Measuring the $a_0 - a_2$ pion scattering lengths through $K \to 3\pi$ decays *

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We discuss the recent Cabibbo's proposal to measure the $\pi\pi$ scattering lengths combination $a_0 - a_2$ from the cusp effect in the $\pi^0\pi^0$ energy spectrum at threshold for $K^+ \to \pi^0\pi^0\pi^+$ and $K_L \to \pi^0\pi^0\pi^0$. We estimate the theoretical uncertainty of the $a_0 - a_2$ determination at NLO in our approach and obtain that it is not smaller than 5 % for $K^+ \to \pi^0\pi^0\pi^+$. One gets similar theoretical uncertainties if the neutral $K_L \to \pi^0\pi^0\pi^0$ decay data below threshold are used instead. For this decay, there are very large theoretical uncertainties above threshold due to cancellations and data above threshold cannot be used to get the scattering lengths.

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1. Introduction

Final state interaction (FSI) phases at next-to-leading order (NLO) in Chiral Perturbation Theory for $K \to 3\pi$ are an important ingredient to obtain the charged CP-violating asymmetries at NLO [1–3]. The dominant contribution to these $K^+ \to 3\pi$ FSI at NLO are from two-pion cuts and they were calculated analytically in [1].

Though to get the full $K \to 3\pi$ amplitudes at order p^6 implies a two-loop calculation, one can get the FSI phases at NLO using the optical theorem within CHPT with the advantage that one just needs to know $\pi\pi$ scattering and $K \to 3\pi$ both at $\mathcal{O}(p^4)$. Notice that NLO in the dispersive part of the amplitude means one-loop and $\mathcal{O}(p^4)$ in CHPT while NLO in the absorptive part of the amplitude means two-loops and $\mathcal{O}(p^6)$ in CHPT.

The study of FSI in $K \to 3\pi$ at NLO also became of relevance after the proposal by Cabibbo [4] to measure the combination $a_0 - a_2$ of $\pi\pi$ scattering lengths using the cusp effect in the $\pi^0\pi^0$ spectrum at threshold in $K^+ \to \pi^0\pi^0\pi^+$ and $K_L \to \pi^0\pi^0\pi^0$ decay rates.[3] Within this proposal, it has been recently presented in [6] the effects of FSI at NLO using formulas dictated by unitarity and analyticity approximated at second order in the $\pi\pi$ scattering lengths, $a_i \sim 0.2$. The error was therefore canonically assumed to be of order of a_i^2, i.e., 5%. There, they used a second order polynomial in the relevant final two-pion invariant energy s_3 fitted to data to describe the $K \to 3\pi$ vertex that enters in the formulas of the cusp effect. It is of interest to check this canonical error and provide a complementary analysis of this theoretical uncertainty estimate.

In Ref. [7], we use our NLO in CHPT results for the real part of $K \to 3\pi$ fitted to data to describe the $K \to 3\pi$ vertex that enters in formulas of the cusp effect. Notice that, as we want to extract the pion scattering lengths, we do *not* want to predict the real part of $K \to 3\pi$ in CHPT at any order but to have the best possible description fitted to data. We treat $\pi\pi$ scattering near threshold as in Cabibbo's original proposal. The advantage of using CHPT formulas for the fit to data of the real part of $K \to 3\pi$ is that it con-

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[3]The cusp effect in SU(2) $\pi\pi$ scattering was discussed in [5].

tains the correct singularity structure at NLO in CHPT which can be systematically improved by going at higher orders.

Contributions from next-to-next-to-leading order (NNLO) SU(3) CHPT in the isospin limit are expected typically to be around $(3 \sim 5)\%$, so that our NLO results are just a first step in order to reduce the theoretical error on the determination of the combination $a_0 - a_2$ to the few per cent level. At NNLO, one can follow a procedure analogous to the one we use in [7] to get a more accurate measurement of $a_0 - a_2$ and check the assumed NNLO uncertainty. At that point, and in order to reach the few per cent level in the theoretical uncertainty, it will be necessary to include full isospin breaking effects at NLO too. These are also expected to be of a few per cent as was found for $K \to 3\pi$ in [8,9].

1.1. Basic notation for charged Kaon decays

Near $\pi^+\pi^-$ threshold, we can decompose the $K^+ \to \pi^0\pi^0\pi^+$ amplitude as follows [4,6]

$$A_{00+} = \begin{cases} \overline{A}_{00+} + \overline{B}_{00+} v_\pm(s_3), & \text{for } s_3 > 4m^2_{\pi^+} \\ \overline{A}_{00+} + i\overline{B}_{00+} v_\pm(s_3), & \text{for } s_3 < 4m^2_{\pi^+}, \end{cases} \quad (1)$$

where $\overline{A}_{00+}$ and $\overline{B}_{00+}$ are in general singular functions except near the $\pi^+\pi^-$ threshold [10] and

$$v_{ij}(s) = \sqrt{\frac{|s - (m_{\pi^{(i)}} + m_{\pi^{(j)}})^2|}{s}}. \quad (2)$$

Notice that these kinematical factors are taken with physical pion masses, in this way one can describe the cusp effect which is generated by the different behavior of $K^+ \to \pi^0\pi^0\pi^+$ for the two neutral pions invariant energy above and below the $s_3 = 4m^2_{\pi^+}$ threshold.

With these definitions, the differential decay rate for this amplitude can be written as [6]

$$|A_{00+}|^2 \equiv \mathrm{Re}\overline{A}^2_{00+} + \Delta_A + v_\pm(s_3)\Delta_{\rm cusp}\,, \quad (3)$$

with

$$\Delta_A \equiv \mathrm{Im}\overline{A}^2_{00+} + v^2_\pm(s_3)\left[\mathrm{Re}\overline{B}^2_{00+} + \mathrm{Im}\overline{B}^2_{00+}\right],$$

$$\Delta_{\rm cusp} \equiv \begin{cases} -2\mathrm{Re}\overline{A}_{00+}\mathrm{Im}\overline{B}_{00+} + 2\mathrm{Im}\overline{A}_{00+}\mathrm{Re}\overline{B}_{00+}\,, \\ \text{for } s_3 < 4m^2_{\pi^+}; \\ 2\mathrm{Re}\overline{A}_{00+}\mathrm{Re}\overline{B}_{00+} + 2\mathrm{Im}\overline{A}_{00+}\mathrm{Im}\overline{B}_{00+}\,, \\ \text{for } s_3 > 4m^2_{\pi^+}. \end{cases}$$

The combination of real and imaginary amplitudes $\Delta_{\rm cusp}$ defined above parametrizes the cusp effect due to the $\pi^+\pi^- \to \pi^0\pi^0$ re-scattering in the $K^+ \to \pi^0\pi^0\pi^+$ decay rate.

2. Discussion

Our method is a variation of the original Cabibbo's proposal that uses NLO CHPT for the real part of $K \to 3\pi$ vertex instead of the quadratic polynomial in s_3 approximation used in [4,6] plus analyticity and unitarity.

Notice that we do not use CHPT to predict the real part of $K \to 3\pi$, but use its exact singularity form at NLO in CHPT to fit it to data above threshold. If the two-loop CHPT singularity structure were known it could be used in order to take this singularity structure exactly in $\mathrm{Re}\overline{A}_{00+}$. The treatment of $\pi\pi$ scattering near threshold is independent of this choice and we treat it in the same way as in [6].

The cusp effect originates in the different contributions to $K^+ \to \pi^0\pi^0\pi^+$ and $K_L \to \pi^0\pi^0\pi^0$ amplitudes above and below threshold of $\pi^+\pi^-$ production in the $\pi^0\pi^0$ pair invariant energy. We obtain these contributions using just analyticity and unitarity, in particular applying Cutkosky rules and the optical theorem above and below threshold to calculate the discontinuity across the physical cut. This allows us to separate $\pi\pi$ scattering –which we want to measure– from the rest of $K^+ \to \pi^0\pi^0\pi^+$ or $K_L \to \pi^0\pi^0\pi^0$.

The validity of the use of Cutkosky rules for in $K \to 3\pi$ decays is commented in [7]. In particular, the real part of the discontinuity has a singularity when any of the s_i invariant energy reaches its pseudo-threshold at $(m_K - m_{\pi^{(i)}})^2$ as described in [10–12]. This singularity affects the description of the cusp effect using the discontinuity as dicussed in [7,10,11].

We would like to remark here that making the same approximations that were done in [6] we fully agree with their analytical results. In particular, we checked that the use of the quadratic polynomial in s_3 in [6] produce negligible differences –around 0.5 %– in $\Delta_{\rm cusp}$ in (4).

We also pointed out that while the presence of that singularity at pseudo-thresholds does not affect $\mathrm{Re}\overline{B}_{00+}(s_3)$ and $\mathrm{Re}\overline{B}'_{000}(s_3)$ when s_3 is around threshold, one needs to take fully into account its effects for $\delta\mathrm{Re}\overline{A}_{00+}$ and $\delta\mathrm{Re}\overline{A}'_{000}$ when s_1 or s_2 is above $(m_K^2 - m_\pi^2)/2$. For a possible solution of this problem which does not simply use the discontinuity to describe the cusp effect see [10]. Another possibility could be to use, instead of $\delta\mathrm{Re}\overline{A}_{00+}$ and $\delta\mathrm{Re}\overline{A}'_{000}$, the full-two loop (not available yet) finite relevant pieces to describe the singularities at thresholds at NLO in $\mathrm{Re}\overline{A}_{00+}$ and $\mathrm{Re}\overline{A}'_{000}$, respectively. This could be fitted to data.

In [7] we have also discussed the approximations done in [6] and the numerical differences they induce in $\Delta_{\rm cusp}$. We have found that though each one of them is individually negligible (between 0.5 % to 1 %) they produce final differences in the $\Delta_{\rm cusp}$ around 3 %. Of course, these approximations can be eliminated.

Concerning the theoretical uncertainties in the determination of $a_0 - a_2$ using our formulas, we concluded that for $K^+ \to \pi^0\pi^0\pi^+$, this uncertainty is somewhat larger than 5 % if uncertainties are added quadratically and 7 % if added linearly. I.e., we essentially agree with the estimate in [6]. Notice that we get our final theoretical uncertainty as the sum of several order 1% to 2% uncertainties to the canonical NNLO 5 % uncertainty.

For the case $K_L \to \pi^0\pi^0\pi^0$, we get that –if one uses just data below threshold– the uncertainty in the determination of $a_0 - a_2$ is of the same order as for $K^+ \to \pi^0\pi^0\pi^+$. Above threshold, we found large numerical cancellations which preclude from using it.

An expansion in the scattering lengths a_i and Feynman diagrams were used in [6] to do the power counting and obtain the cusp effect description of $K^+ \to \pi^0\pi^0\pi^+$ at NLO. In general, when FSI $\pi\pi$ scattering effects are included at n-th order[4], there appear new topologies in $K \to 3\pi$ which give contributions to $\Delta_{\rm cusp}$ of order a_i^n. The canonical uncertainty of the n-th order results is thus a_i^n. Notice that the velocity factors that appear after applying the unitarity cuts can be order one –for instance $v_\pm((m_K^2 - m_\pi^2)/2) \simeq 0.6$ appear in $\mathrm{Re}\,\overline{B}_{00+}(4m_\pi^2)$– and do not suppress largely the naive a_i^n estimate.

Our estimate for the uncertainty from NNLO, $\sim a_i^2$, coincides numerically with the one made in [6], i.e. it is around 5 %. We conclude that one cannot expect to decrease this canonical 5% theoretical uncertainty of the NLO result unless one includes $\pi\pi$ scattering effects at NNLO. If one wants to reach the per cent level in the uncertainty of the determination of $a_0 - a_2$ from the cusp effect, one would need to include those NNLO re-scattering effects. As said above, at NNLO it is possible to follow a procedure analogous to the one we use here to get a more accurate measurement of $a_0 - a_2$ and check the estimated NNLO uncertainty.

We have just included isospin breaking due to the different thresholds using two-pion physical phase spaces in the optical theorem and Cutkosky rules. This is needed to describe the cusp effect. The rest of NLO isospin breaking is expected to be important just at NNLO. At that order, isospin breaking effects in $\pi\pi$ scattering at threshold –both from quark masses and from electromagnetism– will have to be implemented and their uncertainties added.

Finally, we believe that it is interesting to continue investigating the proposal in [4,6] to measure the non-perturbative $\pi\pi$ scattering lengths from the cusp effect in $K^+ \to \pi^0\pi^0\pi^+$ and $K_L \to \pi^0\pi^0\pi^0$. Another interesting direction is to develop an effective field theory in the scattering lengths which could both check the results in [6] and allow to go to NNLO. This type of studies is already underway and firsts results were presented [10,11].

[4] $n = 1$ order are for LO contributions.

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