

Aspects of Conformal Supergravity

Sergio Ferrara^{a,b}, Alex Kehagias^d, Dieter Lüst^{e,f}

^a *CERN, Theory Department,
1211 Geneva 23, Switzerland*

^b *INFN, Laboratori Nazionali di Frascati,
Via Enrico Fermi 40, 00044 Frascati, Italy*

^c *Physics Division, National Technical University of Athens
15780 Zografou Campus, Athens, Greece*

^d *Arnold–Sommerfeld–Center for Theoretical Physics,
Ludwig–Maximilians–Universität, 80333 München, Germany*

^e *Max–Planck–Institut für Physik, Werner–Heisenberg–Institut,
80805 München, Germany*

Abstract

In these lectures we discuss \mathcal{N} -extended conformal supergravity and its spectrum in four dimensions. These theories can be considered as the massless limit of Einstein-Weyl supergravity and by taking into account their enhanced gauge symmetries, we derive their massless spectrum, which in general contains a dipole-ghost graviton multiplet and an \mathcal{N} -fold tripole-ghost gravitino multiplet.

1 Introduction

We would like to report in these lectures on some aspects of the truly “missing (super)symmetry component of Space and Time” (as phrased by G. ’t Hooft [1]) which is the dynamical theory of *conformal supergravity* (or superconformal gravity). The latter is the (local) supersymmetric extension of Weyl gravity and it should be regarded as the gauge theory of the superconformal algebra. It was pioneered in [2, 3] and its quantum structure was explored initially in [4]. The fourth order equations of motion of Weyl gravity make the appearance of ghost-like states (of helicity $\pm 2, \pm 1$) [5] together with ordinary massless graviton, all together making a (massless) dipole (helicity ± 2) ghost [6–8] and in addition, an ordinary massless vector of helicity ± 1 [9]. Analogous properties for conformal supergravity were recently explored in [10].

The Lagrangian structure for higher \mathcal{N} -extended conformal supergravity was exhibited in [11], and lately investigated in [12] based on the recent results of [13], where the possible introduction of a holomorphic prepotential in $\mathcal{N} = 4$ was discovered as suggested in [14]. Results on conformal supergravity [15] related to the ghost structure, its possible role in (supergravity) anomalies [16–18] and its higher \mathcal{N} (extended conformal supergravity) extension include [8, 19–28]. Application of the superconformal formulation to cosmological models has been worked out in [29–31].

The Einstein-Weyl theory [5] is described by the action

$$S = \int_{\mathcal{M}} d^4x \sqrt{-g} \left(M_P^2 R + \frac{1}{2g_W^2} W_{\mu\nu\rho\sigma} W^{\mu\nu\rho\sigma} \right), \quad (1)$$

where $W_{\mu\nu\rho\sigma}$ is the Weyl tensor defined as (in 4D)

$$W_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} + g_{\mu[\sigma} R_{\rho]\nu} + g_{\nu[\rho} R_{\sigma]\mu} + \frac{R}{3} g_{\mu[\rho} g_{\sigma]\nu}. \quad (2)$$

Under conformal transformations of the metric

$$g_{\mu\nu} \rightarrow \widehat{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}, \quad (3)$$

the Weyl tensor is invariant

$$\widehat{W}^{\mu}_{\nu\rho\sigma} = W^{\mu}_{\nu\rho\sigma}, \quad (4)$$

and therefore, the second term in the action (1) is invariant under conformal transformations

$$\sqrt{-\widehat{g}} \widehat{W}^{\mu\nu\rho\sigma} \widehat{W}_{\mu\nu\rho\sigma} = \sqrt{-g} W^{\mu\nu\rho\sigma} W_{\mu\nu\rho\sigma}, \quad (5)$$

in four dimensions. However, the full action (1) fails to be conformal invariant due to the Einstein-term which transforms as

$$\widehat{R} = \Omega^{-2} R - 6\Omega^{-3} g^{\mu\nu} \nabla_{\mu} \nabla_{\nu} \Omega, \quad (6)$$

so that it breaks explicitly conformal invariance. Hence, the Einstein-term plays the role of mass term in this theory and can be regarded as a mass deformation.

The propagator of the Einstein-Weyl theory as follows from the action (1) can be written as

$$\Delta_{\mu\nu\rho\sigma} = \Delta(k)P_{\mu\nu\rho\sigma}, \quad (7)$$

where

$$\Delta(k) = \frac{g_W^2}{k^2(k^2 - g_W^2 M_P^2)}, \quad (8)$$

$$P_{\mu\nu\rho\sigma} = \frac{1}{2}(\theta_{\mu\rho}\theta_{\nu\sigma} + \theta_{\mu\sigma}\theta_{\nu\rho}) - \frac{1}{3}\theta_{\mu\nu}\theta_{\rho\sigma}, \quad (9)$$

and

$$\theta_{\mu\nu} = \eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \quad (10)$$

is the usual transverse vector projection operator. Notice that (7) exhibits the conformal $1/k^4$ behavior for $M_P^2 = 0$ (i.e., pure Weyl theory). For finite M_P , we can write $\Delta(k)$ equivalently as

$$\Delta(k) = -\frac{1}{M_P^2} \frac{1}{k^2} + \frac{1}{M_P^2} \frac{1}{k^2 - g_W^2 M_P^2}, \quad (11)$$

where the propagation of the helicity- ± 2 massless graviton is easily recognized in the first term of (11). In addition, there is also a massive spin-2 state associated to the second term of (11). This massive state has a mass given by the pole at $k^2 = g_W^2 M_P^2$. However, the residue at this pole is opposite to that of the massless graviton, and therefore it describes a ghost spin-2 state. This shows that the theory contains as propagating degrees of freedom (dof) the standard, massless spin-2 graviton $g_{\mu\nu}$ plus an additional massive spin-2 field $w_{\mu\nu}$.

In terms of helicity, the spectrum contains a spin-2 ghost ($s = 2 \rightarrow \pm 2, \pm 1, 0$) and an ordinary massless ($\lambda = \pm 2$) massless graviton (a total of 7 dof). This counting has a nice extension in the case of superconformal case. Actually to show that the Einstein-Weyl action (1) describes both massless and massive spin-2, it is convenient to make a particular bimetric gravity with two spin-2 fields $g_{\mu\nu}$ and $w_{\mu\nu}$ with a new action [32]:

$$S = \int_{\mathcal{M}} d^4x \sqrt{-g} \left(M_P^2 R(g) + 2M_P G_{\mu\nu}(g) w^{\mu\nu} - M_W^2 (w^{\mu\nu} w_{\mu\nu} - w^2) \right), \quad (12)$$

where

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \quad (13)$$

is the Einstein-tensor of the metric $g_{\mu\nu}$. The last term in (12) describes a mass term for the second spin-2 field $w_{\mu\nu}$. It is straightforward to see that (1) and (12) are equivalent [33]. Indeed, the equation of motion for $w_{\mu\nu}$ turns out to be

$$\frac{\delta S}{\delta w_{\mu\nu}} \Rightarrow w_{\mu\nu} = \frac{M_P}{M_W^2} \left(R_{\mu\nu}(g) - \frac{1}{6} g_{\mu\nu} R \right). \quad (14)$$

When the equation of motion is used in (12), one gets the original action (1) by using the identity

$$W_{\mu\nu\rho\sigma} W^{\mu\nu\rho\sigma} = GB + 2(R_{\mu\nu} R^{\mu\nu} - \frac{1}{3} R^2), \quad (15)$$

where

$$GB = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4R_{\mu\nu} R^{\mu\nu} + R^2 \quad (16)$$

is the Gauss-Bonnet tensor. Therefore, the actions (1) and (12) are equivalent up to total derivatives and the theory contains one massless and one massive spin-2 field $g_{\mu\nu}$ and $w_{\mu\nu}$, respectively. However, although the action for $g_{\mu\nu}$ is at non-linear level, the fields $w_{\mu\nu}$ appears just quadratically in (12). Let us also note that although it looks that there is no kinetic term for the second field $w_{\mu\nu}$, the kinetic term of the latter is hidden in the term $G_{\mu\nu}(g)w^{\mu\nu}$. Indeed, for small fluctuation $h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$ around Minkowski spacetime with flat metric $\eta_{\mu\nu}$, we find that, at quadratic level, the action (12) is written as

$$\mathcal{S}_2 = \int d^4x \sqrt{-g} \left\{ -\frac{M_P^2}{2} h^{\mu\nu} \mathcal{E}_{\mu\nu}^{\alpha\beta} h_{\alpha\beta} + \frac{1}{g_W} w^{\mu\nu} \mathcal{E}_{\mu\nu}^{\alpha\beta} h_{\alpha\beta} - \left(w^{\mu\nu} w_{\mu\nu} - w^2 \right) \right\}, \quad (17)$$

where $g_W = M_W/M_P$ and $\mathcal{E}_{\mu\nu}^{\alpha\beta}$ is defined as

$$\mathcal{E}_{\mu\nu}^{\alpha\beta} h_{\alpha\beta} = -\frac{1}{2} \left\{ \square h_{\mu\nu} - 2\partial_{(\mu} \partial_{\alpha} h_{\nu)}^{\alpha} + \partial_{\mu} \partial_{\nu} h - \eta_{\mu\nu} \left(\square h - \partial_{\alpha} \partial_{\beta} h^{\alpha\beta} \right) \right\}. \quad (18)$$

Then, in terms of the new field

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{2}{g_W M_P^2} w_{\mu\nu}, \quad (19)$$

the quadratic action (17) is written as

$$\mathcal{S}_2 = \int d^4x \sqrt{-g} \left\{ -\frac{M_P^2}{2} \bar{h}^{\mu\nu} \mathcal{E}_{\mu\nu}^{\alpha\beta} \bar{h}_{\alpha\beta} + \frac{1}{g_W^2 M_P^2} w^{\mu\nu} \mathcal{E}_{\mu\nu}^{\alpha\beta} w_{\alpha\beta} - \left(w^{\mu\nu} w_{\mu\nu} - w^2 \right) \right\}. \quad (20)$$

Clearly now, the field $\bar{h}_{\mu\nu}$ is a massless graviton and $w_{\mu\nu}$ is a massive spin-2 tensor field. However, the kinetic term of $w_{\mu\nu}$ has a wrong sign (compared to $\bar{h}_{\mu\nu}$) and therefore, it is a ghost field.

There are three interesting limits of the theory one may consider:

(A) *Decoupling of gravity:*

First we consider the infinite mass limit

$$M_P \rightarrow \infty, \quad g_W = \text{fixed}. \quad (21)$$

This is the limit where gravity decouples from the theory as this is the limit where gravity becomes non-dynamical. In particular, if the coupling g_W is finite, then both spin-2 states decouple completely, since in this case, the mass $M_W = g_W M_P$ of $w_{\mu\nu}$ is infinite.

(B) *Massless bigravity limit:*

The limit of vanishing Planck mass with finite g_W

$$M_P \rightarrow 0, \quad g_W = \text{fixed} \quad (22)$$

is the massless limit of the theory. In this limit, the propagator $\Delta(k)$ turns out to be

$$\Delta(k) \rightarrow \frac{g_W^2}{k^4}, \quad (23)$$

in accordance with conformal invariance. Hence, the spin-2 state $w_{\mu\nu}$ turns out to be massless. This is a Higgs effect since for finite M_P the spin-2 state $w_{\mu\nu}$ becomes massive. The theory in this limit is then the massless Weyl gravity with action

$$S = \frac{1}{2g_W^2} \int d^4x \sqrt{-g} W_{\mu\nu\rho\sigma} W^{\mu\nu\rho\sigma}. \quad (24)$$

This theory possesses conformal invariance and it propagates the following degrees of freedom:

- (i) The standard massless graviton $g_{\mu\nu}$, associated to plane-waves in Einstein theory.
- (ii) A massless spin-2 ghost state $w_{\mu\nu}$ in the non-standard sector corresponding to non-planar waves. There is also a massless vector w_μ , coming from the ± 1 helicities of the massive $w_{\mu\nu}$ [9]. Notice that, due to conformal invariance, the helicity-0 state can be gauged away and it do not appear in the physical spectrum.

(C) *Light spin-2 limit:*

One may also consider the double-scaling limit

$$M_P \rightarrow \infty \quad \text{and} \quad g_W \rightarrow 0 \quad \text{with} \quad M_W \ll M_P, \quad (25)$$

where g_W vanishes faster than M_P^{-1} . In this limit, the standard gravitational sector (where the massless graviton $g_{\mu\nu}$ resides) decouples from the non-standard massless spin-2 ($w_{\mu\nu}$) sector and

the theory again describes massless Weyl with action (24). Indeed, the propagator in Eq.(11) can be approximated for $M_W = g_W M_P \rightarrow 0$ by

$$\begin{aligned}\Delta(k) &= -\frac{1}{M_P^2} \frac{1}{k^2} + \frac{1}{M_P^2} \frac{1}{k^2 - M_W^2} \\ &\approx -\frac{1}{M_P^2} \frac{1}{k^2} + \frac{1}{M_P^2} \frac{1}{k^2} \left(1 + \frac{M_W^2}{k^2} + \mathcal{O}(M_W^4)\right) \approx \frac{g_W^2}{k^4} + \mathcal{O}(M_W^4),\end{aligned}\quad (26)$$

which is the propagator (23) for the massless Weyl gravity theory.

2 The \mathcal{N} -extended 4D Superconformal Algebra

For massless theories, it has been shown [34] that the conformal group can be extended to the superconformal group which is generated by:

- The generators $(M_{\mu\nu}, P_\mu, K_\mu, D)$ of the conformal algebra $SO(2,4) = SU(2,2)$, i.e., the generators of the Poincaré algebra, the special conformal boosts K_μ and the dilation D ,
- the $2\mathcal{N}$ spinor generators $Q_\alpha^i, S_{\alpha i}$, ($i = 1, \dots, \mathcal{N}$),
- and \mathcal{N}^2 bosonic generators R, T_j^i , ($T_i^i = 0$)

with (anti)commutation relations

$$\begin{aligned}\{Q_\alpha^i, \bar{Q}_{\dot{\alpha}j}\} &= 2\delta_j^i \sigma_{\alpha\dot{\alpha}}^\mu P_\mu, & \{S_{\alpha j}, \bar{S}_{\dot{\alpha}}^i\} &= 2\delta_j^i \sigma_{\alpha\dot{\alpha}}^\mu K_\mu, \\ \{Q_\alpha^i, S_{\beta j}\} &= -\delta_j^i \sigma_{\alpha\beta}^{\mu\nu} M_{\mu\nu} - 4\epsilon_{\alpha\beta} T_j^i - 2\epsilon_{\alpha\beta} (R + iD), \\ [R, Q_\alpha^i] &= \frac{4-N}{2N} Q_\alpha^i, & [R, S_{\alpha i}] &= -\frac{4-N}{2N} S_{\alpha i}, \\ [D, Q_\alpha^i] &= \frac{i}{2} Q_\alpha^i, & [D, S_{\alpha i}] &= -\frac{i}{2} S_{\alpha i}, \\ [T_j^i, Q_\alpha^k] &= -\frac{1}{2} (\delta_j^k Q_\alpha^i - \delta_j^i Q_\alpha^k), \\ [T_j^i, S_{\alpha k}] &= \frac{1}{2} (\delta_k^i S_{\alpha j} - \delta_j^i S_{\alpha k}).\end{aligned}$$

The bosonic generators T_j^i and R generate $SU(\mathcal{N})$ transformations and axial rotations on the spinors, respectively. In particular, there are no in general central charges, except for $\mathcal{N} = 4$ where R is central. These relations, together with the standard commutation relations of the Poincaré subalgebra form the superconformal algebra $SU(2,2|\mathcal{N})$, which is the supersymmetric extension of the conformal algebra $SU(2,2)$.

The standard superspace for the realization of the $SU(2, 2|\mathcal{N})$ superconformal algebra is the coset

$$\mathbb{R}^{4|4\mathcal{N}} = \frac{SU(2, 2|\mathcal{N})}{\{K, S, \overline{S}, M, D, T, R\}} = (x^\mu, \theta_i^\alpha, \overline{\theta}^{\dot{\alpha}i}), \quad (27)$$

which coincides with the $\mathbb{R}^{4|4\mathcal{N}}$ superspace of the Poincaré superalgebra. Superfields $\Phi(x^\mu, \theta_i^\alpha, \overline{\theta}^{\dot{\alpha}i})$ are superconformal primaries satisfying

$$S_{\alpha i} \Phi = \overline{S}_{\dot{\alpha}i} \Phi = K_\mu \Phi = 0. \quad (28)$$

The UIR's of $SU(2, 2|\mathcal{N})$ are then labeled as

$$D(j_1, j_2, d, r, \{\vec{a}\}), \quad (29)$$

where d is the conformal weight, (j_1, j_2) specify the Lorentz representation, r is the R -charge and $\{\vec{a}\}$ are the Dynkin labels of $SU(\mathcal{N})$. Similarly to the super-Poincaré algebra, $SU(2, 2|\mathcal{N})$ can be realised in a smaller superspace by using chiral superfields $\Phi(x, \theta)$ which are elements of the coset

$$\mathbb{C}^{4|2\mathcal{N}} = \frac{SU(2, 2|\mathcal{N})}{\{K, S, \overline{S}, M, D, T, R, \overline{Q}\}} = (x^\mu, \theta_i^\alpha). \quad (30)$$

Such fields should satisfy

$$\overline{Q}_{\dot{\alpha}i} \Phi = 0, \quad (31)$$

and the compatibility condition

$$\{\overline{Q}_{\dot{\alpha}i}, \overline{S}_{\dot{\beta}j}\} \Phi = 0, \quad (32)$$

gives that the chiral primary Φ is a Lorentz singlet ($(j_1 = j_2 = 0)$ with $d = -r$).

Let us note that the conformal algebras in $D = 3, 4, 5$ and $D = 6$ are $Sp(4, \mathbb{R})$, $SU(2, 2)$, $SO(5, 2)$ and $SO^*(8) = SO(6, 2)$, respectively. These algebras are subalgebras of the bosonic subgroup of the superconformal groups in the corresponding dimensions. The latter are simple superalgebras which are supersymmetry algebras and have been classified in [35]. We have tabulated the simple superalgebras in the following table 1.

| D | Superalgebra | Bosonic subalgebra | Fermionic reps. |
|-----|------------------------|---|--|
| 3 | $Osp(3 \mathcal{N})$ | $Sp(4, \mathbb{R}) \times SO(\mathcal{N})$ | $(\underline{4}, \underline{\mathcal{N}})$ |
| 4 | $SU(2, 2 \mathcal{N})$ | $SU(2, 2) \times SU(\mathcal{N}) \times U(1)$ | $(\underline{4}, \underline{\mathcal{N}}) \oplus (\underline{4}, \underline{\mathcal{N}}), \quad \mathcal{N} \neq 4$ |
| 4 | $SU(2, 2 4)$ | $SU(2, 2) \times SU(4)$ | $(\underline{4}, \underline{4}) \oplus (\underline{4}, \underline{4}), \quad \mathcal{N} = 4$ |
| 5 | $F(4)$ | $SO(5, 2) \times SU(2)$ | $(\underline{8}, \underline{2})$ |
| 6 | $Osp(6 \mathcal{N})$ | $SO(8^*) \times USp(2\mathcal{N})$ | $(\underline{8}, \underline{2\mathcal{N}})$ |

We observe from the above table that the bosonic subalgebra in D -dimensions contains the conformal group in that dimension and the fermionic generators are in the spinorial representation of the conformal group. In fact, this is the condition for a superalgebra to be a supersymmetry algebra, namely, the odd generators should transform in the spinor representation of the even subalgebra of the superalgebra.

The conformal supermultiplets can be realized in Lagrangian theories on chiral and vector multiplets for $\mathcal{N} = 1$, hyper and vector multiplets for $\mathcal{N} = 2$ and vector multiplets for $\mathcal{N} = 3, 4$. However, when coupled to Weyl supergravity, $\mathcal{N} = 3, 4$ are different because massive spin-2 multiplets are different in $\mathcal{N} = 3$ and $\mathcal{N} = 4$ (comprised of 128 and 256 states, respectively). Note that superconformal Lagrangians can be realized only for $\mathcal{N} \leq 4$. This bound is in analogy with the $\mathcal{N} \leq 8$ bound of Poincaré supergravity in $D = 4$. The corresponding bound $D \leq 11$ for Poincaré supergravity corresponds to $D \leq 6$ for the higher dimensional conformal supergravity. The bound $\mathcal{N} \leq 4$ can be understood as the bound of spin-2 massive supermultiplets in extended supersymmetry [36].

Another way to understand the $\mathcal{N} \leq 4$ in $4D$ is to compute the quadratic part of the super-Weyl action using the superconformal Ward identities [3]. It turns out that the Weyl term and the $U(\mathcal{N})$ gauge fields occur as follows

$$\mathcal{L} = 2\alpha \left(1 - \frac{4}{\mathcal{N}}\right) F^{\mu\nu} F_{\mu\nu} - 4\alpha F^{i\mu\nu} F_{\mu\nu}^i + 8\alpha W^{\mu\nu\rho\sigma} W_{\mu\nu\rho\sigma}, \quad (33)$$

where $F_{\mu\nu}$ and $F_{\mu\nu}^i$ ($i = 1, \dots, \mathcal{N}$) are the field strengths of the $U(1)$ and $SU(\mathcal{N})$ subgroups of $U(\mathcal{N})$. Clearly, only for $\mathcal{N} \leq 4$ the $U(\mathcal{N})$ gauge fields have the same sign and so they can belong to the same multiplet. For $\mathcal{N} = 4$, the $U(1)$ gauge field has no kinetic term.

3 Massless Higher Derivative Theories

Let us apply canonical quantization to the (dipole ghost) Lagrangian

$$\mathcal{L}_A = \frac{1}{2} (\partial_\mu \partial_\nu A)^2, \quad (34)$$

which is equivalent to

$$\mathcal{L}_A = A_1 \square A - \frac{1}{2} \lambda^2 A_1^2, \quad (35)$$

with A_1 a Lagrange multiplier field. The equations of motion for the fields A and A_1 are

$$\square A = \lambda^2 A_1, \quad \square A_1 = 0. \quad (36)$$

We can construct the Fock space as usual: we introduce creation and annihilation operators $a(k), a_1(k)$ and $a^\dagger(k), a_1^\dagger(k)$ respectively, where the vacuum is defined as

$$a(k)|0\rangle = a_1(k)|0\rangle = 0. \quad (37)$$

The one-particle states are

$$|k\rangle = a^\dagger(k)|0\rangle, \quad |k\rangle_1 = a_1^\dagger(k)|0\rangle, \quad (38)$$

and they satisfy

$${}_1\langle k|k'\rangle_1 = \langle k|k'\rangle = 0, \quad {}_1\langle k|k'\rangle = \delta_D^{(3)}(k - k'). \quad (39)$$

Only one of these one-particle states is an energy eigenstate

$$P^0|k\rangle_1 = \omega|k\rangle_1, \quad P^0|k\rangle = \omega|k\rangle + \frac{\lambda^2}{2\omega}|k\rangle_1. \quad (40)$$

Eqs.(38) and (39) are typical of a dipole ghost state. By using a linear transformation, the two states $|k\rangle$ and $|k\rangle_1$ can be diagonalized such that, one of them will have positive and one negative norm but none of them would be then an energy eigenstate.

For the fermions we have the Lagrangian

$$\mathcal{L}_\psi = \partial_\mu \bar{\psi} \not{\partial} \partial^\mu \psi, \quad (41)$$

which can be written, with the help of Lagrange-multipliers fermion fields ψ_1, ψ_2 as

$$\mathcal{L}_\psi = \bar{\psi}_1 \not{\partial} \psi_1 + \bar{\psi} \not{\partial} \psi_2 + \lambda \bar{\psi}_2 \psi_1. \quad (42)$$

Note that adding a second scalar B with the same Lagrangian (35) (or similarly B and B_1) and two extra scalars F and G with standard kinetic terms, the full theory with the fields $(A, A_1, B, B_1, \psi, \psi_1, \psi_2, F, G)$ is supersymmetric [6]. In particular, the complex scalar $(A + iB)$ is a dipole ghost and an ordinary scalar, whereas, the Majorana (or Weyl) fermions form a triple ghost fermion, that is a dipole ghost and an ordinary fermion. Only two of the one-particle states of a triple ghost $(|k\rangle, |k\rangle_1, |k\rangle_2)$ are energy eigenstates and they satisfy

$$P^0|k\rangle_1 = \omega|k\rangle, \quad P^0|k\rangle_2 = \omega|k\rangle_2, \quad P^0|k\rangle = \omega|k\rangle + \frac{\lambda^2}{2\omega}|k\rangle_2. \quad (43)$$

The theory describes $6_B + 6_F$ dof. The 6_B bosonic dof correspond to 1 dipole ghost and 1 complex scalar, whereas the 6_F fermionic dof are associated with the dof of the triple ghost, that is a dipole ghost (ψ, ψ_2) and an ordinary fermion (ψ_1) . This is the spectrum of a supersymmetric theory describing one dipole WZ ghost multiplet and one ordinary WZ physical multiplet.

4 From Rigid to Local Supersymmetry

As we have discussed above, the massive Weyl gravity, given in Eq.(1) propagates $7 = 2 + 5$ dof associated to a massless helicity ± 2 graviton states, and 1 massive spin-2 state with mass $g_w M_p$ and $5 = (\pm 2, \pm 1, 0)$ dof. In the massless limit, the Weyl symmetry removes the scalar dof and there remain $6 = (\pm 2 + \pm 2 + \pm 1)$ states where the ± 2 helicities made up a dipole ghost, and the ± 1 helicity corresponds to an ordinary vector.

4.1 $\mathcal{N} = 1$ Massive and Massless Weyl Supergravity

The central superfield in conformal supergravity is the super-Weyl tensor $\mathcal{W}_{\alpha\beta\gamma}$, where $(\alpha, \beta, \gamma = 1, 2)$ are $SL(2, \mathbf{C})$ spinor indices. It is chiral superfield with spin $(\frac{3}{2}, 0)$, so that its highest component has spin-3/2. In addition, $\mathcal{W}_{\alpha\beta\gamma}$ accommodates also the five dof of a massive spin-2 field. The Weyl superfield $\mathcal{W}_{\alpha\beta\gamma}$ is chiral so that it satisfies the conditions

$$\mathcal{W}_{\alpha\beta\gamma} = \mathcal{W}_{(\alpha\beta\gamma)}, \quad (\mathcal{W}_{\alpha\beta\gamma})^* = \overline{\mathcal{W}}_{\dot{\alpha}\dot{\beta}\dot{\gamma}}, \quad \overline{\mathcal{D}}_{\dot{\delta}} \mathcal{W}_{\alpha\beta\gamma} = 0, \quad (44)$$

and it has in its components the Weyl tensor $W_{\mu\nu\rho\sigma}$ and the vector auxiliary A_μ of the $\mathcal{N} = 1$ supergravity [23].

4.1.1 Massive theory

The bosonic part of the massive supersymmetric Weyl theory turns out to be [23, 37]

$$e^{-1} \mathcal{L}_{\mathcal{W}} = \frac{1}{2} R + \frac{1}{3} A_\mu A^\mu - \frac{1}{3} u \bar{u} + \frac{\alpha}{2} \left(\frac{1}{2} R_{HP}^2 - \frac{2}{3} F_{\mu\nu} F_{\rho\sigma} \epsilon^{\mu\nu\rho\sigma} \right) + \frac{1}{2g^2} \left(W^{\mu\nu\rho\sigma} W_{\mu\nu\rho\sigma} - \frac{4}{3} F^{\mu\nu} F_{\mu\nu} \right),$$

where u is the auxiliary scalar, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength of A_μ and $R_{HP}^2 = W_{\mu\nu\kappa\lambda} \epsilon^{\kappa\lambda\rho\sigma} W_{\rho\sigma}{}^{\mu\nu}$ is the Hirzebruch–Pontryagin tensor. It is clear that the auxiliary vector of $\mathcal{N} = 1$ Poincaré supergravity acquires a kinetic term and therefore it is dynamical in conformal supergravity. The spectrum of the massive $\mathcal{N} = 1$ super-Weyl theory contains $n_B + n_F = 20$ dof in the following form:

(i) The standard massless spin-2 graviton multiplet (h, q_R) with $n_B + n_F = 4$ dof, helicity h and q_R $U_R(1)$ charge:

$$g_{\mathcal{N}=1} : (+2, 0) + \left(+\frac{3}{2}, +\frac{1}{2}\right) + \text{CPT}. \quad (45)$$

(ii) The massive spin-2 supermultiplet in the non-standard sector with $n_B + n_F = 16$ dof which form the massive $\mathcal{N} = 1$ super-Weyl multiplet [6]:

$$w_{\mathcal{N}=1} : \left(2, 2\left(\frac{3}{2}\right), 1\right). \quad (46)$$

It should be noted that the massive states in $\mathcal{N} = 1$ supergravity form $USp(2)$ representations.

4.1.2 Massless theory

The spectrum of the massless $\mathcal{N} = 1$ super-Weyl theory is the following:

(i) A standard massless spin-2 (h, q_R) supergravity multiplet with $n_B + n_F = 4$ dof

$$g_{\mathcal{N}=1} : (+2, 0) + \left(+\frac{3}{2}, +\frac{1}{2}\right) + \text{CPT}. \quad (47)$$

(ii) In the non-standard sector, we have a massless ghost-like spin-2 supermultiplet, arising from the massive Weyl multiplet $w_{\mathcal{N}=1}$:

$$\text{spin} - 2 : (+2, 0) + \left(+\frac{3}{2}, +\frac{1}{2}\right) + \text{CPT}, \quad (4 \text{ dof}). \quad (48)$$

In addition, the $w_{\mathcal{N}=1}$ multiplet gives a massless, physical spin-3/2 supermultiplet:

$$\text{spin} - 3/2 : \left(+\frac{3}{2}, -\frac{1}{2}\right) + (+1, 0) + \text{CPT}, \quad (4 \text{ dof}), \quad (49)$$

a massless spin-one vector multiplet:

$$\text{spin} - 1 : (+1, 0) + \left(+\frac{1}{2}, +\frac{1}{2}\right) + \text{CPT}, \quad (4 \text{ dof}) \quad (50)$$

and a chiral spin-1/2 multiplet:

$$\text{spin} - 1/2 : \left(+\frac{1}{2}, -\frac{1}{2}\right) + (0, 0) + \text{CPT}, \quad (4 \text{ dof}). \quad (51)$$

Note that the spin-3/2 fields form a tripole ghost. The latter acts effectively as a physical spin-3/2 multiplet together with a spin-2 dipole ghost multiplet [6]. However, the chiral multiplet is unphysical as it can be gauged away due to the superconformal and $U(1)_R$ symmetries [10].

Recapulating, the massless super-Weyl gravity theory contains $n_B + n_F = 16$ physical, propagating degrees of freedom [8]. In addition, the gravitino action contains ($\mathcal{N} = 1$) eight helicity states $\left(+\frac{3}{2}, +\frac{1}{2}\right), \left(+\frac{3}{2}, +\frac{1}{2}\right), \left(+\frac{3}{2}, -\frac{1}{2}\right)$ (+ CPT conjugates) [8], which is important in anomaly cancellation. The counting of dof is summarized in the following table 2.

| \mathcal{N} | DoF of Massive Weyl | Gauged DoF in Massless Weyl | DoF of Massless Weyl |
|---------------|---------------------|-----------------------------|----------------------|
| 0 | $5 + 2 = 7$ | 1 scalar | $7 - 1 = 6$ |
| 1 | $16 + 4 = 20$ | 1 chiral multiplet | $20 - 4 = 16$ |
| 2 | $48 + 8 = 56$ | 1 vector and 1 hyper | $56 - 16 = 40$ |
| 3 | $128 + 16 = 144$ | 3 vector multiplets | $144 - 48 = 96$ |
| 4 | $256 + 32 = 288$ | 6 vector multiplets | $288 - 96 = 192$ |

In fact, a formula that gives the number of degrees of freedom in “massless” \mathcal{N} -extended supergravity is

$$\begin{aligned}
\mathcal{N} = 0, 1, 2, 3, \quad \text{dof} &= (3 + \mathcal{N}) \cdot 2^{\mathcal{N}+1}; & \text{with matter } (n) \quad \text{dof} &= (3 + \mathcal{N} + n) \cdot 2^{\mathcal{N}+1}, \\
\mathcal{N} = 4, \quad \text{dof} &= 12 \cdot 2^4; & \text{with matter } (n) \quad \text{dof} &= (12 + n) \cdot 2^4, \quad (52)
\end{aligned}$$

without and with (n) matter multiplets. The factor $3 + \mathcal{N}$ denotes the two (dipole) graviton multiplets, the vector multiplet and the \mathcal{N} -gravitini multiplets (all massless). The \mathcal{N} -gravitini multiplets contain the \mathcal{N}^2 gauge bosons, which gauge the $U(N)$ symmetry. For $\mathcal{N} = 4$, the extra vector multiplet is the partner (Weyl vector) of the Weyl graviton that for $\mathcal{N} < 4$ seats in the extra vector multiplet.

5 $\mathcal{N} = 4$ Weyl Supergravity and its Massless (Ghost) Spectrum

As we have seen above, the dof of the massless $\mathcal{N} = 4$ Weyl supergravity has 192 dof which is the original 288 of the massive theory (a massive and a massless spin-2) after gauging away 96 gauge degrees of freedom of six vector multiplets due to superconformal invariance. Then the remaining massless spectrum is given by 2 $\mathcal{N}=4$ graviton multiplets (2×32 dof) and 4 (spin-3/2) gravitini multiplets (4×32 dof) giving a total of $6 \times 32 = 192$ dof. This allows to classify all the states and compare with the corresponding Lagrangian [11]. This reproduces the known structure which is cubic in certain fermions (spin-3/2) and (spin-1/2) and standard in others ($\overline{20}$) [?]. In particular, we have:

- *Bosonic Spectrum:* From the spin-2 (dipole) graviton multiplets we have

$$2 \times \left[(+2, \underline{1}) + (+1, \underline{6}) + (0, \underline{1}) + \text{CPT} \right], \quad (53)$$

whereas from the $\underline{4}$ gravitino multiplets we get

$$(+1, \underline{15} + \underline{1}) + (0, \overline{\underline{10}} + \underline{6}) + \text{CPT}. \quad (54)$$

The vectors and scalars in the $\underline{6}$ correspond to the higher derivative graviphoton contribution, whereas the vectors in the $(\underline{15} + \underline{1})$ correspond to the $SU(4)$ gauge fields and the Weyl vector. In particular, the scalars in the $\overline{\underline{10}}$ have standard two derivatives kinetic terms while the $SU(4)$ singlet has higher derivative (quartic) kinetic term.

- *Fermionic Spectrum:* From the two (dipole) $\mathcal{N} = 4$ gravitino multiplets we have

$$\left(+\frac{3}{2}, 4 \right) + \left(+\frac{3}{2}, 4 \right) + \left(+\frac{1}{2}, \overline{4} \right) + \left(+\frac{1}{2}, \overline{4} \right) + \text{CPT}, \quad (55)$$

and from the $\overline{\mathbb{4}}$ gravitino multiplets we get

$$\left(+\frac{3}{2}, \overline{\mathbb{4}}\right) + \left(+\frac{1}{2}, \overline{\mathbb{20}}\right) + \left(+\frac{1}{2}, \underline{\mathbb{4}}\right) + \left(+\frac{1}{2}, \underline{\mathbb{4}}\right) + \text{CPT}. \quad (56)$$

In addition, the gravitino and the spin-1/2 fields have cubic kinetic terms whereas the spin-1/2 fields in the $\overline{\mathbb{20}}$ have physical standard kinetic terms.

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References

- [1] G. 't Hooft, “Local Conformal Symmetry: the Missing Symmetry Component for Space and Time,” arXiv:1410.6675 [gr-qc].
- [2] M. Kaku, P. K. Townsend and P. van Nieuwenhuizen, “Superconformal Unified Field Theory,” Phys. Rev. Lett. **39** (1977) 1109; “Properties of Conformal Supergravity,” Phys. Rev. D **17** (1978) 3179.
- [3] S. Ferrara, M. Kaku, P. K. Townsend and P. van Nieuwenhuizen, “Gauging the Graded Conformal Group with Unitary Internal Symmetries,” Nucl. Phys. B **129** (1977) 125.
- [4] E. S. Fradkin and A. A. Tseytlin, “One Loop Beta Function in Conformal Supergravities,” Nucl. Phys. B **203** (1982) 157; “Conformal Supergravity,” Phys. Rept. **119** (1985) 233.
- [5] K. S. Stelle, “Renormalization of Higher Derivative Quantum Gravity,” Phys. Rev. D **16** (1977) 953; “Classical Gravity with Higher Derivatives,” Gen. Rel. Grav. **9** (1978) 353.
- [6] S. Ferrara and B. Zumino, “Structure of Linearized supergravity and Conformal Supergravity,” Nucl. Phys. B **134** (1978) 301.
- [7] S. Ferrara, M. T. Grisaru and P. van Nieuwenhuizen, “Poincare and Conformal Supergravity Models With Closed Algebras,” Nucl. Phys. B **138**, 430 (1978).
- [8] S. C. Lee and P. van Nieuwenhuizen, “Counting of States in Higher Derivative Field Theories,” Phys. Rev. D **26**, 934 (1982).
- [9] R. J. Riegert, “The Particle Content Of Conformal Gravity,” Phys. Lett. A **105**, 110 (1984).

- [10] S. Ferrara, A. Kehagias and D. Lüüst, “Aspects of Weyl Supergravity,” *JHEP* **1808** (2018) 197 [arXiv:1806.10016 [hep-th]].
- [11] E. Bergshoeff, M. de Roo and B. de Wit, “Extended Conformal Supergravity,” *Nucl. Phys. B* **182** (1981) 173.
- [12] S. Ferrara, A. Kehagias and D. Lüüst, “Bimetric, Conformal Supergravity and its Superstring Embedding,” *JHEP* **1905** (2019) 100 [arXiv:1810.08147 [hep-th]].
- [13] D. Butter, F. Ciceri, B. de Wit and B. Sahoo, “Construction of all N=4 conformal supergravities,” *Phys. Rev. Lett.* **118**, no. 8, 081602 (2017) [arXiv:1609.09083 [hep-th]].
- [14] N. Berkovits and E. Witten, “Conformal supergravity in twistor-string theory,” *JHEP* **0408** (2004) 009 [hep-th/0406051].
- [15] D. Z. Freedman and A. Van Proeyen, “Supergravity,” Cambridge University Press, 2012.
- [16] D. M. Capper and M. J. Duff, “Conformal Anomalies and the Renormalizability Problem in Quantum Gravity,” *Phys. Lett. A* **53**, 361 (1975).
- [17] E. S. Fradkin and A. A. Tseytlin, “Conformal Anomaly in Weyl Theory and Anomaly Free Superconformal Theories,” *Phys. Lett.* **134B** (1984) 187.
- [18] H. Römer and P. van Nieuwenhuizen, “Axial Anomalies in $N = 4$ Conformal Supergravity,” *Phys. Lett.* **162B**, 290 (1985).
- [19] B. de Wit, J. W. van Holten and A. Van Proeyen, “Transformation Rules of N=2 Supergravity Multiplets,” *Nucl. Phys. B* **167** (1980) 186.
- [20] H. Liu and A. A. Tseytlin, “ $D = 4$ superYang-Mills, $D = 5$ gauged supergravity, and $D = 4$ conformal supergravity,” *Nucl. Phys. B* **533** (1998) 88 [hep-th/9804083].
- [21] A. Kehagias, C. Kounnas, D. Lüüst and A. Riotto, *JHEP* **1505** (2015) 143 [arXiv:1502.04192].
- [22] L. Alvarez-Gaume, A. Kehagias, C. Kounnas, D. Lüüst and A. Riotto, “Aspects of Quadratic Gravity,” *Fortsch. Phys.* **64** (2016) no.2-3, 176 [arXiv:1505.07657 [hep-th]].
- [23] F. Farakos, S. Ferrara, A. Kehagias and D. Lüüst, “Non-linear Realizations and Higher Curvature Supergravity,” *Fortsch. Phys.* **65**, no. 12, 1700073 (2017) [arXiv:1707.06991 [hep-th]].
- [24] A. Salvio, “Quadratic Gravity,” *Front. in Phys.* **6** (2018) 77 [arXiv:1804.09944 [hep-th]].

- [25] A. A. Tseytlin, “On divergences in non-minimal $N = 4$ conformal supergravity,” *J. Phys. A* **50** (2017) no.48, 48LT01 [arXiv:1708.08727 [hep-th]].
- [26] M. Beccaria and A. A. Tseytlin, “Superconformal index of higher derivative $\mathcal{N} = 1$ multiplets in four dimensions,” *JHEP* **1810** (2018) 087 [arXiv:1807.05911 [hep-th]].
- [27] S. Ferrara, R. Kallosh and A. Van Proeyen, “Conjecture on hidden superconformal symmetry of $N = 4$ Supergravity,” *Phys. Rev. D* **87**, no. 2, 025004 (2013) [arXiv:1209.0418 [hep-th]].
- [28] S. Ferrara and D. Lüüst, “Spin-four $N=7$ W-Supergravity: S-fold and Double Copy Construction,” arXiv:1805.10022 [hep-th], accepted for publication in *JHEP*.
- [29] R. Kallosh, L. Kofman, A. D. Linde and A. Van Proeyen, “Superconformal symmetry, supergravity and cosmology,” *Class. Quant. Grav.* **17** (2000) 4269 Erratum: [*Class. Quant. Grav.* **21** (2004) 5017] [hep-th/0006179].
- [30] S. Ferrara, R. Kallosh, A. Linde, A. Marrani and A. Van Proeyen, “Jordan Frame Supergravity and Inflation in NMSSM,” *Phys. Rev. D* **82** (2010) 045003 [arXiv:1004.0712 [hep-th]]; “Superconformal Symmetry, NMSSM, and Inflation,” *Phys. Rev. D* **83** (2011) 025008 [arXiv:1008.2942 [hep-th]].
- [31] R. Kallosh and A. Linde, “Superconformal generalization of the chaotic inflation model $\frac{\lambda}{4}\phi^4 - \frac{\xi}{2}\phi^2 R$,” *JCAP* **1306** (2013) 027 [arXiv:1306.3211 [hep-th]].
- [32] E. A. Bergshoeff, O. Hohm and P. K. Townsend, “Massive Gravity in Three Dimensions,” *Phys. Rev. Lett.* **102**, 201301 (2009). [arXiv:0901.1766 [hep-th]].
- [33] B. Gording and A. Schmidt-May, “Ghost-free infinite derivative gravity,” arXiv:1807.05011.
- [34] R. Haag, J. T. Lopuszanski and M. Sohnius, “All Possible Generators of Supersymmetries of the S-Matrix,” *Nucl. Phys. B* **88** (1975) 257.
- [35] W. Nahm, “Supersymmetries and their Representations,” *Nucl. Phys. B* **135** (1978) 149.
- [36] B. de Wit and S. Ferrara, “On Higher Order Invariants in Extended Supergravity,” *Phys. Lett.* **81B** (1979) 317.
- [37] S. Cecotti, S. Ferrara, M. Porrati and S. Sabharwal, “New Minimal Higher Derivative Supergravity Coupled To Matter,” *Nucl. Phys. B* **306** (1988) 160.
- [38] H. Johansson, G. Mogull and F. Teng, “Unraveling conformal gravity amplitudes,” *JHEP* **1809**, 080 (2018) [arXiv:1806.05124 [hep-th]].