SUPERSYMMETRY ALGEBRAS AND LORENTZ INVARIANCE FOR d = 10 SUPER YANG-MILLS

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ABSTRACT

We consider ways in which conventional supersymmetry can be embedded in the set of more general fermionic transformations proposed recently [1] as a framework in which to study d = 10 super Yang-Mills. Solutions are exhibited which involve closed algebras of various numbers of supersymmetries together with their invariance groups: nine supersymmetries with $G_2 \times SO(1, 1)$ invariance; eight supersymmetries with $SO(7) \times SO(1, 1)$ invariance; four supersymmetries with $SO(3, 1) \times U(3)$ invariance. We recover in this manner all previously known ways of adding finite numbers of bosonic auxiliary fields so as to partially close the d = 10 superalgebra. A crucial feature of these solutions is that the auxiliary fields transform non-trivially under the residual Lorentz symmetry, even though they are originally introduced as Lorentz scalars.

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1. Introduction and overview

Supersymmetric Yang-Mills theories possess a wealth of fascinating physical and mathematical properties which have been extensively studied in the last twenty years.² An infamous outstanding problem is the construction of an off-shell formulation of ten-dimensional super Yang-Mills.³ A solution would be of considerable interest in its own right as well as being likely to offer a new perspective on other difficult and long-standing questions such as the existence of covariant quantum actions for superparticles and superstrings in ten dimensions.⁴ Here we investigate a novel setting for this problem which was suggested recently in [1]. In this first section we clarify some general aspects of this approach and formulate our aims.

The Lagrangian for d = 10 super Yang-Mills can be written

$$L = \text{Tr}\left[-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{i}{2}\psi\Gamma^{\mu}D_{\mu}\psi + \frac{1}{2}G_{i}G_{i}\right].$$
 (1.1)

 A_{μ} is a Yang-Mills gauge field, $D_{\mu} = \partial_{\mu} + A_{\mu}$ is the associated covariant derivative, $F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu} + [A_{\mu}, A_{\nu}]$ is the field strength and ψ is a sixteen-component Majorana-Weyl spinor. The fields A_{μ} and ψ describe equal numbers of propagating *on*-shell modes and the auxiliary scalar fields G_i with $i = 1, \ldots, 7$ are included to balance the bosonic and fermionic *off*-shell degrees of freedom. All fields takes values in the Lie algebra of some gauge group, 'Tr' denotes an invariant inner-product on this algebra and L is, of course, gauge-invariant. The classical gauge coupling constant has been scaled out of the Lagrangian.

The Lagrangian L is invariant under the transformations

$$\delta_{(\epsilon,v_i)}A_{\mu} = i\epsilon\Gamma_{\mu}\psi$$

$$\delta_{(\epsilon,v_i)}\psi = \frac{1}{2}F^{\mu\nu}\Gamma_{\mu\nu}\epsilon + G_iv_i$$

$$\delta_{(\epsilon,v_i)}G_i = -iv_i\Gamma^{\mu}D_{\mu}\psi$$
(1.2)

which depend not only on the usual spinor parameter ϵ but also on an additional seven spinor parameters v_i satisfying the conditions

$$v_i \Gamma_\mu \epsilon = 0, \qquad v_i \Gamma_\mu v_j = \delta_{ij} \epsilon \Gamma_\mu \epsilon.$$
 (1.3)

The parameters ϵ and v_i are taken to be *commuting* spinors, which means that the variation $\delta_{(\epsilon,v_i)}$ is fermionic in character, but spinor fields such as ψ must be treated as *anticommuting*. We shall refer to a transformation (1.2) with the additional restriction (1.3) as a

² General introductions with comprehensive citations can be found in [2,3].

³ The maximally-extended super Yang-Mills theories were formulated in [4]. The auxiliary field problem has been studied using: component fields [5]; conventional superspace in d=4 [6,7], d=6 [8], and d=10 [9]; light-cone superspace [10,11,12]; and harmonic superspace [13].

⁴ See [3] and, for a more detailed and up-to-date account, [14].

generalized supersymmetry. It can be shown by direct computation that any two generalized supersymmetries obey, up to field equations, the standard supersymmetry algebra

$$\{\delta_{(\epsilon,v_i)}, \delta_{(\hat{\epsilon},\hat{v}_i)}\} = -2i\epsilon\Gamma^{\mu}\hat{\epsilon} D_{\mu}$$
(1.4)

when acting on D_{μ} , ψ or G_i (considering the action on D_{μ} rather than A_{μ} just gives a neat way of including the relevant gauge transformations in the algebra) and that this algebra holds independently of field equations when $(\epsilon, v_i) = (\hat{\epsilon}, \hat{v}_i)$. One can use this fact to find larger sets of generalized supersymmetries obeying a closed algebra off shell and it was argued in [1] that a maximum of nine generalized supersymmetries can be found with this property.

The attractive feature of this framework is that the Lagrangian (1.1), the transformations (1.2) and the additional constraints (1.3) are manifestly invariant under the Lorentz group SO(9,1). They are also clearly invariant with respect to an internal symmetry group $O(7)_{aux}$ under which the auxiliary fields G_i and the spinor parameters v_i transform as seven-dimensional vectors. The suffix 'aux' indicates that these latter transformations do not alter physical states of the theory; nevertheless they have an important role to play. For one thing the equations (1.3) are sufficient to determine the spinors v_i from a given ϵ precisely up to a transformation of type $O(7)_{aux}$. We shall be concerned with the connected invariance group $SO(9, 1) \times SO(7)_{aux}$ of the complete set of generalized supersymmetries, and with the fact that this cannot, unfortunately, be preserved in choosing subsets of supersymmetries which obey the algebra (1.4) off shell.

Our aim is to study ways in which conventional supersymmetry transformations can be embedded within the set of generalized supersymmetries. Conventional supersymmetries depend linearly on the single spinor parameter ϵ and so our task is to find solutions of (1.3) of the form

$$v_i = M_i \epsilon \tag{1.5}$$

for some suitable matrices M_i . Any such choice must break Lorentz invariance because in d = 10 there are no non-trivial Lorentz-invariant tensors of the required type. In order to construct solutions we shall find it necessary to restrict the spinor ϵ to some subspace. The choice of this subspace together with the choice of matrices M_i will determine a particular subgroup of SO(9, 1) × SO(7)_{aux} which survives as the invariance group of the solution. In sections 2, 3 and 4 below we present solutions involving various numbers of supersymmetries together with detailed discussions of their residual invariance groups and associated representations.

One subtle aspect of the residual symmetry is worth pointing out in advance. In each of the cases below we shall find the pattern of symmetry breaking⁵ SO(9,1) × SO(7)_{aux} \rightarrow SO(n,1) × H, for some n and some group H. But here H is embedded diagonally in the factors on the left-hand side: the Lorentz factor is broken SO(9,1) \rightarrow SO(n,1) × H and the internal auxiliary symmetry is broken SO(7)_{aux} \rightarrow H in such a way that these copies of the group H get identified. Since SO(7)_{aux} has no direct physical significance we are

⁵ Arrows will always indicate reductions in symmetry groups and decompositions of representations, *never* homomorphisms between groups.

entitled to interpret the surviving symmetries belonging to H as Lorentz transformations. But the auxiliary fields transform non-trivially under SO(7)_{aux} and hence under H. The auxiliary fields in our solutions will therefore behave non-trivially under residual Lorentz transformations even though they are originally introduced as SO(9, 1) scalars.

The most important property of our solutions is that the algebra (1.4) will hold offshell. We mentioned above that (1.4) holds when $(\epsilon, v_i) = (\hat{\epsilon}, \hat{v}_i)$ and that in [1] a method was given to generate other generalized supersymmetries which all obey a closed algebra. This method is more general than we need here, however, because we are considering solutions of (1.3) which are all related by (1.5) with the spinor parameter ϵ restricted to some particular subspace. In these circumstances the corresponding generalized supersymmetries depend linearly on ϵ and, denoting them simply by $\delta_{\epsilon} = \delta_{(\epsilon,M_i\epsilon)}$, it follows that any two such transformations obey $\{\delta_{\epsilon}, \delta_{\epsilon}\} = \frac{1}{2}(\{\delta_{\epsilon+\hat{\epsilon}}, \delta_{\epsilon+\hat{\epsilon}}\} - \{\delta_{\epsilon}, \delta_{\epsilon}\} - \{\delta_{\hat{\epsilon}}, \delta_{\hat{\epsilon}}\})$. We know that (1.4) holds for each term on the right-hand side and it follows that it must hold for the expression on the left-hand side too. Hence, any generalized supersymmetries corresponding to a particular space of solutions of (1.3) and (1.5) will automatically obey a closed algebra.

It is natural to ask how our results compare with previous work in which auxiliary fields have been found for subsets of d = 10 supersymmetry transformations. One possibility is to select just those supersymmetries which preserve a light-cone gauge condition [10,11,12]. In [12] a systematic search was made for bosonic auxiliary fields which would close such a light-cone supersymmetry algebra; it was shown that there is no solution which is covariant with respect to the residual SO(8) part of the light-cone symmetry group, but that a solution exists which is covariant with respect to an SO(7) subgroup. The solution of section 2 generalizes this result. Another option, distinct from the light-cone approach, is to consider the trivial dimensional reduction of super Yang-Mills in d = 10 to theories with N = 2 supersymmetry in d = 6 or N = 4 in d = 4. It is easy to find an N = 1superspace formulation of the theory in d = 4 by coupling three chiral matter multiplets to N = 1, d = 4 Yang-Mills [2]. This case arises as a corollary of the solution given in section 4. It is also possible, after overcoming some highly non-trivial obstacles, to find superspace formulations which make manifest N = 2 supersymmetry in d = 4 [7] or N = 1 supersymmetry in d = 6 [8]. These last possibilities involve fermionic auxiliary fields, however, and so one should not expect to find them appearing in the framework considered here. Finally, in [1] a particular solution to (1.3) was presented using octonionic notation. We recover this in section 3 in conventional notation by modifying the solution of section 2.

The last chore before giving the solutions is to fix some details of notation. Spacetime indices in d = 10 are written $\mu, \nu = 0, \ldots, 9$ and they label the vector representation **10** of SO(9,1). The Minkowski metric is defined by $-\eta_{00} = \eta_{11} = \ldots = \eta_{99} = 1$. Spinor indices will often be suppressed but when they are needed upper and lower indices $A, B = 1, \ldots, 16$ will denote the inequivalent Majorana-Weyl representations $\mathbf{16}_{\pm}$. By definition ψ , ϵ and v_i all belong to the $\mathbf{16}_+$ representation. Upper and lower spinor indices cannot be lowered or raised but they can be contracted invariantly. The corresponding gamma matrices $(\Gamma_{\mu})^{AB}$ and $(\Gamma_{\mu})_{AB}$ are symmetric and obey $(\Gamma_{\mu})^{AC}(\Gamma_{\nu})_{CB} + (\Gamma_{\nu})^{AC}(\Gamma_{\mu})_{CB} = 2\eta_{\mu\nu}\delta^{A}{}_{B}$. Antisymmetrized products are denoted in the usual ways, eg. $(\Gamma_{\mu\nu})^{A}{}_{B} = \frac{1}{2}[(\Gamma_{\mu})^{AC}(\Gamma_{\nu})_{CB} - (\Gamma_{\nu})^{AC}(\Gamma_{\mu})_{CB}]$ is the generator of Lorentz transformations in the $\mathbf{16}_{+}$ representation. Because the spinor representations are Majorana, it is always possible to choose a basis in which all components of spinors and gamma matrices are real. Because the spinor indices label Weyl representations, we can also take the gamma matrices to obey $(\Gamma_{0...9})^{A}{}_{B} = \delta^{A}{}_{B}$.

2. Eight supersymmetries with $SO(7) \times SO(1,1)$ invariance

We start by considering a decomposition of the Lorentz group to a light-cone subgroup $SO(9,1) \rightarrow SO(8) \times SO(1,1)$ in which the first factor acts on the subspace of Minkowski space with coordinates labeled by $\mu = 1, \ldots, 8$ and the second factor acts on the subspace with coordinates labeled by $\mu = 0, 9$. With respect to this subgroup the vector representation decomposes $\mathbf{10} \rightarrow \mathbf{8}^0 \oplus \mathbf{1}^2 \oplus \mathbf{1}^{-2}$ and the spinor representations decompose according to $\mathbf{16}_+ \rightarrow \mathbf{8}_+^1 \oplus \mathbf{8}_-^{-1}$ and $\mathbf{16}_- \rightarrow \mathbf{8}_-^1 \oplus \mathbf{8}_+^{-1}$. Here $\mathbf{8}$ and $\mathbf{8}_{\pm}$ denote the vector and spinor representations of SO(8) respectively and superscripts specify the eigenvalue of a suitably normalized generator of the SO(1,1) factor. We will be concerned mostly with the decomposition of the $\mathbf{16}_+$ representation for which this generator is $\Gamma_{09} = \Gamma_{1...8}$.

We will look for a solution of (1.3) in which the supersymmetry parameter satisfies the restriction

$$\Gamma_{09}\epsilon = \epsilon \tag{2.1}$$

giving eight linearly-independent supersymmetries. To obtain such a solution it appears to be necessary to break the symmetry still further by selecting a particular transverse direction, which may as well be the one labeled by $\mu = 8$. We then define

$$v_i = \Gamma_{i8}\epsilon, \qquad i = 1, \dots, 7 \tag{2.2}$$

and claim that (2.1) and (2.2) provide a solution of (1.3). This can be checked using standard gamma matrix manipulations; of particular use is the fact that $\Gamma_{\mu\nu\rho}$ is antisymmetric in its spinor indices so that $\epsilon\Gamma_{\mu\nu\rho}\epsilon = 0$.

In choosing a particular transverse direction we have clearly compounded the reduction in symmetry by breaking SO(8) \rightarrow SO(7). Under this subgroup the vector representation of course decomposes $\mathbf{8} \rightarrow \mathbf{7} \oplus \mathbf{1}$ while the spinor representations remain irreducible but become isomorphic: $\mathbf{8}_{\pm} \rightarrow \mathbf{8}$ (it should be obvious from the context whether **8** denotes the vector representation of SO(8) or the spinor representation of SO(7)). In line with the general remarks made in the introduction, we see that (2.2) identifies this SO(7) subgroup with the group SO(7)_{aux}, because the index *i* on the left-hand side of (2.2) was originally an internal label for the **7** of SO(7)_{aux} whereas on the right-hand side it labels a direction in spacetime. The residual symmetry group is just the diagonal subgroup of these two SO(7) factors under which the auxiliary fields G_i transform as a vector.

To summarize, we have found a closed algebra of eight supersymmetries with the residual invariance group $SO(7) \times SO(1, 1)$ and representations

$$egin{aligned} &A_{\mu}:\, \mathbf{7}^{0}\oplus\mathbf{1}^{0}\oplus\mathbf{1}^{2}\oplus\mathbf{1}^{-2} \ &\psi\,:\,\mathbf{8}^{1}\oplus\mathbf{8}^{-1} \ &G_{i}:\,\mathbf{7}^{0} \ &\epsilon\,:\,\mathbf{8}^{1} \end{aligned}$$

It is instructive to write this solution explicitly in terms of irreducible representations of the invariance group. To do so we consider the decomposition of a spinor $\chi^A \to (\chi^+_{\alpha}, \chi^-_{\alpha})$, where the indices $\alpha, \beta = 1, \ldots, 8$ label the spinor representation of SO(7) and the superscripts indicate the eigenvalue of Γ_{09} . A suitable corresponding block form for the d = 10gamma matrices is given by

$$-(\Gamma_0)_{AB} = (\Gamma_0)^{AB} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad (\Gamma_9)_{AB} = (\Gamma_9)^{AB} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
$$(\Gamma_8)_{AB} = (\Gamma_8)^{AB} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad (\Gamma_i)_{AB} = (\Gamma_i)^{AB} = \begin{pmatrix} 0 & \lambda_i \\ -\lambda_i & 0 \end{pmatrix}$$
(2.3)

where i = 1, ..., 7. The matrices $(\lambda_i)_{\alpha\beta}$ are real and antisymmetric and they obey

$$\lambda_i \lambda_j + \lambda_j \lambda_i = -2\delta_{ij} \tag{2.4}$$

(an explicit construction of these matrices can be found in the appendix). Denoting antisymmetrized products in the usual way, the matrices $(\lambda_{ij})_{\alpha\beta}$ are antisymmetric and generate the spinor representation of SO(7). The solution (2.1), (2.2) can now be written simply as

$$v_i^+ = \lambda_i \epsilon^+ , \qquad v_i^- = \epsilon^- = 0.$$
(2.5)

If we choose the light-cone gauge $A_0+A_9 = 0$ we recover exactly the construction introduced previously in [12]. In addition to understanding how this construction fits into the framework of generalized supersymmetry, we have now learned that such a light-cone gauge choice is not necessary in order to close the light-cone supersymmetry algebra.

As a last point of interest, we note that one can use triality, *ie.* the S_3 group of outer automorphisms of SO(8), to construct similar solutions invariant under inequivalent SO(7) subgroups. There are essentially three inequivalent SO(7) subgroups of SO(8) whose vector representations sit in the 'obvious' ways inside the vector or spinor representations of SO(8) – see *eg.* [15,16]. The construction of solutions invariant under these alternative SO(7) subgroups is essentially similar to the solution presented above and we omit the details.

3. Nine supersymmetries with $G_2 \times SO(1,1)$ invariance

We now show how a similar solution to that of the last section can be 'enlarged' by one more supersymmetry, but only at the expense of reducing the invariance group. In the notation introduced in the last section, we seek a solution in which ϵ^- is non-zero. We pick out a particular direction in spinor space, which may as well be the one labeled by $\alpha = 8$, and define a diagonal matrix $n_{\alpha\beta}$ by $-n_{11} = \ldots = -n_{77} = n_{88} = 1$. We claim that (1.3) is satisfied if

$$v_i^+ = n\lambda_i n\epsilon^+, \qquad v_i^- = -\lambda_i \epsilon^-, \qquad n\epsilon^- = \epsilon^-.$$
 (3.1)

To check this it is convenient to introduce the matrix $p = \frac{1}{2}(1+n)$ which projects onto the one-dimensional positive eigenspace of n. The solution can then be verified by direct substitution using the expressions in (2.3) and using also the facts $p\lambda_i p = 0$, $p\lambda_i\lambda_j p = -\delta_{ij}p$ and $p\lambda_i\lambda_j\lambda_k p = p\lambda_{ijk}p$ which are all simple consequences of (2.4).

Since ϵ^+ is arbitrary but $n\epsilon^- = \epsilon^-$ we have found a solution with nine independent supersymmetries. It can be shown that this coincides with the solution presented in [1] using octonionic notation but we shall not give the details here; it can be demonstrated in a pedestrian way by writing out in conventional notation the results of all octonionic multiplications in the expressions in [1]. Now we consider the invariance properties of this solution, which were not dealt with in [1]. The residual invariance clearly consists of an SO(1,1) factor together with the subgroup of SO(7) which fixes the particular direction $\alpha = 8$ in the spinor representation, which is equivalent to fixing the matrix $n_{\alpha\beta}$. It is wellknown [17,16] that this subgroup is G₂ and that one can write down explicit expressions for its generators. The combination of SO(7) generators $a_{ij}\lambda_{ij}$ leaves the matrix n inert precisely when $c_{ijk}a_{jk} = 0$, where $c_{ijk} = (\lambda_i)_{jk}$ is completely antisymmetric. These seven conditions on the twenty-one generators of SO(7) leave fourteen independent combinations, as required for G₂. The vector and spinor representations decompose under SO(7) \rightarrow G₂ according to $\mathbf{7} \rightarrow \mathbf{7}$ and $\mathbf{8} \rightarrow \mathbf{7} \oplus \mathbf{1}$ respectively.

By combining these results with those of the last section we can read off the final transformation properties of all the fields. We have found a solution giving a closed algebra of nine supersymmetries with invariance group $G_2 \times SO(1, 1)$ and representations

$$egin{aligned} A_{\mu}:\, \mathbf{7}^{0} \oplus \mathbf{1}^{0} \oplus \mathbf{1}^{2} \oplus \mathbf{1}^{-2} \ \psi\,:\, \mathbf{7}^{1} \oplus \mathbf{1}^{1} \oplus \mathbf{7}^{-1} \oplus \mathbf{1}^{-1} \ G_{i}:\,\, \mathbf{7}^{0} \ \epsilon\,:\, \mathbf{7}^{1} \oplus \mathbf{1}^{1} \oplus \mathbf{1}^{-1} \end{aligned}$$

4. Four supersymmetries with $SO(3,1) \times U(3)$ invariance

We start from the decomposition of the Lorentz group $SO(9, 1) \rightarrow SO(3, 1) \times SU(4)$ where the first factor acts on the subspace with coordinates labeled by $\mu = 0, 1, 2, 3$ and the second factor acts on the subspace labeled by $\mu = 4, \ldots, 9$. The vector representation decomposes $\mathbf{10} \rightarrow (\mathbf{4}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{6})$ and the spinors decompose according to $\mathbf{16}_+ \rightarrow (\mathbf{2}, \mathbf{4}) \oplus (\mathbf{\overline{2}}, \mathbf{\overline{4}})$ and $\mathbf{16}_- \rightarrow (\mathbf{2}, \mathbf{\overline{4}}) \oplus (\mathbf{\overline{2}}, \mathbf{4})$. We will be concerned mostly with the representation $\mathbf{16}_+$ for which the irreducible subspaces are just the eigenspaces of the matrix $-\Gamma_{0123} = \Gamma_{456789}$ with eigenvalues $\pm i$.

As in section 2, we will present the solution first using d = 10 notation. We demand that the supersymmetry parameter satisfies

$$\Gamma_{45}\epsilon = \Gamma_{67}\epsilon = \Gamma_{89}\epsilon. \tag{4.1}$$

This amounts to two linearly independent conditions, each of which halves the dimension

of the spinor, leaving us with four supersymmetries. We then define

$$v_{1} = \Gamma_{68}\epsilon = -\Gamma_{79}\epsilon, \qquad v_{2} = -\Gamma_{69}\epsilon = -\Gamma_{78}\epsilon,$$

$$v_{3} = \Gamma_{84}\epsilon = -\Gamma_{95}\epsilon, \qquad v_{4} = -\Gamma_{85}\epsilon = -\Gamma_{94}\epsilon,$$

$$v_{5} = \Gamma_{46}\epsilon = -\Gamma_{57}\epsilon, \qquad v_{6} = -\Gamma_{47}\epsilon = -\Gamma_{56}\epsilon,$$

$$v_{7} = -\Gamma_{45}\epsilon = -\Gamma_{67}\epsilon = -\Gamma_{89}\epsilon,$$

$$(4.2)$$

and claim that this is a solution of (1.3). These alternative expressions, which all follow from the condition (4.1) on ϵ , enable one to check this claim quite quickly. The idea is that by selecting particular expressions for each of the v_i from the possibilities given above one can easily see that all undesired terms in (1.3) take the form $\epsilon \Gamma_{\mu\nu\rho}\epsilon = 0$. (We noted earlier that this combination of gamma matrices is always antisymmetric in its spinor indices.)

The solution (4.2) is clearly SO(3,1) covariant but it is less obvious that the surviving subgroup of the SU(4) factor is U(3). It is possible, with some effort, to demonstrate this using d = 10 notation but we choose instead to expose this symmetry by passing to a complex basis in which spinors take the form $\chi^A \to (\chi^{\alpha a}, \bar{\chi}_{\dot{\alpha}a})$ and $\chi_A \to (\chi_{\alpha a}, \bar{\chi}^{\dot{\alpha}a})$. Here $\alpha, \dot{\alpha} = 1, 2$ label the **2** and $\bar{\mathbf{2}}$ of SO(3,1) and these indices can be raised and lowered according to $\chi^{\alpha} = \varepsilon^{\alpha\beta}\chi_{\beta}, \chi_{\beta} = \chi^{\alpha}\varepsilon_{\alpha\beta}, \bar{\chi}^{\dot{\alpha}} = \varepsilon^{\dot{\alpha}\dot{\beta}}\bar{\chi}_{\dot{\beta}}, \bar{\chi}_{\dot{\beta}} = \bar{\chi}^{\dot{\alpha}}\varepsilon_{\dot{\alpha}\dot{\beta}}$ where the antisymmetric symbols are defined by $\varepsilon^{12} = \varepsilon_{12} = 1$. The upper and lower indices a = 1, 2, 3, 4 label the **4** and $\bar{\mathbf{4}}$ of SU(4) and cannot be lowered or raised. In such a basis the gamma matrices have the block structure

$$(\Gamma_{\mu})^{AB} = \begin{pmatrix} 0 & (\sigma_{\mu})^{\alpha}{}_{\dot{\beta}}\delta^{a}{}_{b} \\ (\bar{\sigma}_{\mu})_{\dot{\alpha}}{}^{\beta}\delta_{a}{}^{b} & 0 \end{pmatrix} \qquad (\Gamma_{m+3})^{AB} = \begin{pmatrix} \varepsilon^{\alpha\beta}(\Sigma_{m})^{ab} & 0 \\ 0 & \varepsilon_{\dot{\alpha}\dot{\beta}}(\bar{\Sigma}_{m})_{ab} \end{pmatrix}$$
$$(\Gamma_{\mu})_{AB} = \begin{pmatrix} 0 & (\sigma_{\mu})_{\alpha}{}^{\dot{\beta}}\delta_{a}{}^{b} \\ (\bar{\sigma}_{\mu})^{\dot{\alpha}}{}_{\beta}\delta^{a}{}_{b} & 0 \end{pmatrix} \qquad (\Gamma_{m+3})_{AB} = \begin{pmatrix} \varepsilon_{\alpha\beta}(\bar{\Sigma}_{m})_{ab} & 0 \\ 0 & \varepsilon^{\dot{\alpha}\dot{\beta}}(\Sigma_{m})^{ab} \end{pmatrix}$$
$$(4.3)$$

where $\mu = 0, 1, 2, 3$ and m = 1, ..., 6. The matrices $(\sigma_{\mu})^{\alpha\dot{\beta}}$ and $(\bar{\sigma}_{\mu})_{\dot{\alpha}\beta}$ obey $\sigma_{\mu}\bar{\sigma}_{\nu} + \sigma_{\nu}\bar{\sigma}_{\mu} = -2\eta_{\mu\nu}$ and $(\sigma_{\mu})^{\alpha}{}_{\dot{\beta}} = (\bar{\sigma}_{\mu})_{\dot{\beta}}{}^{\alpha}$. The matrices $(\Sigma_m)^{ab}$ and $(\bar{\Sigma}_m)_{ab}$ are antisymmetric, they obey $\Sigma_m \bar{\Sigma}_n + \Sigma_n \bar{\Sigma}_m = -2\delta_{mn}$, and they can be taken to be complex conjugates of one another. Explicit constructions of both sets of matrices are given in the appendix.

Armed with this notation the condition (4.1) can be replaced by

$$\epsilon^{\alpha a} = \bar{\epsilon}_{\dot{\alpha}a} = 0, \qquad a = 1, 2, 3.$$
(4.4)

It is convenient to drop the label a = 4 on the only non-zero components of ϵ , writing these simply as ϵ^{α} and $\bar{\epsilon}_{\dot{\alpha}}$. Now (4.2) can be replaced by the expressions

$$(v_m)^{\alpha a} = (\Sigma_m)^{a4} \epsilon^{\alpha} , \qquad (\bar{v}_m)_{\dot{\alpha} a} = (\bar{\Sigma}_m)_{a4} \bar{\epsilon}_{\dot{\alpha}} , (v_7)^{\alpha 4} = i \epsilon^{\alpha} , \qquad (\bar{v}_7)_{\dot{\alpha} 4} = -i \bar{\epsilon}_{\dot{\alpha}} ,$$

$$(4.5)$$

with m = 1, ..., 6 and a = 1, 2, 3 and all other components zero. The alternative forms of the solution (4.1), (4.2) and (4.4), (4.5) can be shown to coincide using (4.3) and the

explicit expressions for Σ_m and Σ_m given in the appendix. It can also be checked directly that (4.4) and (4.5) provide a solution of (1.3).

The condition (4.4) breaks the Lorentz factor $SU(4) \rightarrow U(3) = SU(3) \times U(1)$ which is then preserved by the full solution (4.5). There is a similar pattern of breaking for the auxiliary internal symmetry $SO(7)_{aux} \rightarrow SU(4) \rightarrow SU(3) \times U(1)$ (where the first reduction occurs as a result of picking out the direction i = 7). Once again the surviving subgroups of the Lorentz and auxiliary symmetries are identified by the solution. For the reduction $SU(4) \rightarrow SU(3)$ the quantities $(\Sigma_m)^{a4}$ and $(\bar{\Sigma}_m)_{a4}$ are exactly the invariant tensors which describe the decomposition $\mathbf{6} \rightarrow \mathbf{3} \oplus \bar{\mathbf{3}}$, and so the SU(3) factor of the final invariance is manifest in (4.5). The U(1) factor is more subtle because it involves a non-trivial relative normalization between the two groups; we simply state below the final result for the various U(1) weights.

We have found a closed algebra of four supersymmetries with invariance group $SO(3, 1) \times SU(3) \times U(1)$ and representations

$$egin{aligned} &A_{\mu}:\, (\mathbf{4},\mathbf{1})^0\oplus(\mathbf{1},\mathbf{3})^{-2}\oplus(\mathbf{1},ar{\mathbf{3}})^2 \ &\psi:\, (\mathbf{2},\mathbf{1})^{-3}\oplus(ar{\mathbf{2}},\mathbf{1})^3\oplus(\mathbf{2},\mathbf{3})^1\oplus(ar{\mathbf{2}},ar{\mathbf{3}})^{-1} \ &G_i:\, (\mathbf{1},\mathbf{1})^0\oplus(\mathbf{1},\mathbf{3})^4\oplus(\mathbf{1},ar{\mathbf{3}})^{-4} \ &\epsilon:\, (\mathbf{2},\mathbf{1})^{-3}\oplus(ar{\mathbf{2}},\mathbf{1})^3 \end{aligned}$$

Given the residual symmetry group of this solution, it is natural to ask what happens if we perform a trivial dimensional reduction from d = 10 to d = 4. The answer is that we obtain off-shell N = 1 Yang-Mills in the Wess-Zumino gauge coupled to three chiral N = 1matter multiplets [2]. The components of the former can be taken to be A_{μ} ($\mu = 0, 1, 2, 3$), $\psi^{\alpha 4}$, G_7 , while the components of the latter can be taken to be $\phi^a = (\Sigma_m)^{a4} A_{m+3}, \psi^{\alpha a},$ $K^a = (\Sigma_m)^{a4} G_m$ (a = 1, 2, 3). The fields K^a are not quite the usual matter auxiliary fields (because their equations of motion are $K^a = 0$) but they are related to them in a simple way. In d = 4 the residual SU(3) × U(1) invariance becomes an internal symmetry with a rather nice interpretation. The SU(3) factor acts on the three chiral multiplets in an obvious way; the U(1) factor is an example of an *R*-symmetry and it can be checked that the weights given above can be obtained by applying the general prescription of [18] to this model.

5. Concluding remarks

In this paper we have clarified how off-shell algebras of conventional supersymmetries can exist within the framework of generalized supersymmetry. In doing so we have recovered all previously known ways of adding finite numbers of bosonic auxiliary fields so as to partially close the d = 10 superalgebra. The auxiliary fields in these solutions must eventually transform non-trivially under remnants of the d = 10 Lorentz group, even though they are introduced as SO(9,1) scalars. We have seen in each case how this is made possible by the existence of the internal auxiliary symmetry SO(7)_{aux}. Our results lend further weight to the idea of generalized supersymmetry introduced in [1] and provide strong motivation for its future study. It would be interesting to try and include non-propagating fermionic degrees of freedom so as to reproduce the more complicated sets of auxiliary fields given in [7,8,12]. It would also be very interesting to find some superspace description of generalized supersymmetry transformations with the exciting possibility that this might lead to new covariant actions for superparticles and superstrings.

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APPENDIX

Let τ_1, τ_2, τ_3 be the usual Pauli matrices. For the matrices λ_i of section 2 we can take

$$\begin{split} \lambda_1 &= i\tau_2 \otimes i\tau_2 \otimes i\tau_2 \,, \quad \lambda_2 &= \tau_1 \otimes i\tau_2 \otimes 1 \,, \quad \lambda_3 &= i\tau_2 \otimes 1 \otimes \tau_1 \,, \quad \lambda_4 &= -i\tau_2 \otimes 1 \otimes \tau_3 \,, \\ \lambda_5 &= 1 \otimes \tau_1 \otimes i\tau_2 \,, \qquad \lambda_6 &= -\tau_3 \otimes i\tau_2 \otimes 1 \,, \qquad \lambda_7 &= -1 \otimes \tau_3 \otimes i\tau_2 \,. \end{split}$$

For the matrices $(\sigma_{\mu})^{\alpha\dot{\alpha}}$ and $(\bar{\sigma}_{\mu})_{\dot{\alpha}\alpha}$ of section 4 we can take

$$\sigma^0 = \bar{\sigma}^0 = 1, \qquad \sigma_a = -\bar{\sigma}_a = \tau_a, \quad a = 1, 2, 3.$$

For the matrices $(\Sigma_m)^{AB}$ and $(\overline{\Sigma}_m)_{AB}$ of section 4 we can take

$\Sigma_1 = i\tau_2 \otimes \tau_1 ,$	$\Sigma_2 = -\tau_1 \otimes \tau_2 ,$	$\Sigma_3 = -i au_2 \otimes au_3 ,$
$\Sigma_4 = -\tau_2 \otimes 1,$	$\Sigma_5 = i1 \otimes au_2 ,$	$\Sigma_6 = \tau_3 \otimes \tau_2 ,$

with Σ_m the complex conjugate of Σ_m .

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