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AUXILIARY FIELDS FOR SUPER YANG-MILLS FROM DIVISION ALGEBRAS

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ABSTRACT

Division algebras are used to explain the existence and symmetries of various sets of auxiliary fields for super Yang-Mills in dimensions $d = 3, 4, 6, 10$.

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1. Introduction

The simplest supersymmetric Yang-Mills theories are those in which the physical degrees of freedom are described by a vector gauge field A_μ ($\mu = 0, \ldots, d-1$) and a spinor field ψ, both defined on d-dimensional Minkowski space and taking values in the Lie algebra of some gauge group. A necessary condition for supersymmetry is that the number of physical bosons and fermions should be equal, which leads to the possibilities $d = 3, 4, 6, 10$ with the spinor being Majorana, Majorana or Weyl, Weyl, Majorana and Weyl respectively. To show that this equality of bosons and fermions is sufficient for supersymmetry as well as necessary, one can take the detailed expressions for the supersymmetry transformations—which are fixed up to irrelevant constants by gauge-invariance and dimensional considerations—and check that they obey the standard algebra up to terms involving field equations (this is in fact equivalent to requiring invariance of the natural action in which the spinor is minimally coupled to the gauge field). The condition for the supersymmetry algebra to close *on-shell* in this way is a certain gamma-matrix identity which is indeed satisfied precisely for the values of d and the types of spinor listed above [1].

It is always desirable in a supersymmetric theory to try to promote the on-shell symmetry algebra to one which holds *off-shell*, that is, without the use of field equations. For $d = 3$ super Yang-Mills the superalgebra closes automatically; for $d = 4, 6$ one can introduce auxiliary fields to close the superalgebra in a Lorentz-covariant way [2,3]; but for $d = 10$ the best which can be done with a finite set of auxiliary fields is to partially close the superalgebra, breaking the manifest $d = 10$ Lorentz symmetry to some subgroup in the process. An interesting new perspective on these matters was provided recently in [4] with the introduction of more general fermionic transformations which include the conventional supersymmetry transformations of $d = 10$ super Yang-Mills as special cases. It was then shown in [5] that all previously known sets of auxiliary fields for the $d = 10$ theory could be recovered within this framework.

Our aim here is to show how the possible auxiliary fields for each of the allowed super Yang-Mills theories can be understood using the language of division algebras [6-8]. Connections between super Yang-Mills and the division algebras have been established in the past by interpreting the gamma-matrix identity necessary for on-shell supersymmetry from a number of closely related points of view using division algebra-valued spinors [9,10], Jordan algebras [10,11] and trialities [12]. The role of division algebras in understanding off-shell supersymmetry was emphasized first in [7] and then in [4] where octonions were used to find a new set of auxiliary fields in $d = 10$. We shall indicate here how the solutions given in [4,5] for $d = 10$, together with analogous solutions for the lower-dimensional theories, can all be understood using this language; we shall also explain from this point of view the residual symmetries of these various sets of auxiliary fields.

2. Division Algebra Notation

We denote by $\mathbb{K}_n$ with $n = 1, 2, 4, 8$ the division algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$ of real numbers, complex numbers, quaternions and octonions respectively; for background see papers such

as [6-12]. We denote by e_a $(a = 1, \ldots, n)$ an orthonormal basis for $\mathbb{K}_n$ with $e_n = 1$ and all other basis elements pure-imaginary. Bars will denote conjugation and daggers will denote hermitian conjugation.

The basic idea is to define the action of certain spin representations of SO(n+1, 1) on two-component objects with entries in $\mathbb{K}_n$ [7,8]. The cases $n = 1, 2, 4, 8$ obviously correspond to $d = 3, 4, 6, 10$ and the spinor appearing in each of the allowed super Yang-Mills theories in these dimensions can then be written as an object ψ^α $(\alpha = 1, 2)$ or as its conjugate $\bar{\psi}^{\dot{\alpha}}$ $(\dot{\alpha} = 1, 2)$. There are also dual spinor representations acting on objects which we denote by χ_α and $\bar{\chi}_{\dot{\alpha}}$. When spinor indices are suppressed we will regard ψ, $\bar{\psi}$ as row vectors and χ, $\bar{\chi}$ as column vectors. Spinor indices are never raised or lowered and the usual Lorentz-invariant inner-product is given by the real expression $\psi\chi + \chi^\dagger\psi^\dagger = \psi^\alpha\chi_\alpha + \bar{\chi}_{\dot{\alpha}}\bar{\psi}^{\dot{\alpha}}$.

The gamma-matrices needed to construct the spin representations are invariant tensors $(\Gamma_\mu)_{\alpha\dot{\alpha}}$ and $(\Gamma^\mu)^{\dot{\alpha}\alpha}$. We define these in a particular basis in which their components are equal and are given by the hermitian matrices:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ for } \mu = 0; \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ for } \mu = n{+}1; \quad \begin{pmatrix} 0 & e_a \\ \bar{e}_a & 0 \end{pmatrix} \text{ for } \mu = a = 1, \ldots, n.$$

The order of indices on gamma-matrices indicates whether they should multiply spinors from the left or from the right with the convention that only adjacent upper and lower indices of the same type can be contracted. For example, the matrices $(\Gamma_\mu)_{\alpha\dot{\alpha}}$ act by right-multiplication on ψ^α and left-multiplication on $\bar{\psi}^{\dot{\alpha}}$.

It will be important for us to understand how certain subgroups of SO(n+1, 1) appear in this formalism. With the basis introduced above, the two $\mathbb{K}_n$-valued components of any spinor clearly carry irreducible representations of the light-cone subgroup SO(1,1)×SO(n). In fact these two copies of $\mathbb{K}_n$ carry the inequivalent spin representations of SO(n) for $n = 2, 4, 8$. If we restrict further to the subgroup SO(n−1) which fixes the direction $\mu = n$ then these spin representations become equivalent. The resulting representation is realized on $\mathbb{K}_n$ in two particularly simple ways: the relevant gamma-matrices or invariant tensors are just the imaginary basis elements e_i $(i = 1, \ldots, n{-}1)$ and these multiply components of type ψ^α from the right and components of type χ_α from the left.

Notice that the elements of unit modulus in $\mathbb{C}$ and $\mathbb{H}$ form groups U(1) and SU(2) respectively. They act naturally by multiplication on complex and quaternionic spinors so as to commute with Lorentz transformations; we shall therefore refer to these operations on spinors as internal transformations. The generators of such transformations are unit imaginary elements e_i multiplying components of type ψ^α from the left and components of type χ_α from the right (compare with the last paragraph). The internal groups U(1) and SU(2) appear as symmetries of the super Yang-Mills theories in $d = 4$ and $d = 6$; no such symmetries arise in $d = 3$ or $d = 10$.

Finally, it can be shown that the algebras $\mathbb{H}$ and $\mathbb{O}$ have non-trivial continuous automorphism groups SO(3) and G_2 respectively [6,8]. The action of these automorphisms on spinor components can always be reproduced by combinations of Lorentz and internal transformations. The special feature of the octonionic case is that the automorphism group is contained entirely within the Lorentz group—in fact within the SO(7) subgroup mentioned above.

3. Generalized and Conventional Supersymmetry

The generalized supersymmetry transformations proposed in [4] involve real bosonic auxiliary fields G_i $(i = 1, \ldots, d-3)$ which balance the off-shell bosonic and fermionic degrees of freedom. They can be written

$$\begin{aligned} \delta A_\mu &= \epsilon(\Gamma_\mu \psi^\dagger) + (\psi \Gamma_\mu)\epsilon^\dagger \,, \\ \delta\psi &= \tfrac{1}{2}(\epsilon\Gamma_\nu)\Gamma_\mu F^{\mu\nu} + G_i v_i \,, \\ \delta G_i &= -v_i(\Gamma^\mu D_\mu \psi^\dagger) - (D_\mu \psi \Gamma^\mu) v_i^\dagger \,, \end{aligned} \tag{1}$$

where $D_\mu = \partial_\mu + A_\mu$ is the usual covariant derivative and $F_{\mu\nu} = [D_\mu, D_\nu]$ is the field strength. (We have defined δ so as to remove some factors of i compared to [4,5] and we have normalized spinors so as to remove some factors of $\frac{1}{2}$.) The parameters of the transformations are commuting spinors ϵ^α and v_i^α which must satisfy the additional relations

$$\begin{aligned} \epsilon(\Gamma_\mu v_i^\dagger) + (v_i \Gamma_\mu)\epsilon^\dagger &= 0 \,, \\ v_i(\Gamma_\mu v_j^\dagger) + (v_j \Gamma_\mu) v_i^\dagger &= \delta_{ij}(\, \epsilon(\Gamma_\mu \epsilon^\dagger) + (\epsilon\Gamma_\mu)\epsilon^\dagger \,) \,. \end{aligned} \tag{2}$$

These ensure that the standard supersymmetry algebra still holds up to field equations despite the introduction of extra parameters.

To recover conventional supersymmetry transformations from those written in (1) above one must solve the equations (2) with v_i depending linearly on ϵ and with the value of ϵ restricted to some subspace if necessary. The subset of conventional supersymmetry transformations obtained in this way will automatically obey a closed algebra. However, such a solution will, in general, break the full Lorentz invariance of equations (1) and (2) down to a subgroup determined both by the subspace to which ϵ is confined and by the precise definitions of the quantities v_i. These points are explained in more detail in [5].

4. Solutions and their Symmetries

Solutions to (2) can be written very simply in division algebra notation. We consider first the possibility

$$v_i = e_i \epsilon \tag{3}$$

with the spinor ϵ confined to some subspace. The transformations (1) can now be re-expressed in a more compact way by combining the $n-1$ auxiliary fields into the pure-imaginary object $G = G_i e_i$ with the result

$$\begin{aligned} \delta A_\mu &= \epsilon(\Gamma_\mu \psi^\dagger) + (\psi \Gamma_\mu)\epsilon^\dagger \,, \\ \delta\psi &= \tfrac{1}{2}(\epsilon\Gamma_\nu)\Gamma_\mu F^{\mu\nu} + G\epsilon \,, \\ \delta G &= \epsilon(\Gamma^\mu D_\mu \psi^\dagger) - (D_\mu \psi \Gamma^\mu)\epsilon^\dagger \,. \end{aligned} \tag{4}$$

Before discussing the individual solutions it will be best to make some general remarks concerning the possible symmetries of these equations.

In the associative cases the equations (4) are invariant under any Lorentz transformation if the auxiliary fields behave as scalars (and provided that the transformation respects whatever restriction is placed on ϵ of course). This can be checked by calculating explicitly the transformation of the expression given for δG and by noting also that if G is inert then the term $G\epsilon$ transforms in the same way that ϵ does; all other terms are covariant by construction. In the octonionic case, however, this statement must be qualified: we find that a Lorentz transformation is a symmetry of (4) with G inert only if it is constructed from gamma-matrices whose entries have vanishing associators with the components of ϵ. Aside from such Lorentz transformations, the equations (4) are also invariant under the internal transformations which arise in the complex and quaternionic cases in the manner discussed earlier. Lastly, the equations (4) are invariant under any automorphism of $\mathbb{K}_n$ (provided once again that it respects the restriction on ϵ) and then the auxiliary fields will transform in some non-trivial way. From the remarks made previously we know that automorphisms give genuinely new symmetries only in the octonionic case. We are now in a position to explain the symmetries which appear in each of our solutions.

It is easy to show that in the associative cases (3) provides a solution of (2) for any spinor ϵ and the formulas (4) then reproduce the standard, Lorentz-covariant, off-shell supersymmetry transformations for $d = 4$ [2] and $d = 6$ [3,7] super Yang-Mills with scalar auxiliary fields. In $d = 4$ the off-shell supersymmetries are also invariant under the U(1) internal group with the auxiliary field being inert. In $d = 6$ they are invariant under the SU(2) internal group with the auxiliary fields transforming as a triplet.

The lack of associativity of the octonions means that in $d = 10$ the formula (3) is no longer a solution of (2) for all ϵ. However, there are at least two interesting ways of restricting ϵ in (3) so as to obtain closed subalgebras of supersymmetry transformations of type (4). Both solutions require for their verification several lines of octonionic manipulations which we shall omit.

The first case in which (3) provides a solution of (2) is where one of the components of ϵ is restricted to be real, giving a closed algebra of nine supersymmetries [4]. To preserve the form of ϵ under a Lorentz transformation we must confine attention to a subgroup SO(1,1)×SO(7). The generator of SO(1,1) is real and so never gives rise to an associator. But the additional transformations are symmetries of (4) only if they lie in the subgroup of automorphisms G_2 within SO(7), with the auxiliary fields transforming in its seven-dimensional representation. In this way we find exactly the residual invariance SO(1,1)×G_2 and the pattern of representations given in section 3 of [5].

The second case in which (3) provides a solution of (2) is where both components of ϵ lie in some copy of the complex numbers $\mathbb{C}$ within $\mathbb{O}$, yielding a closed algebra of four supersymmetries. This solution is clearly invariant under a subgroup SO(3,1) generated by combining gamma-matrices with entries in this same copy of $\mathbb{C}$, since all the relevant associators then vanish. The other pairs of gamma-matrices which lead to vanishing associators give $d = 10$ Lorentz generators which coincide when acting on ϵ and which, when exponentiated, correspond simply to multiplication by an arbitrary phase within our chosen copy of $\mathbb{C}$. Lastly, the solution is invariant under the subgroup of G_2 which sends our particular copy of $\mathbb{C}$ to itself, and it is well-known that the subgroup of automorphisms which fix a given imaginary element is SU(3) [6]. The residual invariance is therefore

SO(3,1)×U(1)×SU(3) and closer examination of the details of the representations gives complete agreement with those found in section 4 of [5].

So far we have discussed solutions of (2) of type (3); it is natural to ask whether there exist similar solutions of the form

$$v_i = \epsilon e_i \tag{5}$$

with ϵ restricted to some subspace. The transformations (1) can once again be re-written, after a little work, in terms of $G = G_i e_i$ and the result is

$$\begin{aligned} \delta A_\mu &= \epsilon(\Gamma_\mu \psi^\dagger) + (\psi \Gamma_\mu)\epsilon^\dagger \,, \\ \delta \psi &= \tfrac{1}{2}(\epsilon \Gamma_\nu)\Gamma_\mu F^{\mu\nu} + \epsilon G \,, \\ \delta G &= (\overline{D_\mu \psi \Gamma^\mu})\bar{\epsilon}^\dagger - \bar{\epsilon}(\overline{\Gamma^\mu D_\mu \psi^\dagger}) \,. \end{aligned} \tag{6}$$

In $d = 4$ the formulas (5) and (6) coincide with (3) and (4) because the complex numbers are commutative; but for $d = 6$ and $d = 10$ we obtain new possibilities.

In the non-commutative cases (5) fails to satisfy (2) for general ϵ, but it does provide a solution if we restrict ϵ to having only one non-zero component. This restriction clearly breaks the Lorentz group at least to the light-cone subgroup SO(1,1)×SO(n) and in fact the compact part of the surviving symmetry is SO($n-1$). This can be seen from the way in which the pure-imaginary elements e_i appear in (6) through G, because these quantities act as invariant tensors for the subgroup SO($n-1$) when multiplying ϵ from the right, as we have already mentioned. Thus in $d = 6$ we obtain a closed algebra of four supersymmetries of type (6) with residual Lorentz invariance SO(1,1)×SO(3) and auxiliary fields transforming as a three-dimensional vector. There is also an SU(2) internal symmetry under which the auxiliary fields are inert. In $d = 10$ we find a closed algebra of eight supersymmetries of type (6) with residual invariance SO(1,1)×SO(7) and auxiliary fields transforming as a seven-dimensional vector. This is just the solution presented in section 2 of [5].

5. Comments

We have seen that division algebras provide an elegant language in which to understand the occurrence and symmetries of bosonic auxiliary fields for super Yang-Mills, with the lack of a covariant set in $d = 10$ being traced directly to the non-associativity of the octonions [4]. It is natural to wonder to what extent the solutions we have discussed here are exhaustive. It would also be interesting to investigate whether division algebras could be used to understand fermionic auxiliary fields (see the papers cited in [5]) in a similar fashion.

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