On exact and numerical solutions of the one-dimensional planar Bratu problem

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June 12, 2003

The nonlinear eigenvalue problem $\Delta u + \lambda e^u = 0$ in the unit square with $u = 0$ on the boundary is often referred to as "the Bratu problem" or "Bratu's problem." The Bratu problem in 1-dimensional planar coordinates, $u'' + \lambda e^u = 0$ with $u(0) = u(1) = 0$ has two known, bifurcated, exact solutions for values of $\lambda < \lambda_c$ and no solutions for $\lambda > \lambda_c$. The value of λ_c is simply $8(\alpha^2 - 1)$ where α is the fixed point of the hyperbolic cotangent function. Numerical approximations to the exact solution of the one-dimensional planar Bratu problem are computed using various numerical methods. Of particular interest is the application of nonstandard finite-difference schemes known as Mickens finite differences to solve the problem. In addition, standard finite-differences, Boyd collocation and Adomian polynomial decomposition are employed to generate numerical solutions to this Bratu problem and the results compared.

Keywords: Mickens difference, nonstandard finite-difference scheme, Bratu problem, bifurcation, nonlinear eigenvalue problem

Introduction

In this paper the results of applying a nonstandard ("Mickens") finite-difference scheme to a specific boundary value problem related to the classical Bratu problem [4] shall be presented. The Bratu problem is an elliptic partial differential equation which comes from a simplification of the solid fuel ignition model in thermal combustion theory [9]. It is also a nonlinear eigenvalue problem that is often used as a benchmarking tool for numerical methods ([2], [3], [7]). In [11], Jacobsen and Schmitt provide an excellent summary of the significance and history of the Bratu problem. In this paper, a Mickens finite difference [14] is applied to the 1-dimensional planar Bratu problem. Similar work applied to the Bratu

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problem in cylindrical coordinates, more commonly known as the Bratu-Gel'fand problem [10], has been presented elsewhere [6]. These two versions of the Bratu problem were selected because they have known exact solutions which can be used to check the accuracy of solutions generated by numerical experiments. The goal of this paper will be to compare the solutions to the planar one-dimensional Bratu problem produced by Mickens differences to solutions produced by other numerical techniques.

The classical Bratu problem is

$$\Delta u + \lambda e^u = 0 \qquad \text{on } \Omega : \{(x,y) \in 0 \le x \le 1, 0 \le y \le 1\} \tag{1}$$

$$\text{with } u = 0 \qquad \text{on } \partial\Omega \tag{2}$$

The 1-dimensional (planar) version of this problem is

$$u''(x) + \lambda e^{u(x)} = 0 \qquad 0 \le x \le 1, \tag{3}$$

$$\text{with } u(0) = 0 \qquad \text{and } u(1) = 0 \tag{4}$$

In section 1 of this paper the exact solution of the one-dimensional planar Bratu problem will be presented. Details of the bifurcated nature of the solution are given. In Section 2 brief explanations of the various methods chosen to solve the will be presented. In Section 3 numerical solutions generated using nonstandard finite differences will be compared to solutions produced using different numerical methods: standard finite differences, a pseudospectral method due to Boyd [3] and the Adomian polynomial decomposition algorithm [7]. All of the approximate solutions are compared to the exact solution. The paper shall conclude with some overall comments and observations based on the numerical results.

1 The 1-dimensional Planar Bratu Problem

The exact solution to (3) is given in [2] and [7] and presented here as

$$u(x) = -2\ln\left[\frac{\cosh((x-\frac{1}{2})\frac{\theta}{2})}{\cosh(\frac{\theta}{4})}\right] \tag{5}$$

where θ solves

$$\theta = \sqrt{2\lambda}\cosh\left(\frac{\theta}{4}\right). \tag{6}$$

There are two solutions to (6) for values of $0 < \lambda < \lambda_c$. For $\lambda > \lambda_c$ there are no solutions. The solution (5) is only unique for a critical value of $\lambda = \lambda_c$ which solves

$$1 = \sqrt{2\lambda_c}\sinh\left(\frac{\theta_c}{4}\right)\frac{1}{4}. \tag{7}$$

By graphing the line $y = \theta$ and the curve $y = \sqrt{2\lambda}\cosh(\frac{\theta}{4})$ for fixed values of $\lambda = 1, 2, 3, 4$ and 5 the solutions of (6) can be seen as the points of intersections of the curve and the line in Figure 1. Clearly, there is only one solution when the $y = \theta$ line is exactly tangential to the $y = \sqrt{2\lambda}\cosh(\frac{\theta}{4})$ curve, which leads to the condition given in (7).

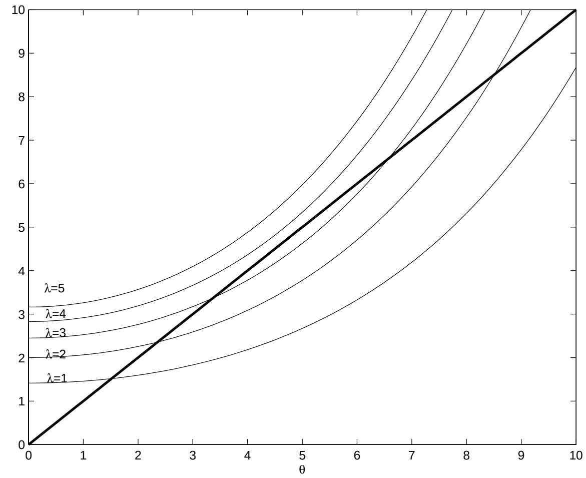


Figure 1: Graphical depiction of dependence of solutions of (6) upon λ

Dividing (7) by (6) produces:

$$\begin{aligned} \frac{4}{\theta_c} &= \tanh\left(\frac{\theta_c}{4}\right) \\ \Rightarrow \frac{\theta_c}{4} &= \coth\left(\frac{\theta_c}{4}\right) \\ \Rightarrow \alpha &= \coth(\alpha) \end{aligned}$$

The critical value θ_c is four times α, which is the positive fixed point of the hyperbolic cotangent function, 1.19967864.

$$\theta_c = 4.79871456 \tag{8}$$

The exact value of θ_c can therefore be used in (7) to obtain the exact value of λ_c.

$$\lambda_c = \frac{8}{\sinh^2\left(\frac{\theta_c}{4}\right)} = 8(\alpha^2 - 1) \Rightarrow \lambda_c = 3.513830719 \tag{9}$$

The relationship between λ and θ for some values of λ less than λ_c are given in Table 1. Obviously, when $\lambda = \lambda_c$ then $\theta_1 = \theta_2 = \theta_c$ and when $\lambda > \lambda_c$ there are no solutions to (6).

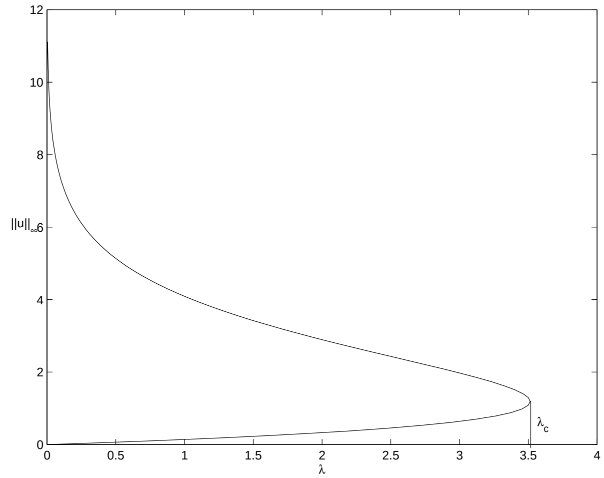


Figure 2: Bifurcated nature of the exact solution to the Bratu problem

Figure 2 shows how the maximum value of the solution function (5) depends on the nonlinear eigenvalue λ with the critical value of λ_c highlighted at the "turning point." Table 1 and Figure 2 are two different ways of depicting the property of the solution that it is double-valued for $\lambda < \lambda_c$. In the next section, numerical methods to compute these solutions to (3).

2 Numerical Methods

In this section of the paper, the details of various numerical methods used to compute solutions to (3) shall be given. The first method involves approximating the differential equation with finite differences. Both standard and nonstandard (Mickens) finite-difference schemes are used. In addition to the methods which use finite-differences, two pseduospectral methods are used. The first, due to Boyd [3], uses Gegenbauer polynomials as basis functions. The second, due to Adomian [1] assumes the solution can be represented as an infinite series of polynomials. Lastly the problem was also solved using a shooting method easily available in `Matlab`.

λ	θ_1	θ_2
0.5	1.0356946	13.0382393
1.0	1.5171645	10.9387028
1.5	1.9397652	9.5816998
2.0	2.3575510	8.5071995
2.5	2.8115549	7.5480981
3.0	3.3735077	6.5765692
3.5	4.5518536	5.0543427
λ_c	4.7987146	4.7987146

Table 1: Corresponding values of θ for various $\lambda \leq \lambda_c$

2.1 Finite Difference Methods

To solve a boundary value problem using finite differences involves discretizing the differential equation and boundary conditions. This method transforms the problem into a system of simultaneous nonlinear equations which are then usually easily solved using Newton's method. There are many choices for how to approximate the derivatives which appear in a differential equation. In this section of the paper standard finite differences and nonstandard finite differences will be deployed. Nonstandard finite differences have been extensively studied by Professor Ronald E. Mickens of Clark Atlanta University ([15], [16], [17]). The first step in the computation of the numerical solution of (3) using a finite-difference method is to approximate the continuous domain of the problem with a discrete grid. The grid chosen was $\{x_j\}_{j=0}^N$ on the interval $0 \le x \le 1$ where

$$0 = x_0 < x_1 < x_2 < \ldots < x_j < \ldots < x_N = 1. \tag{10}$$

For a uniform grid, the grid separation parameter h is constant and $h = 1/N$ with $x_k = 0 + kh$ for $k = 0$ to N. Using a standard finite-difference scheme, the discrete version of the planar Bratu problem (3) will be

$$\frac{u_{j+1} - 2u_j + u_{j-1}}{h^2} + \lambda e^{u_j} = 0, \qquad j = 1, 2, \ldots, N-1 \tag{11}$$

A nonstandard finite-difference scheme for (3) is

$$\frac{u_{j+1} - 2u_j + u_{j-1}}{2\ln[\cosh(h)]} + \lambda e^{u_j} = 0, \qquad j = 1, 2, \ldots, N-1 \tag{12}$$

The boundary conditions given in (4) become

$$u_0 = u_N = 0. \tag{13}$$

The discretization given in (12) is an example of a Mickens discretization. Mickens has repeatedly shown that one can find nonstandard finite difference schemes which produce *exact* discrete solutions of a differential equation [17]. For example, in [13] the following Mickens scheme

$$\frac{u_{j+1} - u_j}{\left(\dfrac{1 - e^{-\alpha h}}{\alpha}\right)} = -\alpha u_j \tag{14}$$

is an exact nonstandard finite difference scheme for the differential equation $\dfrac{du}{dx} = -\alpha u$. Also found in [13] is the following exact Mickens discretization for $\dfrac{du}{dx} = -u^3$.

$$\frac{u_{j+1} - u_j}{h} = -\left(\frac{2u_{j+1}}{u_{j+1} + u_j}\right) u_{j+1} u_j^2 \tag{15}$$

A Mickens difference is a nonstandard finite-difference scheme which (1) approximates a derivative using a nonlinear denominator function and/or (2) uses "non-local" or "off-grid" representations of expressions in the differential equation.

The scheme given in (14) is an example of the use of a nonlinear denominator function in a Mickens finite difference. Note that the denominator function in (14), $\phi(h) = \dfrac{1-e^{-\alpha h}}{\alpha}$ has the property that in the limit as $h \to 0$, $\phi(h) \to h$. In general, the denominator function ϕ in a Mickens finite-difference for the first derivative

$$u' \approx \frac{u_{j+1} - u_j}{\phi(h)}$$

has the property that $\phi(h) = h + o(h)$.

The scheme given in (15) is an example of a "non-local" discretization appearing in a Mickens difference. The standard discrete representation of u^3 would be expected to be simply u_j^3. However the unexpectedly florid discretization of this cubic term that appears on the right-hand side of (15) leads to an *exact* discrete solution to the differential equation.

The nonstandard finite difference scheme given in (12) is a Mickens difference for a second derivative

$$u'' \approx \frac{u_{j+1} - 2u_j + u_{j-1}}{\phi(h)}$$

where the denominator function $\phi(h) = 2\ln[\cosh(h)] = h^2 + o(h^2)$. Thus, in the limit as $h \to 0$ the standard finite-difference scheme (11) and the Mickens-difference scheme (12) will be identical. However, for the finite values of h at which numerical computations are conducted the hypothesis is that the nonstandard form of the denominator function $\phi(h)$ will lead to improved accuracy.

2.2 Boyd collocation

Boyd [3] developed a pseudospectral method to produce approximate solutions to the classical two-dimensional planar Bratu problem

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \lambda e^u = 0 \text{ on } \{(x,y) \in -1 \le x \le 1, -1 \le y \le 1\} \tag{16}$$

with $u = 0$ on the boundary of the square. The basic idea is that the unknown solution $u(x,y)$ can be completely represented as an infinite series of spectral basis functions

$$u(x,y) = \sum_{k=1}^{N} a_k \phi_k(x,y). \tag{17}$$

The basis functions $\phi_k(x,y)$ are chosen so that they obey the boundary conditions and have the property that the more terms of the series that are kept, the more accurate the representation of the solution $u(x,y)$ is. In other words, as $N \to \infty$ the error diminishes to zero. For finite N the series expansion in (17) is substituted into (16) to produce the residual R. The residual function will depend on the spatial variables (x,y), the unknown coefficients a_k and the parameter λ. The goal of Boyd's pseudospectral method is to find a_k so that the residual function R is zero at N "collocation points." The collocation points are usually chosen to be the roots of orthogonal polynomials that fall into the same family as the basis functions $\phi_k(x,y)$. Boyd [3] uses the Gegenbauer polynomials to define the collocation points. The Gegenbauer polynomials [8] are orthogonal on the interval $[-1,1]$ with respect to the weight function $w(x) = (1-x^2)^b$ where $b = -1/2$ corresponds to the Chebyshev polynomials and $b = 1$ is the choice Boyd uses. The second-order Gegenbauer polynomial is

$$G_2(x) = \frac{3}{2}(5x^2 - 1), \qquad -1 \le x \le 1. \tag{18}$$

Using a 1-point collocation method at the point $x_1 = \left(\frac{1}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$ and the choice of $\phi_1(x,y) = (1-x^2)(1-y^2)$ Boyd is able to obtain an approximation to the value of λ_c with a relative error of 8% [3]. Note that this choice for $\phi_1(x,y)$ satisfies the boundary conditions (2) since $\phi(1,y) = \phi(-1,y) = \phi(x,-1) = \phi(x,1) = 0$.

In the rest of this section Boyd's collocation method described above for Bratu's problem in planar two-dimensional coordinates (16) shall be extrapolated to solve the planar 1-d Bratu probem (3). The most obvious difference is the change in the domain from a square $[-1,1] \times [-1,1]$ to an interval $[0,1]$. Using a linear transformation of $z = \frac{x+1}{2}$ the Gegenbauer polynomial $G_2(x)$ defined on $[-1,1]$ found in (18) becomes

$$G_2(2z-1) = 6(1 - 5z + 5z^2), \qquad 0 \le z \le 1. \tag{19}$$

The corresponding collocation point to x_1 becomes $z_1 = \frac{1}{10}(5+\sqrt{5})$ with $\phi_1(z) = z(1-z)$ and assuming $u(z) = A\phi_1(z)$. (Note this form of $\phi(z)$ satisfies the boundary conditions that $\phi(0) = \phi(1) = 0$.) Substituting z_1 and $\phi_1(z)$ into the one-dimensional planar Bratu problem (replace x by z) produces an equation for the residual which is constrained to be zero.

$$R[z_1; \lambda, A] = R[\frac{1}{10}(5+\sqrt{5}); \lambda, A] = -2A + \lambda e^{\frac{1}{5}A} = 0 \tag{20}$$

Solving (20) for λ produces

$$\lambda_1(A) = 2Ae^{-0.2A} \tag{21}$$

The expression (21) attains its maximum value of λ_c at $A = 5$. Using 1-point Boyd collocation produces an estimate of $\lambda_c = 10e^{-1} = 3.67879441$ which is 4.7% greater than the exact value of $\lambda_c = 3.513830719$.

To increase accuracy the number of collocation points is increased. However, the number of residual equations (and their complexity) will simultaneously also increase. Using 2-point collocation the form of $u(x)$ is assumed to be

$$u(z) = A\phi_1(z) + B\phi_2(z) = Az(1-z) + Bz(1-z)(2z-1)^2, \qquad 0 \le z \le 1. \tag{22}$$

The above two-point collocation expansion corresponds to the expansion $u(x) = A(1-x^2) + Bx^2(1-x^2)$ which would be valid on $[-1,1]$. The fourth-order Gegenbauer polynomial, defined on $[-1,1]$ is

$$G_4(x) = \frac{15}{8}(1 - 14x^2 + 21x^4), \qquad -1 \le x \le 1 \tag{23}$$

which on transformation to $[0,1]$ becomes

$$G_4(2z-1) = 15(1-14z+56z^2-84z^3+42z^4), \qquad 0 \le z \le 1 \tag{24}$$

$G_4(x)$ has four roots on the interval $[-1,1]$ symmetrically distributed around the origin. The two largest roots are selected as the collocation points for the 2-point Boyd collocation method. The two residual equations are formed by substituting (22) into the planar Bratu equation at the collocation points.

$$\begin{aligned} R\left[\frac{1}{42}\left(21+\sqrt{21(7-2\sqrt{7})}\right);\lambda,A,B\right] &= -8A-8B+\frac{32B}{\sqrt{7}}+\lambda e^{\frac{2}{63}(21A+3\sqrt{7}A+5B-\sqrt{7}B)}=0 \\ R\left[\frac{1}{42}\left(21+\sqrt{21(7+2\sqrt{7})}\right);\lambda,A,B\right] &= -8A-8B-\frac{32B}{\sqrt{7}}+\lambda e^{\frac{2}{63}(21A-3\sqrt{7}A+5B+\sqrt{7}B)}=0 \end{aligned}$$

The method of solution is to find closed-form expressions for λ and B in terms of either A. This is not easy to do with the system as currently constituted. However by eliminating terms in the exponentials which are significantly smaller than the other terms it turns out that a closed form expression for $\lambda(A)$ obtained from the 2-point Boyd collocation method can be found.

$$\lambda_2(A) = \frac{64\sqrt{7}Ae^{\frac{2}{21}(-7+\sqrt{7})A}}{-7+4\sqrt{7}+(7+4\sqrt{7})e^{\frac{4A}{3\sqrt{7}}}} \tag{25}$$

The maximum value of the expression (25) is $\lambda_c = 3.45611039$, which is 1.64% smaller than the exact value (9).

2.3 Adomian polynomial decomposition

Adomian [1] developed a "polynomial decomposition" method of representing solutions to boundary value problems of the form

$$\begin{aligned} u'' &= -F(u) \\ u(0)=\alpha \quad &\text{and} \quad u(1)=\beta. \end{aligned}$$

The exact solution to (2.3) can be represented by a Green's Function

$$u(x) = \lambda\int_0^1 g(x,s)F(u(s))ds + (1-x)\alpha + \beta x \tag{26}$$

where

$$g(x,s) = \begin{cases} s(1-x), & 0 \le s \le x \\ x(1-s), & x \le s \le 1. \end{cases} \tag{27}$$

Adomian's decomposition method assumes that the unknown solution $u(x)$ and the given nonlinear functional $F(u)$ can each be represented as infinite series.

$$u = \sum_{i=0}^{\infty} u_i = u_0 + u_1 + u_2 + \ldots \tag{28}$$

and

$$F(u) = \sum_{i=0}^{\infty} A_i = A_0 + A_1 + A_2 + \ldots. \tag{29}$$

In the case of $F(u)$ the infinite series is a Taylor Expansion about u_0. In other words

$$F(u) = F(u_0) + F'(u_0)(u-u_0) + F''(u_0)\frac{(u-u_0)^2}{2} + F^{(3)}(u_0)\frac{(u-u_0)^3}{3} + \ldots \tag{30}$$

By re-writing (28) as $u - u_0 = u_1 + u_2 + u_3 + \ldots$, substituting it into (30) and then equating the two expressions for $F(u)$ found in (30) and (29) defines formulas for the "Adomian polynomials."

$$F(u(s)) = A_0+A_1+A_2+\ldots = F(u_0)+F'(u_0)(u_1+u_2+u_3+\ldots)+F''(u_0)\frac{(u_1+u_2+u_3+\ldots)^2}{2!}+\ldots \tag{31}$$

By equating terms in (31) the first few Adomian polynomials A_0, A_1, A_2 are given...

$$\begin{aligned}
A_0 &= F(u_0)\\
A_1 &= u_1 F'(u_0)\\
A_2 &= u_1^2 F''(u_0)/2! + u_2 F'(u_0)\\
A_3 &= u_1^3 F^{(3)}(u_0)/3! + 2u_1u_2F''(u_0)/2! + u_3F'(u_0)\\
A_4 &= u_1^4 F^{(4)}(u_0)/4! + 3u_1^2u_2F^{(3)}(u_0)/3! + (2u_1u_3 + u_2^2)F''(u_0)/2! + u_4F'(u_0)
\end{aligned}$$

Now that the $\{A_k\}_{k=0}^{\infty}$ are known, (29) can be substituted in (26) to specify the terms in the expansion for the solution (28).

$$\begin{aligned}
u(x) &= \lambda \int_0^1 g(x,s)\sum_{i=0}^{\infty} A_i\; ds + (1-x)\alpha + \beta x\\
\sum_{i=0}^{\infty} u_i &= \alpha(1-x) + \beta x + \lambda \sum_{i=0}^{\infty}\int_0^1 g(x,s)A_i\; ds
\end{aligned}$$

Equating the terms yields

$$\begin{aligned}
u_0 &= \alpha(1-x) + \beta x\\
u_1 &= \lambda \int_0^1 g(x,s)A_0(s)ds\\
u_2 &= \lambda \int_0^1 g(x,s)A_1(s)ds\\
\vdots\; & \quad \vdots\\
u_k &= \lambda \int_0^1 g(x,s)A_{k-1}(s)ds
\end{aligned}$$

Now the $\{u_k\}_{k=0}^{\infty}$ are known, so the solution is given by $u = u_0 + u_1 + u_2 + u_3 + \ldots$.

To apply the Adomian polynomial decomposition method to solve the one-dimensional planar Bratu problem (3) involves setting $\alpha = \beta = 0$ and $F(u) = e^u$. A happy accident is that the k^{th} derivative of $F(u)$, $F^{(k)}(u) = e^u$ so that choosing $u_0 = 0$ greatly simplifies the formulas for the Adomian polynomials $\{A_k\}$ since $e^{u_0} = 1$.

$$
\begin{aligned}
A_0 &= 1 \\
A_1 &= u_1 \\
A_2 &= u_1^2/2! + u_2 \\
A_3 &= u_1^3/3! + u_1 u_2 + u_3 \\
A_4 &= u_1^4/4! + u_1^2 u_2/2 + u_1 u_3 + u_2^2/2! + u_4 \\
&\vdots \quad \vdots
\end{aligned}
$$

Knowing the $\{A_k\}$ terms leads to the calculation of the $\{u_k\}$ terms

$$
\begin{aligned}
u_0 &= 0 \\
u_1 &= \lambda \int_0^1 g(x,s) \cdot 1 \; ds = \lambda \int_0^x s(1-x) \; ds + \lambda \int_x^1 x(1-s) \; ds \\
&= \frac{1}{2}(1-x)x\lambda \\
u_2 &= \lambda \int_0^1 g(x,s) \cdot \frac{1}{2}(1-s)s\lambda \; ds \\
&= \lambda^2(1-x) \int_0^x \frac{1}{2}(1-s)s^2 \; ds + \lambda^2 x \int_x^1 \frac{1}{2}(1-s)^2 s \; ds \\
&= \frac{1}{24}(1-2x^2+x^3)x\lambda^2 \\
u_3 &= \lambda \int_0^1 g(x,s) \cdot A_2(s) \; ds \\
&= \frac{1}{1440}(9-10x^2-15x^3+24x^4-8x^5)x\lambda^3 \\
u_4 &= \lambda \int_0^1 g(x,s) \cdot A_3(s) ds \\
&\vdots \quad \vdots
\end{aligned}
$$

2.4 Shooting Method

The last and probably the most obvious method used to obtain a numerical solution of the planar Bratu problem is the nonlinear shooting method. This involves converting the nonlinear boundary value problem (3) into a system of nonlinear initial value problems which look like

$$\frac{d}{dt}\vec{y} = \vec{f}(\vec{y}), \qquad \vec{y}(0) = \vec{y}_0,$$

with the choice of $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$, $\vec{f} = \begin{bmatrix} f_1(\vec{y}) \\ f_2(\vec{y}) \\ f_3(\vec{y}) \\ f_4(\vec{y}) \end{bmatrix} = \begin{bmatrix} y_2 \\ -e^{y_1} \\ y_4 \\ -e^{y_3} \end{bmatrix}$ and $\vec{y_0} = \begin{bmatrix} 0 \\ s_0 \\ 0 \\ 1 \end{bmatrix}$.

The shooting method works by choosing a value s_0 for $u'(0)$, solving the initial value problem (using a standard ODE solver like Runge-Kutta) and then comparing the value of $y_1(b)$ with the expected value of $u(b) = 0$. A new value of s_k is chosen by using Newton's Method, where

$$s_{k+1} = s_k - \frac{y_1(b) - u(b)}{y_3(b)} \tag{32}$$

The method is said to converge when the difference between subsequent values of s_k fall below a given tolerance, in other words $y_1(b)$ is very close to $u(b)$.

In the next section, the results of using the numerical methods detailed in this section are given.

3 Numerical Results

The results of applying various numerical methods to produce solutions of the planar one-dimensional Bratu problem (3) are given in this section. We shall begin with considering the results obtained using finite differences. A comparison of the errors generated using Mickens finite differences and standard finite differences are illustrated in Figure 3. By examining Figure 3 it can be observed that the error due to each finite difference method decreases proportionally to with $h^2 = \frac{1}{N^2}$. Also note that the Mickens discretization error (solid line) is consistently smaller than the standard discretization error (dashed line). The value of the parameter λ shall be taken to be one.

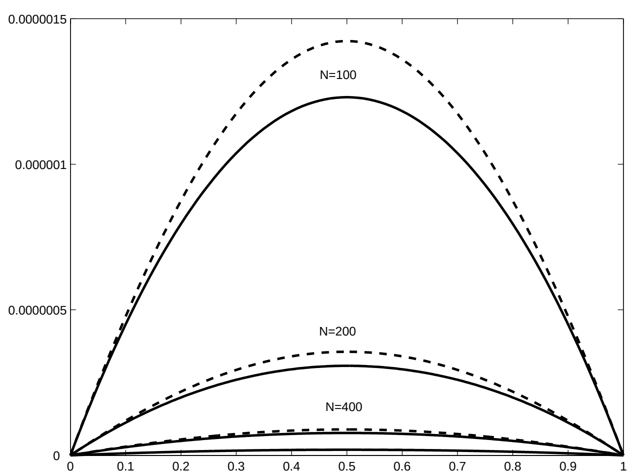


Figure 3: Comparison of standard error and Mickens error for $N = 100, 200, 400$ and 800 when $\lambda = 1$

Interestingly, despite the fact that there are two solutions to (3) at $\lambda = 1$ as shown in Figure 4, the standard finite difference scheme will only converge to one of them, the "lower" solution, i.e. the one below the $\lambda = \lambda_c$ solution. The Mickens discretization will converge to either solution, depending on the initial guess chosen for all values $0 < \lambda < \lambda_c$. Neither discretization method will converge to the unique solution at $\lambda = \lambda_c$.

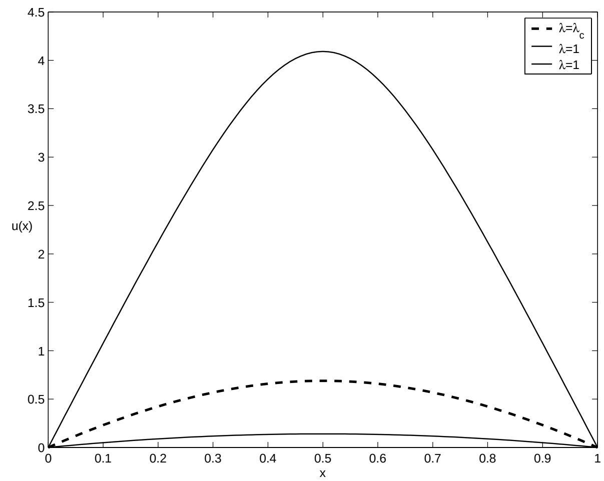


Figure 4: The two solution curves for $\lambda = 1$ and the unique solution curve for $\lambda = \lambda_c$

The solution produced by Boyd's pseudospectral method does not have the deficiency of being unable to converge to both solutions of the Bratu problem for $\lambda < \lambda_c$ which the standard finite-difference method and Adomian decomposition method both have. Boyd's method is able to produce continuous expressions for λ versus the maximum value of the solution. In Figure 5 the behavior of Boyd solutions produced using 1-point and 2-point collocation is compared with the exact solution's bifurcated behavior (as depicted in Figure 2), which indicates the multivalued nature of the Boyd solutions.

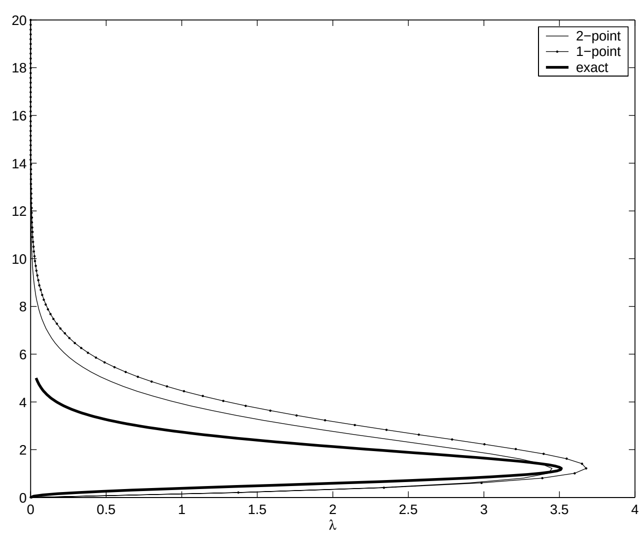
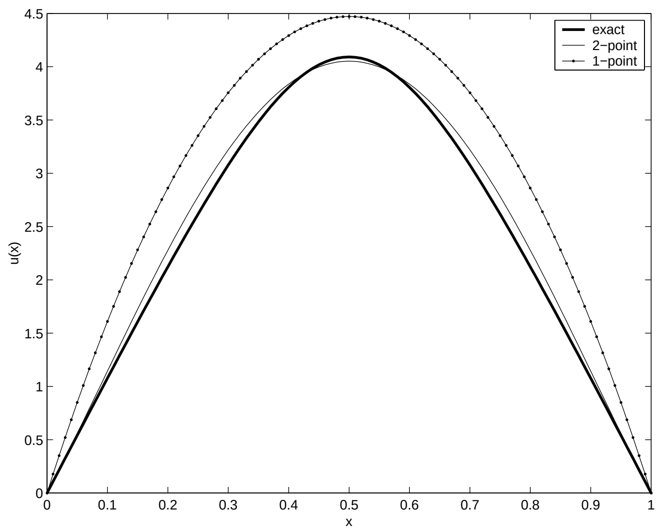


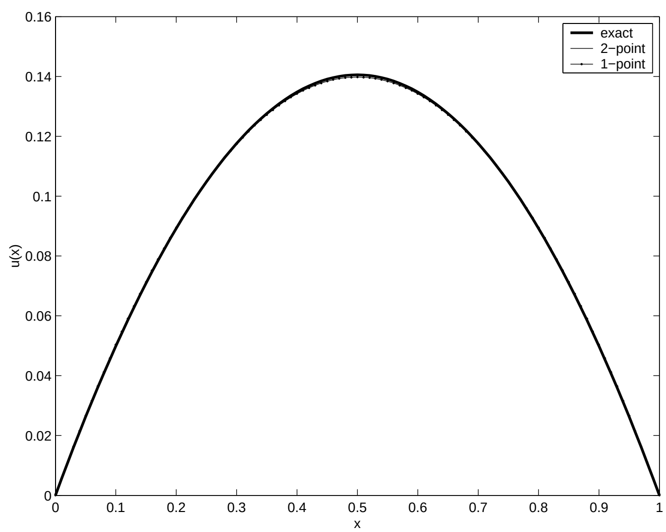
Figure 5: Dependence of Boyd pseudospectral solutions on λ

When $\lambda = 1$ there are two solutions to the Bratu problem (3), which are depicted in Figure 4 and called the "upper" solution and the "lower" solution. In Figure 6 the results of producing solutions using Boyd's pseudospectral method to the Bratu problem when

$\lambda = 1$ are depicted. The exact solution is the dark solid line, with the solution from the 1-point collocation depicted using a continuous dotted line and the solution from the 2-point collocation depicted using a continuous solid line. Interestingly, Boyd's method does very well with just 1-point collocation to approximate the lower solution. The 1-point Boyd collocation method doesn't do a very good job of approximating the solution to the "upper" Bratu solution, though the 2-point Boyd collocation does much better, as seen in Figure 6. This is not a surprise, since the expectation is that using more collocation points will decrease the error. By looking at Figure 5 it is clear that at $\lambda = 1$ the three curves (exact solution, 1-point and 2-point) are close together at the lower arc of the bifurcation curve corresponding to the "lower solution" and they are not close together at the upper arc of the bifurcation curve corresponding to the "upper solution." The proximity of the curves is indicative of the numerical error, and the error in approximating the lower solution is smaller than the error in approximating the upper solution.



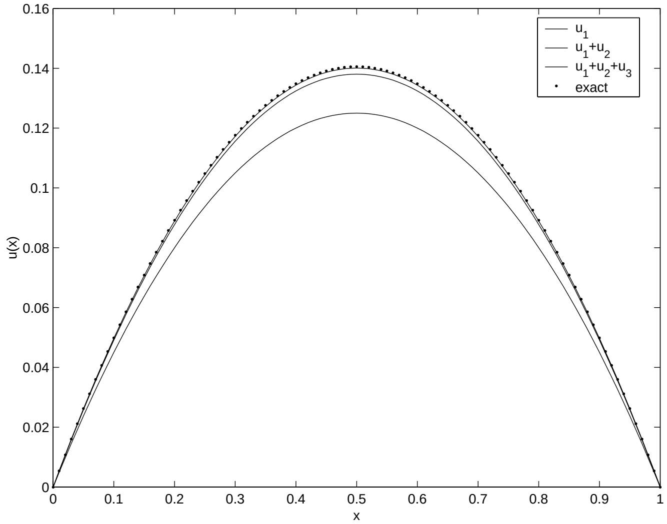
(a) Upper solution when $\lambda = 1$



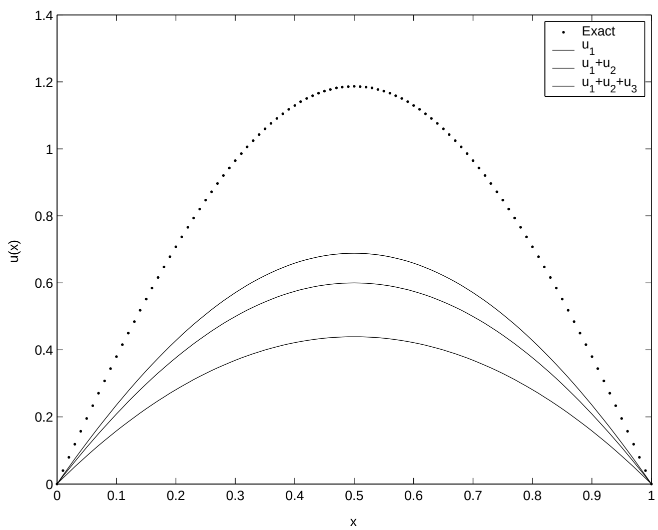
(b) Lower solution when $\lambda = 1$

Figure 6: Comparison of Boyd solutions generated by one-point and two-point collocation

The solutions generated by the Adomian polynomial decomposition only approach the exact solution for small values of $\lambda \leq 1$. Like the standard discretization method, the Adomian method's solution only converges to the "lower" solution at $\lambda = 1$. In Figure 7 the first three nonzero terms of the Adomian polynomial expansion (solid curves) are depicted next to the exact solution (unconnected dotted line). Clearly, these terms $(u_1 + u_2 + u_3)$ are enough to approximate the exact solution relatively accurately when $\lambda = 1$. However, if $\lambda = \lambda_c$ is selected one needs far more than three terms of the series $\{u_k\}_{k=0}^{\infty}$ to converge to the exact solution, as can be seen in Figure 7. Deeba et. al. [7] obtained identical results when they applied Adomian's polynomial decomposition method to the same boundary value problem (3).



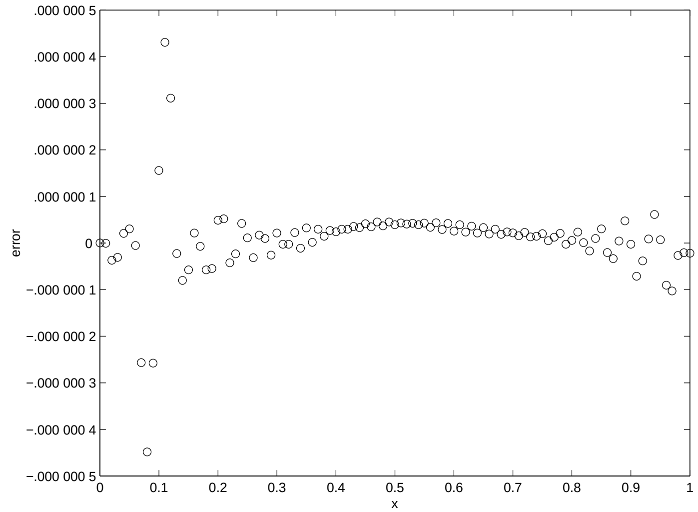
(a) Adomian polynomial solution for $\lambda = 1$



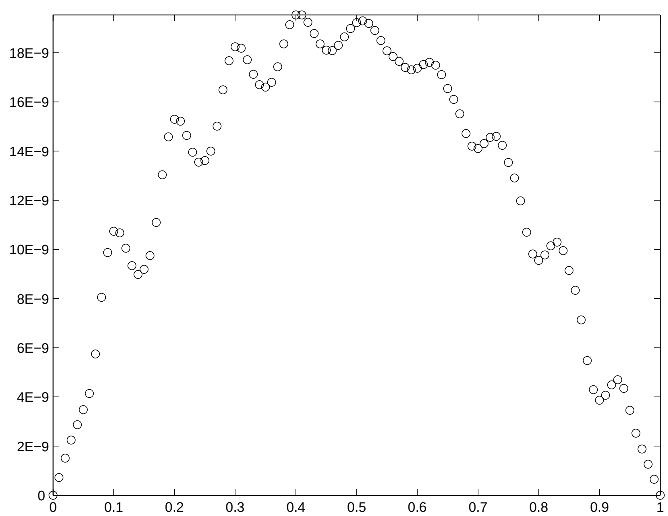
(b) Adomian polynomial solution for $\lambda = \lambda_c$

Figure 7: Comparison of using first three non-zero terms of Adomian polynomial solution for $\lambda = 1$ and $\lambda = \lambda_c$

The shooting method was implemented using the MATLAB routine `ode45` and produces accurate numerical solutions rapidly for values of $\lambda < \lambda_c$. The shooting method will converge to both the upper and lower solutions depicted in Figure 4 by carefully choosing the value of the initial slope s_0 in (32). However, when $\lambda = \lambda_c$, like the finite-difference methods in Section 2.1, the shooting method will not converge to a solution within the given tolerance. Figure 8 depicts the numerical error produced by the nonlinear shooting method as it approximates both Bratu solutions at $\lambda = 1$. Since the numerical error of the shooting method depends on the tolerance of the ODE solver, and not the grid separation, N was chosen to be 100 with a `RelTol` of 10^{-10}.



(a) Error due to nonlinear shooting for $\lambda = 1$ "upper" solution



(b) Error due to nonlinear shooting for $\lambda = 1$ "lower" solution

Figure 8: Errors generated by the nonlinear shooting method when $\lambda = 1$ using $N = 100$

4 Conclusion

Five different methods were used to generate numerical solutions of the planar one-dimensional Bratu problem. The five methods were, two finite-difference methods, two spectral methods and a nonlinear shooting method. The methods were chosen for their ease of use for relative error generated. This is why only a few terms (two in the case of the Boyd pseudospectral method and three in the case of the Adomian polynomial decomposition) were used. Only the Mickens discretization and the nonlinear shooting method had no difficulty handling the bifurcated nature of the solution for subcritical values of the parameter λ. The Adomian and Boyd methods do successfully approximate the "lower" of the multiple solutions when the value of λ is small using very few collocation points. However to increase their accuracy would require many more collocation points and would no longer make these "simple" methods to implement. It is worthwhile to note that the Mickens discretization method performs as well as the nonlinear shooting method, and is also very easy to implement.

Acknowledgements

The author would like to thank Tom Witelski for first mentioning the Bratu problem to me. The numerical calculations for this paper were performed using `Matlab` and `Mathematica` and the figures were mostly generated using `Matlab`.

This paper is dedicated to Professor Ronald Mickens to celebrate the occasion of his 60th birthday (February 7, 2003).

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