The Bimodal Logic of Commuting Difference Operators Is Decidable

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Abstract

In this paper, we show that the bimodal logic of two commuting difference operators $[\textbf{Diff}, \textbf{Diff}]$ is decidable (in N2ExpTime), despite lacking the finite model property and thereby remaining impervious to standard filtration techniques. The proof of decidability involves an exponential-time reduction from the satisfiability problem of the commutator $[\textbf{Diff}, \textbf{Diff}]$ to that of the product logic $\textbf{Diff} \times \textbf{Diff}$, via an intermediate quasimodel construction. To the best of the author's knowledge, this marks the first example of a (non-trivial) reduction between a commutator $[L_h, L_v]$ and its respective product logic $L_h \times L_v$, where the two logics do not coincide. By adapting this same technique, we are able to establish the finite model property for $[\textbf{S5}, \textbf{Diff}]$, without recourse to standard filtration techniques that break down for logics that are not Horn-axiomatizable, such as that of the difference operator.

Keywords: Two-dimensional modal logics, difference operator, product logic, commutator, decidable, fmp

1 Introduction

The logic of the difference operator $\textbf{Diff}$, first described by von Wright [23] as the logic of 'elsewhere', is the set of all propositional unimodal formulas that are valid in all *difference frames* $\mathfrak{F} = (W, \neq)$, in which each possible world is accessible from every other *distinct* possible world. In isolation, the logic of the difference operator has been extensively studied [5,6,21], and shares connections with *nominals* [9] as well as *graded modalities* [1], each of which finds applications in description logics [2].

Segerberg [21] proved that $\textbf{Diff}$ is sound and complete with respect to the class of all frames that are symmetric and *weakly-transitive* [1] :

$$\forall x \forall y \forall z \big(xRy \wedge yRz \rightarrow (x = z \vee xRz) \big)$$

Consequently, $\textbf{Diff}$ can be axiomatized as the smallest normal modal logic containing the formulas:

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[1] Segerberg refers to this property as *alio-transitivity*.

$$(B) := p \to \Box\Diamond p \qquad \text{and} \qquad (w4) := \Diamond\Diamond p \to (p \lor \Diamond p),$$

corresponding to symmetry and weak-transitivity, respectively. It is a routine exercise to show that, like the above axiomatization, any axiomatization for **Diff** must contain some formulas that are not equivalent to any universal Horn-formula [4, Proposition 6.2.2]; that is to say **Diff** is not *Horn-axiomatizable.*

As a unimodal logic, **Diff** shares many similarities with the logic **S5** of all equivalence relations: the two logics are finitely axiomatisable (with **S5** being axiomatized over **Diff** with the addition of the axiom $(T) := \Box p \to p$), both logics have NP-complete satisfiability problems (the latter can be polynomially reducible to the former), and the structure of their frames is quite similar (with the class of frames **Diff** extending the class of frames for **S5** by allowing frames with some irreflexive points). However, these similarities do not extend to their bimodal counterparts, where their interactions with other logics often differ considerably [12,13].

In what follows, we will be interested in combining unimodal logics, expressed over the bimodal language $\mathcal{ML}_2$ whose formulas are given by the following grammar:

$$\varphi ::= p_i \mid \neg\varphi \mid (\varphi_1 \land \varphi_2) \mid \Diamond_h\varphi \mid \Diamond_v\varphi$$

where $p_i \in \mathsf{Prop}$ ranges over a countably infinite set of propositional variables. The other Boolean connectives and $\Box_j\varphi$ are defined in the usual way with the addition of $\Box_j^+\varphi := \varphi \land \Box_j\varphi$ and $\Diamond_j^+\varphi := \varphi \lor \Diamond_j\varphi$, for $j = h, v$.

We define the size of an $\mathcal{ML}_2$-formula to be it's length $\|\varphi\|$, taken to be the number of symbols it comprises, and note that $|\mathsf{sub}\,(\varphi)| \leq \|\varphi\|$, where $\mathsf{sub}\,(\varphi)$ denotes the set of all subformulas of φ. Formulas of $\mathcal{ML}_2$ are interpreted over Kripke models $\mathfrak{M} = (\mathfrak{F}, \mathfrak{V})$, in which $\mathfrak{F} = (W, R_h, R_v)$ is a *bimodal Kripke frame*, where $R_h, R_v \subseteq W^2$ are binary relations on W, and $\mathfrak{V} : \mathsf{Prop} \to 2^W$ is a propositional valuation. Satisfiability is defined in the usual way with $\Diamond_j\varphi$ being interpreted over R_j, for $j = h, v$. For convenience, we also write $R_j^+ := R_j \cup \{(w, w) : w \in W\}$ so that $\Diamond_j^+\varphi$ can be interpreted over R_j^+, for $j = h, v$.

The product construction, first investigated by Segerberg [20] and later extended by Shehtman [22], provides a natural semantic way of combining unimodal logics and has been extensively studied since its inception [8,7,16]. Given two unimodal Kripke frames $\mathfrak{F}_h = (W_h, R_h)$ and $\mathfrak{F}_v = (W_v, R_v)$, we define their *product frame* to be the bimodal Kripke frame $\mathfrak{F}_h \times \mathfrak{F}_v = (W_h \times W_v, \overline{R_h}, \overline{R_v})$, where $W_h \times W_v$ is the Cartesian product of W_h and W_v, and where $\overline{R_h}$ and $\overline{R_v}$ act component-wise on $W_h \times W_v$, such that:

$$(u, v)\overline{R_h}(u', v') \iff \quad uR_hu' \text{ and } v = v',$$

$$(u, v)\overline{R_v}(u', v') \iff \quad u = u' \text{ and } vR_vv',$$

for all $(u, v), (u', v') \in W_h \times W_v$.

The *product logic* $L_h \times L_v$ of two unimodal logic is characterised by all those formulas that are valid in every product frame $\mathfrak{F}_h \times \mathfrak{F}_v$, in which $\mathfrak{F}_j$ is a frame for L_j, for $j = h, v$.

Among those formulas that common to all product logics are those of the two constituent logics L_h and L_v (rewritten with the appropriate operators: $\Diamond_h$ and $\Diamond_v$, respectively), together with the formulas $\Diamond_v \Diamond_h p \to \Diamond_h \Diamond_v p$, $\Diamond_h \Diamond_v p \to \Diamond_v \Diamond_h p$, and $\Diamond_h \Box_v p \to \Box_v \Diamond_h p$, corresponding to the following frame properties inherent to the structure of all product frames:

– *Left-commutativity:* $\quad \forall x \forall y \forall z \big(x R_v y \wedge y R_h z \to \exists u (x R_h u \wedge u R_v z) \big)$

– *Right-commutativity:* $\quad \forall x \forall y \forall z \big(x R_h y \wedge y R_v z \to \exists u (x R_v u \wedge u R_h z) \big)$

– *Church-Rosser property:* $\quad \forall x \forall y \forall z \big(x R_h y \wedge x R_v z \to \exists u (y R_v u \wedge z R_h u) \big).$

We define the *commutator* of L_h and L_v, denoted $[L_h, L_v]$, to be the smallest modal logic axiomatized by these formulas. It follows that $[L_h, L_v] \subseteq L_h \times L_v$ for any choice of L_h and L_v. Furthermore, in the case where both L_h and L_v are Horn-axiomatizable, the two logics are known to coincide; in which case we say that L_h and L_v are *product matching*.

Theorem 1.1 (Gabbay–Shehtman [8, Theorem 7.12]) *Let L_h and L_v be any two Kripke complete, Horn-axiomatizable unimodal logics. Then L_h and L_v are product-matching.*

However, in general, the product logic may admit many more formulas than its commutator sublogic. Indeed, it is straightforward to see that $[\mathbf{Diff}, \mathbf{Diff}] \neq \mathbf{Diff} \times \mathbf{Diff}$, as can be evidenced by the following frame $\mathfrak{F} = (\{a_1, a_2, a_3, b_1, b_2\}, R_h, R_v)$, where

$$R_h = \{(u, v) : u \neq v\} \qquad \text{and} \qquad R_v = \{(a_i, a_j), (b_i, b_j) : i \neq j\}.$$

Despite being a frame for $[\mathbf{Diff}, \mathbf{Diff}]$, it is straightforward to check that $\mathfrak{F}$ is not the p-morphic image of *any* product frame for $\mathbf{Diff} \times \mathbf{Diff}$. Hence, the Jankov-Fine frame formula (see [3]) for $\mathfrak{F}$ is therefore an example of a formula that is satisfiable with respect to $[\mathbf{Diff}, \mathbf{Diff}]$ but not with respect to $\mathbf{Diff} \times \mathbf{Diff}$.

Indeed, in a forthcoming paper [14], it is shown that every bimodal logic between $\mathbf{K} \times \mathbf{Diff}$ and $\mathbf{S5} \times \mathbf{Diff}$ cannot be axiomatized using only finitely many variables and, thus, there are infinitely many logics separating $[\mathbf{Diff}, \mathbf{Diff}]$ from $\mathbf{Diff} \times \mathbf{Diff}$.

Theorem 1.2 (Hampson et al. [14]) *Let L be any Kripke complete bimodal logic such that $\mathbf{K} \times \mathbf{Diff} \subseteq L \subseteq \mathbf{S5} \times \mathbf{Diff}$. Then L cannot be axiomatized using only finitely many propositional variables.*

Consequently, although the satisfiability problem for $\mathbf{Diff} \times \mathbf{Diff}$ can be reduced to that of the decidable ([18]) two-variable fragment of first-order logic with counting quantifiers[2], we cannot appeal to this result directly in order to establish the decidability of $[\mathbf{Diff}, \mathbf{Diff}]$.

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[2] By exploiting the ability to express 'elsewhere' with $\exists^{\neq} x \varphi := \exists_{>1} x \varphi \vee (\neg \varphi \wedge \exists_{>0} x \varphi)$.

Furthermore, as demonstrated below, the commutator $[\mathbf{Diff}, \mathbf{Diff}]$ lacks the finite model property and so is impervious to standard filtration techniques that are commonly employed to establish decidability of commutators [8].

2   Main Results

2.1 $[\mathbf{Diff}, \mathbf{Diff}]$ lacks the finite model property

In this section, we first establish that the logic of two commuting difference operators lacks the finite model property, thereby necessitating the need for the alternative approach taken in this paper. To this end, let φ_∞ be the conjunction of the following formulas:

$$\diamondsuit_h \diamondsuit_v (p \wedge \neg q \wedge \square_h \neg p \wedge \square_v \neg q), \tag{1}$$
$$\square_h \diamondsuit_v (p \wedge \neg q \wedge \square_h \neg p), \tag{2}$$
$$\square_v \diamondsuit_h (q \wedge \neg p \wedge \square_v \neg q), \tag{3}$$

where $p, q \in \mathsf{Prop}$ are propositional variables.

Lemma 2.1 *Let $\mathfrak{F} = (W, R_h, R_v)$ be any bimodal Kripke frame such that:*

(i) *R_h and R_v commute,*

(ii) *R_h and R_v are both* weakly-Euclidean*:*

$$\forall x \forall y \forall z \; \big(x R_j y \wedge x R_j z \rightarrow (y = z \vee y R_j z)\big), \qquad \text{for } j = h, v.$$

If φ_∞ is satisfiable in $\mathfrak{F}$ then $\mathfrak{F}$ must be infinite.

Proof. Let $\mathfrak{F} = (W, R_h, R_v)$ be as described and suppose that $\mathfrak{M}, r \models \varphi_\infty$, for some model $\mathfrak{M} = (\mathfrak{F}, \mathfrak{V})$ based on $\mathfrak{F}$, with $r \in W$. We define, inductively, four infinite sequences:

$$\langle x_k \in W : k < \omega \rangle, \qquad \langle y_k \in W : k < \omega \rangle,$$
$$\langle u_k \in W : k < \omega \rangle, \quad \text{and} \quad \langle v_k \in W : k < \omega \rangle,$$

such that $\mathfrak{M}, u_0 \models \square_v \neg q$, and for all $k < \omega$:

(**inf1**) $r R_h x_k$ and $r R_v y_k$,

(**inf2**) $x_k R_v u_k$ and $x_{k+1} R_v v_k$,

(**inf3**) $y_k R_h v_k$ and $y_k R_h u_k$,

(**inf4**) $\mathfrak{M}, u_k \models p \wedge \neg q \wedge \square_h \neg p$,

(**inf5**) $\mathfrak{M}, v_k \models q \wedge \neg p \wedge \square_v \neg q$.

At this stage we do not assume that all the points are distinct from one another.

Firstly, by (1), there is some $x_0, u_0 \in W$ such that $r R_h x_0$, $x_0 R_v u_0$ and $\mathfrak{M}, u_0 \models p \wedge \neg q \wedge \square_h \neg p \wedge \square_v \neg q$. Then by (i), there is some $y_0 \in W$ such that $r R_v y_0$ and $y_0 R_h u_0$. Whence, it follows from (3) that there is some $v_0 \in W$ such that $y_0 R_h v_0$ and $\mathfrak{M}, v_0 \models q \wedge \neg p \wedge \square_v \neg q$.

Now, suppose we have already defined x_k, y_k, u_k, v_k, for some $k < \omega$. By **(inf1)** and **(inf3)**, we have that $rR_v y_k$ and $y_k R_h v_k$. Then, by (i), there is some $x_{k+1} \in W$ such that $rR_h x_{k+1}$ and $x_{k+1} R_v v_k$. Whence, it follows from (2) that there is some $u_{k+1} \in W$ such that $x_{k+1} R_v u_{k+1}$ and $\mathfrak{M}, u_{k+1} \models p \wedge \neg q \wedge \Box_h \neg p$.

Now, by (i), there is some $y_{k+1} \in W$ such that $rR_v y_{k+1}$ and $y_{k+1} R_h u_{k+1}$. Whence, it follows from the formula given in (3) that there is some $v_{k+1} \in W$ such that $y_{k+1} R_h v_{k+1}$ and $\mathfrak{M}, v_{k+1} \models q \wedge \neg p \wedge \Box_v \neg q$. Hence, by induction on the length, we may extend each of the four sequences indefinitely, as illustrated in Figure 1.
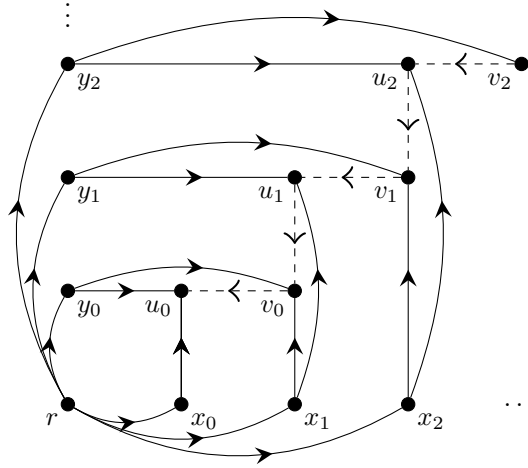

Fig. 1. Illustration of the model generated by φ_∞.

To show that each of the u_k are distinct, consider the following sequence of formulas $\langle \psi_k : k < \omega \rangle$, defined inductively, by taking $\psi_0 := \Box_v \neg q$, and

$$\psi_{k+1} \; := \; \Diamond_v \big(q \wedge \Diamond_h (p \wedge \psi_k) \big),$$

for $k < \omega$. We claim that, for all $k < \omega$:

$$\mathfrak{M}, u_k \models \psi_k \qquad \text{and} \qquad \mathfrak{M}, u_\ell \not\models \psi_k \quad \text{for } \ell < k. \qquad (4)$$

Indeed, it is immediate from the definition of u_0 that $\mathfrak{M}, u_0 \models \psi_0$. So suppose that $\mathfrak{M}, u_k \models \psi_k$, for some $k < \omega$, and that $\mathfrak{M}, u_\ell, \not\models \psi_k$ for all $\ell < k$. By **(inf4)**–**(inf5)**, we must have that $u_i \neq v_j$, for all $i, j < \omega$. Therefore, it follows from **(inf2)**–**(inf3)** and (ii), that $v_k R_h u_k$ and $u_{k+1} R_v v_k$. Whence, by **(inf4)**–**(inf5)**, we deduce that $\mathfrak{M}, u_{k+1} \models \psi_{k+1}$.

Now suppose, to the contrary, that $\mathfrak{M}, u_\ell \models \psi_{k+1}$ for some $\ell \leq k$. Then there is some $u, v \in W$ such that $u_\ell R_v v, v R_h u, \mathfrak{M}, v \models q$, and $\mathfrak{M}, u \models p \wedge \psi_k$. It then follows from **(inf2)**–**(inf5)** and (ii) that $v = v_{\ell-1}$ and $u = u_{\ell-1}$. This is to

say that $\mathfrak{M}, u_{\ell-1} \models \psi_k$, where $\ell - 1 < k$, contrary to our induction hypothesis. Hence, $\mathfrak{M}, u_\ell \not\models \psi_{k+1}$, for all $\ell < k + 1$.

It then follows from (4) that each of the $u_k \in W$ must be distinct. Being such, we must have that $\mathfrak{F}$ is infinite, as required. $\square$

It is straightforward to check that every frame for $[\mathbf{Diff}, \mathbf{Diff}]$ satisfied conditions (i)–(ii) of Lemma 2.1. Furthermore, we note that φ_∞ is satisfiable with respect to $[\mathbf{Diff}, \mathbf{Diff}]$, as evidenced by the model $\mathfrak{M} = (\mathfrak{F}, \mathfrak{V})$, where $\mathfrak{F} = (\omega, \neq) \times (\omega, \neq)$ and $\mathfrak{V}$ is such that

$$\mathfrak{V}(p) = \{(k, k) : 0 < k < \omega\} \qquad \text{and} \qquad \mathfrak{V}(q) = \{(k+1, k) : 0 < k < \omega$$

for which it is straightforward to check that $\mathfrak{M}, (0, 0) \models \varphi_\infty$. Hence, it follows that $[\mathbf{Diff}, \mathbf{Diff}]$ does not possess the finite model property and, as such, does not admit filtration.

Theorem 2.2 *The bimodal logic of two commuting difference operators* $[\mathbf{Diff}, \mathbf{Diff}]$ *does not possess the finite model property.*

It is known that $\mathbf{Diff} \times \mathbf{Diff}$ lacks the finite *product* model property, as it can be viewed as a syntactic variant of the two-variable fragment of first-order logic with counting quantifiers $\{\exists_{>0}, \exists_{>1}\}$, which is known to lack the fmp [10]. However, since $\mathfrak{F}_\infty$ is the product of two difference frames, as a further corollary of Lemma 2.1, we have the stronger result that φ_∞ is satisfiable with respect to $\mathbf{Diff} \times \mathbf{Diff}$ but cannot be satisfied in *any* finite frame for $\mathbf{Diff} \times \mathbf{Diff}$; product or otherwise!

Corollary 2.3 *The product logic* $\mathbf{Diff} \times \mathbf{Diff}$ *does not posses the* (abstract) *finite model property.*

2.2   Quasimodels for $[\mathbf{Diff}, \mathbf{Diff}]$

Despite the lack of any finite model property for $[\mathbf{Diff}, \mathbf{Diff}]$, we are still able to obtain a N2ExpTime upper-bound on the complexity of its satisfiability problem by a exponential-time reduction to that of $\mathbf{Diff} \times \mathbf{Diff}$, whose NExpTime-completeness follows from that of the two-variable fragment of first-order logic with counting quantifiers [18,19].

Theorem 2.4 *The satisfiability problem for the bimodal logic of two commuting difference operators* $[\mathbf{Diff}, \mathbf{Diff}]$ *is decidable in* N2ExpTime.

To facilitate this reduction, we employ a variation of the quasimodel technique [24,7] as an intermediary stage in the reduction. By limiting the size of the constituent *quasistates* to 'small' states, we incur at most an exponential increase in the complexity.

Definition 2.5 (Types and Quasistates)   Let $\varphi \in \mathcal{ML}_2$ be an arbitrary bimodal formula of size n, and define a *type* for φ to be any Boolean-saturated subset $t \subseteq \mathsf{sub}(\varphi)$, which is to say that:

(tp1) $\neg\psi \in t$ if and only if $\psi \notin t$, for all $\neg\psi \in \mathsf{sub}(\varphi)$,

(tp2) $(\psi_1 \wedge \psi_2) \in t$ if and only if $\psi_1 \in t$ and $\psi_2 \in t$, for all $(\psi_1 \wedge \psi_2) \in \mathsf{sub}\,(\varphi)$.

Note that the set of all types for φ can be constructing in time that is at most exponential in the size of $\mathsf{sub}\,(\varphi)$.

A *quasistate* for φ is a tuple (T, S_h, S_v) such that:

(qs1) T is a non-empty set of types for φ, and $S_h, S_v \subseteq T^2$ are binary relations on T.

(qs2) For all $t, t' \in T$, if $t \neq t'$ then $tS_h t'$ and $tS_v t'$,

(qs3) *(internal coherence)* For all $t \in T$ and $\diamondsuit_j \psi \in \mathsf{sub}\,(\varphi)$,

$$\exists t' \in T; \ tS_j t' \text{ and } \psi \in t' \qquad \Longrightarrow \qquad \diamondsuit_j \psi \in t,$$

for $j = h, v$,

Let $\mathfrak{Q}$ denote the set of all possible quasistates for φ, of which there can be at most finitely many; albeit of the order of 2^{2^n}, where $n = |\mathsf{sub}\,(\varphi)|$.

Given a quasistate $q = (T, S_h, S_v) \in \mathfrak{Q}$, we say that $\diamondsuit_i \psi \in \mathsf{sub}\,(\varphi)$ is a *defect* of q if there is some $t \in T$ such that $\diamondsuit_j \psi \in t$, and there is no $t' \in T$ such that $tS_j t'$ and $\psi \in T$. Let $D_q \subseteq \mathsf{sub}\,(\varphi)$ denote the set of all defects of q. For convenience, we write $\psi \in \bigcup q$ if there is some $t \in T$ such that $\psi \in t$, and $\psi \in \bigcap q$ if $\psi \in t$ for all $t \in T$.

Definition 2.6 (Quasimodels) Given a set of quasistates $\mathfrak{S} \subseteq \mathfrak{Q}$, we define a $\mathfrak{S}$-*quasimodel* for φ to be a triple (X, Y, q) such that

(qm1) X and Y are non-empty sets, and q is a function associating each pair $(x, y) \in X \times Y$ with a quasistate $q(x, y) = (T^{x,y}, S_h^{x,y}, S_v^{x,y}) \in \mathfrak{S}$,

(qm2) There is some $x_0 \in X$ and $y_0 \in Y$ such that $\varphi \in \bigcup q(x_0, y_0)$,

(qm3) *($\diamondsuit_h$-coherence)* For all $x \in X$, $y \in Y$ and $\diamondsuit_h \psi \in \mathsf{sub}\,(\varphi)$,

$$\exists x' \in X; \ x \neq x' \text{ and } \psi \in \bigcup q(x', y) \qquad \Longrightarrow \qquad \diamondsuit_h \psi \in \bigcap q(x, y),$$

(qm4) *($\diamondsuit_v$-coherence)* For all $x \in X$, $y \in Y$ and $\diamondsuit_v \psi \in \mathsf{sub}\,(\varphi)$,

$$\exists y' \in Y; \ y \neq y' \text{ and } \psi \in \bigcup q(x, y') \qquad \Longrightarrow \qquad \diamondsuit_v \psi \in \bigcap q(x, y),$$

(qm5) *($\diamondsuit_h$-saturation)* For all $x \in X$, $y \in Y$ and $\diamondsuit_h \psi \in \mathsf{sub}\,(\varphi)$,

$$\diamondsuit_h \psi \in D_{q(x,y)} \qquad \Longrightarrow \qquad \exists x' \in X; \ x \neq x' \text{ and } \psi \in \bigcup q(x', y),$$

(qm6) *($\diamondsuit_v$-saturation)* For all $x \in X$, $y \in Y$ and $\diamondsuit_v \psi \in \mathsf{sub}\,(\varphi)$,

$$\diamondsuit_v \psi \in D_{q(x,y)} \qquad \Longrightarrow \qquad \exists y' \in Y; \ y \neq y' \text{ and } \psi \in \bigcup q(x, y'),$$

(qm7) For all $x, x' \in X$ and $y, y' \in Y$,

$$h\text{-}\mathsf{size}(q(x,y)) = 1 \qquad \Longrightarrow \qquad h\text{-}\mathsf{size}(q(x, y')) = 1,$$

$$v\text{-}\mathsf{size}(q(x,y)) = 1 \qquad \Longrightarrow \qquad v\text{-}\mathsf{size}(q(x', y)) = 1,$$

where j-$\mathsf{size}(\boldsymbol{q}(x,y)) = |T^{x,y}| + \big|\{t \in T^{x,y} : tS_j^{x,y}t\}\big|$ is a measure of the horizontal and vertical 'size' of each quasistate, for $j = h, v$ respectively.

We first show that by taking $\mathfrak{S} = \mathfrak{Q}$ to be the set of all possible quasistates for φ, our $\mathfrak{Q}$-quasimodels adequately capture the notion of satisfiability with respect to $[\mathbf{Diff}, \mathbf{Diff}]$.

Lemma 2.7 *A formula φ is satisfiable with respect to $[\mathbf{Diff}, \mathbf{Diff}]$ if and only if there is a $\mathfrak{Q}$-quasimodel for φ, where $\mathfrak{Q}$ is the set of all quasistates for φ.*

Proof.

($\Rightarrow$) Suppose that $\mathfrak{M}, r \models \varphi$ for some model $\mathfrak{M} = (\mathfrak{F}, \mathfrak{V})$, where $\mathfrak{F} = (W, R_h, R_v)$ is a frame for $[\mathbf{Diff}, \mathbf{Diff}]$. We define an equivalence relation on W by taking

$$u \sim v \qquad \Longleftrightarrow \qquad uR_h^+v \quad \text{and} \quad uR_v^+v$$

for all $u, v \in W$. Now let X and Y be defined by taking

$$X = \{[u] : rR_h^+u\} \qquad \text{and} \qquad Y = \{[v] : rR_v^+v\},$$

where $[u]$ denotes the $\sim$-equivalence class containing $u \in W$. We define an intermediary function $h : X \times Y \to W/\sim$ as follows: For each $[u] \in X$ and $[v] \in Y$, we have that rR_h^+u and rR_v^+v. Hence, by the Church-Rosser property, there is some $w \in W$ such that uR_v^+w and vR_h^+w. Moreover, for all $w' \in W$ such that uR_v^+w' and vR_h^+w' we must have that wR_h^+w' and wR_v^+w', since both R_h^+ and R_v^+ are equivalence relations, which is to say that $w \sim w'$. Hence, we may uniquely define $h([u], [v]) = [w]$ to be the equivalence class containing w.

With every $w \in W$ we associate a type

$$\boldsymbol{t}(w) = \{\psi \in \mathsf{sub}\,(\varphi) : \mathfrak{M}, w \models \psi\},$$

and for each $(x, y) \in X \times Y$, we may associate the quasistate $\boldsymbol{q}(x, y) = (T^{x,y}, S_h^{x,y}, S_v^{x,y})$ by taking:

– $T^{x,y} = \{\boldsymbol{t}(w) : w \in h(x, y)\}$, and

– $\boldsymbol{t}(u)S_i^{x,y}\boldsymbol{t}(v)$ if and only if there is some $u', v' \in h(x, y)$ such that $\boldsymbol{t}(u) = \boldsymbol{t}(u')$, $\boldsymbol{t}(v) = \boldsymbol{t}(v')$ and $u'R_iv'$, for $i = h, v$.

It is straightforward to verify that $(X, Y, \boldsymbol{q})$ is a $\mathfrak{Q}$-quasimodel for φ (the details for which can be found in the Appendix.

($\Leftarrow$) Conversely, suppose that $(X, Y, \boldsymbol{q})$ is a $\mathfrak{Q}$-quasimodel for φ. We define a model $\mathfrak{M} = (\mathfrak{F}, \mathfrak{V})$, where $\mathfrak{F} = (W, R_h, R_v)$, by taking

$$W = \{(x, y, t) : x \in X, \ y \in Y \text{ and } t \in T^{x,y}\},$$

where $\boldsymbol{q}(x, y) = (T^{x,y}, S_h^{x,y}, S_v^{x,y}) \in \mathfrak{Q}$, and defining $R_h, R_v \subseteq W \times W$, such that

$$(x, y, t)R_h(x', y', t') \quad \Longleftrightarrow \quad y = y' \text{ and } \big(x \neq x' \text{ or } tS_h^{x,y}t'\big),$$

$$(x, y, t)R_v(x', y', t') \quad \Longleftrightarrow \quad x = x' \text{ and } \big(y \neq y' \text{ or } tS_v^{x,y}t'\big)$$

for all $(x, y, t), (x', y', t') \in W$.

Since $S_h^{x,y}$ and $S_v^{x,y}$ are such that $t S_i^{x,y} t'$ for all $t \neq t'$, it is straightforward to verify that R_h and R_v commute and are both symmetric and weakly-transitive, which is to say that $\mathfrak{F}$ is a frame for $[\textbf{Diff}, \textbf{Diff}]$.

Finally, for each propositional variable $p \in \mathsf{sub}\,(\varphi)$, take

$$\mathfrak{V}(p) = \{(x, y, t) \in W \; : \; p \in t\}.$$

It remains to show that $\mathfrak{M}$ is a model for φ.

We claim that for all $(x, y, t) \in W$ and $\psi \in \mathsf{sub}\,(\varphi)$,

$$\mathfrak{M}, (x, y, t) \models \psi \qquad \Longleftrightarrow \qquad \psi \in t. \qquad \text{(I.H.)}$$

The cases where ψ is a propositional variable or a Boolean combination of smaller formulas are trivial and follow immediately from the definitions.

So suppose that $\mathfrak{M}, (x, y, t) \models \Diamond_h \alpha$, for some $\alpha \in \mathsf{sub}\,(\varphi)$. It follows that there is some $(x', y', t') \in W$ such that $(x, y, t) R_h (x', y', t')$ and $\mathfrak{M}, (x', y', t') \models \alpha$. By the induction hypothesis (I.H.) we find that $\alpha \in t'$. Furthermore, by the definition of R_h we have that $y = y'$ and either $x \neq x'$ or else $x = x'$ and $t S_h^{x,y} t'$.

- Suppose that $x \neq x'$, then by **(qm3)** we have that $\Diamond_h \alpha \in \bigcap \boldsymbol{q}(x, y)$ and therefore $\Diamond_h \alpha \in t$, by definition.
- Otherwise, we must have that $t S_h^{x,y} t'$ and so it follows from **(qs3)** that $\Diamond_h \alpha \in t$.

Conversely, suppose that $\Diamond_h \alpha \in t$. Then we have two cases to consider, depending on whether or not $\Diamond_h \alpha$ is a defect of $\boldsymbol{q}(x, y)$:

- If $\Diamond_h \alpha$ is not a defect of $\boldsymbol{q}(x, y)$, then by definition there is some $t' \in \boldsymbol{q}(x, y)$ such that $t S_h^{x,y} t'$ and $\alpha \in t'$. By the induction hypothesis we have that $\mathfrak{M}, (x, y, t') \models \alpha$. Moreover, by definition, we have that $(x, y, t) R_h (x, y, t')$ and so $\mathfrak{M}, (x, y, t) \models \Diamond_h \alpha$, as required.
- If $\Diamond_h \alpha$ is a defect of $\boldsymbol{q}(x, y)$, then by **(qm5)** there is some $x' \in X$ such that $x \neq x'$ and $\alpha \in \bigcup \boldsymbol{q}(x', y)$, which is to say that $\alpha \in t'$ for some $t' \in \boldsymbol{q}(x', y)$. Again, by the induction hypothesis, we have that $\mathfrak{M}, (x', y, t') \models \alpha$. Furthermore, by definition, we have that $(x, y, t) R_h (x', y, t')$ and so $\mathfrak{M}, (x, y, t) \models \Diamond_h \alpha$, as required.

The case where ψ is of the form $\Diamond_v \alpha$, for some $\alpha \in \mathsf{sub}\,(\varphi)$, is analogous. Hence we have that $\mathfrak{M}, (x, y, t) \models \psi$ if and only if $\psi \in t$, for all $(x, y, t) \in W$ and $\psi \in \mathsf{sub}\,(\varphi)$. In particular, it follows from **(qm2)** that there is some $(x_0, y_0, t_0) \in W$ such that $\mathfrak{M}, (x_0, y_0, t_0) \models \varphi$, as required.

$\square$

Thus we have reduced the problem of deciding whether φ is satisfiable with respect to $[\textbf{Diff}, \textbf{Diff}]$ to that of checking whether φ has a suitable quasimodel. This exercise is fruitless, however, unless we have some means by which we can effectively search for quasimodels, which may still be infinite!

Fortunately, owing to the rigid grid-like structure of our quasimodels, we may further reduce the problem of checking whether φ has a quasimodel to that of satisfiability with respect to $\textbf{Diff} \times \textbf{Diff}$, whose satisfiability

problem is known to be NExpTime-complete [18]. This approach is similar to the one described in [16, Lemma 32] in which the problem of identifying $(\mathbf{K4.3} \times \mathbf{K})$-quasimodels is reduced to that of satisfiability for some monadic second-order formula qm^m, defined therein.

First, given a set of quasistates $\mathfrak{S} \subseteq \mathfrak{Q}$, we associate with each quasistate $q \in \mathfrak{S}$ some propositional variable $\mathsf{QS}_q \in \mathsf{Prop}$. We then define, for each $\psi \in \mathsf{sub}\,(\varphi)$, the following abbreviations

$$\mathsf{SOME}_\psi^{\mathfrak{S}} := \bigvee \left\{ \mathsf{QS}_q : q \in \mathfrak{S} \text{ and } \psi \in \bigcup q \right\}, \tag{5}$$

$$\mathsf{ALL}_\psi^{\mathfrak{S}} := \bigvee \left\{ \mathsf{QS}_q : q \in \mathfrak{S} \text{ and } \psi \in \bigcap q \right\}, \tag{6}$$

$$\mathsf{DEFECT}_\psi^{\mathfrak{S}} := \bigvee \left\{ \mathsf{QS}_q : q \in \mathfrak{S} \text{ and } \psi \in D_q \right\}. \tag{7}$$

Furthermore, we define

$$\mathsf{SIZE}_j^{\mathfrak{S}} := \bigvee \{ \mathsf{QS}_q \ : \ \boldsymbol{q} \in \mathfrak{S} \text{ and } j\text{-size}(q) = 1 \}, \tag{8}$$

for $j = h, v$.

Take $\mathfrak{S}\text{-}\mathsf{qm}_\varphi$ to be the conjunction of the following formulas:

$$\boxplus \bigvee_{q \in \mathfrak{S}} \mathsf{QS}_q \wedge \boxplus \bigwedge_{\substack{q,q' \in \mathfrak{S} \\ q \neq q'}} \neg(\mathsf{QS}_q \wedge \mathsf{QS}_{q'}) \wedge \oplus \mathsf{SOME}_\varphi^{\mathfrak{S}}, \tag{9}$$

$$\boxplus \bigwedge_{\diamondsuit_j \psi \in \mathsf{sub}(\varphi)} (\diamondsuit_i \mathsf{SOME}_\psi^{\mathfrak{S}} \to \mathsf{ALL}_{\diamondsuit_j \psi}^{\mathfrak{S}}), \qquad \text{for } j = h, v, \tag{10}$$

$$\boxplus \bigwedge_{\diamondsuit_j \psi \in \mathsf{sub}(\varphi)} (\mathsf{DEFECT}_{\diamondsuit_j \psi}^{\mathfrak{S}} \to \diamondsuit_j \mathsf{SOME}_\psi^{\mathfrak{S}}), \qquad \text{for } j = h, v, \tag{11}$$

$$\boxplus (\mathsf{SIZE}_h^{\mathfrak{S}} \to \Box_v \mathsf{SIZE}_h^{\mathfrak{S}}) \quad \wedge \quad \boxplus (\mathsf{SIZE}_v^{\mathfrak{S}} \to \Box_h \mathsf{SIZE}_v^{\mathfrak{S}}), \tag{12}$$

where $\boxplus \varphi := \Box_h^+ \Box_v^+ \varphi$ and $\oplus \varphi := \diamondsuit_h^+ \diamondsuit_v^+ \varphi$. With these formulas, we are able to establish an equivalence between the existence of a $\mathfrak{S}$-quasimodel for φ and the satisfiability of $\mathfrak{S}\text{-}\mathsf{qm}_\varphi$ with respect to $\mathbf{Diff} \times \mathbf{Diff}$, as is demonstrated below in Lemma 2.8.

Lemma 2.8 *The formula $\mathfrak{S}\text{-}\mathsf{qm}_\varphi$ is satisfiable with respect to $\mathbf{Diff} \times \mathbf{Diff}$ if and only if φ has an $\mathfrak{S}$-quasimodel, for all $\mathfrak{S} \subseteq \mathfrak{Q}$.*

Proof.

$(\Rightarrow)$ Suppose that $\mathfrak{M}, (r_h, r_v) \models \mathfrak{S}\text{-}\mathsf{qm}_\varphi$, for some product model $\mathfrak{M} = (\mathfrak{F}_h \times \mathfrak{F}_v, \mathfrak{V})$, where $\mathfrak{F}_j = (W_j, \neq)$ is a difference frame, for $j = h, v$. We define a quasimodel $(X, Y, \boldsymbol{q})$ by taking

$$X = W_h, \qquad Y = W_v, \qquad \text{and} \qquad \boldsymbol{q}(x,y) = q \iff (x,y) \in \mathfrak{V}(\mathsf{QS}_q),$$

for all quasistates $q \in \mathfrak{S}$.

By (9), we can be assured that $\boldsymbol{q}(x, y)$ is well-defined, for all $(x, y) \in X \times Y$, and that there is some $x_0 \in X$, $y_0 \in Y$ such that $\varphi \in \bigcup \boldsymbol{q}(x_0, y_0)$, as required for condition **(qm2)**. Conditions **(qm3)**–**(qm4)** are satisfied by (10), while conditions **(qm5)**–**(qm6)** are satisfied by (11). Finally, (12) ensures that condition **(qm7)** is satisfied. Hence $(X, Y, \boldsymbol{q})$ is an appropriate $\mathfrak{S}$-quasimodel for φ, as required.

($\Leftarrow$) Conversely, suppose that $(X, Y, \boldsymbol{q})$ is an $\mathfrak{S}$-quasimodel for φ. Let $\mathfrak{F}_h = (X, \neq)$ and $\mathfrak{F}_v = (Y, \neq)$ be difference frames on X and Y, respectively, and define a new model $\mathfrak{M} = (\mathfrak{F}_h \times \mathfrak{F}_v, \mathfrak{V})$ over $\mathfrak{F}_h \times \mathfrak{F}_v$, by taking

$$(x, y) \in \mathfrak{V}(\mathsf{QS}_q) \qquad \Longleftrightarrow \qquad \boldsymbol{q}(x, y) = q,$$

for all $x \in X$, $y \in Y$, and all quasistates $q \in \mathfrak{S}$. The following are then immediate consequences of the definitions:

$$\mathfrak{M}, (x, y) \models \mathsf{SOME}_\psi^{\mathfrak{S}} \qquad \Longleftrightarrow \qquad \psi \in \bigcup \boldsymbol{q}(x, y),$$
$$\mathfrak{M}, (x, y) \models \mathsf{ALL}_\psi^{\mathfrak{S}} \qquad \Longleftrightarrow \qquad \psi \in \bigcap \boldsymbol{q}(x, y),$$
$$\mathfrak{M}, (x, y) \models \mathsf{DEFECT}_\psi^{\mathfrak{S}} \qquad \Longleftrightarrow \qquad \psi \in D_{\boldsymbol{q}(x, y)},$$
$$\mathfrak{M}, (x, y) \models \mathsf{SIZE}_h^{\mathfrak{S}} \qquad \Longleftrightarrow \qquad h\text{-}\mathsf{size}(\boldsymbol{q}(x, y)) = 1,$$
$$\mathfrak{M}, (x, y) \models \mathsf{SIZE}_v^{\mathfrak{S}} \qquad \Longleftrightarrow \qquad v\text{-}\mathsf{size}(\boldsymbol{q}(x, y)) = 1.$$

It is then straightforward to check that each of the conjuncts (9)–(12) reflect the conditions **(qm2)**–**(qm8)** on $(X, Y, \boldsymbol{q})$ being a quasimodel for φ. Hence we must have that $\mathfrak{S}\text{-}\mathsf{qm}_\varphi$ is satisfiable with respect to $\mathbf{Diff} \times \mathbf{Diff}$, as required.

$\square$

Hence, it follows from Lemmas 2.7–2.8 that φ is satisfiable with respect to $[\mathbf{Diff}, \mathbf{Diff}]$ if and only if $\mathfrak{Q}\text{-}\mathsf{qm}_\varphi$ is satisfiable with respect to $\mathbf{Diff} \times \mathbf{Diff}$, where $\mathfrak{Q}$ is the set of all quasistates. Since the satisfiability problem for the product logic $\mathbf{Diff} \times \mathbf{Diff}$ is decidable, so too must be the satisfiability problem for the commutator $[\mathbf{Diff}, \mathbf{Diff}]$. However, the size of $\mathfrak{Q}\text{-}\mathsf{qm}_\varphi$ is *doubly-exponential* in the size of $\mathsf{sub}(\varphi)$. Together with the optimal NExpTime upper-bound on the satisfiability problem for $\mathbf{Diff} \times \mathbf{Diff}$, this would provide only a N3ExpTime upper-bound on the satisfiability problem for $[\mathbf{Diff}, \mathbf{Diff}]$.

2.3   Small Quasimodels for $[\mathbf{Diff}, \mathbf{Diff}]$

To redress the issue highlighted above, we can restrict our attention to a much smaller set of quasistates $\mathfrak{Q}_{\mathsf{sm}} \subseteq \mathfrak{Q}$ that is at most (singly-) exponential in the size of φ, thereby reducing the upper-bound on the satisfiability problem for $[\mathbf{Diff}, \mathbf{Diff}]$ from N3ExpTime to N2ExpTime.

Definition 2.9 (Small Quasistates) A quasistate $q = (T, S_h, S_v)$ for φ is said to be *small* if it satisfies the condition that:

(sm1) $|T| \leq 2n^2$ is at most quadratic in $n = |\mathsf{sub}\,(\varphi)|$.

Let $\mathfrak{Q}_{\mathsf{sm}} \subseteq \mathfrak{Q}$ denote the set of all small quasistates for φ.

Note that $|\mathfrak{Q}_{\mathsf{sm}}|$ is a most exponential in the size of φ, since there can be at most 2^n types for φ, and at most $\sum_{k=1}^{N}(2^n)^k \leq N \cdot 4^{n^3}$ possible candidates for T, where $N = 2n^2$ is the maximum size of T. Furthermore, there are at most $2^{|T|} \leq 4^{n^2}$ candidates for S_h and S_v, since, by **(qs2)**, they are both completely defined by their reflexive elements. Since the set of all types can be constructed in exponential-time, so too can be the set of all small quasistates.

Lemma 2.10 φ *has a* $\mathfrak{Q}$*-quasimodel if and only if* φ *also has a* $\mathfrak{Q}_{\mathsf{sm}}$*-quasimodel, comprising only* small *quasistates.*

Proof. The right-to-left direction is trivial, since $\mathfrak{Q}_{\mathsf{sm}} \subseteq \mathfrak{Q}$, and so every $\mathfrak{Q}_{\mathsf{sm}}$-quasimodel is also a $\mathfrak{Q}$-quasimodel. For the converse, it is sufficient to show that every quasistate $q \in \mathfrak{Q}$ can be replaced with small quasistate $q' \in \mathfrak{Q}_{\mathsf{sm}}$ such that

$$\bigcup q \; = \; \bigcup q', \qquad \bigcap q \; = \; \bigcap q', \qquad \text{and} \qquad D_q = D_{q'}$$

since each of the conditions **(qm1)**–**(qm7)** makes reference only to these properties of its constituent quasistates.

To this end, let $q = (T, S_h, S_v)$ be an arbitrary quasistate, and for each $\psi \in \bigcup q$, fix some $t_\psi \in T$ such that $\psi \in t_\psi$. Let $T_0 \subseteq T$ be the subset comprising all such types. For each $t \in T_0$ and $\Diamond_j \alpha \in t$ such that $\Diamond_j \alpha \notin D_q$, fix some $s_{(t,\alpha)} \in T$ such that $t S_j s_{(t,\alpha)}$ and $\alpha \in s_{(t,\alpha)}$, and take T_1 to be the set of all such types. We may then define $q' = (T', S_h', S_v')$ by taking

$$T' \; := \; T_0 \cup T_1 \qquad \text{and} \qquad t S_j' t' \quad \Longleftrightarrow \quad t S_j t'$$

for all $t, t' \in T'$ and $j = h, v$. It clear that q' is quasistate for φ and that, by construction $\bigcap q = \bigcap q'$ and $\bigcup q = \bigcup q'$. Furthermore, it is clear from the construction that $|T| \leq (n + n^2) \leq 2n^2$, and so $q' \in \mathfrak{Q}_{\mathsf{sm}}$. All that remains is to show that $D_q = D_{q'}$.

– It is straightforward to verify that $D_q \subseteq D_{q'}$, since T' is a subset of T and so cannot provide any remedies to any of the defects of q. For the other direction, suppose that $\Diamond_j \alpha \notin D_q$, and suppose that $t \in T'$ is such that $\Diamond_j \alpha \in t$. If $t \in T_0$ then by construction there is some $s_{(t,\alpha)} \in T_1 \subseteq T'$ such that $t S_j s_{(t,\alpha)}$ and $\alpha \in s_{(t,\alpha)}$, which is to say that $\Diamond_j \alpha \notin D_{q'}$. On the other hand, if $t \in T_1$ then there is some $t' \in T_0$ such that $t' S_j t$. Moreover, since $\Diamond_j \alpha \notin D_q$, there must be some $t'' \in T$ such that $t S_j t''$ and $\alpha \in t''$. By **(qs2)**, either $t'' = t' \in T'$ or $t'' S_j t$. In the latter case, by **(qs3)**, we must have that $\Diamond_j \alpha \in t''$ and so, by construction, there is some $s_{(t'',\alpha)} \in T_1 \subseteq T'$ such that $\alpha \in s_{(t'',\alpha)}$. In both cases, there is some $s \in T'$ such that $t S_j s$ and $\alpha \in s$, which is to say that $\Diamond_j \alpha \notin D_{q'}$, as required.

Hence, it follows that every $\mathfrak{Q}$-quasimodel for φ can be transformed into a $\mathfrak{Q}_{\mathsf{sm}}$-quasimodel for φ, in which each quasistate is small. $\qquad \square$

It then follows from Lemmas 2.7, 2.8 and 2.10 that φ is satisfiable with respect to $[\mathbf{Diff}, \mathbf{Diff}]$ if and only if $\mathfrak{Q}_{\mathsf{sm}}\text{-}\mathsf{qm}_\varphi$ is satisfiable with respect to $\mathbf{Diff} \times \mathbf{Diff}$, where $\mathfrak{Q}_{\mathsf{sm}}$ is the set of all small quasistates for φ. Furthermore, since the size of $\mathfrak{Q}_{\mathsf{sm}}\text{-}\mathsf{qm}_\varphi$ is at most exponential in the size of $\mathsf{sub}\,(\varphi)$ and can be constructed in exponential-time, the above reduction from commutator $[\mathbf{Diff}, \mathbf{Diff}]$ to its product $\mathbf{Diff} \times \mathbf{Diff}$ incurs, at most, an *exponential* increase in complexity. As the satisfiability problem for $\mathbf{Diff} \times \mathbf{Diff}$ can be decided in NExpTime, so it follows that the satisfiability problem for $[\mathbf{Diff}, \mathbf{Diff}]$ can be decided in N2ExpTime, thereby completing the proof of Theorem 2.4.

It is worth noting that this is in-line with what is typically achieved from standard filtration techniques which place double-exponential bounds on the size of the filtered models (see, for example, Gabbay et al. [7, Theorem 5.27]).

2.4 The Finite Model Property of $[\mathbf{S5}, \mathbf{Diff}]$

One immediate consequence of Theorem 2.4 is that the the satisfiability problem for $[\mathbf{S5}, \mathbf{Diff}]$ is also decidable, since it can be identified with a term-definable fragment of $[\mathbf{Diff}, \mathbf{Diff}]$, by rewriting $\Diamond_h \psi := \psi \vee \Diamond_h \psi$.

However, in this case we are able to prove a stronger result, by appealing to the fact that, unlike $\mathbf{Diff} \times \mathbf{Diff}$, the product $\mathbf{S5}$ and $\mathbf{Diff}$ possess the exponential *product* fmp [11], which is to say that every formula φ that is satisfiable with respect to $\mathbf{S5} \times \mathbf{Diff}$ can be satisfied in a *product* model for $\mathbf{S5} \times \mathbf{Diff}$ that is at most exponential in the size of φ. This affords us the possibility of adapting the above strategy by finitizing the resulting product model and thereby placing an upper-bound on the size of the satisfying models for $[\mathbf{S5}, \mathbf{Diff}]$, despite the lack of a known method of filtration.

Theorem 2.11 *The commutator* $[\mathbf{S5}, \mathbf{Diff}]$ *has the doubly-exponential sized finite model property.*

Proof. Let φ be an $\mathcal{ML}_2$ formula, and define $\mathfrak{Q}_{\mathsf{sm}}^\bullet \subseteq \mathfrak{Q}$ to be the set of all small *horizontally reflexive* quasistates, in which $tS_h t$, for all $t \in T$. We claim that φ is satisfiable with respect to $[\mathbf{S5}, \mathbf{Diff}]$ if and only if there is an $\mathfrak{Q}_{\mathsf{sm}}^\bullet$-quasimodel for φ (the proof is analogous to that of Lemmas 2.7).

Since every type belonging to a horizontally reflexive quasistate is reflexive, if $\Diamond_h \psi \in D_{\boldsymbol{q}(x,y)}$ then we must necessarily have that $\psi \notin \bigcup \boldsymbol{q}(x,y)$. Hence, for $\mathfrak{Q}_{\mathsf{sm}}^\bullet$-quasimodels, condition **(qm5)** of Definition 2.6 is equivalent to:

(qm5′) For all $x \in X$, $y \in Y$ and $\Diamond_h \psi \in \mathsf{sub}\,(\varphi)$,

$$\Diamond_h \psi \in D_{\boldsymbol{q}(x,y)} \qquad \Longrightarrow \qquad \exists x' \in X;\ \psi \in \bigcup \boldsymbol{q}(x',y)$$

From here it is not difficult to adapt the proof of Lemma 2.8, to show that φ has a $\mathfrak{Q}_{\mathsf{sm}}^\bullet$-quasimodel if and only if $\mathfrak{Q}_{\mathsf{sm}}^\bullet\text{-}\mathsf{qm}_\varphi$ is satisfiable with respect to $\mathbf{S5} \times \mathbf{Diff}$. Moreover, the size of the resulting quasimodel is at most (singly-) exponential in the size of the satisfying model.

Hence, if φ is satisfiable with respect to $[\mathbf{S5}, \mathbf{Diff}]$ then $\mathfrak{Q}_{\mathsf{sm}}^\bullet\text{-}\mathsf{qm}_\varphi$ is satisfiable with respect to $\mathbf{S5} \times \mathbf{Diff}$. Furthermore, since $\mathbf{S5} \times \mathbf{Diff}$ has the exponential *product* fmp [11], $\mathfrak{Q}_{\mathsf{sm}}^\bullet\text{-}\mathsf{qm}_\varphi$ can be satisfied in a product model that is at most

exponential in the size of $\mathfrak{Q}_{\mathsf{sm}}^{\bullet}\text{-}\mathsf{qm}_{\varphi}$. This, in turn, can be converted back to a model for φ that is at most exponential in the size of $\mathfrak{Q}_{\mathsf{sm}}^{\bullet}\text{-}\mathsf{qm}_{\varphi}$ and at most *doubly-exponential* in the size of φ, as required.   □

Again, it is worth noting that in [8], this same upper-bound is achieved through a method of filtration for a large collection of commutators of the form $[L, \mathbf{S5}]$, where L can be axiomatized exclusively with proposition-free formulas, and formulas from among $\{p \to \Box\Diamond p, \Diamond^k p \to \Diamond p : k > 0\}$. However, neither this nor any other method of filtration is known for the case where L is the logic of the difference operator.

3   Discussion

This paper provides a first glance into the behaviour of some commutators of modal logics which are neither product matching nor are both Horn-axiomatizable. We conclude with a discussion of some open problems and directions for future work:

- The satisfiability problem for $[\mathbf{Diff}, \mathbf{Diff}]$ is known to be NExpTime-hard [17], and so it remains open as to where lies the precise complexity? Is it possible to improve upon the N2ExpTime upper-bound on the complexity of the satisfiability problem for $[\mathbf{Diff}, \mathbf{Diff}]$, or is it, perhaps, possible to exploit the infinite 'grid'-like structure of Lemma 2.1 to encode some N2ExpTime-hard problem?

- A natural generalization of the logic of the difference operator is provided by Jansana's [15] family of logics $\mathbf{Kn.4B}$, for $n > 1$, axiomatized by the following formulas:

$$(n.4) \; := \; [n]p \to [n+1]p \qquad \text{and} \qquad (n.B) \; := \; p \to [n]\langle n\rangle p,$$

  where $\langle 0\rangle p := p$, $\langle n\rangle p := \Diamond^n p \vee \langle n-1\rangle p$, and $[n]\varphi := \neg\langle n\rangle\neg\varphi$. These logics are characterised by the class of frames in which every possible world is reachable from every other in fewer than n transitions; in particular, we have that $\mathbf{Diff} = \mathbf{K1.4B}$. Can the above techniques be adapted to construct a reduction between $[\mathbf{Kn.4B}, \mathbf{Km.4B}]$ and $\mathbf{Kn.4B} \times \mathbf{Km.4B}$, for $n, m \geq 1$?

- Finally, if the techniques employed here could be suitably extended, this would serve to limit the search for any examples of Kripke complete modal logics L_h and L_v such that one and only one of the logics $[L_h, L_v]$ and $L_h \times L_v$ is decidable; a question posed in [7].

A   Appendix

Claim A.1 *The triple $(X, Y, \boldsymbol{q})$ of Lemma 2.7 is a $\mathfrak{Q}$-quasimodel for φ.*

Proof. First, we must verify that each $\boldsymbol{q}(x,y) \in \mathfrak{Q}$ is indeed a quasistate for φ:

– Clearly $T^{x,y}$ is a non-empty set of types, since $h(x,y)$ is non-empty by construction, as required for **(qs1)**.

– Suppose that $\boldsymbol{t}(u), \boldsymbol{t}(v) \in T^{x,y}$ are such that $\boldsymbol{t}(u) \neq \boldsymbol{t}(v)$, for some $u,v \in h(x,y)$. In particular, we have that $uR_h v$ and $uR_v v$. It then follows immediately from the definition that $\boldsymbol{t}(u)S_j^{x,y}\boldsymbol{t}(v)$ for $j = h,v$, as required for **(qs2)**.

– Finally, suppose that $\boldsymbol{t}(u), \boldsymbol{t}(v) \in T^{x,y}$ and $\Diamond_j \psi \in \mathsf{sub}\,(\varphi)$ are such that $\boldsymbol{t}(u)S_j^{x,y}\boldsymbol{t}(v)$ and $\psi \in \boldsymbol{t}(v)$. It follows by definition that there is some $u',v' \in h(x,y)$ such that $\boldsymbol{t}(u) = \boldsymbol{t}(u')$, $\boldsymbol{t}(v) = \boldsymbol{t}(v')$ and $u'R_j v'$. Hence we have that $\alpha \in \boldsymbol{t}(v')$ and consequently, that $\Diamond_j \psi \in \boldsymbol{t}(u') = \boldsymbol{t}(u)$, as required for **(qs3)**.

Next, we must check that $(X,Y,\boldsymbol{q})$ satisfies all the conditions **(qm1)**–**(qm7)** to be a suitable $\mathfrak{Q}$-quasimodel for φ:

– By definition, $\mathfrak{M}, r \models \varphi$, and by definition we have that $[r] \in X$ and $[r] \in Y$. Therefore, we may take $x_0 = y_0 = r \in h([r],[r])$ such that $\varphi \in \boldsymbol{t}(r)$ and hence, by construction, $\varphi \in \bigcup \boldsymbol{q}(x_0, y_0)$, as required for **(qm2)**.

– For **(qm3)**, suppose that $x, x' \in X$, $y \in Y$ and $\Diamond_h \psi \in \mathsf{sub}\,(\varphi)$ are such that $x \neq x'$ and $\psi \in \bigcup \boldsymbol{q}(x',y)$, which is to say that $\psi \in \boldsymbol{t}(w')$ for some $w' \in h(x',y)$. By construction we have that $h(x',y) \neq h(x,y)$, since R_h^+ is an equivalence relation. It then follows, again from the fact that R_h^+ is an equivalence relation, that $wR_h w'$ for all $w \in h(x,y)$ and $w' \in h(x',y)$. Hence, we have that $\Diamond_h \psi \in \boldsymbol{t}(w)$ for all $w \in h(x,y)$, which is to say that $\Diamond_h \psi \in \bigcap \boldsymbol{q}(x,y)$, as required. Condition **(qm4)** is analogous.

– For **(qm5)**, suppose that $x \in X$, $y \in Y$ and $\Diamond_h \psi \in D_{\boldsymbol{q}(x,y)}$. By definition, there is some $u \in h(x,y)$ such that $\Diamond_h \psi \in \boldsymbol{t}(u)$ and there is no $v \in h(x,y)$ such that $\boldsymbol{t}(u)S_h^{x,y}\boldsymbol{t}(v)$ and $\psi \in \boldsymbol{t}(v)$. However, since $\mathfrak{M}, u \models \Diamond_h \psi$, there must be some $v' \in W$ such that $uR_h v'$ and $\psi \in \boldsymbol{t}(v')$. It then follows that there is some $x' \in X$ such that $x \neq x'$ and $v' \in h(x',y)$. Hence we have that $\psi \in \bigcup \boldsymbol{q}(x',y)$, as required. Condition **(qm6)** is analogous.

– For **(qm7)**, suppose that $x \in X$ and $y,y' \in Y$ are such that $y \neq y'$ and $h\text{-size}(\boldsymbol{q}(x,y)) > 1$ then by definition there are some $t,t' \in T^{x,y}$ such that $tS_h^{x,y}t'$ (note that t and t' may or may not be identical). Hence there are some $u,v \in h(x,y)$ such that $\boldsymbol{t}(u) = t$, $\boldsymbol{t}(v) = t$ and $uR_h v$. Let $u' \in h(x,y')$ then since $y \neq y'$ and R_v^+ is an equivalence relation, we have that $uR_v u'$. Hence, by the Church-Rosser property, there is some $v' \in W$ such that $u'R_h v'$ and $vR_v v'$. Moreover, since R_v^+ is an equivalence relation, we have that $uR_v^+ v'$, and thus $v' \in h(x,y')$. Hence there are $\boldsymbol{t}(u'), \boldsymbol{t}(v') \in T^{x,y'}$ such that $\boldsymbol{t}(u')S_h^{x,y'}\boldsymbol{t}(v')$. It then follows that $h\text{-size}(\boldsymbol{q}(x,y')) > 1$, as required. The case for $v\text{-size}(\boldsymbol{q}(x,y))$ is analogous.

Hence it follows that $(X,Y,\boldsymbol{q})$ is a $\mathfrak{Q}$-quasimodel for φ, as required. $\square$

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