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IMPROVED CALCULATION WITH THE AGS PROGRAM FOR THE OFF-MOMENTUM ORBITS

by

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Abstract

The routine MATRIX in the AGS program has been modified in order to compute more precisely the transfer matrices of multipole magnets ($n \geqq 2$) having a radialy variable length, on off-momentum orbits.

The details of this modification are described as well as the consequences of the use of the modified routine for predictions on the beam dynamics for the present and future low-β insertions of the ISR.

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1. INTRODUCTION

Problems have been encountered with the AGS program[1] in finding closed orbits for large momentum deviations on certain machines. An alternative search method was proposed (see Appendix I), which made it possible to find the closed orbits in almost all cases (this method has recently been added as option "VE" inside the AGS program). However, with this new tool, a second type of problem was revealed. Particularly on the ISR machine with ELSA type working lines ($Q_h \sim 8.9$), the calculated closed orbits went locally to infinity for a certain negative value of the momentum deviation. This was in contradiction with the fact that protons with this momentum deviation were observed to circulate in the ISR.

As aperture calculation and collimation studies for future low-β insertions required reliable calculations of the off-momentum orbits, an analysis of the AGS calculation was made.

It was found that errors existed in the calculation of the off-momentum transfer matrices due to an incomplete treatment of the edge effect for those multipole magnets, whose magnetic length depends on the radial position. Although the effect of the radial length variation of the dipole field was included in the program, the edge focusing due to higher order multipoles was not taken into account. Off central orbit, these higher order effects are not at all negligible and their absence results in appreciable errors in the calculated chromaticity. An example is given in paragraph 5.1 which explains the above-mentioned orbit anomaly.

In this report a new version of the AGS routine MATRIX (calculation of the transfer matrices) is proposed, which takes into account the edge effects due to multipole components up to the octupole. Equally the momentum dependence of the bending radius has been added which influences considerably the momentum dependence of gamma transition and gives a better agreement with measured values. Some tests of the consistency of the new computation are presented as well as the consequences of its use to compute the properties of the present ISR and of the future superconducting low-β insertion.

2. TREATMENT OF EDGE EFFECTS IN THE AGS PROGRAM

2.1 Use of the transfer matrices in AGS

The MATRIX routine in AGS prepares the transfer matrices of magnet units and drift spaces for use in two cases :

i) To calculate betatron parameters on the centre line, or (if $\Delta p/p \neq 0$) on an off momentum closed orbit. The quantities used for the computation of the matrix elements are magnetic normalised gradients K. (K is equal to the gradient divided by the magnetic rigidity of the particles[3]; in order to simplify, K will be referred to as gradient.)

ii) To calculate trajectories which are to be used in the closed orbit search for $\Delta p/p \neq 0$. In this case the routine switches over from gradient to field. This procedure is approximate (see Appendix II) for a non-linear machine, since the transfer matrices express the solution of the linear equation of motion and do not in any case give the solution of the non-linear equation of motion.

2.2 Differences between the present and the proposed version of the MATRIX routine

In the following, the existing routine will be referred to as MATRIX 1, and the proposed modified version as MATRIX 2.

The difficulties in calculating trajectories and focusing with combined function magnets originate from two sources :

i) Their length L may be large, (i.e. the β function may vary strongly across the magnets).

ii) L can vary across the aperture of the magnet, i.e. the derivatives of L with respect to the radial position x, L' and L" may be non-zero.

In MATRIX 1 these complications are treated together (p. 12 of ref. 1). In MATRIX 2 the two complications are separated as follows :

a) For long magnets with gradient K and gradient derivatives with respect to the radial position x, K' and K", the transfer matrix is calculated according to ref. 1, but using rectangular magnets with length L equal to the magnetic length on central orbit (L' = L" = 0).

b) The edge effects resulting from the difference between the real and the rectangular magnet are built in by adding thin lenses at the extremities of the long magnet.

The edge effect due to the length variation of the dipole field is treated as in MATRIX 1. The edge effects of the remaining multipole field (only on off momentum orbits) are treated as follows :

The radial length variation is described by :

$$L(x) = \sum_n L^{(n)} \cdot x^n/n! \qquad \text{with } n = 1, 2$$

The value of the magnetic field is :

$$F(x) = x \sum_m K^{(m)} \cdot x^m/(m+1)! \qquad \text{with } m = 0, 1, 2$$

The integrated field, which is used to calculate trajectories, may then be written as :

$$L(x) \cdot F(x) = x \cdot \sum_{n,\, m} \frac{L^{(n)}\ K^{(m)}\ x^{n+m}}{n!\ (m+1)!}, \tag{1}$$

and the integrated gradient, used to calculate the focusing along a given trajectory, may be written as :

$$\frac{d}{dx}\,[L(x) \cdot F(x)] = \sum_{n,\, m} \frac{L^{(n)}\ K^{(m)}\ x^{n+m}}{n!\ (m+1)!}\,(n+m+1). \tag{2}$$

In MATRIX 2 this is written in the Fortran code as :

$$\ldots\ L^{(n)} \cdot K^{(m)} \cdot x^{(n+m)} \cdot f_{(n+m)} \cdot (n+m+1), \tag{3}$$

where $f_i = 1$ for focusing calculation,

$f_i = 1/(i+1)$ for trajectory calculation.

At this point the new version differs considerably from the old version. In MATRIX 1 the f-coefficients are placed in a different way, which would be written in the thin lens approximation as :

$$\ldots\ L^{(n)} . K^{(m)} . x^{(n+m)} . f_m . (m+1). \tag{4}$$

In the trajectory calculation, equations (3) and (4) are identical. However, in the case of focusing calculation, (4) is different from (3), and corresponds to :

$L(x)\,\dfrac{dF(x)}{dx}$ rather than $\dfrac{d}{dx}\,[L(x) \cdot F(x)]$; i.e. the terms

$\dfrac{dL(x)}{dx}\ F(x) = (L' + L''\,x)(Kx + \dfrac{K'x^2}{2} + \dfrac{K''x^3}{6})$ are missing.

3. CORRECTION TO THE BENDING RADIUS

The equation of horizontal motion for large momentum deviations has been established in ref. 2 (equation 2.52 a). It is :

$$x'' - \left[K_p - \frac{1}{\rho_p^{\,2}} \right] x = \frac{1}{\rho_p} \cdot \frac{\Delta p}{p} ,$$

where $K_p = \dfrac{K}{1 + \dfrac{\Delta p}{p}}$ and $\rho_p = \rho \left(1 + \dfrac{\Delta p}{p}\right)$

In MATRIX 1 K_p is used, but ρ (value at the centre orbit) is used instead of ρ_p.

The consequence is an incorrect dependance of the position of the closed orbit, and hence gamma transition, on $\Delta p/p$ (see fig. 1).

In MATRIX 2 ρ has been replaced by ρ_p.

4. COMPARISON BETWEEN THE RESULTS OBTAINED USING THE PRESENT AND THE PROPOSED VERSION OF THE MATRIX ROUTINE

4.1 Agreement between "trajectory" and "focusing" Q_h

Once a linear machine has been computed, it is common practice to introduce non-linear field components in certain magnets (e.g. main magnets of the ISR) in order to obtain specified variations of the betatron parameters with the momentum deviation.

In this process an incorrect focusing calculation on off-momentum orbits results in wrong calculated values of the non-linear field components, which has two consequences :

i) The positions of the off-momentum closed orbits are incorrect;

ii) The values of Q_h obtained from the closed orbit calculation as shown in appendix I ("trajectory Q_h" computed from the Twiss matrix which is obtained in the course of the orbit computation) are different from those computed by the focusing calculation ("focusing Q_h") which is not consistent.

For example for the ISR with the steel low-β insertion, when the chromaticity correction is made by means of MATRIX 1, the trajectory Q_h is very different from the focusing Q_h (see 5.1.1).

Using MATRIX 2 on this machine, the two methods of computing Q_h produce values which differ by less than 0.001 for $\Delta p/p$ as large as $\pm$ 0.02.

For machines in which the multipole components do not have radially variable lengths, MATRIX 1 and MATRIX 2 obviously give the same results. In this case, the values of Q_h computed by the two methods differ by less than 10^{-5} in the useful range of momentum deviation, in machines as different as LEP and ISR.

4.2 Correction of the chromaticity to the first order

In the case of a machine consisting of combined function magnets only, the sextupole component K' needed to cancel the chromaticity to the first order in $\Delta p/p$ is, neglecting $\frac{1}{\rho^2}$ with respect to K :

$$K' = K / \bar{\alpha}p \qquad (5)$$

K is the quadrupole component and $\bar{\alpha}p$ the average value of the dispersion function in the elements which have the same K and K'.

If the quadrupole component has a radially variable length with derivative L', it introduces an integrated sextupole component equal to :

$$2\ K\ L'.$$

Assuming that $\bar{\alpha}p$ is not very different from the mean value between the entrance and the exit value of αp, the sextupole component needed to cancel the chromaticity is now :

$$K'_c = K' - 2\ K\ L' / L \qquad (6)$$

where K' is given by (5) and L is the magnetic length of the element on the central orbit. Table 1 gives the values of the different terms in (6), in the case of a 4-fold symmetric ISR which only contains combined function magnets.

Table 1

Cancellation of the linear chromaticity in the ISR, $\bar{\alpha}p$ is computed for the two first magnets of the superperiod.

Magnet type	$K' = K/\bar{\alpha}p$	K' AGS computed L'= 0	$K'_{AGS} - 2KL'/L$	K'_c computed by AGS2
D	0.02550	0.025336	0.013186	0.012690
F	- 0.01914	- 0.018906	- 0.03118	- 0.03111

The agreement between the two values of K'_c is satisfactory since there are approximations both in formula (6) and in the AGS formalism.

4.3 Comparison between field components calculated by AGS and measured values

At the ISR the working lines are operationally corrected, if necessary, by adjusting the current in the pole face windings in the main magnet units. The working line is then measured with a circulating beam, by means of a high precision Q-meter. This procedure is iterated until the absolute difference between measured and theoretical working line is smaller than 3.10^{-3}.

The multipole field components of the main magnets of the ISR are measured on reference units and processed to produce the equivalent of K' in equation (6), i.e. the edge effects are included.

The values of K and its derivatives are given in table 2 for the main magnets of the present ISR (ring 2) with the steel low-β insertion (LBAC machine). Since AGS computes K_c' (eq. 6) the K' have been obtained by means of eq. (6) and the K" have been computed by the same formula, replacing K and K' by K' and K". The K' calculated by means of MATRIX 2 are closer to the measured values than those calculated by means of MATRIX 1.

Table 2

Field components in the main magnets for the ring 2 of the ISR on the LBAC machine

	K_F	K_F'	K_F''	K_D	K_D'	K_D''
MATRIX 1	- 0.039702	- 0.0095464	0.03195	0.038568	0.031856	- 0.056626
MATRIX 2	- 0.039702	- 0.01420	0.00064	0.038568	0.02600	0.0082
measured	- 0.039893	- 0.01506	0.00394	0.038734	0.023907	0.022367

The discrepancy between the measured values of K' and the MATRIX 2 computed ones represent a relative difference smaller than 10^{-4} of the field of the main magnet over its useful aperture. This is about the limit of the accuracy of the field measurements.

The computed octupole components K" are very different from the measured ones because the second order expansion of the trajectories is inaccurate in AGS as shown in appendix II, and because the field associated with those components is within the uncertainty of the measurement.

5. CONSEQUENCES OF THE USE OF THE NEW VERSION OF MATRIX

The new version MATRIX 2 has made it possible essentially to assign the correct horizontal betatron wavenumber to the off-momentum closed orbits for machines with $L' \neq 0$. The consequences of the use of MATRIX 2 are summarised below for the ISR machine with and without the existing low-β insertion, and with the future superconducting low-β insertion.

5.1 ISR with steel low-β insertion ("ELSA" line)

5.1.1 Calculation of machine parameters

The working line ELSA ($Q_h = 8.902$, $Q'_h = +\ 2.5$) is shown in fig. 2, as calculated with MATRIX 1. If the magnetic parameters resulting from the chromaticity correction performed with MATRIX 1 are used as input in the new version MATRIX 2, the result is a working line with $Q' = -\ 4.1$ (see fig. 2).

With MATRIX 1 orbit problems occurred at $\Delta p/p \cong -\ 0.020$. The calculation using MATRIX 2 shows that the "trajectory Q_h" is equal to 9.0 for $\Delta p/p = -\ 0.0185$, which can explain why the injection orbit ($\Delta p/p = -\ 0.021$) could not be calculated using MATRIX 1.

5.1.2 Consequences for machine operation

The absence of sextupole components in the low-β quadrupoles creates a mismatch of the orbits of the second order in $\Delta p/p$. Therefore the positions of the closed orbits for $\Delta p/p = -\ 0.021$ and for $+\ 0.021$ are quite different from $\alpha p \times (\Delta p/p)$.

Before MATRIX 2 was written, no betatron parameters were available for the injection orbit. Instead, the values of the central orbit were used, or a linear extrapolation between the central orbit data and the (available, but incorrect) data for the outside orbit at $\Delta p/p = +\ 0.021$.

This resulted in inaccuracies in the calculated operational positions of collimators, and made it impossible to perform precise aperture calculations.

With MATRIX 2 reliable data for the injection orbit have become available, which considerably speeds up the setting up of the collimator system, the scrapers and the injection kicker position.

5.2 Application to the variation of γ_t with $\Delta p/p$ in the ISR

The variation of γ_t with $\Delta p/p$ was measured[4)] on the "8C" working line ($Q_H = 8.614$, $Q'_H = +\ 1.6$), without low-β insertion.

The γ_t values are calculated[4] according to :

$$\frac{1}{\gamma_t^2} = \frac{p}{R} \cdot \frac{dR}{dp} ,$$

with R = average machine radius of the orbit,

p = momentum;

rather than :

$$\frac{1}{\gamma_t^2} = \frac{p_c}{R_c} \cdot \frac{(R - R_c)}{(p - p_c)} .$$

R_c and p_c are radius and momentum on central orbit. The latter definition is used in the AGS program. The values of γ_t calculated by means of the first formula with MATRIX 2 agree well with the measurements (fig. 1), whereas with MATRIX 1 the slope of the curve which represents $\gamma_t(\Delta p/p)$ is too large by a factor of 3.

5.3 ISR with the future superconducting low-β insertion

The predictions which could be affected by the modification of AGS are those associated with non-linear elements; they concern the useful aperture, the feasibility of the bare machine and the sextupole components in the insertion.

5.3.1 The useful aperture

The above is determined by comparison to the useful aperture of a known machine, assuming that it is proportional to the momentum bite of the stacked beam. The aperture reduction induced by the superconducting scheme computed by MATRIX 2 differs by about 1°/₀₀ from that computed by MATRIX 1 because great care has always been taken with the matching of the off-momentum orbits : this means that the aperture reduction only results from the increase of the amplitudes of the horizontal betatron oscillations due to their bad matching and this phenomenon is equally well described using MATRIX 1 or MATRIX 2.

5.3.2 Feasibility of the "bare machine" (i.e. insertion off)

Once the ELSA working line has been built for the machine with the insertion, the insertion is switched off and the remaining bare machine has to be built with the present ISR in order to check the feasibility of the scheme. The bare machines obtained according to this procedure, either using MATRIX 1 or MATRIX 2 are very similar and thus the feasibility established previously remains valid.

5.3.3 Sextupole components in the insertion

With MATRIX 1 the value of Q_h associated with the orbit at the top of the stack was about 8.7 and the matching of the orbit to the second order in $\Delta p/p$ was not critical, so that the optimisation of the sextupole components was achieved by acting more on the value of β_h than on the position of the orbit in order to maximize the useful aperture.

With MATRIX 2 the value of Q_h for the orbit is correct (Q_h = 8.947) and the matching of the orbit to the second order in $\Delta p/p$ is important. Thus one sextupole component had to be strongly modified in order to satisfy this latter procedure and this is the most important consequence of the use of MATRIX 2 to analyse the superconducting scheme.

The importance of the introduction of sextupole components in the scheme was also established : with MATRIX 1 the suppression of the sextupoles led to a loss of luminosity of about 4 %, since the off-momentum orbits were not sensitive to a mismatch; with MATRIX 2 the loss becomes 10 %.

6. CONCLUSION

It is important that the wave number associated with a given off-momentum closed orbit be the same as the wave number calculated from the gradients along this orbit since the common practice is to specify the latter wave number in the process of chromaticity correction. It has been shown that a discrepancy between the two wave numbers may indeed lead to a bad description of the off-momentum closed orbits and cause problems in machine design and operation.

Therefore it is proposed that the modified version MATRIX 2 of MATRIX is used in AGS for the calculations of the ISR beam optics. The binary version of MATRIX 2 can be found in the permanent file : NEWAGS, ID=IS170RISS in the CDC computer MFA.

The predictions concerning the performance of the superconducting low-β scheme are not dramatically different when they are remade by means of MATRIX 2. The major change concerns the value of one of the sextupole components in the scheme.

For machines in which the magnetic length of the multipole components do not depend on the radial position, the new calculation does not change the values of Q but substantially changes the values of γ_t on off-momentum orbits.

REFERENCES

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2. K.G. Steffen, High Energy Beam Optics, Interscience Publishers, John Wiley & Sons, (1965).

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APPENDIX I

Computation of the coordinates of the closed orbit and the Twiss matrix around it

The principle of the method is to track a sufficiently large number of trajectories in the vicinity of the closed orbit for computing the unknowns which are the coordinates of the closed orbit and the Twiss parameters.

The position of a particle with respect to the closed orbit around which it oscillates is described by the vector :

$$X_{osc} = \begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix}$$

If the initial position is $X_{osc,o}$, the position after one machine period is :

$$X_{osc,1} = T\, X_{osc,o}$$

where T is the Twiss matrix, according to the linear theory of the betatron oscillation[3].

Now let X be the position of the particle with respect to an arbitrary system of coordinates and X_{orb} be the position of the closed orbit with respect to this system :

$$X = X_{orb} + X_{osc} \qquad (1)$$

After one period we have :

$$X_1 = X_{orb} + X_{osc,1} = X_o - X_{osc,o} + T\, X_{osc,o}$$

where the vectors indexed o are the initial vectors.

After i periods, if we put $D_i = X_i - X_o$, we see that :

$$D_i = (T^i - I)\, X_{osc,o} \qquad (2)$$

which gives a set of equations which do not contain X_{orb}.

We now eliminate $X_{osc,o}$ by subtracting D from the previous equation :

$$D_i - D_1 = (T^i - I)\ X_{osc,o} - (T - I)\ X_{osc,o}$$

$$= T\ (T^{i-1} - I)\ X_{osc} = T\ D_{i-1}$$

Which leads to the system :

$$D_2 - D_1 = TD_1$$

$$D_3 - D_1 = TD_2 \qquad (3)$$

$$D_n - D_1 = TD_{n-1}$$

in which the D_i's may be obtained from the trajectory tracking and the unknowns are the coefficients of the Twiss matrix T. If the motion in the two oscillation planes with coupling is considered, there are 16 elements in the Twiss matrix; since each equality in (3) is equivalent to 4 equations, 4 equalities are needed and n must be equal to 5. Once the Twiss matrix is determined, the coordinates of the oscillation are computed by means of system (2). Then equation (1) makes it possible to compute the coordinates of the closed orbit.

If we only consider the closed orbit in the horizontal plane, there are only 4 elements in the Twiss matrix, which only need n = 3 in (3) (a tracking of trajectories over 3 periods of the machine).

The latter procedure has been implemented in AGS as option "VE".

The presentation of the computation in terms of matrix equations (3) was proposed by B. Autin (private communication).

APPENDIX II

Verification of the computation of the trajectories by AGS up to the second order

This verification is based upon an analytical calculation of the solution of the non-linear equations of motion up to the second order in amplitude for combined function magnets (the terms associated with the expression in power series of $\Delta p/p$ are not considered). This calculation is made by the perturbation method.

1. Properties of the solutions of the linear equation of motion

(The formalism below has been used in ref. 2 and some of the results below are also in this reference.)

The equation of motion is : $x'' + Kx = 0$ (1)

where K is constant and x is a function of the abscissa s. It is convenient for the following computations to have general properties of the particular solutions C(s) and S(s) which satisfy the initial conditions :

$$C(o) = S'(o) = 1 \qquad C'(o) = S(o) = 0,$$

since the general solution of (1) will be :

$$X(s) = x_o\, C(s) + x'_o\, S(s)$$

x_o and x'_o being the initial conditions of the motion.

Integrating (1) we obtain :

$$\int_o^s X ds = -\frac{1}{K}(X' - x'_o).$$

Multiplying (1) by 2X' and integrating :

$$X'^2 - x'^2_o + K\,(X^2 - x^2_o) = 0 \tag{2}$$

Dividing the latter by X^n and integrating by part :

$$(x'^2_o + Kx^2_o)\int_o^s \frac{ds}{X^n} = \frac{x'_o}{(n-1)x_o^{\,n-1}} - \frac{X'}{(n-1)X^{n-1}} + \frac{K(n-2)}{n-1}\int_o^s \frac{ds}{X^{n-2}} \tag{3}$$

From (2) we can deduce :

$$C'^2 + K\,(C^2 - 1) = 0 \qquad S'^2 - 1 + K\,S^2 = 0.$$

Writing (1) for S and C and eliminating K we obtain :

$$SC'' - CS'' = 0 \qquad CS' - SC' = 1$$

Dividing the latter by C^2, integrating and comparing with (3) gives :

$$C' = - KS \qquad \text{hence} \quad S' = C$$

which leads to $C^2 + KS^2 = 1$

Many other integral relationships can be obtained for S and C irrespective to the sign of K. Those established above are sufficient for the subsequent calculations. Finally the expression of the derivatives of C and S makes it possible to obtain the expression of the Taylor series of S and C :

$$C(s) = C(so) - (s - so)KS(so) - \frac{(s - so)^2}{2} KC(so) + \frac{(s - so)^3}{3!} K^2S(so) + \ldots$$

$$S(s) = S(so) - (s - so)C(so) - \frac{(s - so)^2}{2} KS(so) - \frac{(s - so)^3}{3!} KC(so) + \ldots$$

2. Computation of the trajectories up to the 2nd order in amplitude for a combined function magnet

Let us consider a magnet with a quadrupole and a sextupole component. Taking into account the quadrupole only, the trajectory of a particle is described by the function X(s) which has been defined in paragraph 1.

If we now introduce the sextupole component, equation (1) becomes :

$$x'' + K x + K' \frac{x^2}{2} = 0 \qquad (4)$$

This new equation will be solved by the perturbation method putting $x = X(s)\ [1 + y(s)]$, where y(s) is small with respect to 1. Introducing this function in (4) and only keeping the first order terms in y and K', we obtain :

$$(X^2y')' = - \frac{K'}{2} X^3$$

X^3 can be integrated by parts, using (2) and the integral of (1).
This gives :

$$X^2y' = \frac{K'}{6K} [\ X^2X' - x_o^2x_o' + \frac{2}{K} (X' - x_o')(Kx_o^2 + x_o'^2)]$$

y' can be integrated easily thanks to the formuls (3), we thus obtain :

$$y = \frac{K'}{6K} \left[X - x_o + 2(Kx_o^2 + x_o'^2)\frac{(X - x_o)}{KXx_o} - \frac{x_o'(3x_o^2K + x_o'^2)(x_o'X - x_oX'}{Kx_oX(Kx_o^2 + x_o'^2)} \right]$$

the approximate solution of the non-linear equation (4) is then :

$$x = X + \frac{K'}{6K}\left[X(X - x_o) + \frac{2}{Xx_o}(X - x_o)(Kx_o^2 + x_o'^2) - \frac{x_o'(3x_o^2k + x_o'^2)(Kx_o^2 + x_o'^2)}{Kx_o(Kx_o^2 + x_o'^2)}\right]$$

This solution can be expressed in a simpler form by substituting $X = Cxo + Sx_o'$:

$$x = X + \frac{K'(C-1)}{6K}\left[(C + 2)x_o^2 + 2Sx_ox_o' - (C-1)\frac{x_o'^2}{K}\right] \qquad (6)$$

If $\sqrt{K}\,\ell$ is small with respect to 1 (ℓ being the length of the magnet considered),

this function becomes : $x(\ell) = X(\ell) - \frac{K'\ell^2}{4}\left(x_o^2 + \frac{2}{3}x_ox_o'\ell + \frac{1}{6}x_o'^2\ell^2\right)$,

which can be obtained directly by integrating (4) with K = 0.

3. Expansion of the transfer matrices for the trajectories in AGS

In AGS the coordinates of the trajectories at the exit of an element of length ℓ are :

$$x = x_o\, C_{\bar K}(\ell) + x_o'\, S_{\bar K}(\ell)$$

$$x' = -K\, S_{\bar K}(\ell)\, x_o + C_{\bar K}(\ell)\, x_o' \qquad (4)$$

Where $C_{\bar K}$ means that this function is the solution of (1) with $K = \bar K$,

$$\bar K = K + K'\frac{\bar x}{2} \qquad \text{with} \quad \bar x = \frac{x + x_o}{2} + \frac{x_o' - x'}{12}\,\ell$$

The latter formula comes for the description of the trajectory x(s) by a 3rd degree polynom. $C_{\bar K}$ and $S_{\bar K}$ can be expressed once we have noticed that the relevant variable in (1) is $\sqrt{K}\,\ell$:

$$\sqrt{\bar K}\,\ell \cong \sqrt{K}\,\ell\left(1 + \frac{K'\bar x}{4K}\right)$$

$$\cong \sqrt{K}\left(\ell + \frac{K'\bar x\,\ell}{4K}\right)$$

From the expansion of C(s) and S(s) in power of s (see end of paragraph 1) we have :

$$C_{\bar K}(\ell) \cong C(\ell) - \frac{K'\bar x\ell}{4}\,S(\ell) \qquad S_{\bar K}(\ell) \cong S(\ell) + C(\ell)\,\frac{K'\bar x\ell}{4K}$$

Putting back those expansions in (4) we obtain :

$$x = x_o \left(C - \frac{SK'\bar{x}\ell}{4}\right) + x'_o \left(S + \frac{CK'\bar{x}\ell}{4K}\right)$$

$$x' = - x_o K \left(S + \frac{CK'\bar{x}\ell}{4K}\right) + x'_o \left(C - \frac{SK'\bar{x}\ell}{4}\right)$$

Using the notations of paragraph (1) this can be written :

$$x = X + \frac{X'K'\ell}{8K} \left(x + x_o + \frac{x' - x'_o}{6}\right)$$

$$x' = X' - \frac{XK'\ell}{8} \left(x + x_o + \frac{x' - x'_o}{6}\right)$$

which is a linear system with 2 unknowns x and x'. The solution is :

$$x = X + \frac{K'\ell}{8K} \left[- KSx_o^2 \,(1 + C + \frac{KS\ell}{6}) + x_o x'_o \,(C^2 + C - KS^2 - \frac{KS\ell}{6} - \frac{CSK\ell}{6}) + x_o'^2 (SC + \frac{C\ell}{6} - \frac{C^2\ell}{6}) \right]$$

$$x' = X' - \frac{K'\ell}{8} \left[Cx_o^2 \,(1 + C + KS) + x_o x'_o \,(S + 2SC + KS^2 - \frac{C\ell}{6} + \frac{C^2\ell}{6}) + x_o'^2 S (S + \frac{\ell}{6} - \frac{C\ell}{6}) \right]$$

4. <u>Comparison between the analytical expansion and that of AGS</u>

It is sufficient to know the values of K and ℓ to compute the coefficients of the above 2nd order polynoms. For the F main magnets of the ISR (where the position of the orbit x is the largest) K is about 0.04 m^{-2} and ℓ is about 2.5 m : the coefficients of the expansion of x associated with those two numerical values are given in table 3 :

<u>Table 3</u>

Expansion of x to the second order in x_o and x'_o

	coefficients of the second order terms			
2nd order terms	analytical calculation	numerical value	AGS	numerical value
$\frac{K'}{K} x_o^2$	$(C-1)(C+2)/6$	-0.0578	$\underline{5}K\ell(C + 1 + \frac{KS\ell}{6})/8$	-0.0575
$\frac{K'}{K} x_o x'_o$	$S(C-1)/3$	-0.0978 m	$\ell(C^2 + C - KS^2 - \frac{KS\ell}{6} + \frac{KSC\ell}{3})/8$	$+0.453$ m
$\frac{K'}{K} x_o'^2$	$-(C-1)^2/6K$	$-0.0624\ m^2$	$\ell(SC + \frac{C}{6} - \frac{C^2\ell}{6})/8$	$+0.671\ m^2$

The coefficient of x_o^2 is almost exact in AGS but not the other ones. However, if we consider the F magnets of the ISR and the orbit at the top of the stack, we have : $x_o \cong 45$ mm, $x'_o \cong 36$ mrad, $K' \cong 0.02\ m^{-3}$. In those conditions the second order contribution to the position of the horizontal trajectory at the exit of a F magnet of the ISR is - 0.068 mm for the analytical expansion and - 0.017 mm for AGS.

The consequences of the error made by AGS are the following : there exists the equivalent of a field error on the off-momentum orbits. The associated closed orbit distorsion can be evaluated as follows : if Δx is the trajectory deviation at the end of the magnet considered, the associated deflection error is:

$$\delta = \frac{2\Delta x}{\ell}.$$

The closed orbit distorsion induced by n identical kickers at places separated by the phase advance μ_c is :

$$x = \frac{\sqrt{\beta}}{2\ \sin \pi Q} \sum_{i=1}^{n} \sqrt{\beta_i}\,\delta \cos(\pi Q - |\mu - \mu_i|) \quad \text{with } \mu_i = \mu_o + i\mu_c$$

The sum of the cosine functions is of the type :

$$\sum_{i=1}^{n} \cos\ (\emptyset + i\mu_c) = \frac{\cos\ (\emptyset + \frac{n-1}{2}\mu_c)\ \sin \frac{n}{2}\mu_c}{\sin \frac{\mu_c}{2}}$$

An upper limit of x is then :

$$\hat{x} = \frac{\beta_{max}}{\sin \frac{\pi Q}{n}} \times \frac{\Delta x}{\ell}$$

when considering a machine made from n identical cells, in which $n\mu = 2\pi Q$. For the ISR $Q_h = 8.95$ at the top of the stack and $n = 44$, then $\hat{x} = 0.7$ mm for $\Delta x = 0.05$ mm. It is useless to describe the position of the closed orbit inside the magnets by a polynom of the 3rd degree since this does not make the 2nd order expansion more accurate (if the factor 1/6 in the AGS expansion (see table 3) is replaced by zero the expansion is as accurate).

It is useless to compare the experimental and theoretical octupole components since the 2nd order effects are too approximate in AGS.

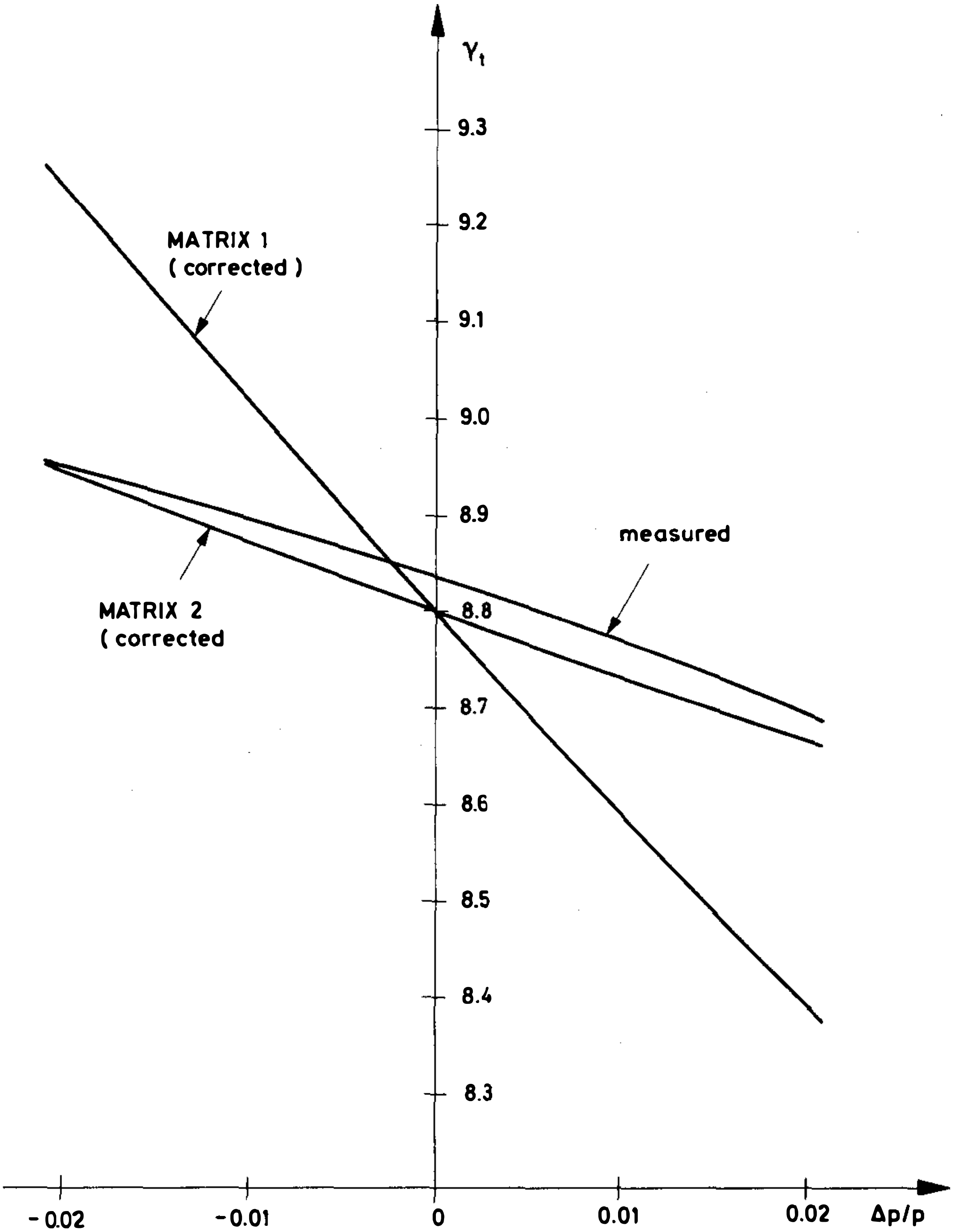

Fig. 1 Variation of γ across the aperture of the ISR.

For the injection orbit $\Delta p/p$ is about -0.02; for the top of the stack $\Delta p/p$ is about $+0.021$.

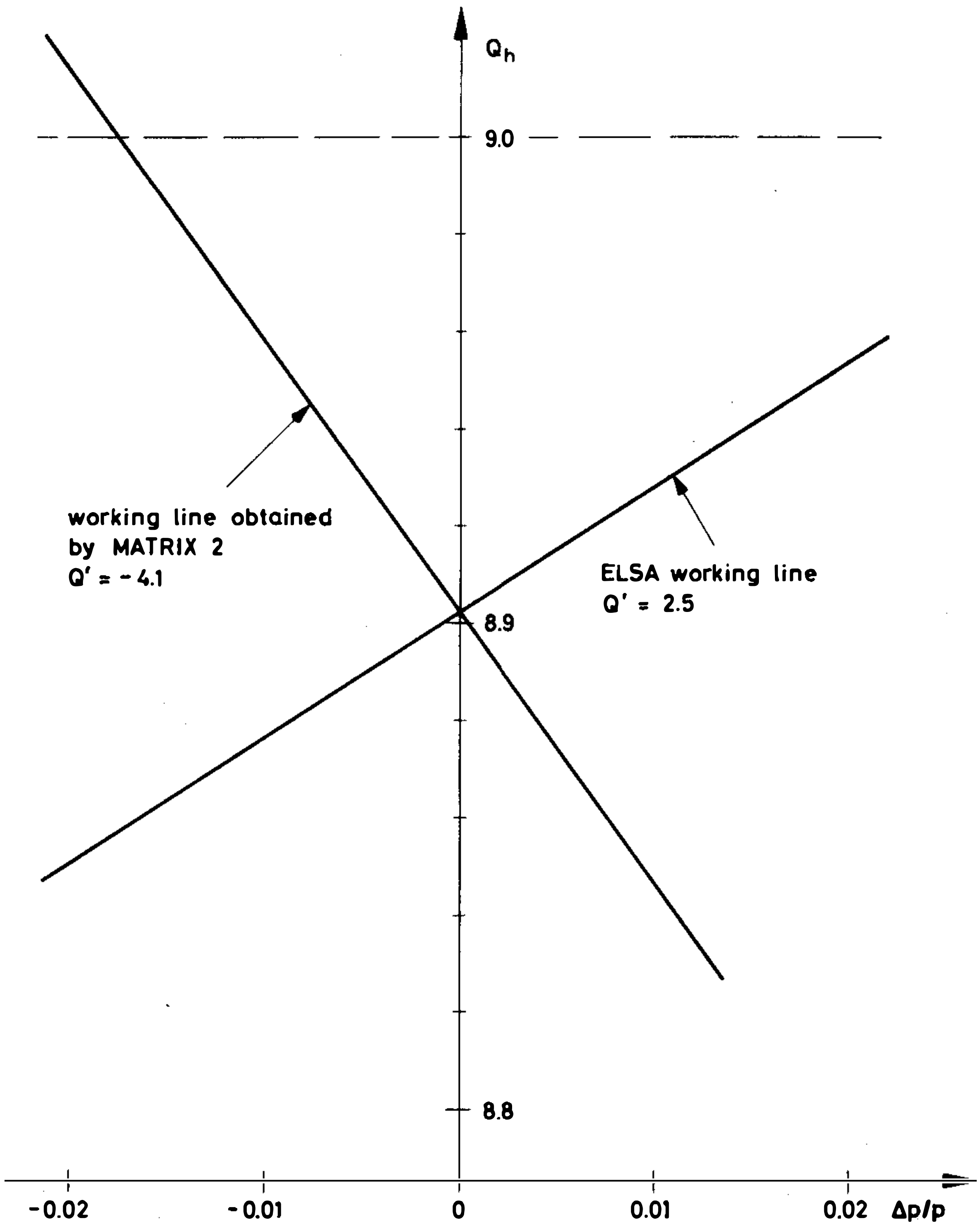

Fig. 2 Working line computed by MATRIX 2, with the magnetic parameters computed by MATRIX 1 in order to obtain the ELSA line.