prod_{k=1}^{\alpha-1} n_k \\ &\quad + \frac{1}{n_j - 1} \prod_{k=1}^{\alpha-1} (n_k - 1) \\ &= \sharp\alpha(G_k) - \frac{n_0 - 1}{n_j} \prod_{k=1}^{\alpha-1} n_k + \frac{1}{n_j} \prod_{k=1}^{\alpha-1} n_k. \end{aligned}$$

Since, in this case, we have $x, y_k, x_j \in C_j$, we obtain $n_0 \geq n_j \geq 3$, which implies the contradiction $\sharp\alpha(G_{k-1}) < \sharp\alpha(G_k)$.

Next, we assume that $\frac{n}{\alpha} < 2$. This implies that G_k arises from $F(n,\alpha)$ by adding edges incident with the special cutvertex x_0 of $F(n,\alpha)$. Since x and y_k are true twins and no cutvertices of G_k, was may assume, by symmetry, that $C_1 = \{x, y_k\}$, and that x_0 is adjacent to x and y_k. Since y_k is a neighbor of x in G_{k-1}, the graph G_{k-1} arises from $F(n,\alpha)$ by adding an edge between y_k and some vertex distinct from x_0, which easily implies the contradiction $\sharp\alpha(G_{k-1}) < \sharp\alpha(G_k)$.

This completes the proof. □

References

[1] T. Derikvand and M.R. Oboudi, On the number of maximum independent sets of graphs, Transactions on Combinatorics 3 (2014) 29-36.

[2] J.R. Griggs, C.M. Grinstead, and D.R. Guichard, The number of maximal independent sets in a connected graph, Discrete Mathematics 68 (1988) 211 - 220.

[3] M.-J. Jou and G.J. Chang, The number of maximum independent sets in graphs, Taiwanese Journal of Mathematics 4 (2000) 685-695.

[4] E. Mohr and D. Rautenbach, On the maximum number of maximum independent sets, arXiv:1805.02519.

[5] J.W. Moon and L. Moser, On cliques in graphs, Israel Journal of Mathematics 3 (1965) 23-28.

[6] B.E. Sagan and V.R. Vatter, Maximal and maximum independent sets in graphs with at most r cycles. Journal of Graph Theory 53 (2006) 283-314.

[7] P. Turán, Eine Extremalaufgabe aus der Graphentheorie, Matematikai és Fizikai Lapok 48 (1941) 436-452.

[8] J. Zito, The structure and maximum number of maximum independent sets in trees, Journal of Graph Theory 15 (1991) 207-221.

[9] A.A. Zykov, On some properties of linear complexes, Matematicheskiĭ Sbornik. Novaya Seriya 24(66) (1949) 163-188.