

tt^* equations, localization and exact chiral rings in 4d $\mathcal{N} = 2$ SCFTs

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ABSTRACT: We compute exact 2- and 3-point functions of chiral primaries in four-dimensional $\mathcal{N} = 2$ superconformal field theories, including all perturbative and instanton contributions. We demonstrate that these correlation functions are nontrivial and satisfy exact differential equations with respect to the coupling constants. These equations are the analogue of the tt^* equations in two dimensions. In the $SU(2)$ $\mathcal{N} = 2$ SYM theory coupled to 4 hypermultiplets they take the form of a semi-infinite Toda chain. We provide the complete solution of this chain using input from supersymmetric localization. To test our results we calculate the same correlation functions independently using Feynman diagrams up to 2-loops and we find perfect agreement up to the relevant order. As a spin-off, we perform a 2-loop check of the recent proposal of arXiv:1405.7271 that the logarithm of the sphere partition function in $\mathcal{N} = 2$ SCFTs determines the Kähler potential of the Zamolodchikov metric on the conformal manifold. We also present the tt^* equations in general $SU(N)$ $\mathcal{N} = 2$ superconformal QCD theories and comment on their structure and implications.

KEYWORDS: Supersymmetric gauge theory, Extended Supersymmetry, Nonperturbative Effects

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Contents

1	Introduction	2
2	Marginal deformations and the chiral ring	4
2.1	The chiral ring of $\mathcal{N} = 2$ theories	4
2.2	Marginal deformations	5
2.3	The exact Zamolodchikov metric from supersymmetric localization	6
3	tt^* equations in four-dimensional $\mathcal{N} = 2$ SCFTs	6
3.1	tt^* equations and the connection on the bundles of chiral primaries	6
3.2	Differential equations for 2- and 3-point functions of chiral primaries	10
3.3	Global issues	11
3.4	Solving the tt^* equations	12
3.5	Extremal correlators	12
3.6	$\mathcal{N} = 4$ theories	13
4	$\mathcal{N} = 2$ superconformal QCD as an instructive example	16
4.1	Definitions	16
4.2	SU(2) with 4 hypermultiplets	17
4.2.1	tt^* equations and exact 2- and 3-point functions	18
4.2.2	Perturbative expressions	19
4.2.3	Comments on $SL(2, \mathbb{Z})$ duality	21
4.3	SU(N) with $2N$ hypermultiplets	22
4.3.1	The structure of the SU(N) tt^* equations	23
4.3.2	3-point functions	24
5	Checks in perturbation theory	24
5.1	SU(2) SCQCD	25
5.1.1	Tree-level	25
5.1.2	Quantum corrections up to 2 loops	26
5.2	SU(N) SCQCD at tree level	31
5.3	SU(3) examples	31
5.4	SU(N) observations	32
6	Summary and prospects	34
A	Collection of useful facts about S^4 partition functions	35
B	Conventions in SU(N) $\mathcal{N} = 2$ SCQCD	36

1 Introduction

In this paper we are interested in four-dimensional theories with $\mathcal{N} = 2$ superconformal invariance. There are many well known examples of $\mathcal{N} = 2$ quantum field theories (with or without a known Lagrangian description) that exhibit manifolds of superconformal fixed points (specific examples will be discussed in the main text). Although particular neighborhoods of these manifolds can sometimes be described by a conventional weakly coupled Lagrangian, the generic fixed point is a superconformal field theory (SCFT) at finite or strong coupling. It is of considerable interest to determine how the physical properties of these theories vary as we change the continuous parameters (moduli) that parametrize these manifolds.¹ A well-studied maximally supersymmetric example with a (complex) one-dimensional conformal manifold is $\mathcal{N} = 4$ super-Yang-Mills (SYM) theory. Large classes of examples are also known in theories with minimal ($\mathcal{N} = 1$) supersymmetry (see e.g. [1]). Four-dimensional superconformal field theories with $\mathcal{N} = 2$ supersymmetry are particularly interesting because they are less trivial than the $\mathcal{N} = 4$ theories, but are considerably more tractable compared to the $\mathcal{N} = 1$ theories.

A particularly interesting subsector of $\mathcal{N} = 2$ dynamics is controlled by chiral primary operators. These are special operators in short multiplets annihilated by all supercharges of one chirality. They form a chiral ring structure under the operator product expansion (OPE). The exact dependence of this structure on the marginal coupling constants is currently a largely open interesting problem.

In two spacetime dimensions the application of the ‘topological anti-topological fusion’ method gives rise to a set of differential equations, called tt^* equations, which were employed successfully in the past [2, 3] to determine the coupling constant dependence of correlation functions in the $\mathcal{N} = (2, 2)$ chiral ring. An analogous set of tt^* equations in four-dimensional $\mathcal{N} = 2$ theories was formulated using superconformal Ward identities in [4].² In four dimensions, however, it is less clear how to solve these differential equations without further input.

More recently, a different line of developments has led to the proposal that the exact quantum Kähler potential on the $\mathcal{N} = 2$ superconformal manifold is given by the S^4 partition function of the theory [7]. The latter can be determined non-perturbatively with the use of localization techniques [8]. As a result, it is now possible to compute exactly the Zamolodchikov metric on the manifold of superconformal deformations of $\mathcal{N} = 2$ theories via second derivatives of the S^4 partition function. Equivalently, the two-point function of

¹The moduli of the conformal manifold in this paper should be distinguished from the moduli space of vacua, e.g. Coulomb or Higgs branch moduli, of a given conformal field theory.

²In a different direction, tt^* geometry techniques have also been applied to higher dimensional quantum field theories more recently in [5, 6].

scaling dimension 2 chiral primaries is expressed in terms of second derivatives of the S^4 partition function. We review the relevant statements in section 2.

In the present work we take a further step and argue that, when combined with the tt^* equations of [4], the exact Zamolodchikov metric is a very useful datum that leads to exact information about more general properties of the chiral ring structure of $\mathcal{N} = 2$ SCFTs. Specifically, it provides useful input towards an exact solution of the tt^* equations, which encodes the non-perturbative dependence of 2- and 3-point functions of chiral primary operators on the marginal couplings of the SCFT. In this solution, correlation functions of chiral primaries with scaling dimension greater than two are expressed in terms of more than two derivatives of the S^4 partition function. A review of the relevant concepts with the precise form of the tt^* equations is presented in section 3.

Such results can have wider implications. In subsection 3.5 we demonstrate that a solution of the 2- and 3-point functions in the $\mathcal{N} = 2$ chiral ring has immediate implications for a larger class of n -point ‘extremal’ correlation functions. Moreover, it is not unreasonable to expect that 2- and 3-point functions in the chiral ring may eventually provide useful input towards a more general solution of the theory using conformal bootstrap techniques.

In section 4 we demonstrate the power of these observations in an interesting well-known class of theories: $\mathcal{N} = 2$ superconformal QCD defined as $\mathcal{N} = 2$ SYM theory with gauge group $SU(N)$ coupled to $2N$ fundamental hypermultiplets. This theory has a (complex) one-dimensional manifold of exactly marginal deformations parametrized by the complexified gauge coupling constant $\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g_{YM}^2}$. For the $SU(2)$ theory, which has a single chiral ring generator, we demonstrate that the tt^* equations take the form of a semi-infinite Toda chain.³ Solving this chain in terms of the $SU(2)$ S^4 partition function provides the exact 2- and 3-point functions of the *entire* chiral ring. Unlike the $\mathcal{N} = 4$ SYM case, where these correlation functions are known not to be renormalized [9–18], in $\mathcal{N} = 2$ theories they turn out to have very nontrivial, and at the same time exactly computable, coupling constant dependence that we determine. In section 4 we also comment on the transformation properties of these results under $SL(2, \mathbb{Z})$ duality.

In the more general $SU(N)$ case, the presence of additional chiral ring generators makes the structure of the tt^* equations considerably more complicated. A recursive use of the tt^* equations is now less powerful and appears to require information beyond the Zamolodchikov metric (e.g. information about the exact 2-point functions of the additional chiral ring generators) which is not currently available. We present the $SU(N)$ tt^* equations and provide preliminary observations about their structure.

Independent evidence for these statements is provided in section 5 with a series of computations in perturbation theory up to two loops. Already at tree-level, agreement with the predicted results is a non-trivial exercise, where the generic correlation function comes from a straightforward, but typically involved, sum over all possible Wick contractions. We find evidence that there are compact expressions for general classes of tree-level correlation functions in the $SU(N)$ theory. The next-to-leading order contribution arises at two loops.

³We remind that in certain two-dimensional examples with $\mathcal{N} = (2, 2)$ supersymmetry the tt^* equations give a periodic Toda chain [3].

We provide an explicit 2-loop check for the general correlation function in the $SU(2) \mathcal{N} = 2$ superconformal QCD theory. As a by-product of this analysis we present a 2-loop check of a recently proposed relation [7] between the quantum Kähler potential on the superconformal manifold and the S^4 partition function.

Some of the wider implications of the tt^* equations and interesting open problems are discussed in section 6. Useful facts, conventions and more detailed proofs of several statements are collected for the benefit of the reader in four appendices at the end of the paper.

A companion note [19] contains a concise presentation of some of the main results of this work with emphasis on the $SU(2) \mathcal{N} = 2$ superconformal QCD theory.

2 Marginal deformations and the chiral ring

2.1 The chiral ring of $\mathcal{N} = 2$ theories

The R-symmetry of 4d $\mathcal{N} = 2$ SCFTs is $SU(2)_R \times U(1)_R$. We concentrate on (scalar) *chiral primary operators* defined as superconformal primary operators annihilated by all supercharges of one chirality. These operators belong to short multiplets of type “ $\mathcal{E}_{\frac{R}{2}(0,0)}$ ” in the notation of [20].⁴ As was shown there, these must be singlets of the $SU(2)_R$ and must have nonzero charge R under $U(1)_R$. We work in conventions⁵ where the unitarity bound is

$$\Delta \geq \frac{|R|}{2}. \tag{2.1}$$

Superconformal primaries saturating the bound $\Delta = \frac{R}{2}$ are annihilated by all right-chiral supercharges \bar{Q}_α^i . We call them chiral primaries and denote them by ϕ_I . Their conjugate, which obey $\Delta = -\frac{R}{2}$, are annihilated by Q_α^i . We call them anti-chiral primaries and denote them as $\bar{\phi}_I$. We write the 2-point functions of chiral primaries as

$$\langle \phi_I(x) \bar{\phi}_J(0) \rangle = \frac{g_{I\bar{J}}}{|x|^{2\Delta}}. \tag{2.2}$$

By the symbol $g^{\bar{J}I}$ we denote the inverse matrix i.e. $g_{I\bar{J}} g^{\bar{J}K} = \delta_I^K$.

It is well known that the OPE of chiral primaries is non-singular

$$\phi_I(x) \phi_J(0) = C_{IJ}^K \phi_K(0) + \dots, \tag{2.3}$$

where ϕ_K is also chiral primary and C_{IJ}^K are the chiral ring OPE coefficients [22]. We also define the 3-point function of chiral primaries

$$\langle \phi_I(x) \phi_J(y) \bar{\phi}_K(z) \rangle = \frac{C_{IJK}}{|x-y|^{\Delta_I+\Delta_J-\Delta_K} |x-z|^{\Delta_I+\Delta_K-\Delta_J} |y-z|^{\Delta_J+\Delta_K-\Delta_I}}, \tag{2.4}$$

⁴For an interesting recent discussion of other higher-spin chiral primary operators see [21].

⁵In these conventions the supercharges Q_α^i have $U(1)_R$ charge equal to -1 and \bar{Q}_α^i have $+1$. The $\alpha, \dot{\alpha}$ are Lorentz spinor indices, while the i is an $SU(2)_R$ index.

and we have the obvious relation between OPE and 3-point coefficients

$$C_{IJK} = C_{IJ}^L g_{L\bar{K}} . \tag{2.5}$$

So far we have defined the chiral ring for one particular $\mathcal{N} = 2$ SCFT. In general, such SCFTs may have exactly marginal coupling constants. In that case the elements of the chiral ring (i.e. the corresponding 2- and 3-point functions) will become functions of the coupling constants. The goal of our paper is to analyze this (typically non-trivial) coupling-constant dependence of the chiral ring.

2.2 Marginal deformations

We are interested in $\mathcal{N} = 2$ SCFTs with exactly marginal deformations. We parametrize the space of marginal deformations (conformal manifold), called \mathcal{M} from now on, by complex coordinates $\lambda^i, \bar{\lambda}^i$. Under an infinitesimal marginal deformation the action changes by

$$S \rightarrow S + \frac{\delta\lambda^i}{4\pi^2} \int d^4x \mathcal{O}_i(x) + \frac{\delta\bar{\lambda}^i}{4\pi^2} \int d^4x \bar{\mathcal{O}}_i(x) . \tag{2.6}$$

It can be shown that the marginal deformation preserves $\mathcal{N} = 2$ superconformal invariance, if and only if the marginal operators are descendants of (anti)-chiral primaries with $\Delta = 2$ and $R = \pm 4$, more specifically

$$\mathcal{O}_i = \mathcal{Q}^4 \cdot \phi_i , \quad \bar{\mathcal{O}}_i = \bar{\mathcal{Q}}^4 \cdot \bar{\phi}_i , \tag{2.7}$$

where ϕ_i is chiral primary of charge $R = 4$. The notation $\mathcal{O}_i = \mathcal{Q}^4 \cdot \phi_i$ means that \mathcal{O}_i can be written as the nested (anti)-commutator of the four supercharges of left chirality. Their Lorentz and $SU(2)_R$ indices of the supercharges are combined to give a Lorentz and $SU(2)_R$ singlet. The overall normalization of factors of 2 etc. is fixed so that equation (2.10) holds. Notice that since the \mathcal{Q} 's have $U(1)_R$ charge equal to -1 the marginal operators are $U(1)_R$ neutral, as they should.

From now on in this section and the next we use lowercase indices i, j, \dots to indicate chiral primaries of R-charge equal to ± 4 . These are special since, via (2.7), they correspond to marginal deformations. We use uppercase indices I, J, \dots to denote general chiral primaries of any R -charge.

The Zamolodchikov metric is defined by the 2-point function⁶

$$\langle \mathcal{O}_i(x) \bar{\mathcal{O}}_j(0) \rangle = \frac{G_{i\bar{j}}}{|x|^8} . \tag{2.8}$$

The conformal manifold \mathcal{M} equipped with this metric is a complex Kähler manifold (possibly with singularities). The corresponding “metric” for the chiral primaries is

$$\langle \phi_i(x) \bar{\phi}_j(0) \rangle = \frac{g_{i\bar{j}}}{|x|^4} . \tag{2.9}$$

⁶Notice that 2-point functions of the form $\langle \mathcal{O}_i \mathcal{O}_j \rangle$ or $\langle \bar{\mathcal{O}}_i \bar{\mathcal{O}}_j \rangle$ are zero, as can be easily shown by superconformal Ward identities.

We define the normalization of (2.7) in such a way that $\langle \mathcal{O}_i(x) \overline{\mathcal{O}}_j(0) \rangle = \nabla_x^2 \nabla_x^2 \langle \phi_i(x) \overline{\phi}_j(0) \rangle$, which implies

$$g_{i\bar{j}} = \frac{G_{i\bar{j}}}{192} . \tag{2.10}$$

2.3 The exact Zamolodchikov metric from supersymmetric localization

In [7] it was shown that the partition function of an $\mathcal{N} = 2$ theory on the four-sphere S^4 , regulated in a scheme that preserves the massive supersymmetry algebra $OSp(2|4)$, computes the Kähler potential for the Zamolodchikov metric. The result is

$$G_{i\bar{j}} = \partial_i \partial_{\bar{j}} \mathcal{K} , \tag{2.11}$$

where⁷

$$\mathcal{K} = 192 \log Z_{S^4} . \tag{2.12}$$

Combining this result with (2.10) we conclude that

$$g_{i\bar{j}} = \partial_i \partial_{\bar{j}} \log Z_{S^4} . \tag{2.13}$$

The partition function Z_{S^4} can be computed exactly for a certain class of $\mathcal{N} = 2$ SCFTs, using supersymmetric localization [8]. Via (2.13) this immediately provides the 2-point functions of chiral primaries with scaling dimension $\Delta = 2$.

Our strategy will be to use these 2-point functions and the tt^* equations that we derive in the following section to compute the 2-point functions of chiral primaries of higher R-charge. In turn, this will allow us to compute the *exact, non-perturbative* 3-point functions of chiral primaries over the conformal manifold.

3 tt^* equations in four-dimensional $\mathcal{N} = 2$ SCFTs

In this section we review the analogue of the tt^* equations for 4d $\mathcal{N} = 2$ SCFTs, which were derived in [4]. We omit proofs, which can be found there.

3.1 tt^* equations and the connection on the bundles of chiral primaries

We parametrize the conformal manifold \mathcal{M} by complex coordinates $\lambda^i, \bar{\lambda}^i$. In general, the chiral primary 2- and 3-point functions are non-trivial functions of the coupling constants. In order to discuss the coupling constant dependence of correlators we have to address issues related to operator mixing. This mixing is an intrinsic property of the theory, similar to the (in general, non-abelian) Berry phase, which appears in perturbation theory in Quantum Mechanics.⁸ The operator mixing in conformal perturbation theory has been discussed in several earlier works, here we mention those that are most relevant for our approach [4, 23–28].

⁷In [7] the marginal operators are normalized in a different way, namely $\mathcal{O}_{here} = 4\mathcal{O}_{there}$, so various coefficients have been adjusted accordingly. For instance this explains the factor $192 = 12 \times 4 \times 4$ as opposed to 12 in [7].

⁸In fact, by considering the state-operator map, it becomes possible to relate more precisely the connection on the space of operators to the Berry phase of quantum states of the CFT on $S^3 \times \text{time}$.

In order to describe the operator mixing, it is useful to think of local operators as being associated to vector bundles over the conformal manifold. These bundles are equipped with a natural connection that we denote by $(\nabla_\mu)_K^L = \delta_K^L \partial_\mu + (A_\mu)_K^L$. This connection encodes the mixing of operators with the same quantum numbers under conformal perturbation theory. The curvature of this connection can be defined in terms of an integrated 4-point function in conformal perturbation theory, by the expression

$$(F_{\mu\nu})_K^L \equiv [\nabla_\mu, \nabla_\nu]_K^L = \frac{1}{(2\pi)^2} \int d^4x d^4y \langle \phi^L(\infty) \mathcal{O}_{[\mu}(x) \mathcal{O}_{\nu]}(y) \phi_K(0) \rangle . \quad (3.1)$$

The index L is raised with the inverse of the matrix of 2-point functions. The reason that the r.h.s. is not identically zero, despite the antisymmetrization in the indices μ, ν , is that the integral on the r.h.s. has to be regularized to remove divergences from coincident points. The need for regularization is one way to understand why we end up with nontrivial operator mixing. A very thorough explanation of the regularization procedure needed to do the double integral is given in [27].⁹

In the case of $\mathcal{N} = 2$ SCFTs, and when considering operators in the chiral ring, this double integral can be dramatically simplified, given that the marginal operators are descendants of chiral primaries of the form $\mathcal{O}_i = \mathcal{Q}^4 \cdot \phi_i$ and similarly for the antiholomorphic deformations. As was shown in [4], we can use the superconformal Ward identities to move the supercharges from one insertion to the other, and using the SUSY algebra $\{\mathcal{Q}_\alpha^i, \bar{\mathcal{Q}}_\beta^j\} = 2P_{\alpha\beta} \delta^{ij}$ repeatedly, we get derivatives inside the integral. Then, by integrations by parts the integral simplifies drastically, and only picks up contributions which are determined by chiral ring 2- and 3-point functions and the CFT central charge c . The interested reader should consult [4] for details. The final result is that in $\mathcal{N} = 2$ SCFTs the curvature of bundles of chiral primaries is given by

$$[\nabla_i, \nabla_j]_K^L = [\bar{\nabla}_i, \bar{\nabla}_j]_K^L = 0, \quad (3.2a)$$

$$[\nabla_i, \bar{\nabla}_j]_K^L = -[C_i, \bar{C}_j]_K^L + g_{i\bar{j}} \delta_K^L \left(1 + \frac{R}{4c} \right) . \quad (3.2b)$$

The equations on the first line express the fact that the bundles of chiral primaries are (at least locally¹⁰) holomorphic vector bundles over the conformal manifold.

In the second line, R is the $U(1)_R$ charge of the bundle, c the central charge of the CFT and $g_{i\bar{j}}$ is the 2- point function of chiral primaries of $\Delta = 2$, whose descendants are the marginal operators (2.7). These equations are the analogue of the tt^* equations derived in [2] for the Berry phase of the Ramond ground states and the chiral ring of $\mathcal{N} = (2, 2)$ theories in two dimensions.

⁹In [27] only 2d CFTs are discussed but several of their statements can be generalized to 4d conformal perturbation theory.

¹⁰From now on, whenever we say ‘holomorphic bundle’, ‘holomorphic section’, ‘holomorphic function’ these terms should be understood in the sense of ‘locally holomorphic’, since the equations we derived are local and we have not analyzed global issues. There may be obstructions in extending the holomorphic dependence globally.

Moreover, it can be shown [4] that the OPE coefficients of chiral primaries are covariantly holomorphic

$$\bar{\nabla}_j C_{JK}^I = 0 \tag{3.3}$$

and that OPE coefficients obey the analogue of the WDVV equations [29–31] which have the form

$$\nabla_i C_{jK}^L = \nabla_j C_{iK}^L . \tag{3.4}$$

Here, and according to our notation, the indices i, j run over the marginal deformations, while K, L , can be any chiral primary.

Finally, the supercharges and supercurrents are associated to a holomorphic line bundle \mathcal{L} over the conformal manifold, whose curvature is given by¹¹

$$\begin{aligned} F_{ij} &= F_{\bar{i}\bar{j}} = 0 , \\ F_{\bar{i}\bar{j}} &= \frac{1}{4c} g_{\bar{i}\bar{j}} . \end{aligned} \tag{3.5}$$

The bundle \mathcal{L} encodes the ambiguity of redefining the phases of the supercharges as $Q_\alpha^i \rightarrow e^{i\theta} Q_\alpha^i$ and $\bar{Q}_\alpha^i \rightarrow e^{-i\theta} \bar{Q}_\alpha^i$ (the superconformal generators transform as $S \rightarrow e^{-i\theta} S$ and $\bar{S} \rightarrow e^{i\theta} \bar{S}$, while the bosonic generators remain invariant). It is clear that this transformation is an automorphism of the $\mathcal{N} = 2$ superconformal algebra. The equations (3.5) are saying that in the natural connection defined by conformal perturbation theory, the choice of this phase varies as we move on the conformal manifold. As we see from (3.5) the curvature of the corresponding bundle \mathcal{L} is proportional to the Kähler form of the Zamolodchikov metric.

The statements above are covariant in the sense that they hold independent of how we select the normalization/basis of chiral primaries as a function of the coupling constants. However, it is more practical to select a particular scheme, where we will see that the equations above reduce to standard partial differential equations for the 2- and 3-point functions, without any reference to the connection A on the bundles.

A natural choice would be to select a basis of chiral primaries over the conformal manifold that consists of holomorphic sections of the corresponding bundles. Furthermore, from (3.2a) we see that it is possible to go to a holomorphic gauge $(A_{\bar{j}})^I_K = 0$, where $\nabla_{\bar{j}} = \partial_{\bar{j}}$. In this gauge, the condition (3.3) simply becomes $\partial_{\bar{j}} C_{JK}^I = 0$, so the OPE coefficients are holomorphic functions of the couplings. Let us denote the chiral primaries in the gauge where they are holomorphic sections as ϕ'_I and the corresponding 2-point functions as $\langle \phi'_I \bar{\phi}'_J \rangle = g'_{I\bar{J}}$. In terms of these holomorphic sections, the curvature of the underlying holomorphic bundles can be simply expressed as

$$[\nabla_i, \bar{\nabla}_j]^L_K = -\partial_{\bar{j}}(g'^{ML} \partial_i g'_{K\bar{M}}) , \tag{3.6}$$

and there is no longer any explicit dependence on the connection A . Here we used the compatibility of the connection and the metric on the bundle, see [27] for explanations.

¹¹This can be shown [4] by considering the general formula (3.1) and applying it to the case where the operators ϕ_K, ϕ^L are the supercurrents. Since $[\text{supercharge}] = \int d^3x [\text{supercurrent}]_0$ it is clear that the holonomy (phase) that the supercharges pick up under conformal perturbation theory is the same as that of the supercurrents.

We could continue working with these holomorphic sections, but we need to pay attention to the following technical detail. The marginal operators \mathcal{O}_i can be related to the chiral primaries ϕ'_i with $\Delta = 2$ by an expression of the form $\mathcal{O}_i = \mathcal{Q}'^4 \cdot \phi'_i$. The supercharges \mathcal{Q}' can be viewed as sections of the holomorphic bundle \mathcal{L} mentioned in equations (3.5). Having chosen a convention for \mathcal{O}_i and ϕ'_i we have also chosen the conventions for the section \mathcal{Q}' . Assuming \mathcal{O}_i is holomorphic (from (2.6)), the above choice of the holomorphic section ϕ'_i implies that \mathcal{Q}' is a holomorphic section of \mathcal{L} . These conventions for the supercharges are not the standard ones following from the supersymmetry algebra. In the standard conventions, although the overall phase of the supercharges can be redefined in a coupling-constant dependent way due to the U(1) automorphism of the algebra, the “magnitude” of the normalization of the supercharges is fixed in order to satisfy the standard supersymmetry algebra $\{\mathcal{Q}_\alpha^i, \overline{\mathcal{Q}}_\beta^j\} = 2P_{\alpha\beta}\delta^{ij}$. Equivalently, the normalization of the 2-point function of the corresponding supercurrents is independent of the coupling constant. Since the supercharges \mathcal{Q} with this standard choice have constant magnitude, they cannot be a holomorphic section of the bundle \mathcal{L} .¹² Hence, the standard \mathcal{Q} and the \mathcal{Q}' above are different types of sections. What is the precise relation between them?

Equation (3.5) implies that the combination

$$\mathcal{Q}' = e^{\frac{\mathcal{K}}{c'}} \mathcal{Q}, \quad c' = 8 \times 192 \times c \tag{3.7}$$

can be a holomorphic section for an appropriate choice of the (coupling-constant dependent) phase of \mathcal{Q} . \mathcal{K} is the Kähler potential of the Zamolodchikov metric. Notice that the appropriate choice of the phase of \mathcal{Q} depends on the choice of Kähler gauge. Under a Kähler transformation, $\mathcal{K} \rightarrow \mathcal{K} + f + \bar{f}$ (where f (\bar{f}) is (anti)holomorphic), the section \mathcal{Q}' in (3.7) becomes

$$e^{\frac{2f}{c'}} e^{i\frac{2\text{Im}f}{c'}} \mathcal{Q}' .$$

There is an overall holomorphic factor $e^{\frac{2f}{c'}}$ and the original phase of \mathcal{Q} has been shifted. With these specifications (3.7) is the relation between \mathcal{Q} and \mathcal{Q}' that we are looking for.

This suggests the following choice of conventions: select chiral primaries ϕ_I at any level of R -charge R so that $\phi'_I = e^{-\frac{R}{c'}\mathcal{K}}\phi_I$ are holomorphic sections. Equivalently, if we have already a choice of holomorphic sections ϕ'_I (as above), then we define a new non-holomorphic basis by $\phi_I = e^{\frac{R}{c'}\mathcal{K}}\phi'_I$.¹³ The corresponding 2-point functions obey the relation $g_{I\bar{J}} = e^{\frac{2R}{c'}\mathcal{K}}g'_{I\bar{J}}$. This choice ensures that $\mathcal{O}_i = \mathcal{Q}'^4 \cdot \phi'_i = \mathcal{Q}^4 \cdot \phi_i$, where \mathcal{Q} are supercharges with the standard normalization. The non-holomorphicity of ϕ_i precisely cancels the non-holomorphicity of \mathcal{Q} . In addition, the general OPE coefficients are the same in the two bases, $C^I_{JK} = C^{I'}_{J'K'}$, as a consequence of R -charge conservation.

¹²Had they been holomorphic sections with constant magnitude, we would conclude from (3.6) that the curvature of \mathcal{L} is zero, which is inconsistent with the direct computation leading to (3.5).

¹³Again, this definition of ϕ_I depends on the Kähler gauge and the resulting 2-point function $g_{I\bar{J}}$ transforms as $g_{I\bar{J}} \rightarrow e^{\frac{2R}{c'}(f+\bar{f})}g_{I\bar{J}}$ under Kähler transformations. Happily, this dependence drops out of the final equation (3.10), which is indeed invariant under Kähler transformations. We are grateful to M. Buican, for discussions which led us to an investigation of the invariance of our statements under Kähler transformations.

In the ϕ_I -basis the curvature of the bundles becomes

$$[\nabla_i, \bar{\nabla}_j]_K^L = -\partial_{\bar{j}}(g'^{\bar{M}L} \partial_i g'_{K\bar{M}}) = -\partial_{\bar{j}}(g^{\bar{M}L} \partial_i g_{K\bar{M}}) + \frac{R}{4c} g_{i\bar{j}} \delta_K^L. \quad (3.8)$$

Inserting into (3.2b) we obtain the partial differential equations¹⁴

$$\partial_{\bar{j}}(g^{\bar{M}L} \partial_i g_{K\bar{M}}) = [C_i, \bar{C}_j]_K^L - g_{i\bar{j}} \delta_K^L. \quad (3.9)$$

3.2 Differential equations for 2- and 3-point functions of chiral primaries

The result of this choice of gauge (scheme) is that the tt^* equations reduce to differential equations for the 2- and 3-point functions, where there is no explicit appearance of the connection on the bundles. For the sake of clarity we summarize here the detailed form of the equations with all indices written out

$$\frac{\partial}{\partial \lambda^{\bar{j}}} \left(g^{\bar{M}L} \frac{\partial}{\partial \lambda^i} g_{K\bar{M}} \right) = C_{iK}^P g_{P\bar{Q}} C_{\bar{j}\bar{R}}^{*\bar{Q}} g^{\bar{R}L} - g_{K\bar{N}} C_{\bar{j}\bar{U}}^{*\bar{N}} g^{\bar{U}V} C_{iV}^L - g_{i\bar{j}} \delta_K^L. \quad (3.10)$$

As we can see these differential equations relate the coupling constant dependence of 2- and 3-point functions of various chiral primaries. They have to be supplemented by equation (3.3), which in this gauge takes the simpler form

$$\frac{\partial}{\partial \lambda^{\bar{j}}} C_{IJ}^K = 0, \quad (3.11)$$

and the WDVV equations (3.4)

$$\frac{\partial C_{jK}^L}{\partial \lambda^i} - \frac{\partial C_{iK}^L}{\partial \lambda^j} = g^{\bar{Q}L} \partial_i g_{P\bar{Q}} C_{jK}^P - C_{jP}^L g^{\bar{Q}P} \partial_i g_{K\bar{Q}} - (i \leftrightarrow j). \quad (3.12)$$

In the examples that we will study later the conformal manifold is 1-(complex) dimensional, hence the WDVV equations are trivially obeyed and that is why we do not discuss them any further. In other $\mathcal{N} = 2$ theories with higher dimensional conformal manifolds they may be nontrivial.

Let us elaborate a little further on the notation in equation (3.10). The lowercase indices i, \bar{j} run over (anti)-chiral primaries of $\Delta = 2, R = \pm 4$, or equivalently, over the marginal directions along the conformal manifold. We remind that chiral primaries of $R = \pm 4$ and dimension $\Delta = 2$ are those whose descendants are the marginal operators corresponding to $\lambda^i, \lambda^{\bar{j}}$ on the l.h.s.. The capital indices run over general chiral primaries of any R-charge. These equations can be applied for each possible sector of chiral primaries. The function $g_{K\bar{M}}$ is the 2-point function of chiral primaries of charge R . The OPE coefficients C_{iK}^P relate the chiral primaries of charge R (corresponding to the index K) to the chiral primaries of charge $R + 4$ (corresponding to the index P). The indices U, V correspond to chiral primaries of charge $R - 4$. Finally by $C_{\bar{j}\bar{R}}^{*\bar{Q}}$ we mean $(C_{jR}^Q)^*$.

¹⁴The reader familiar with the 2d tt^* equations should notice that the last term $-g_{i\bar{j}} \delta_K^L$ can be effectively removed by a slight redefinition, see the discussion around (4.9) for an example.

Remark on the curvature of the Zamolodchikov metric. If we consider equation (3.10) specifically for the bundle of chiral primaries of R-charge 4 (whose descendants are the marginal operators) and using (2.10) and the general formula for the Riemann tensor of a Kähler manifold we get the equation

$$R_{i\bar{j}k}^l = -C_{ik}^M g_{M\bar{N}} C_{j\bar{q}}^{*N} g^{\bar{q}l} + g_{k\bar{j}} \delta_i^l + g_{i\bar{j}} \delta_k^l \tag{3.13}$$

We notice that the curvature of the conformal manifold obeys an equation, which is reminiscent of the one for the moduli space of 2d $\mathcal{N} = (2, 2)$ SCFTs with general values of the central charge, as some sort of generalization of special geometry [25, 26].

Note on normalization conventions. We emphasize once again that the differential equations (3.10) hold in a particular choice of normalization conventions described near the end of section 3.1. The benefit of this choice is that it allows us to circumvent the details of a non-trivial connection on the chiral primary bundles. These normalization conventions are typically different from the more common ones in conformal field theory where one diagonalizes the 2-point functions of conformal primary fields,

$$\langle \phi_K(x) \bar{\phi}_L(0) \rangle = \frac{\delta_{K\bar{L}}}{|x|^{2\Delta}}. \tag{3.14}$$

In the conventions (3.14) the OPE coefficients C_{IJ}^K are no longer holomorphic functions of the marginal couplings and therefore do not obey (3.11) (but they still obey (3.3)).

In the examples of section 4 a natural basis of chiral primaries will lead to the holomorphic gauge of equation (3.11). Once there is a solution of the tt^* equations in this basis, it is not hard to rotate to the more conventional basis (3.14).

3.3 Global issues

When studying the equations (3.10) it is important and interesting to explore certain global issues¹⁵ of the bundles of chiral primaries over the conformal manifold \mathcal{M} . The equations are *local*, since they were derived in conformal perturbation theory, but the conformal manifold may have special points (e.g. the weak coupling point $g_{YM} = 0$) and nontrivial topology like in the class \mathcal{S} theories [32, 33], where the conformal manifold is related to the moduli space of punctured Riemann surfaces. Because of these global issues, it is conceivable that in certain theories, the connection on the space of operators is not entirely determined by the local curvature expression (3.10), but there may be additional “Wilson line”-like configurations around the special points/nontrivial cycles on the conformal manifold. Moreover, whether we can find global holomorphic sections or not and if we can set $\bar{\partial}C = 0$ globally, may be a nontrivial question. In this paper, since we are dealing mostly with the simpler superconformal QCD theories, we will not go into these global issues but we are planning to return to them in future work.

¹⁵We are grateful to M. Buican for discussions on this.

3.4 Solving the tt^* equations

The resulting equations (3.10) are a set of coupled differential equations for the 2- and 3- point functions of chiral primaries. In certain 2d $\mathcal{N} = (2, 2)$ QFTs the tt^* equations could be solved [2, 3] just from the requirement that the 2-point functions must be positive and from knowing the correlators in the weak coupling region. For this to work it was important that the chiral ring in 2d is finite dimensional. For example, in $\mathcal{N} = (2, 2)$ SCFTs a unitarity bound constrains the R-charge by $|q| \leq \frac{c}{3}$, which shows that in theories with reasonable spectrum the chiral ring is truncated. In 4d $\mathcal{N} = 2$ SCFTs the chiral ring has no known upper bound in R-charge and if we try to apply these equations we end up with an infinite set of coupled differential equations. For instance, while in certain 2d examples one gets equations corresponding to the periodic Toda chain [3], in 4d $\mathcal{N} = 2$ SCFTs we find equations similar to the semi-infinite Toda-chain (this will become more clear in section 4). Unlike what happened to 2d examples [2, 3], we have not been able to find a way to uniquely determine a solution of these equations, just from the requirement of positivity of the 2-point functions and the boundary conditions at weak coupling.

On the other hand, in certain 4d $\mathcal{N} = 2$ SCFTs, these equations have a recursive structure: if we somehow fix the coupling constant dependence of the lowest nontrivial chiral primaries, then the equations predict the 2- and 3- point functions of higher-charge chiral primaries. As we explained in section 2, the 2-point functions of chiral primaries of R-charge 4, are proportional to the Zamolodchikov metric on the conformal manifold.

Hence, if we knew the exact Zamolodchikov metric as a function of the coupling, we would also know the 2-point function of chiral primaries of R-charge 4, and then by plugging this into the sequence of tt^* equations we would be able to compute the 2- and 3-point functions of an infinite number of other chiral primaries. Progress in this direction becomes possible after the recent proposal [7], which relates the partition function of $\mathcal{N} = 2$ SCFTs on S^4 computed by localization in the work of Pestun [8], to the Kähler potential of the Zamolodchikov metric on the moduli space.

While this strategy allows us to partly solve the tt^* equations, it would be interesting to explore whether it is possible to determine the relevant solution of these equations without input from localization. This could perhaps be possible by demanding positivity of all 2-point functions of chiral primaries over the conformal manifold supplemented by some weak coupling perturbative data, in analogy to what was done in [3]. This is a very speculative possibility, which if true, would in principle lead to an alternative computation of the nontrivial information encoded in the sphere partition function, without the use of localization. We plan to investigate this further in future work.

3.5 Extremal correlators

By computing the 2- and 3-point functions of chiral primaries we can also get exact results for more general “extremal correlators”. These are correlators of the form

$$\langle \phi_{I_1}(x_1) \dots \phi_{I_n}(x_n) \bar{\phi}_J(y) \rangle, \tag{3.15}$$

where ϕ_{I_k} are chiral primaries and $\bar{\phi}_J$ is antichiral, with R-charges related as $R_J = -\sum_k R_{I_k}$.

First, it is convenient to use a conformal transformation of the form

$$x^{\mu'} = \frac{x^\mu - y^\mu}{|x - y|^2} \tag{3.16}$$

to write the correlator as

$$\langle \phi_{I_1}(x_1) \dots \phi_{I_n}(x_n) \bar{\phi}_J(y) \rangle = \frac{\langle \phi_{I_1}(x'_1) \dots \phi_{I_n}(x'_n) \bar{\phi}_J(\infty) \rangle}{|x_1 - y|^{2\Delta_1} \dots |x_n - y|^{2\Delta_n}}, \tag{3.17}$$

where the x' 's on the r.h.s. are related to x 's by (3.16).

For an extremal correlator in $\mathcal{N} = 2$ SCFT, the superconformal Ward identities imply that

$$\langle \phi_{I_1}(x_1) \dots \phi_{I_n}(x_n) \bar{\phi}_J(\infty) \rangle \tag{3.18}$$

is independent of the positions x_i . Consequently, we are free to evaluate it in any particular limit. Let us define a new chiral primary ϕ_I by fusing together all the chiral primaries

$$\phi_I(0) \equiv \lim_{\{x_i\} \rightarrow 0} \phi_{I_1}(x_1) \times \dots \times \phi_{I_n}(x_n), \tag{3.19}$$

where the symbol \times refers to an OPE. Notice that, since all operators are chiral primaries, this multi-OPE is non-singular and associative, so the limit is well defined and it is simply given by a chiral primary ϕ_I of charge $R_I = \sum_k R_{I_k}$. Then we find that

$$\langle \phi_{I_1}(x_1) \dots \phi_{I_n}(x_n) \bar{\phi}_J(\infty) \rangle = \langle \phi_I(0) \bar{\phi}_J(\infty) \rangle = g_{I\bar{J}}, \tag{3.20}$$

where on the last step we got the usual 2-point functions of chiral primaries (2.2). Due to the associativity of the chiral ring we can also write

$$g_{I\bar{J}} = C_{I_1 I_2}^{M_1} C_{M_1 I_3}^{M_2} \dots C_{M_{n-2} I_n}^{M_{n-1}} g_{M_{n-1} \bar{J}} \tag{3.21}$$

Re-instating the full coordinate dependence from (3.17), we can write the following formula for extremal correlators

$$\langle \phi_{I_1}(x_1) \dots \phi_{I_n}(x_n) \bar{\phi}_J(y) \rangle = \frac{g_{I\bar{J}}}{|x_1 - y|^{2\Delta_1} \dots |x_n - y|^{2\Delta_n}}. \tag{3.22}$$

So according to our argument, extremal correlators can be uniquely determined by the chiral ring 2- and 3-point functions, which were used in formulae (3.19) (OPE coefficients) and (3.20) (2-point functions).

3.6 $\mathcal{N} = 4$ theories

Until this point we considered general theories with $\mathcal{N} = 2$ supersymmetry. It is interesting to ask parenthetically how the formalism captures the properties of $\mathcal{N} = 4$ theories. An $\mathcal{N} = 4$ theory can also be written as an $\mathcal{N} = 2$ theory, so our formalism should apply. The R-symmetry $SU(2)_R \times U(1)_R$ of the $\mathcal{N} = 2$ viewpoint, is embedded inside the underlying $SO(6)_R$ of the full $\mathcal{N} = 4$ theory. We proceed to flesh out the pertinent details and verify that the tt^* equations work correctly in $\mathcal{N} = 4$ theories.

Consider an $\mathcal{N} = 4$ gauge theory with semi-simple gauge group \mathcal{G} . The theory has 6 real scalars $\Phi^I, I = 1, \dots, 6$. It is useful to define the complex combination

$$\varphi = \Phi^1 + i\Phi^2 \tag{3.23}$$

which is the bottom component of an $SU(3)$ highest weight $\mathcal{N} = 1$ superfield. The $U(1)_R$ symmetry that rotates this field corresponds to rotations on the 1-2 plane. The chiral primary, whose descendant is the $\mathcal{N} = 4$ marginal operator, has the form

$$\phi_2 \propto \text{Tr}[\varphi^2] . \tag{3.24}$$

From the $\mathcal{N} = 4$ viewpoint this is the superconformal primary of the $\frac{1}{2}$ -BPS short representation of $\mathcal{N} = 4$ which contains, among other operators, the R-symmetry currents, stress tensor and marginal operators.

General chiral primaries of charge R in $\frac{1}{2}$ -BPS representations can be deduced from multitrace operators of the form

$$\phi_K \propto \text{Tr}[\varphi^{n_1}] \dots \text{Tr}[\varphi^{n_k}], \tag{3.25}$$

where $2 \sum n_i = R$. The trace is taken in the adjoint of \mathcal{G} .

The conformal manifold of this theory is parametrized by the complexified coupling

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g_{YM}^2} \tag{3.26}$$

up to global identifications due to S-duality transformations. θ denotes the θ -angle and g_{YM} the Yang-Mills coupling. An important point is that for $\mathcal{N} = 4$ theories the Zamolodchikov metric on the conformal manifold does not receive any quantum corrections and in our conventions is equal to

$$G_{\tau\bar{\tau}}^{\mathcal{N}=4} = 96 \frac{c}{\text{Im}\tau^2} . \tag{3.27}$$

This means that the conformal manifold is locally a two-dimensional homogeneous space of constant negative curvature. The marginal operators $\mathcal{O}_\tau, \bar{\mathcal{O}}_\tau$ can be thought of as holomorphic and antiholomorphic tangent vectors to the conformal manifold. Since the manifold (3.27) has nonzero curvature, the marginal operators have a nontrivial connection.

On the other hand, we will argue that the bundles encoding the connection for chiral primaries have vanishing curvature in $\mathcal{N} = 4$ theories. This can be seen as follows: while from the $\mathcal{N} = 2$ point of view the chiral primaries are only charged under $U(1)_R$, in the underlying $\mathcal{N} = 4$ theory they belong to representations of $SO(6)_R$. Since the conformal manifold is one-complex dimensional and the holonomy of the tangent bundle is only $U(1)$, it is not possible to have nontrivial $SO(6)$ -valued curvature for bundles over the conformal manifold, without breaking the $SO(6)$ invariance of the theory.

Hence we conclude that the bundles of chiral primaries for $\mathcal{N} = 4$ theories must have vanishing curvature. One might wonder, how this statement can be consistent with the fact that the tangent bundle has nontrivial curvature and the fact that the marginal operators are descendants of the chiral primaries. The resolution is simple. Recalling the relation

$\mathcal{O}_\tau = \mathcal{Q}^4 \cdot \phi_2$, we can see that the curvature corresponding to \mathcal{O}_τ is given by the sum of the curvature of the supercurrents plus that of ϕ_2 . Since the latter is vanishing, we learn that the curvature of the tangent bundle comes entirely from that of the supercharges (3.5). It is easy to check that, using (3.5), the relation $\mathcal{O}_\tau = \mathcal{Q}^4 \cdot \phi_2$ and comparing with the curvature of the tangent bundle of (3.27), all factors work out correctly.

Alternatively, we can verify the fact that the chiral primaries in $\mathcal{N} = 4$ have vanishing curvature directly from the tt^* equations. This can be done in two steps. The first step is to observe that in $\mathcal{N} = 4$ theories, we have a non-renormalization theorem for 3-point functions [9–18], which can be expressed in equations as

$$\nabla_\tau C = \bar{\nabla}_{\bar{\tau}} C = 0 . \tag{3.28}$$

The second step requires taking the covariant derivative (either ∇ or $\bar{\nabla}$) of both sides of the tt^* equation (3.2b). The covariant derivative of the r.h.s., which involves the two-point function coefficients g and the 3-point function coefficients C , vanishes from (3.28) and the compatibility of g with the connection, which implies $\nabla g = \bar{\nabla} g = 0$. The vanishing of the covariant derivative of the r.h.s. implies that the covariant derivative of the l.h.s. also vanishes, from which we deduce that the bundles must have covariantly constant curvature. This allows a direct evaluation of the curvature in the weak coupling limit. Hence, in order to show that the curvature vanishes in $\mathcal{N} = 4$ theories for all values of the coupling, it is enough to show that the r.h.s. of the tt^* equations (3.2b) vanishes in the weak coupling limit.

All ingredients on the r.h.s. of (3.2b) can be evaluated — in principle — by standard, alas rather involved in general, Wick contractions. In appendix C we provide an alternative derivation of the following general combinatoric/group theoretic identity

$$\left\{ -[C_2, \bar{C}_2]_K^L + g_{2\bar{2}} \delta_K^L \left(1 + \frac{R}{\dim \mathcal{G}} \right) \right\}_{\text{tree}} = 0 . \tag{3.29}$$

This is an identity¹⁶ for free-field contractions between traces that should hold for any semi-simple group \mathcal{G} . The subscript 2 refers to the chiral primary $\phi_2 = \text{Tr}[\varphi^2]$.

Using this identity, we can demonstrate the desired result, i.e. that the r.h.s. of the tt^* equation vanishes for $\mathcal{N} = 4$ theories: in standard $\mathcal{N} = 4$ gauge theories the central charge is related to $\dim \mathcal{G}$ by

$$c = \frac{\dim \mathcal{G}}{4} .$$

Inserting this formula into (3.29) we find

$$- [C_2, \bar{C}_2]_K^L + g_{2\bar{2}} \delta_K^L \left(1 + \frac{R}{4c} \right) = 0 \tag{3.30}$$

which is precisely what we wanted to show.

As a final comment we would like to clarify a possibly confusing point. The tt^* equations (3.10) predict that the chiral primaries in $\mathcal{N} = 2$ theories have nonzero curvature

¹⁶It is quite possible that this equation corresponds to a natural group-theoretic statement, but we have not yet investigated this in detail. See also section 5.2 for related explicit tree-level 2-point functions.

even in the limit of weak coupling. Indeed, the relation between c and $\dim\mathcal{G}$ is different for $\mathcal{N} = 2$ theories compared to $\mathcal{N} = 4$ theories and as a result (3.30) does not hold in $\mathcal{N} = 2$ theories ((3.29), however, does hold). On the other hand, we argued that the curvature of operators in conformal perturbation theory is computed by (3.1). In the free limit the 4-point function inside the double integral, relevant for the computation of the curvature of chiral primaries, is the same in $\mathcal{N} = 2$ and $\mathcal{N} = 4$ theories. How can it then be, that in $\mathcal{N} = 2$ the bundle of primaries has nonzero curvature even in the weak coupling limit, while in $\mathcal{N} = 4$ the curvature vanishes?

The answer is that the two processes, of taking the zero coupling limit and of doing the double regularized integral, do not commute. In principle, the correct computation is to first compute the integral at some finite value of the coupling, and then send the coupling to zero. If one (wrongly) first takes the zero coupling limit inside the integral, then operators whose conformal dimension takes “accidentally” small value at zero coupling, start to contribute to the double integral. At infinitesimally small coupling these operators lift and their contribution discontinuously drops out of the double integral. Such operators are different between $\mathcal{N} = 2$ and $\mathcal{N} = 4$, thus resolving the aforementioned paradox.

4 $\mathcal{N} = 2$ superconformal QCD as an instructive example

4.1 Definitions

The $\mathcal{N} = 2$ SYM theory with gauge group $SU(N)$ coupled to $2N$ hypermultiplets (in short, $\mathcal{N} = 2$ superconformal QCD or SCQCD) is a well known superconformal field theory for any value of the complexified gauge coupling constant (3.26). This theory will serve as a testing ground for the general ideas presented above. The bosonic field content of the theory comprises of: (a) the gauge field A_μ and a complex scalar field φ in the adjoint representation of the gauge group (both are part of the $\mathcal{N} = 2$ vector multiplet), and (b) $2N$ doublets of complex scalars $Q_{\mathcal{I}}$ ($\mathcal{I} = \pm$) in the fundamental representation of the gauge group, that belong to $2N$ $\mathcal{N} = 2$ hypermultiplets. The global symmetry group is $U(2N) \times SU(2)_R \times U(1)_R$. $U(2N)$ is a flavor symmetry rotating the hypermultiplets and $SU(2)_R \times U(1)_R$ is the $\mathcal{N} = 2$ R-symmetry. More details about the theory are summarized in appendix B.

The generators of the $\mathcal{N} = 2$ chiral ring, as defined in section 2.1, are the single-trace superconformal primaries

$$\phi_\ell \propto \text{Tr} [\varphi^\ell], \quad \ell = 2, 3, \dots, N. \tag{4.1}$$

The proportionality constant is convention-dependent (specific convention choices will be made below). The remaining fields of the chiral ring are generated by products of the fields (4.1); in the weak-coupling formulation of the theory chiral primaries with $\ell > N$ are related to the primaries with $\ell \leq N$ by polynomial equations dictated by the Cayley-Hamilton theorem of $N \times N$ matrices.

$\mathcal{N} = 2$ superconformal QCD has a single (complex) exactly marginal deformation (2.6) with coupling τ (3.26). The exactly marginal operator \mathcal{O}_τ is a descendant of the chiral

primary field ϕ_2

$$\mathcal{O}_\tau = \mathcal{Q}^4 \cdot \phi_2 . \tag{4.2}$$

We note in passing that the chiral ring defined in terms of an $\mathcal{N} = 1$ subalgebra contains the additional mesonic superconformal primaries

$$\mathcal{M}_{3\mathcal{I}}^{\mathcal{J}} \propto \left(Q_{\mathcal{I}j} \bar{Q}^{\mathcal{J}j} \right) - \frac{1}{2} \left(Q_{\mathcal{K}j} \bar{Q}^{\mathcal{K}j} \right) \delta_{\mathcal{I}}^{\mathcal{J}} . \tag{4.3}$$

A sum over the gauge group indices is implicit, the index $j = 1, \dots, 2N$ runs over the number of hypermultiplets, $\mathcal{I}, \mathcal{J}, \mathcal{K} = \pm$ are $SU(2)_R$ indices, and the subindex **3** denotes that this particular combination belongs in a triplet representation of the $SU(2)_R$.¹⁷ Such primaries are not part of the $\mathcal{N} = 2$ chiral ring defined in section 2.1 and therefore will not be part of our analysis.

4.2 $SU(2)$ with 4 hypermultiplets

We begin the discussion with the $SU(2)$ case which provides a simple clear demonstration of the general ideas in section 3. In this case, ϕ_2 is the *single* chiral ring generator. We normalize ϕ_2 by requiring the validity of the conventions (2.6), (2.7), (2.10) (see also section 5.1.1 for an explicit tree-level implementation of these conventions). We notice that since \mathcal{O}_τ is, by this definition, related to a holomorphic section of the tangent bundle of the conformal manifold, then as explained in section 3, $\phi_2 \propto \text{Tr}[\varphi^2]$ (with a normalization that is a holomorphic function of τ) is a *non-holomorphic* section of the bundle of chiral primaries. A holomorphic ϕ_2 arises by multiplying $\text{Tr}[\varphi^2]$ with the non-holomorphic factor $e^{-\frac{\mathcal{K}}{384c}}$, where \mathcal{K} is the Kähler potential for the Zamolodchikov metric.

In addition, the chiral ring includes a unique chiral primary $\phi_{2n} \propto (\text{Tr}[\varphi^2])^n$ at each scaling dimension $\Delta = 2n$ ($n \in \mathbb{Z}_+$) (generated by ϕ_2 with repeated multiplication). We normalize the higher order chiral primaries ϕ_{2n} ($n > 1$) by requiring the OPE

$$\phi_2(x)\phi_{2n}(0) = \phi_{2n+2}(0) + \dots \tag{4.4}$$

which fixes the OPE coefficients

$$C_{2n}^{2n+2} = 1 . \tag{4.5}$$

Notice that this choice is consistent with the holomorphic gauge (3.11). Moreover, as a straightforward consequence of the associativity of the chiral ring all the non-vanishing OPE coefficients are fixed to one; namely, one can further show that

$$C_{2n\ 2m}^{2(n+m)} = 1 . \tag{4.6}$$

¹⁷For a complete analysis of the shortening conditions of the $\mathcal{N} = 2$ superconformal algebra in general theories we refer the reader to [20]. For an application to the $\mathcal{N} = 2$ superconformal QCD theories see for example [34].

4.2.1 tt^* equations and exact 2- and 3-point functions

In these conventions the 2-point functions of the chiral primaries ϕ_{2n}

$$\langle \phi_{2n}(x) \bar{\phi}_{2n}(0) \rangle = \frac{g_{2n}(\tau, \bar{\tau})}{|x|^{4n}} \tag{4.7}$$

have a non-trivial dependence on the modulus τ . Our purpose is to determine the exact form of the functions $g_{2n}(\tau, \bar{\tau})$. This will immediately provide information about 3-point functions as well.

Since we have a one-dimensional sequence of chiral primaries without any non-trivial degeneracies, the tt^* equations (3.10) assume the following particularly simple form

$$\partial_\tau \partial_{\bar{\tau}} \log g_{2n} = \frac{g_{2n+2}}{g_{2n}} - \frac{g_{2n}}{g_{2n-2}} - g_2, \tag{4.8}$$

where $n = 1, 2, \dots$ and $g_0 = 1$ by definition. This infinite sequence of differential equations can be recast as the more familiar semi-infinite Toda chain

$$\partial_\tau \partial_{\bar{\tau}} q_n = e^{q_{n+1} - q_n} - e^{q_n - q_{n-1}}, \quad n = 2, \dots \tag{4.9}$$

by setting $g_{2n} = \exp(q_n - \log Z_{S^4})$. A reality condition on q_n implies that g_{2n} are positive, which is expected by unitarity. In section 5 we collect several perturbative checks of equations (4.8).

It may be interesting to classify the most general solution of the equations (4.8), subject to positivity over the entirety of the conformal manifold, but this is beyond the scope of the current paper.¹⁸ Instead, in what follows we will use these equations to solve recursively for the 2-point functions as follows

$$g_{2n+2} = g_{2n} \partial_\tau \partial_{\bar{\tau}} \log g_{2n} + \frac{g_{2n}^2}{g_{2n-2}} + g_2 g_{2n}, \quad n = 1, 2, \dots \tag{4.10}$$

Knowledge of a single 2-point function, e.g. g_2 , implies recursively the precise form of all the rest. As we show now, for $SU(2)$ this provides the complete non-perturbative determination of the 2- and 3-point functions of all chiral primary operators.

Exact 2-point functions. We can use supersymmetric localization on S^4 and the formula (2.13) to determine the exact coupling constant dependence of g_2 . For the $SU(2)$ SCQCD theory an integral expression for the sphere partition function gives [8]

$$Z_{S^4}(\tau, \bar{\tau}) = \int_{-\infty}^{\infty} da e^{-4\pi \text{Im}(\tau) a^2} (2a)^2 \frac{H(2ia)H(-2ia)}{(H(ia)H(-ia))^4} |Z_{\text{inst}}(a, \tau)|^2. \tag{4.11}$$

H is a function on the complex plane defined in terms of the Barnes G -function [35] as

$$H(z) = G(1+z)G(1-z). \tag{4.12}$$

¹⁸We do not expect positivity alone to fix the solution uniquely. It is worth exploring the possibility that positivity, in combination with the data of higher order perturbative corrections around the point weak coupling point $\text{Im}\tau = \infty$, might lead to a unique solution, in analogy to 2d examples [3].

Further details are summarized for the convenience of the reader in appendix A. Z_{inst} is the Nekrasov partition function [36] that incorporates the contribution from all instanton sectors.

Consequently, implementing (2.13) we obtain the exact 2-point function of the lowest chiral primary ϕ_2 as

$$g_2 = \partial_\tau \partial_{\bar{\tau}} \log Z_{S^4} . \tag{4.13}$$

The 2-point functions of the higher order chiral primaries can be computed recursively using (4.10). We will return to the resulting expressions momentarily.

Exact 3-point functions. The general non-vanishing 3-point function

$$\langle \phi_{2m}(x_1) \phi_{2n}(x_2) \bar{\phi}_{2m+2n}(y) \rangle = \frac{C_{2m\ 2n\ \overline{2m+2n}}}{|x_1 - y|^{4m} |x_2 - y|^{4n}} \tag{4.14}$$

follows immediately from the above data since

$$C_{2m\ 2n\ \overline{2m+2n}} = C_{2m\ 2n}^{2(m+n)} g_{2(m+n)} = g_{2(m+n)} . \tag{4.15}$$

In the second equality we made use of the OPE coefficients (4.6). This formula provides the non-perturbative 3-point functions of chiral primaries as a function of the modulus τ , including all instanton corrections. Following section 3.5 it is straightforward to extend this result to any extremal correlator of chiral primaries.

While the above normalization of the chiral primaries is very convenient for the type of computations of the previous section, it is common in conformal field theory to work with orthonormal fields $\hat{\phi}_I$ for which

$$\langle \hat{\phi}_I(x) \bar{\hat{\phi}}_J(0) \rangle = \frac{\delta_{I\bar{J}}}{|x|^{2\Delta}} . \tag{4.16}$$

In these conventions, the OPE coefficients \hat{C}_{IJ}^K depend non-trivially on the moduli. Converting to this normalization in the case at hand we find the structure constants

$$\hat{C}_{2m\ 2n\ \overline{2m+2n}} = \sqrt{\frac{g_{2m+2n}}{g_{2m} g_{2n}}} . \tag{4.17}$$

4.2.2 Perturbative expressions

The tt^* equations have allowed us to obtain exact results for 2- and 3-point functions of the chiral primary fields. The resulting expressions depend implicitly on the S^4 partition function of the SU(2) theory, which is given in terms of an one-dimensional integral (4.11). It is interesting to work out the first few orders in the perturbative expansion of the exact expressions. This will be useful later on in section 5 when we compare against independent computations in perturbation theory.

0-instanton sector. Consider the perturbative contributions around the weak coupling regime $g_{YM} \rightarrow 0$, or equivalently $\tau \rightarrow +i\infty$. Working with the perturbative (0-instanton) part of the S^4 partition function we obtain

$$Z_{S^4}^{(0)} = \int_{-\infty}^{\infty} da e^{-4\pi \text{Im}(\tau) a^2} (2a)^2 \frac{H(2ia)H(-2ia)}{(H(ia)H(-ia))^4} . \tag{4.18}$$

The mathematical identity

$$\log \left(\frac{H(2ia)H(-2ia)}{(H(ia)H(-ia))^4} \right) = -8 \sum_{k=2}^{\infty} \frac{\zeta(2k-1)}{k} (2^{2k-2} - 1) (-1)^k a^{2k} \quad (4.19)$$

implies the perturbative expansion (see also [37])

$$Z_{S^4}^{(0)} = \frac{1}{4\pi(\text{Im}\tau)^{3/2}} \left(1 - \frac{45\zeta(3)}{(4\pi\text{Im}\tau)^2} + \frac{525\zeta(5)}{(4\pi\text{Im}\tau)^3} + \dots \right). \quad (4.20)$$

Then, employing (4.13) and the recursive tt^* equations (4.10) we deduce the perturbative expansion of the 2-point functions of any chiral primary. For the first five chiral primaries the specific expressions are

$$g_2^{(0)} = \frac{3}{8} \frac{1}{(\text{Im}\tau)^2} - \frac{135\zeta(3)}{32\pi^2} \frac{1}{(\text{Im}\tau)^4} + \frac{1575\zeta(5)}{64\pi^3} \frac{1}{(\text{Im}\tau)^5} + \dots, \quad (4.21)$$

$$g_4^{(0)} = \frac{15}{32} \frac{1}{(\text{Im}\tau)^4} - \frac{945\zeta(3)}{64\pi^2} \frac{1}{(\text{Im}\tau)^6} + \frac{7875\zeta(5)}{64\pi^3} \frac{1}{(\text{Im}\tau)^7} + \dots, \quad (4.22)$$

$$g_6^{(0)} = \frac{315}{256} \frac{1}{(\text{Im}\tau)^6} - \frac{76545\zeta(3)}{1024\pi^2} \frac{1}{(\text{Im}\tau)^8} + \frac{1677375\zeta(5)}{2048\pi^3} \frac{1}{(\text{Im}\tau)^9} + \dots, \quad (4.23)$$

$$g_8^{(0)} = \frac{2835}{512} \frac{1}{(\text{Im}\tau)^8} - \frac{280665\zeta(3)}{512\pi^2} \frac{1}{(\text{Im}\tau)^{10}} + \frac{1913625\zeta(5)}{256\pi^3} \frac{1}{(\text{Im}\tau)^{11}} + \dots, \quad (4.24)$$

$$g_{10}^{(0)} = \frac{155925}{4096} \frac{1}{(\text{Im}\tau)^{10}} - \frac{91216125\zeta(3)}{16384\pi^2} \frac{1}{(\text{Im}\tau)^{12}} + \frac{2982065625\zeta(5)}{32768\pi^3} \frac{1}{(\text{Im}\tau)^{13}} + \dots. \quad (4.25)$$

In section 5 we verify independently the validity of the first two orders of these expressions (for arbitrary $g_{2n}^{(0)}$) in perturbation theory. For each of these 2-point functions, the leading order term comes from a tree-level computation. The one-loop contribution is always vanishing and the next-to-leading order contribution comes from a two-loop computation.

The corresponding 3-point functions follow immediately from equation (4.15). In the alternative conventions (4.16) they follow from a straightforward application of equation (4.17). The first few coefficients are

$$\hat{C}_{224}^{(0)} = \sqrt{\frac{10}{3}} \left(1 - \frac{9\zeta(3)}{2\pi^2} \frac{1}{(\text{Im}\tau)^2} + \frac{525\zeta(5)}{8\pi^3} \frac{1}{(\text{Im}\tau)^3} + \dots \right), \quad (4.26)$$

$$\hat{C}_{246}^{(0)} = \sqrt{7} \left(1 - \frac{9\zeta(3)}{\pi^2} \frac{1}{(\text{Im}\tau)^2} + \frac{675\zeta(5)}{4\pi^3} \frac{1}{(\text{Im}\tau)^3} + \dots \right), \quad (4.27)$$

$$\hat{C}_{268}^{(0)} = 2\sqrt{3} \left(1 - \frac{27\zeta(3)}{2\pi^2} \frac{1}{(\text{Im}\tau)^2} + \frac{2475\zeta(5)}{8\pi^3} \frac{1}{(\text{Im}\tau)^3} + \dots \right), \quad (4.28)$$

$$\hat{C}_{2810}^{(0)} = \sqrt{\frac{55}{3}} \left(1 - \frac{18\zeta(3)}{\pi^2} \frac{1}{(\text{Im}\tau)^2} + \frac{975\zeta(5)}{2\pi^3} \frac{1}{(\text{Im}\tau)^3} + \dots \right), \quad (4.29)$$

$$\hat{C}_{448}^{(0)} = 3\sqrt{\frac{14}{5}} \left(1 - \frac{18\zeta(3)}{\pi^2} \frac{1}{(\text{Im}\tau)^2} + \frac{825\zeta(5)}{2\pi^3} \frac{1}{(\text{Im}\tau)^3} + \dots \right), \quad (4.30)$$

$$\hat{C}_{4610}^{(0)} = \sqrt{66} \left(1 - \frac{27\zeta(3)}{\pi^2} \frac{1}{(\text{Im}\tau)^2} + \frac{2925\zeta(5)}{4\pi^3} \frac{1}{(\text{Im}\tau)^3} + \dots \right). \quad (4.31)$$

1-Instanton sector. The contribution of instantons can be deduced from known expressions of Z_{inst} without much additional effort. For example, in the 1-instanton sector¹⁹ the first few orders in the perturbative expansion of Z_{S^4} are

$$Z_{S^4}^{(1)} = \cos \theta \exp\left(-\frac{8\pi^2}{g_{YM}^2}\right) \left(-\frac{3}{4\pi(\text{Im}\tau)^{3/2}}\right) \left[1 - \frac{1}{8\pi\text{Im}\tau} - \frac{45\zeta(3)}{16\pi^2(\text{Im}\tau)^2} + \frac{105(\zeta(3)+10\zeta(5))}{128\pi^3(\text{Im}\tau)^3} + \dots\right]. \quad (4.32)$$

We have written out $\theta = \pi(\tau + \bar{\tau})$ and g_{YM} explicitly in some of the terms, to make the expression more intuitive. The corresponding corrections $g_{2n}^{(1)}$ of g_{2n} can be computed by starting with (4.13)

$$g_2 = \partial_\tau \partial_{\bar{\tau}} \log \left(Z_{S^4}^{(0)} + Z_{S^4}^{(1)} + \dots \right), \quad (4.33)$$

recursively applying (4.10)

$$g_{2n+2} = g_{2n} \partial_\tau \partial_{\bar{\tau}} \log g_{2n} + \frac{g_{2n}^2}{g_{2n-2}} + g_2 g_{2n}, \quad n = 1, 2, \dots \quad (4.34)$$

and finally isolating the $\exp\left(-\frac{8\pi^2}{g_{YM}^2}\right)$ contribution $g_{2n}^{(1)}$ at every level g_{2n} . For the first terms we find

$$g_2^{(1)} = \cos \theta \exp\left(-\frac{8\pi^2}{g_{YM}^2}\right) \left(\frac{3}{8(\text{Im}\tau)^2} + \frac{3}{16\pi(\text{Im}\tau)^3} - \frac{135\zeta(3)}{32\pi^2(\text{Im}\tau)^4} + \dots\right), \quad (4.35)$$

$$g_4^{(1)} = \cos \theta \exp\left(-\frac{8\pi^2}{g_{YM}^2}\right) \left(\frac{15}{16(\text{Im}\tau)^4} + \frac{15}{32\pi(\text{Im}\tau)^5} - \frac{945\zeta(3)}{32\pi^2(\text{Im}\tau)^6} + \dots\right) \quad (4.36)$$

It is straightforward to continue with higher n if desired. Analogous results can be obtained likewise for the general ℓ -instanton sector. From these 2-point functions we can also express the exact instanton corrections to chiral primary 3-point functions.

It would be interesting to confirm these results with an independent perturbative computation in the ℓ -instanton sector.

4.2.3 Comments on $\text{SL}(2, \mathbb{Z})$ duality

It is interesting to explore the transformation properties of correlators of chiral primaries in $\mathcal{N} = 2$ SCQCD under non-perturbative $\text{SL}(2, \mathbb{Z})$ transformations²⁰

$$\tau' = \frac{a\tau + b}{c\tau + d}, \quad a, b, c, d \in \mathbb{Z}, \quad ad - bc = 1. \quad (4.37)$$

We expect that the Zamolodchikov metric obeys the identity

$$G_{\tau' \bar{\tau}'} d\tau' d\bar{\tau}' = G_{\tau \bar{\tau}} d\tau d\bar{\tau}, \quad (4.38)$$

¹⁹By this we mean contributions of 1 instanton or 1 anti-instanton, i.e. the part that scales like $\exp\left(-\frac{8\pi^2}{g_{YM}^2}\right)$.

²⁰In this subsection we denote by τ a holomorphic coordinate on the conformal manifold which transforms in a simple way (4.37) under $\text{SL}(2, \mathbb{Z})$. We would like to thank B. van Rees for discussions on this issue.

or equivalently

$$G_{\tau\bar{\tau}}\left(\frac{a\tau+b}{c\tau+d}, \frac{a\bar{\tau}+b}{c\bar{\tau}+d}\right) = |c\tau+d|^2 G_{\tau\bar{\tau}}(\tau, \bar{\tau}) . \tag{4.39}$$

A similar transformation property holds for the 2-point function $g_2 = G_{\tau\bar{\tau}}/192$.

Given the relation between the Zamolodchikov metric and the S^4 partition function

$$G_{\tau\bar{\tau}} = 192 \partial_{\tau} \partial_{\bar{\tau}} Z_{S^4} \tag{4.40}$$

and taking into account the transformation (4.37), we notice that the validity of (4.39) requires the partition function Z_{S^4} to be $SL(2, \mathbb{Z})$ invariant up to Kähler transformations

$$\log Z_{S^4}(\tau') = \log Z_{S^4}(\tau) + f(\tau) + \overline{f(\bar{\tau})} . \tag{4.41}$$

The issue we would like to address here is the following: suppose that we have verified the correct $SL(2, \mathbb{Z})$ transformation of g_2 . What is the $SL(2, \mathbb{Z})$ behavior of the 2-point functions g_{2n} of the higher order chiral primaries?

The tt^* equations provide a specific answer. Assuming $g'_2 = |c\tau+d|^2 g_2$, it is easy to verify recursively from (4.10) that

$$g'_{2n} = |c\tau+d|^{2n} g_{2n} . \tag{4.42}$$

Alternatively, in the normalization (4.16), equations (4.17) and (4.42) imply that the 3-point functions are $SL(2, \mathbb{Z})$ invariant

$$\hat{C}'_{2m\ 2n\ \overline{2m+2n}} = \hat{C}_{2m\ 2n\ \overline{2m+2n}} , \tag{4.43}$$

which is consistent with expectations. See [38] for a related discussion of the S -duality properties of chiral primary correlation functions in $\mathcal{N} = 4$ SYM theory.

4.3 $SU(N)$ with $2N$ hypermultiplets

The case of general $SU(N)$ gauge group can be analyzed in a similar fashion. Unfortunately, for general $N \geq 3$ it is less clear under which conditions we can identify the relevant solution of the tt^* equations. We proceed to discuss the detailed structure of the $SU(N)$ tt^* equations.

The general $SU(N)$ $\mathcal{N} = 2$ SCQCD theories possess $N - 1$ chiral ring generators represented by the single-trace operators

$$\text{Tr}[\varphi^2], \text{Tr}[\varphi^3], \dots, \text{Tr}[\varphi^N] . \tag{4.44}$$

The general element of the chiral ring is freely generated from these operators and can be viewed as a linear combination of the primaries

$$\phi_{(n_1, n_2, \dots, n_{N-1})} \propto \prod_{i=1}^{N-1} (\text{Tr}[\varphi^{i+1}])^{n_i} . \tag{4.45}$$

The operator that gives rise to the single exactly marginal direction \mathcal{O}_{τ} of the theory is

$$\phi_2 \equiv \phi_{(1, 0, \dots, 0)} . \tag{4.46}$$

We notice that the scaling dimension of the generic chiral primary (4.45) is $\Delta = \sum_{i=1}^{N-1} (i+1)n_i$. Obviously, there are values of Δ where more than one chiral primary can have the same scaling dimension. Such chiral primaries can mix non-trivially with each other to exhibit non-diagonal τ -dependent 2-point function matrices. We verify this mixing explicitly in specific examples at tree-level in subsection 5.3.

The OPE of the chiral primaries (4.45) can be chosen to take the form

$$\phi_{(n_1, \dots, n_{N-1})}(x) \phi_{(m_1, \dots, m_{N-1})}(0) = \phi_{(n_1+m_1, \dots, n_{N-1}+m_{N-1})}(0) + \dots, \quad (4.47)$$

or in more compact notation

$$\phi_K(x) \phi_L(0) = \phi_{K+L}(0) + \dots. \quad (4.48)$$

This choice allows us to fix the non-vanishing OPE coefficients to

$$C_{KL}^{K+L} = 1, \quad (4.49)$$

in analogy to the SU(2) equation (4.6). In this way, once we choose the normalization of the chiral ring generators (4.44) the normalization of all the chiral primary fields is uniquely determined. We will consider a normalization of ϕ_2 that adheres to the conventions (2.6), (2.10). The remaining chiral primaries in (4.44) are chosen with an arbitrary normalizing factor $\mathcal{N}_K(\tau)$ that is a holomorphic function of the complex coupling τ .

4.3.1 The structure of the SU(N) tt^* equations

In these conventions the tt^* equations (3.10) become

$$\partial_{\bar{\tau}} \left(g^{\bar{M}\Delta L\Delta} \partial_{\tau} g_{K\Delta \bar{M}\Delta} \right) = g_{K\Delta+2, \bar{R}\Delta+2} g^{\bar{R}\Delta L\Delta} - g_{K\Delta \bar{R}\Delta} g^{\bar{R}\Delta-2, L\Delta-2} - g_2 \delta_{K\Delta}^L. \quad (4.50)$$

The addition of 2 in the index notation $K+2$ refers to the element $\phi_2 \phi_K$. The subindex Δ on the indices has been added here to flesh out the scaling dimension of the corresponding chiral primaries. Sample tree-level checks of equations (4.50) (that exhibit the non-trivial mixing of chiral primaries) are collected in section 5.3.

Similar to the SU(2) case the equations (4.50) relate 2-point functions of chiral primaries at three different scaling dimensions and can be recast in the recursive form

$$g_{K\Delta+2, \bar{N}\Delta+2} = g_{L\Delta \bar{N}\Delta} \partial_{\bar{\tau}} \left(g^{\bar{M}\Delta L\Delta} \partial_{\tau} g_{K\Delta \bar{M}\Delta} \right) + g_{K\Delta \bar{M}\Delta} g^{\bar{M}\Delta-2, L\Delta-2} g_{L\Delta \bar{N}\Delta} + g_2 g_{K\Delta \bar{N}\Delta}. \quad (4.51)$$

However, unlike the situation of the SU(2) gauge group, the complicated degeneracy pattern of the general SU(N) theory and the corresponding non-trivial mixing of the chiral primary fields makes this system of differential equations a far more complicated one to solve explicitly in terms of a few externally determined data (like the Zamolodchikov metric).

Most notably, the l.h.s. of equation (4.51) involves primaries that belong in a subsequence generated by multiplication with the field ϕ_2 . In contrast, the r.h.s. involves in general 2-point functions of all available chiral primaries. This feature complicates the recursive solution of the system of equations (4.51). As we move up in scaling dimension with

the action of ϕ_2 the number of degenerate fields will stay the same or increase. Increases are due to the appearance of additional degenerate chiral primary fields that involve the action of the extra chiral ring generators other than ϕ_2 , i.e. $\text{Tr}[\varphi^3]$ etc. In such cases, there are seemingly new 2-point function coefficients that have not been determined recursively from the previous lower levels and represent new data that need to be provided externally. It is an interesting open question whether other properties (like the property of positivity over the entire moduli space) are strong enough to reduce the number of unknowns and fix the full solution uniquely.

Despite the apparent complexity of (4.51), it is quite likely that this system has a hidden structure that allows to simplify its description. For example, in section 5.4 we find preliminary evidence at tree-level that one can isolate differential equations that form a closed system on the subsequence of the chiral primary fields $(\phi_2)^n$. If true, the data of such subsequences could be determined solely in terms of the $SU(N)$ S^4 partition function in direct analogy to the $SU(2)$ case. Such possibilities are currently under investigation.

4.3.2 3-point functions

The non-vanishing 3-point structure constants of the $SU(N)$ theory are

$$C_{K\Delta_1 L\Delta_2 \bar{M}\Delta_1+\Delta_2} = C_{K\Delta_1 L\Delta_2}^{(K+L)\Delta_1+\Delta_2} g_{(K+L)\Delta_1+\Delta_2, \bar{M}\Delta_1+\Delta_2} = g_{(K+L)\Delta_1+\Delta_2, \bar{M}\Delta_1+\Delta_2} . \quad (4.52)$$

This relation is the $SU(N)$ generalization of (4.14), (4.15). Consequently, a solution of the tt^* equations (4.50) determines immediately also the 3-point functions (4.52).

The conversion of the above results into the language of the common alternative normalization (4.16)

$$\langle \hat{\phi}_K(x) \bar{\hat{\phi}}_{\bar{L}}(0) \rangle = \frac{\delta_{K\bar{L}}}{|x|^{2\Delta_K}} \quad (4.53)$$

requires a transformation

$$\hat{\phi}_K = \sum_L \mathcal{N}_K^L \phi_L \quad (4.54)$$

at each scaling dimension Δ , where the matrix elements \mathcal{N}_K^L are suitable functions of the 2-point coefficients $g_{K\bar{L}}$. Once the matrix elements \mathcal{N}_K^L are determined the 3-point structure constants $\hat{C}_{IJ\bar{K}}$ in the basis (4.53) can be written as

$$\hat{C}_{IJ\bar{K}} = \sum_{L_1, L_2, \bar{L}_3} \mathcal{N}_I^{L_1} \mathcal{N}_J^{L_2} \bar{\mathcal{N}}_{\bar{K}}^{\bar{L}_3} g_{L_1+L_2, \bar{L}_3} . \quad (4.55)$$

5 Checks in perturbation theory

In this section we perform a number of independent checks of the above statements in perturbation theory. These checks provide a concrete verification of the validity of the general formal proof of the tt^* equations in [4], and allow us to verify that the tt^* equations were applied correctly in the previous section. In the process, we encounter and comment on several individual properties of correlation functions in $\mathcal{N} = 2$ SCQCD. We work in the conventions listed in appendix B.

5.1 SU(2) SCQCD

We begin with a perturbative computation up to 2 loops of the 2-point coefficients g_{2n} in the SU(2) $\mathcal{N} = 2$ SCQCD theory.

5.1.1 Tree-level

Let us start with a comment about normalizations in the general SU(N) theory. At leading order in the weak coupling limit, $g_{YM} \ll 1$, (and the conventions summarized in appendix B) the 2-point function of the adjoint scalars $\varphi = \varphi^a T^a$ is

$$\langle \varphi^a(x) \bar{\varphi}^b(0) \rangle = \delta^{ab} \frac{1}{\pi \text{Im}\tau} \frac{1}{|x|^2}. \quad (5.1)$$

T^a ($a = 1, \dots, N^2 - 1$) is a basis of the SU(N) Lie algebra. Normalizing the chiral primary operator ϕ_2 as

$$\phi_2 = \frac{\pi}{4N} \text{Tr}[\varphi^2] = \frac{\pi}{4} \varphi^a \varphi^a \quad (5.2)$$

we obtain

$$\langle \phi_2(x) \bar{\phi}_2(0) \rangle = \frac{N^2 - 1}{8} \frac{1}{(\text{Im}\tau)^2} \frac{1}{|x|^4}. \quad (5.3)$$

On the other hand, the exactly marginal operator \mathcal{O}_τ , (4.2), has the explicit form presented in equation (B.11) of appendix B. A tree-level computation yields²¹

$$\langle \mathcal{O}_\tau(x) \bar{\mathcal{O}}_\tau(0) \rangle = 24(N^2 - 1) \frac{1}{(\text{Im}\tau)^2} \frac{1}{|x|^8}, \quad (5.4)$$

which is consistent with the conventions (2.8), (2.9), (2.10). This is important for the validity of the tt^* equations (3.10), or the equations (4.8) in the SU(2) case of this subsection.

Specializing now to the SU(2) case we find that the 2-point function (5.3) has the tree-level coefficient

$$g_2 = \frac{3}{8} \frac{1}{(\text{Im}\tau)^2}. \quad (5.5)$$

We can read off the 2-point function coefficients g_{2n} of the higher chiral primary operators $\phi_{2n} = (\phi_2)^n$ from free field Wick contractions in the 2-point correlation function

$$\langle \phi_{2n}(x) \bar{\phi}_{2n}(0) \rangle = \langle (\phi_2)^n(x) (\bar{\phi}_2)^n(0) \rangle. \quad (5.6)$$

A brute-force computation gives

$$g_{2n} = \frac{(2n+1)!}{6^n} g_2^n. \quad (5.7)$$

With this result the tt^* equations (4.8)

$$\partial_\tau \partial_{\bar{\tau}} \log g_{2n} = \frac{g_{2n+2}}{g_{2n}} - \frac{g_{2n}}{g_{2n-2}} - g_2 \quad (5.8)$$

reduce at tree-level to the differential equation

$$\partial_\tau \partial_{\bar{\tau}} \log g_2 = \frac{4}{3} g_2, \quad (5.9)$$

which is found to hold for the g_2 given in equation (5.5).

²¹At tree-level only the gauge part $\frac{i\pi}{16} F_{\mu\nu}^a F^{\mu\nu+a}$ of \mathcal{O}_τ in (B.11) contributes. The auxiliary fields contribute only contact terms and the cubic interactions are subleading in g_{YM} . The boson and fermion kinetic terms vanish on-shell. A similar observation was made in [12].

5.1.2 Quantum corrections up to 2 loops

We proceed to compute the first non-vanishing quantum corrections to g_{2n} in perturbation theory. This will allow us to reproduce the Zamolodchikov metric derived from localization [7] at g_{YM}^4 order and will provide a test of the tt^* equations at the quantum level. Furthermore, due to the discussion in section 3.2, this provides a g_{YM}^4 check of the chiral primary three-point functions in a diagonal basis as well. We will use the techniques of [39], namely we will exploit the fact that quantum corrections for $\mathcal{N} = 4$ SYM vanish at each order in perturbation theory,²² so that we only need to compute the diagrammatic difference between the $\mathcal{N} = 2$ and $\mathcal{N} = 4$ theories.

Following [39], it is easy to see that the diagrammatic difference between $\mathcal{N} = 2$ and $\mathcal{N} = 4$ at order g_{YM}^2 vanishes. It immediately follows that the theory does not receive quantum corrections to this order, consistent with the results from localization (4.21)–(4.25).

We now examine the diagrams that contribute to order g_{YM}^4 to the 2-point function

$$\langle \phi_{2n}(x) \bar{\phi}_{2n}(0) \rangle = \langle (\phi_2)^n(x) (\bar{\phi}_2)^n(0) \rangle = \frac{g_{2n}}{|x|^{4n}}. \quad (5.10)$$

To understand what type of diagrams can contribute to this order, it is convenient to temporarily regard the adjoint scalar φ lines as external and change the normalization of the fields so that the coupling constant dependence is on the vertices. Diagrams which differ between $\mathcal{N} = 2$ and $\mathcal{N} = 4$ must involve hypermultiplets running in the internal lines. After a brief inspection of the $\mathcal{N} = 2$ SCQCD Lagrangian it is not too hard to convince oneself that the only possible types of diagrams that can contribute to order g_{YM}^4 (and which differ between $\mathcal{N} = 2$ and $\mathcal{N} = 4$) come from two types of topologies, when trying to connect the $2n$ ‘external lines’ of φ at point x to the $2n$ ‘external lines’ of $\bar{\varphi}$ at point 0:

- a) diagrams where one external φ line is connected to one external $\bar{\varphi}$ line by a 2-loop-corrected $\varphi - \bar{\varphi}$ propagator, while all others lines are connected by free propagators
- b) diagrams where two external φ lines and two external $\bar{\varphi}$ lines are all connected together by a nontrivial 4-leg subdiagram, while the remaining φ and $\bar{\varphi}$ lines are connected by free propagators.

Let us examine the former first. We denote the quantum corrected propagator as

$$\langle \varphi^a(x) \bar{\varphi}^b(y) \rangle = \delta^{ab} S(x-y) = \delta^{ab} S^{(0)}(x-y) (1 + f_1 g_{YM}^4 + \dots), \quad (5.11)$$

where $S^{(0)}(x-y)$ is the tree-level propagator (5.1) and we have used the fact that the g_{YM}^4 corrections are proportional to the tree-level propagator [39]. f_1 is a numerical constant that we will determine in the following.

²²See [40, 41] for perturbative computations of 2-point functions of chiral primaries in $\mathcal{N} = 4$ SYM.

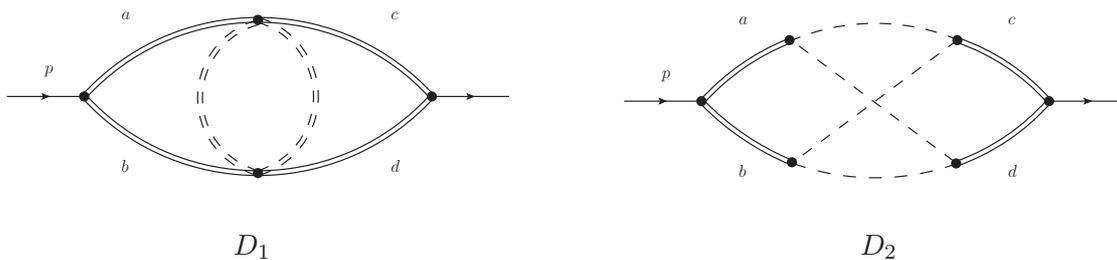


Figure 1. The diagrams D_1 and D_2 . Solid double lines represent φ propagators, dashed double lines correspond to hyperscalars and dashed lines to hyperfermions. a, b, c and d are adjoint gauge indices and p is the incoming momentum.

Regarding diagrams of type b), there are only two diagrams²³ that can contribute to this order, which are shown in figure 1. In the diagram D_1 hyperscalars run in the internal loop, while the diagram D_2 corresponds to the exchange of hyperfermions. In more detail, we define $D_1(x, y)$ and $D_2(x, y)$ as

$$D_1(x, y) = \frac{1}{2} \left\langle \varphi^a(x) \varphi^b(x) \bar{\varphi}^c(y) \bar{\varphi}^d(y) (\Xi_1)^2 \right\rangle_{\text{connected}}, \quad (5.12)$$

$$D_2(x, y) = \frac{1}{4!} \left\langle \varphi^a(x) \varphi^b(x) \bar{\varphi}^c(y) \bar{\varphi}^d(y) (\Xi_2)^4 \right\rangle_{\text{connected}}, \quad (5.13)$$

where Ξ_1 and Ξ_2 are the interaction actions associated to the terms in the Lagrangian (B.6) coupling the vector sector to the hypermultiplet sector, namely

$$\Xi_1 = \int d^4x Q_{\mathcal{I}} (\bar{\varphi} \varphi + \varphi \bar{\varphi}) Q^{\mathcal{I}}, \quad (5.14)$$

$$\Xi_2 = i\sqrt{2} \int d^4x (\tilde{\psi} \varphi \psi - \bar{\psi} \bar{\varphi} \bar{\psi}), \quad (5.15)$$

and we take Wick’s contractions that correspond to connected diagrams only.

It is easy to see that all the other diagrams either vanish or are identical to their $\mathcal{N} = 4$ counterparts. We start by examining the gauge structure of these diagrams. Both are proportional to $\text{Tr}(T^a T^c T^b T^d)$ (or permutations thereof), so the difference between the $\mathcal{N} = 2$ and $\mathcal{N} = 4$ color factors reads

$$4 \text{Tr}(T^a T^c T^b T^d)_{\text{fundamental}} - \text{Tr}(T^a T^c T^b T^d)_{\text{adjoint}} = -\frac{1}{2} \left(\delta^{ac} \delta^{bd} + \delta^{ad} \delta^{bc} + \delta^{ab} \delta^{cd} \right), \quad (5.16)$$

where the factor of 4 in the equation above comes from the fact that the $\mathcal{N} = 2$ theory has 4 hypermultiplets. It is thus convenient to define the quantity

$$\mathcal{C} \equiv \delta^{ac} \delta^{bd} + \delta^{ad} \delta^{bc} + \delta^{ab} \delta^{cd}, \quad (5.17)$$

²³We remind the reader that we are only considering diagrams which differ between $\mathcal{N} = 2$ SCQCD and $\mathcal{N} = 4$ SYM with the same gauge group. Also the statement that these are the only diagrams is true only for $\text{SU}(2)$ gauge group. See [39] for useful background.

and parametrize the contribution from these two diagrams as

$$D_1(x, y) + D_2(x, y) = \mathcal{C} S^{(0)}(x - y)^2 f_2 g_{YM}^4, \quad (5.18)$$

where f_2 is a numerical constant that we will determine momentarily.

With these results, it is straightforward to work out the combinatorics and find the g_{YM}^4 corrections to the correlation functions g_{2n} as a function of the two contributions f_1 and f_2 . After some work we find that the result is

$$\langle \phi_{2n}(x) \bar{\phi}_{2n}(y) \rangle = \left(\frac{\pi}{4}\right)^{2n} (2n + 1)! S^{(0)}(x - y)^{2n} \left[1 + \frac{n}{2} (4f_1 + (6n - 1)f_2) g_{YM}^4\right], \quad (5.19)$$

where f_1 and f_2 are defined in equations (5.11) and (5.18) respectively. In order to derive the expression above, one has to consider all the possible ways to connect the propagators associated to ϕ_{2n} with those associated to $\bar{\phi}_{2n}$, with the insertion of g_{YM}^4 corrections coming from the diagrams described above. We notice that the contribution coming from g_{YM}^4 diagrams with two external φ lines has a different dependence on n compared to the one coming from diagrams with four external φ lines, reflecting the different combinatorial properties of these graphs.

It is important to notice that the equation above is *not* automatically consistent with the tt^* equations. In fact, we find that demanding that (5.19) satisfies the tt^* equations leads to the non-trivial condition

$$f_2 = \frac{2}{5} f_1. \quad (5.20)$$

We conclude that the tt^* equations do encode non-trivial information about the quantum corrections to chiral primary correlation functions, as they are sensitive to the ratio f_2/f_1 . Determining this ratio by explicitly computing the relevant Feynman diagrams will thus provide us with a stringent test of these equations at the quantum level.

We will now determine the value of f_1 and f_2 by computing the Feynman diagrams D_1 and D_2 . We will show that their ratio is precisely the one predicted by the tt^* equations. Furthermore, the result will allow us to compute the g_{YM}^4 correction to the Zamolodchikov metric, providing thus a perturbative check of the results of [7].

Computation of f_1 and f_2 . Recall that the tree-level propagator (5.1) reads

$$S^{(0)}(x - y) = \frac{g_{YM}^2}{4\pi^2(x - y)^2} = g_{YM}^2 \int \frac{d^4 p}{(2\pi)^4} \frac{e^{-ip(x-y)}}{p^2}. \quad (5.21)$$

As is customary, we work in momentum space and in dimensional regularization, where the spacetime dimension d is $d = 4 - 2\epsilon$.

The g_{YM}^4 correction to the propagator $S^{(1)}(x - y)$ was computed in [39], and is given by

$$S^{(1)}(x - y) = -\frac{15\zeta(3)}{64\pi^4} g_{YM}^4 S^{(0)}(x - y), \quad (5.22)$$

which in turn implies that

$$f_1 = -\frac{15\zeta(3)}{64\pi^4}. \quad (5.23)$$

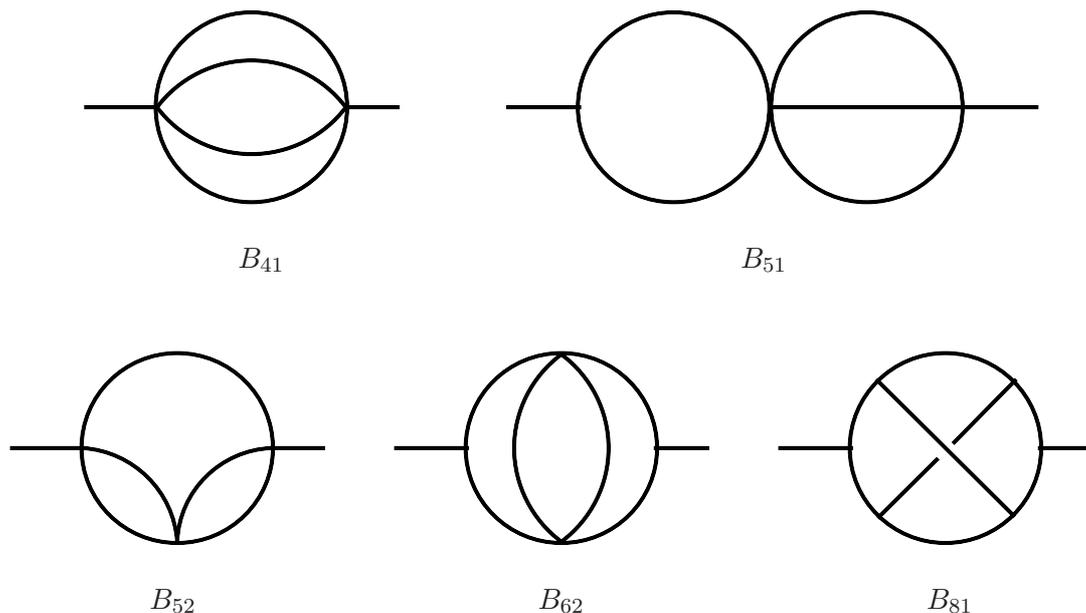


Figure 2. Master integrals appearing in the reduction of the Feynman diagrams of figure 1. They are associated to loop integrals consisting of scalar propagators only (i.e. without non-trivial numerators).

To compute the remaining two diagrams, we employ standard techniques [42] to reduce any 3-loop loop integral to a linear combination of “master integrals”, whose ϵ -expansion can be found in the literature. We will see in a moment that the only master integrals that we need are those that correspond to the topologies shown in figure 2. For the convenience of the reader, we report here their ϵ -expansion up to the order needed for our computation. We use the conventions of [43]

$$B_{41} = p^{4-6\epsilon} \frac{(4\pi)^{3\epsilon-6}}{\Gamma(1-\epsilon)^3} \left(\frac{1}{36\epsilon} + \frac{71}{216} + \frac{3115}{1296}\epsilon - \left(\frac{7\zeta(3)}{9} - \frac{109403}{7776} \right) \epsilon^2 + \dots \right), \quad (5.24)$$

$$B_{51} = p^{2-6\epsilon} \frac{(4\pi)^{3\epsilon-6}}{\Gamma(1-\epsilon)^3} \left(-\frac{1}{4\epsilon^2} - \frac{17}{8\epsilon} - \frac{183}{16} + \left(3\zeta(3) - \frac{1597}{32} \right) \epsilon + \dots \right), \quad (5.25)$$

$$B_{52} = p^{2-6\epsilon} \frac{(4\pi)^{3\epsilon-6}}{\Gamma(1-\epsilon)^3} \left(-\frac{1}{3\epsilon^2} - \frac{10}{3\epsilon} - \frac{64}{3} + \left(\frac{22\zeta(3)}{3} - 112 \right) \epsilon + \dots \right), \quad (5.26)$$

$$B_{62} = p^{-6\epsilon} \frac{(4\pi)^{3\epsilon-6}}{\Gamma(1-\epsilon)^3} \left(\frac{1}{3\epsilon^3} + \frac{7}{3\epsilon^2} + \frac{31}{3\epsilon} + \left(\frac{8\zeta(3)}{3} + \frac{103}{3} \right) + \dots \right), \quad (5.27)$$

$$B_{81} = p^{-4-6\epsilon} \frac{(4\pi)^{3\epsilon-6}}{\Gamma(1-\epsilon)^3} \left(20\zeta(5) + \dots \right). \quad (5.28)$$

The contributions coming from the diagrams D_1 and D_2 in momentum space will be denoted by $\tilde{D}_1(p)$ and $\tilde{D}_2(p)$ respectively. At the end of the computation, we transform back to position space using the formula

$$\int \frac{d^d p}{(2\pi)^d} \frac{e^{-ipx}}{(p^2)^{2-k+\alpha\epsilon}} = \frac{2^{2k-4}}{\pi^2} (-1)^k (k-1)!(k-2)! \alpha \frac{\epsilon}{(x^2)^{k-(\alpha+1)\epsilon}} \left(1 + O(\epsilon) \right). \quad (5.29)$$

This formula tells us that we only need to determine the $1/\epsilon$ term in the Feynman diagrams of interest, since they are the only ones that can contribute to the finite part of the position space correlator (see also [40, 41]). We will explicitly show that all the higher-order poles cancel exactly between the two diagrams D_1 and D_2 , as expected from extended supersymmetry.

We first examine the diagram D_1 . We find that its contribution in momentum space is given by

$$\tilde{D}_1(p) = -8 g_{YM}^8 \mathcal{C} B_{62}, \tag{5.30}$$

where B_{62} is the master integral associated to the topology of the corresponding diagram in figure 2 and \mathcal{C} was defined in (5.17). Since the diagram is already in the “master integral” form, we do not need to further reduce it and we can directly use the result in equation (5.27).

The Feynman diagram D_2 is more complicated, but can also be reduced to a linear combination of master integrals as explained above. We used the MATHEMATICA package FIRE [44] to perform the reduction. The result turns out to be

$$\begin{aligned} \tilde{D}_2(p) = 2 g_{YM}^8 \mathcal{C} & \left(\frac{4(2d-5)(3d-8)(43d^2-288d+480)}{(d-4)^3(2d-7)p^4} B_{41} \right. \\ & + \frac{14(d-3)(3d-10)(3d-8)}{(d-4)^2(2d-7)p^2} B_{51} - \frac{96(d-3)^2}{(d-4)^2 p^2} B_{52} \\ & \left. - \frac{(7d^2-35d+38)}{(d-4)(2d-7)} B_{62} + \frac{(d-4)p^4}{14-4d} B_{81} \right). \end{aligned} \tag{5.31}$$

Combining the results in equations (5.24)–(5.28), we obtain

$$\tilde{D}_1(p) + \tilde{D}_2(p) = \left(-\frac{8\zeta(3)}{(4\pi)^6 \epsilon} g_{YM}^8 \mathcal{C} + \dots \right) \frac{1}{p^{6\epsilon}}, \tag{5.32}$$

where the ellipses denote terms of order ϵ^0 or higher. It is pleasing to see that the $1/\epsilon^3$ and $1/\epsilon^2$ poles precisely cancel, as well as all the non- $\zeta(3)$ contributions to the simple pole. Finally, we use equation (5.29) to transform back to position space, so our final result reads

$$D_1(x, y) + D_2(x, y) = -\frac{6\zeta(3)}{64\pi^4} g_{YM}^4 \mathcal{C} S^{(0)}(x-y)^2. \tag{5.33}$$

Comparing with (5.18), we immediately get

$$f_2 = -\frac{6\zeta(3)}{64\pi^4}. \tag{5.34}$$

Using the results (5.23) and (5.34) we can confirm the relation (5.20), which — as was explained around equation (5.19) — implies the validity of the tt^* equations for the entire chiral ring 2-point functions g_{2n} up to the relevant order!

Moreover, using equation (5.19) we are able to provide an independent derivation of the g_{YM}^4 perturbative correction to the Zamolodchikov metric

$$\langle \phi_2(x) \bar{\phi}_2(y) \rangle = \frac{3\pi^2}{8} S^{(0)}(x-y)^2 \left(1 - \frac{45\zeta(3)}{4\pi^2} \frac{1}{(\text{Im}\tau)^2} + \dots \right). \tag{5.35}$$

Recalling that the tree-level propagator is given by equation (5.1), we find perfect agreement with the result from localization (4.21) and the prediction of [7].

5.2 SU(N) SCQCD at tree level

We continue with a tree-level investigation of the tt^* equations for the general SU(N) group. The 2- and 3-point functions entering in (4.50) can be computed directly by straightforward Wick contractions. Examples of such computations will be provided below.

However, before we enter these examples it is worth making first the following general point. Although the explicit implementation of Wick contractions can be rather cumbersome with complicated combinatorics, it is trivial to obtain the τ -dependence of the 2-point function at leading order in the weak coupling limit. In general,

$$g_{K\bar{M}} \Big|_{\text{tree}} = \frac{1}{(\text{Im}\tau)^{\Delta_K}} \tilde{g}_{K\bar{M}} \tag{5.36}$$

where $\tilde{g}_{K\bar{M}}$ is coupling constant independent and contains the combinatorics from the contractions of the traces. From this expression the l.h.s. of the tt^* equations (4.50) follows trivially as

$$\partial_{\bar{\tau}} \left(g^{\bar{M}L} \partial_{\tau} g_{K\bar{M}} \right) \Big|_{\text{tree}} = -\frac{\Delta_K}{(\tau - \bar{\tau})^2} \delta_K^L = \frac{R}{8(\text{Im}\tau)^2} \delta_K^L \tag{5.37}$$

where we set $\Delta_K = R/2$.

The r.h.s. of the tt^* equations (4.50) has the form

$$[C_2, \bar{C}_2]_K^L - g_2 \delta_K^L . \tag{5.38}$$

Notice that the *tree level* 2- and 3-point functions in this expression are exactly the same as the ones we encountered in section (3.6) in the context of $\mathcal{N} = 4$ SYM theory. As a result, we can use the identity (3.29) to recast (5.38) into the simpler form

$$[C_2, \bar{C}_2]_K^L - g_2 \delta_K^L \Big|_{\text{tree}} = \delta_K^L \frac{R}{\dim \mathcal{G}} g_2 \Big|_{\text{tree}} = \frac{R}{N^2 - 1} \frac{N^2 - 1}{8} \frac{1}{(\text{Im}\tau)^2} \delta_K^L = \frac{R}{8(\text{Im}\tau)^2} \delta_K^L . \tag{5.39}$$

We used the fact that for the SU(N) theories $\dim \mathcal{G} = N^2 - 1$. Comparing the l.h.s. (5.37) and the r.h.s. (5.39) we find that the tt^* equations are obeyed at tree level for any SU(N) $\mathcal{N} = 2$ SCFT and for all sectors of charge R in the chiral ring.

The reader should appreciate that the short argument we have just presented is simpler than the general proof of the tt^* equations in [4] because it makes explicit use of the special properties of correlators in a free CFT, such as the tree-level identity (3.29), and its proof in appendix C.

5.3 SU(3) examples

To illustrate the content of the above equations and the new features of the SU(N) tt^* equations ($N \geq 3$) (compared to the SU(2) case) we consider a few sample tree-level computations in the SU(3) theory.

The SU(3) $\mathcal{N} = 2$ SCQCD theory possesses two chiral ring generators, ϕ_2 and ϕ_3 . We normalize ϕ_2 as in (5.3) and ϕ_3 as

$$\phi_3 = \frac{\mathcal{N}_3}{8} \text{Tr}[\varphi^3] \tag{5.40}$$

with an arbitrary τ -independent normalization constant \mathcal{N}_3 .

The tt^* equation (4.51) applied to scaling dimension $\Delta = 3$ gives

$$g_5 = g_3 \partial_\tau \partial_{\bar{\tau}} \log g_3 + g_2 g_3 . \quad (5.41)$$

g_5 is the 2-point function coefficient for the single chiral primary $\phi_5 = \phi_2 \phi_3$ at $\Delta = 5$. The explicit tree-level computation gives

$$g_3 = 5 \mathcal{N}_3^2 \frac{1}{(\text{Im}\tau)^3} , \quad (5.42)$$

$$g_5 = \frac{35}{4} \mathcal{N}_3^2 \frac{1}{(\text{Im}\tau)^5} \quad (5.43)$$

in agreement with the differential equation (5.41) for any \mathcal{N}_3 .

More involved examples with non-trivial degeneracy arise as we move up in scaling dimension. For instance, at scaling dimension $\Delta = 6$ there are two degenerate chiral primary fields

$$\phi_{(3,0)} = (\phi_2)^3 , \quad \phi_{(0,2)} = (\phi_3)^2 . \quad (5.44)$$

A tree-level computation shows that these fields are not orthogonal. The 2×2 matrix of 2-point function coefficients is

$$G_6 = \begin{pmatrix} g_{(3,0)\overline{(3,0)}} & g_{(3,0)\overline{(0,2)}} \\ g_{(0,2)\overline{(3,0)}} & g_{(0,2)\overline{(0,2)}} \end{pmatrix} = \frac{1}{4(\text{Im}\tau)^6} \begin{pmatrix} 45 & 15\mathcal{N}_3^2 \\ 15\mathcal{N}_3^2 & 425\mathcal{N}_3^4 \end{pmatrix} . \quad (5.45)$$

Similarly, at scaling dimension $\Delta = 8$ there are two degenerate fields

$$\phi_{(4,0)} = (\phi_2)^4 , \quad \phi_{(1,2)} = \phi_2 (\phi_3)^2 \quad (5.46)$$

with the 2-point function coefficient matrix

$$G_8 = \begin{pmatrix} g_{(4,0)\overline{(4,0)}} & g_{(4,0)\overline{(1,3)}} \\ g_{(1,3)\overline{(4,0)}} & g_{(1,3)\overline{(1,3)}} \end{pmatrix} = \frac{1}{4(\text{Im}\tau)^8} \begin{pmatrix} 315 & 105\mathcal{N}_3^2 \\ 105\mathcal{N}_3^2 & 1085\mathcal{N}_3^4 \end{pmatrix} . \quad (5.47)$$

The tt^* equation (4.50) at $\Delta = 6$ is a matrix equation of the form

$$\frac{3}{2} \frac{1}{(\text{Im}\tau)^2} \delta_K^L = (G_8)_{(1,0)+K, \overline{(1,0)+\bar{R}}} (G_6^{-1})^{\bar{R}L} - (G_6)_{K\overline{(3,0)}} (g_4)^{-1} \delta_{(3,0)}^L - g_2 \delta_K^L . \quad (5.48)$$

One can verify that the algebraic equations (5.48) are satisfied by the tree-level expressions (5.45), (5.47) and (5.50) for $N = 3$, $n = 1, 2$.

5.4 $SU(N)$ observations

After the implementation of (5.36) the $SU(N)$ tt^* equations (4.50) take the following algebraic form at tree-level

$$\frac{\Delta}{4} \tilde{g}^{\bar{M}_\Delta L_\Delta} \tilde{g}_{K_\Delta \bar{M}_\Delta} = \tilde{g}_{K_\Delta+2, \bar{R}_\Delta+2} \tilde{g}^{\bar{R}_\Delta L_\Delta} - \tilde{g}_{K_\Delta \bar{R}_\Delta} \tilde{g}^{\bar{R}_\Delta-2, L_\Delta-2} - \tilde{g}_2 \delta_{K_\Delta}^{L_\Delta} . \quad (5.49)$$

This equation as an explicit index version of (5.39) is the $SU(N)$ generalization of the $SU(3)$ matrix equation (5.48) above. Although it is just a simple tree-level version of the full equations (5.36) it continues to carry much of their complexity and encodes non-trivial information about the combinatorics of free field Wick contractions of 2-point functions of arbitrary multi-trace operators in the chiral ring.

Focusing on the 2-point functions g_{2n} of the chiral primary fields $\phi_2 \propto (\text{Tr}[\phi^2])^n$ we have observed experimentally (by direct MATHEMATICA computation of free field Wick contractions in a considerable range of values of n, N), that the following mathematical identity holds²⁴

$$\begin{aligned}
 g_{2n} &= \frac{1}{16^n (\text{Im}\tau)^{2n}} \sum_{a_1, \dots, a_{2n}=1}^{N^2-1} \delta_{a_1 a_2} \cdots \delta_{a_{2n-1} a_{2n}} \sum_{\sigma \in \mathcal{S}_{2n}} \delta^{\sigma(a_1)\sigma(a_2)} \cdots \delta^{\sigma(a_{2n-1})\sigma(a_{2n})} \\
 &= \frac{n!}{4^n} \binom{N^2-1}{2}_n \frac{1}{(\text{Im}\tau)^{2n}}.
 \end{aligned}
 \tag{5.50}$$

In this formula $(x)_n$ denotes the Pochhammer symbol

$$(x)_n = x(x+1) \cdots (x+n-1), \tag{5.51}$$

\mathcal{S}_{2n} refers to the group of permutations of $2n$ elements and σ is the generic permutation in this group.

Although currently we do not have an analytic proof of this formula, we expect that it holds generally for any value of the positive integers $n \geq 1, N > 1$. For example, for $N = 2$ (the $SU(2)$ case, where there are no degeneracies and equations (5.8) make up the full set of tt^* equations) one can easily see that the Pochhammer formula (5.50) reproduces the result (5.5), (5.7). As another explicit check, notice that all the values of g_{2n} (for $n = 1, 2, 3, 4$) in the previous $SU(3)$ section are consistent with (5.50).

The intriguing fact about (5.50) is that it predicts values of g_{2n} (at all $N > 1$) that obey the tree-level version of the same semi-infinite Toda chain

$$\partial_\tau \partial_{\bar{\tau}} \log g_{2n} = \frac{g_{2n+2}}{g_{2n}} - \frac{g_{2n}}{g_{2n-2}} - g_2 \tag{5.52}$$

that followed directly from the tt^* equations in the $SU(2)$ case. This is not an obvious property of the matrix equations (5.49) at arbitrary N and hints at a hidden underlying structure that will be useful to understand further. Moreover, if (5.52) holds for g_{2n} at all N beyond tree-level it would allow us to use the $SU(N)$ S^4 partition function to obtain a complete non-perturbative solution of the two-point functions $\langle (\phi_2)^n(x) (\phi_2)^n(0) \rangle$ in the $SU(N)$ theory similar to the $SU(2)$ case above. These issues and their implications for the structure of the $SU(N)$ tt^* equations (as well as possible extensions to more general chiral primary fields) are currently under investigation.

²⁴We are not aware of a previous appearance of this identity in the literature. Related work that may be useful in proving it has appeared in [45].

6 Summary and prospects

We argued that the combination of supersymmetric localization techniques and exact relations like the tt^* equations opens the interesting prospect for a new class of exact non-perturbative results in superconformal field theories.

In this paper we focused on four-dimensional $\mathcal{N} = 2$ superconformal field theories. Combining the tt^* equations of ref. [4] with the recent proposal [7] that relates the Zamolodchikov metric on the moduli space of $\mathcal{N} = 2$ SCFTs to derivatives of the S^4 partition function we found useful exact relations between 2- and 3-point functions of $\mathcal{N} = 2$ chiral primary operators. In some cases, like the case of $SU(2)$ SCQCD, the tt^* equations form a semi-infinite Toda chain and a unique solution can be determined easily in terms of the well-known S^4 partition function of the $SU(2)$ theory. The solution provides exact formulae for the 2- and 3-point functions of all the chiral primary fields of this theory as a function of the (complexified) gauge coupling. We verified independently several aspects of this result with explicit computations in perturbation theory up to 2-loops.

In more general situations, e.g. the $SU(N)$ SCQCD theory, the structure of the tt^* equations is further complicated by the non-trivial mixing of degenerate chiral primary fields. We provided preliminary observations of an underlying hidden structure in these equations that is worth investigating further. The minimum data needed to determine a unique complete solution of the general $SU(N)$ tt^* equations, and the structure of that solution, remains an interesting largely open question. It would be useful to know if a few fundamental general properties, like positivity of 2-point functions over the entire conformal manifold, combined with some ‘boundary’ data, e.g. weak coupling perturbative data, are enough to specify a unique solution.

An exact solution of the tt^* equations would have several important implications. In section 3.5 we argued that the explicit knowledge of 2- and 3-point functions of chiral primary operators can be used to determine also the generic extremal n -point correlation function of these operators. In a different direction these results can also be used as input in a general bootstrap program in $\mathcal{N} = 2$ SCFTs to determine wider classes of correlation functions, spectral data *etc.* Interesting work along similar lines appeared recently in [46]. For the case of $\mathcal{N} = 2$ SCQCD we note that the methods developed in [46] (e.g. the correspondence with two-dimensional chiral algebras) are best suited for a discussion of the mesonic (Higgs branch) chiral primaries and are less useful for the $\mathcal{N} = 2$ (Coulomb branch) chiral primaries analyzed in the present paper. As a result, our approach can be viewed in this context as a different method providing useful complementary input.

In the main text we considered mostly the case of $\mathcal{N} = 2$ SCQCD theories as an illustrative example. It would be interesting to extend the analysis to other four-dimensional $\mathcal{N} = 2$ theories, e.g. other Lagrangian theories, or the class \mathcal{S} theories [32, 33]. Eventually, one would also like to move away from $\mathcal{N} = 2$ supersymmetry and explore situations with less supersymmetry where quantum dynamics are known to exhibit a plethora of new effects. Two obvious hurdles in this direction are the following: (i) it is known that the S^4 partition function of $\mathcal{N} = 1$ theories is ambiguous [7]; (ii) it is currently unknown whether there is any useful generalization of the tt^* equations to $\mathcal{N} = 1$ theories [4]. A related

question has to do with the extension of these techniques to theories of diverse amounts of supersymmetry in different dimensions, e.g. three-dimensional SCFTs.

Originally, topological-antitopological fusion and the tt^* equations [2, 3] were also useful in analyzing two-dimensional $\mathcal{N} = (2, 2)$ massive theories. Therefore, another interesting direction is to explore whether a similar application of the tt^* equations is also possible in four dimensions. Massive four-dimensional $\mathcal{N} = 2$ theories, like $\mathcal{N} = 2$ SYM theory, would be an interesting example. Related questions were discussed in [5].

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A Collection of useful facts about S^4 partition functions

In section 4 we make use of the S^4 partition function of $\mathcal{N} = 2$ $SU(N)$ SYM theories coupled to $2N$ hypermultiplets. Some of the pertinent details of this partition function are summarized for the convenience of the reader in this appendix.

The S^4 partition function of $\mathcal{N} = 2$ gauge theories was computed using supersymmetric localization in [8] and the general result takes the form

$$Z_{S^4}(\tau, \bar{\tau}) = \frac{1}{|\mathcal{W}|} \int da \Delta(a) Z_{\text{tree}}(a) Z_{1\text{-loop}}(ia) |Z_{\text{inst}}(ia, r^{-1}, r^{-1}, q)|^2, \tag{A.1}$$

where the integral is performed over the Cartan subalgebra of the gauge group \mathcal{G} ,

$$\Delta(a) = \prod_{\alpha \in \text{roots of } G} \alpha(a) \tag{A.2}$$

is the Vandermond determinant, Z_{tree} is the classical tree-level contribution, $Z_{1\text{-loop}}$ is the 1-loop contribution and Z_{inst} is Nekrasov’s instanton partition function [36]. r denotes the radius of S^4 and $q = e^{2\pi i\tau}$. $|\mathcal{W}|$ is the order of the Weyl group \mathcal{G} .

In the case of the $SU(N)$ $\mathcal{N} = 2$ SCQCD theories the elements of the Cartan subalgebra are parametrized by N real parameters a_i ($i = 1, \dots, N$) satisfying the zero-trace condition

$\sum_{i=1}^N a_i = 0$, and

$$\Delta(a) = \prod_{i \neq j}^N (a_i - a_j), \quad (\text{A.3})$$

$$Z_{\text{tree}} = e^{-2\pi \text{Im}(\tau) \sum_{i=1}^N a_i^2}, \quad (\text{A.4})$$

$$Z_{1\text{-loop}} = \frac{\prod_{i \neq j} H(ir(a_i - a_j))}{\prod_{j=1}^N (H(ir a_j) H(-ir a_j))^{2N}}. \quad (\text{A.5})$$

The instanton factor Z_{inst} has a more complicated form. General expressions can be found in [33, 36, 49]. In the main text we set $r = 1$ for the radius of S^4 .

The special function H that appears in the one-loop contribution is related to the Barnes G -function [35]

$$G(1+z) = (2\pi)^{\frac{z}{2}} e^{-((1+\gamma z^2)+z)/2} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right)^n e^{-z + \frac{z^2}{2n}} \quad (\text{A.6})$$

(γ is the Euler constant) through the defining equation

$$H(z) = G(1+z)G(1-z) = e^{-(1+\gamma)z^2} \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)^n \prod_{n=1}^{\infty} e^{\frac{z^2}{n}}. \quad (\text{A.7})$$

B Conventions in $SU(N)$ $\mathcal{N} = 2$ SCQCD

Here we collect our conventions for the $\mathcal{N} = 2$ SCQCD theories with gauge group $SU(N)$.

The $\mathcal{N} = 2$ chiral ring of the $SU(N)$ SCQCD theory is generated by the single-trace operators

$$\phi_\ell \propto \text{Tr}[\varphi^\ell], \quad \ell = 2, 3, \dots, N. \quad (\text{B.1})$$

The descendant

$$\mathcal{O}_\tau = Q^4 \cdot \phi_2 \quad (\text{B.2})$$

of the chiral primary ϕ_2 , that has the lowest scaling dimension $\Delta = 2$, controls the exactly marginal deformation

$$\delta S = \frac{\delta\tau}{4\pi^2} \int d^4x \mathcal{O}_\tau(x) + \frac{\delta\bar{\tau}}{4\pi^2} \int d^4x \bar{\mathcal{O}}_\tau(x). \quad (\text{B.3})$$

The complex marginal coupling is $\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g_{YM}^2}$, and we normalize the elementary fields of the theory so that the full Lagrangian in components takes the form

$$\mathcal{L} = \mathcal{L}_{\text{vector}} + \mathcal{L}_{\text{hyper}}, \quad (\text{B.4})$$

$$\begin{aligned} \mathcal{L}_{\text{vector}} = & -\frac{1}{g_{YM}^2 N} \text{Tr} \left(\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{g_{YM}^2 \theta}{32\pi^2} F_{\mu\nu} \tilde{F}^{\mu\nu} + i \bar{\lambda}_I \bar{\sigma}^\mu \mathcal{D}_\mu \lambda^I + \mathcal{D}_\mu \varphi \mathcal{D}^\mu \bar{\varphi} \right. \\ & \left. + i\sqrt{2} (\epsilon_{I\mathcal{J}} \lambda^I \lambda^{\mathcal{J}} \bar{\varphi} - \epsilon^{\mathcal{I}\mathcal{J}} \bar{\lambda}_I \bar{\lambda}_{\mathcal{J}} \varphi) + \frac{1}{2} [\varphi, \bar{\varphi}]^2 \right), \end{aligned} \quad (\text{B.5})$$

$$\begin{aligned} \mathcal{L}_{hyper} = & - \left(\mathcal{D}^\mu \bar{Q}^{\mathcal{I}} \mathcal{D}_\mu Q_{\mathcal{I}} + i \bar{\psi} \bar{\sigma}^\mu \mathcal{D}_\mu \psi + i \tilde{\psi} \bar{\sigma}^\mu \mathcal{D}_\mu \tilde{\psi} \right. \\ & + i \sqrt{2} \left(\epsilon^{\mathcal{I}\mathcal{J}} \bar{\psi} \bar{\lambda}_{\mathcal{I}} Q_{\mathcal{J}} - \epsilon_{\mathcal{I}\mathcal{J}} \bar{Q}^{\mathcal{I}} \lambda^{\mathcal{J}} \psi + \tilde{\psi} \lambda^{\mathcal{I}} Q_{\mathcal{I}} - \bar{Q}^{\mathcal{I}} \bar{\lambda}_{\mathcal{I}} \tilde{\psi} + \tilde{\psi} \varphi \psi - \bar{\psi} \bar{\varphi} \tilde{\psi} \right) \\ & \left. + \bar{Q}_{\mathcal{I}} (\bar{\varphi} \varphi + \varphi \bar{\varphi}) Q^{\mathcal{I}} + g_{YM}^2 \mathcal{V}(Q) \right), \end{aligned} \quad (\text{B.6})$$

with

$$\begin{aligned} \mathcal{V}(Q) = & (\bar{Q}_a^{\mathcal{I} i} Q_{\mathcal{I}^a j}) (\bar{Q}_b^{\mathcal{J} j} Q_{\mathcal{J}^b i}) - \frac{1}{2} (\bar{Q}_a^{\mathcal{I} i} Q_{\mathcal{J}^a j}) (\bar{Q}_b^{\mathcal{J} j} Q_{\mathcal{I}^b i}) \\ & + \frac{1}{N} \left(\frac{1}{2} (\bar{Q}_a^{\mathcal{I} i} Q_{\mathcal{I}^a i}) (\bar{Q}_b^{\mathcal{J} j} Q_{\mathcal{J}^b j}) - (\bar{Q}_a^{\mathcal{I} i} Q_{\mathcal{J}^a i}) (\bar{Q}_b^{\mathcal{J} j} Q_{\mathcal{I}^b j}) \right) \end{aligned} \quad (\text{B.7})$$

the D-term potential for the hypermultiplet complex scalars Q .

We use standard notation where

$$F_{\mu\nu} = F_{\mu\nu}^a T^a, \quad F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + f^{abc} A_\mu^b A_\nu^c, \quad \tilde{F}_{\mu\nu}^a = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma} \quad (\text{B.8})$$

and T^a ($a = 1, \dots, N^2 - 1$) is a basis of the Lie algebra generators of $SU(N)$ with the normalization

$$\text{Tr} [T_{adj}^a T_{adj}^b] = N \delta^{ab}, \quad \text{Tr} [T_{\square}^a T_{\square}^b] = \frac{1}{2} \delta^{ab} \quad (\text{B.9})$$

in the adjoint and fundamental representations respectively. The gauge-covariant derivatives are

$$\mathcal{D}_\mu = \partial_\mu + i A_\mu. \quad (\text{B.10})$$

$\mathcal{I}, \mathcal{J}, \dots = \pm$ are $SU(2)_R$ indices raised and lowered with the antisymmetric symbols $\epsilon_{\mathcal{I}\mathcal{J}}$. $i, j = 1, \dots, 2N$ in (B.7) are flavor indices. The $\mathcal{N} = 2$ vector fields in the adjoint representation include the bosons A_μ, φ and the fermions $\lambda^{\mathcal{I}}$. The $2N$ $\mathcal{N} = 2$ hypermultiplet fields in the fundamental representation include $2N$ complex bosons $Q_{\mathcal{I}}$ and $2N$ fermion doublets $(\psi, \tilde{\psi})$.

In this normalization all the τ dependence is loaded on the vector part of the Lagrangian.²⁵ This is consistent with (B.2), (B.3) and the identification

$$\begin{aligned} \mathcal{O}_\tau = & \frac{i\pi}{2N} \text{Tr} \left[\frac{1}{8} F_{\mu\nu+} F^{\mu\nu+} + i \bar{\lambda}_{\mathcal{I}} \bar{\sigma}^\mu \mathcal{D}_\mu \lambda^{\mathcal{I}} - \bar{\varphi} \mathcal{D}_\mu \mathcal{D}^\mu \varphi - \frac{1}{2} D^2 - \bar{F} F \right. \\ & \left. + \sqrt{2} (\epsilon_{\mathcal{I}\mathcal{J}} \lambda^{\mathcal{I}} \lambda^{\mathcal{J}} \bar{\varphi} - \epsilon^{\mathcal{I}\mathcal{J}} \bar{\lambda}_{\mathcal{I}} \bar{\lambda}_{\mathcal{J}} \varphi) - D[\varphi, \bar{\varphi}] \right]. \end{aligned} \quad (\text{B.11})$$

$F_{\mu\nu\pm} = F_{\mu\nu} \mp i \tilde{F}_{\mu\nu}$ is the (anti)self-dual part of the gauge field strength. D and F are respectively the D - and F -auxiliary fields of the $\mathcal{N} = 1$ vector and $\mathcal{N} = 1$ chiral multiplet that make up the $\mathcal{N} = 2$ vector multiplet.

²⁵The last term of the hypermultiplet interactions, $g_{YM}^2 \mathcal{V}(Q)$, appears to be g_{YM} -dependent, but this is only so after we integrate out the D auxiliary field. Before integrating out D the Lagrangian \mathcal{L}_{vector} has a term $\frac{1}{2g_{YM}^2} D^2$ and \mathcal{L}_{hyper} has no explicit g_{YM} -dependence.

C An (eccentric) proof of equation (3.29)

In this section we will give a proof of equation (3.29). Instead of giving a direct combinatoric proof, we will proceed as follows. Consider the $\mathcal{N} = 4$ SYM theory with gauge group \mathcal{G} , in the free limit. This can also be thought of as an $\mathcal{N} = 2$ SCFT. This theory has 6 real scalars $\Phi^I, I = 1, \dots, 6$. We consider the complex combination

$$\varphi = \Phi^1 + i\Phi^2. \tag{C.1}$$

The chiral primary, whose descendant is the marginal operator, has the form

$$\phi_2 = \mathcal{N} \text{Tr}[\varphi^2]. \tag{C.2}$$

where the normalization constant \mathcal{N} was determined in previous sections. We define the 2-point function

$$g_{2\bar{2}} = \langle \phi_2 \bar{\phi}_2 \rangle. \tag{C.3}$$

A general chiral primary of charge R can be written as a multitrace operator of the form

$$\phi_K \propto \text{Tr}[\varphi^{n_1}] \dots \text{Tr}[\varphi^{n_k}], \tag{C.4}$$

where $2 \sum n_i = R$. The trace is taken in the adjoint of \mathcal{G} . Similarly we define the anti-chiral primaries and the matrix of 2-point functions $g_{K\bar{L}} = \langle \phi_K \bar{\phi}_L \rangle$. Notice that the matrix of 2-point functions $g_{K\bar{L}}$ is not diagonal in the basis of multitrace operators and is somewhat cumbersome to compute by considering Wick contractions.

Our starting point is to consider the following 4-point function

$$A = \langle \phi_2(x_1) \bar{\phi}_2(x_2) \phi_K(x_3) \bar{\phi}_L(x_4) \rangle. \tag{C.5}$$

Here K, L can be different chiral primaries, but by R-charge conservation this 4-point function is nonzero only if K, L have the same R-charge. By Wick contractions it is not hard to see that there are only three possible structures of the coordinate dependence for this correlator. So the general form is

$$A = \frac{p_1}{|x_{12}|^4 |x_{34}|^{2\Delta_K}} + \frac{p_2}{|x_{12}|^2 |x_{14}|^2 |x_{23}|^2 |x_{34}|^{2\Delta_K-2}} + \frac{p_3}{|x_{14}|^4 |x_{23}|^4 |x_{34}|^{2\Delta_K-4}}. \tag{C.6}$$

In principle we can compute the constants p_1, p_2, p_3 by working out the combinatorics of the Wick contractions, however we will try to avoid this. By considering the double OPE in the (13) \rightarrow (24) channel we learn that

$$p_1 + p_2 + p_3 = C_{2K}^P g_{P\bar{Q}} C_{\bar{2}L}^{*\bar{Q}}. \tag{C.7}$$

By considering the OPE in the (12) \rightarrow (34) channel we have

$$p_1 = g_{2\bar{2}} g_{K\bar{L}}. \tag{C.8}$$

Finally from the OPE in the (14) \rightarrow (23) channel we find

$$p_3 = g_{K\bar{N}} C_{\bar{2}U}^{*\bar{N}} g^{\bar{U}V} C_{2V}^R g_{R\bar{L}}. \tag{C.9}$$

Using these results we have completely fixed the 4-point function (C.5) in the free limit, in terms of the 2- and 3-point function coefficients which enter the tt^* equations.

However, the desired equation (3.29) expresses a nontrivial relation among these coefficients. We will now argue that the consistency of the underlying CFT implies the desired relation.

We will establish the relation by the following argument. The tree level correlator (C.5) can be thought of as a correlator in a theory of only $\dim \mathcal{G}$ complex scalar fields.²⁶ This by itself is a consistent conformal field theory with a central charge c_{scalar} which is related to $\dim \mathcal{G}$ by

$$c_{\text{scalar}} = \frac{8}{3} \dim \mathcal{G}. \tag{C.10}$$

To derive equation (C.8) we considered the OPE in the channel (12) \rightarrow (34) and only kept the leading term, i.e. the identity operator. One of the subleading contributions involves conformal block of the stress energy tensor. In any consistent CFT the contribution of this block is fully determined using Ward identities, by the central charge of the CFT and by the conformal dimension of the external operators [20]. Our strategy is to:

- a) isolate the contribution of the conformal block of the stress energy tensor for the 4-point function (C.5), (C.6) written in terms of the data (C.7), (C.8), (C.9) and
- b) demand that this contribution is the same as that predicted by general arguments based on the Ward identities for CFTs. We will discover that this requirement leads to the desired formula (3.29).

We write equation (C.6) in a notation which is somewhat more convenient to perform the conformal block expansion

$$A = \frac{1}{|x_{12}|^4 |x_{34}|^{2\Delta_K}} \left(p_1 + p_2 \frac{u}{v} + p_3 \frac{u^2}{v^2} \right), \tag{C.11}$$

where we have introduced the conformal cross ratios

$$u = \frac{|x_{12}|^2 |x_{34}|^2}{|x_{13}|^2 |x_{24}|^2}, \quad v = \frac{|x_{14}|^2 |x_{23}|^2}{|x_{13}|^2 |x_{24}|^2}. \tag{C.12}$$

It is easy to see that the term $p_1 = g_{2\bar{2}} g_{K\bar{L}}$ is coming from the exchange of the identity operator (the reason that it is not equal to 1 is because our 2-point functions are not normalized to be $\propto \frac{1}{|x|^{2\Delta}}$). With a little work on the conformal block expansion in the $u \rightarrow 0, v \rightarrow 1$ channel, we find that the block of the stress tensor comes with the coefficient

$$A = \frac{1}{|x_{12}|^4 |x_{34}|^{2\Delta_K}} \left(\dots + \frac{2}{3} p_2 u G^{(2)}(1, 1, 4, u, v) + \dots \right), \tag{C.13}$$

where the function $G^{(2)}$ is defined in [20].

²⁶Since we are in the free limit the presence of the other fields does not make any difference to the counting of the Wick combinatorics.

On the other hand, the Ward identities predict [20] that for any consistent CFT if we have the 4-point function $\langle\phi(x_1)\phi(x_2)\phi'(x_3)\phi'(x_4)\rangle$ and we expand it in the (12) \rightarrow (34) channel, then the stress tensor must contribute like

$$\langle\phi(x_1)\phi(x_2)\phi'(x_3)\phi'(x_4)\rangle = \frac{1}{|x_{12}|^4|x_{34}|^{2\Delta_K}} \left(\dots + \frac{16\Delta\Delta'}{9c} g_{\phi\phi} g_{\phi'\phi'} u G^{(2)}(1, 1, 4, u, v) + \dots \right), \tag{C.14}$$

where c is the central charge of the CFT and $g_{\phi\phi}, g_{\phi'\phi'}$ is the normalization of the 2-point functions, which in [20] was taken to be 1. In our case we have $\Delta = 2, \Delta' = \Delta_K = R/2$ and $c \equiv c_{\text{scalar}} = \frac{8}{3} \dim \mathcal{G}$. Putting everything together we find that what we *expect* in a consistent CFT for the 4-point function (C.6) is that the stress-tensor conformal block comes with the coefficient

$$A = \frac{1}{|x_{12}|^4|x_{34}|^{2\Delta_K}} \left(\dots + \frac{2}{3} \frac{R}{\dim \mathcal{G}} p_1 u G^{(2)}(1, 1, 4, u, v) + \dots \right). \tag{C.15}$$

Comparing this to what we found in (C.13) we conclude that consistency of the CFT demands the relation

$$p_2 = \frac{R}{\dim \mathcal{G}} p_1. \tag{C.16}$$

Using the expression (C.7), (C.8) and (C.9) it is straightforward to show that this implies

$$- [C_2, \bar{C}_2]_K^L + g_{2\bar{2}} \delta_K^L \left(1 + \frac{R}{\dim \mathcal{G}} \right) = 0. \tag{C.17}$$

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