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Current correlators and AdS/CFT away from the conformal point*

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Abstract

Using the AdS/CFT correspondence we study vacua of $\mathcal{N} = 4$ SYM for which part of the gauge symmetry is broken by expectation values of scalar fields. A specific subclass of such vacua can be analyzed with gauged supergravity and the corresponding domain wall solutions lift to continuous distributions of D3-branes in type IIB string theory. Due to the non-trivial expectation value of the scalars, the $SO(6)$ $\mathcal{R}$-symmetry is spontaneously broken and field theory predicts the existence of Goldstone bosons. We explicitly show that, in the dual supergravity description, these emerge as massless poles in the current two-point functions, while the bulk gauge fields which are dual to the broken currents become massive via the Higgs mechanism. We find agreement with field theory expectations and, hence, provide a non-trivial test of the AdS/CFT correspondence far away from the conformal point.

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1 Introduction

Up to this day the AdS/CFT correspondence [1, 2, 3] is the most concrete proposal for a realization of holography. Although this duality allows to perform numerous explicit calculations in the supergravity limit an honest proof of this conjecture directly from string theory is still elusive. Therefore, the best we can do, is to take the results obtained from supergravity as predictions and make qualitative and, more desirably, quantitative comparisons with field theory. Indeed, it is an interesting task in itself to study known phenomena of quantum field theories and try to understand how they are realized on the dual string/gravity side. Most of the comparisons have been performed for the $\mathcal{N} = 4$ SYM theory at the conformal point. Naturally, one may wonder whether the AdS/CFT correspondence can be checked away from conformality as well. This question has been answered positively in [4] where we computed 2-point functions of currents in the Coulomb branch of the $\mathcal{N} = 4$ SYM theory and provided a non-trivial test of the AdS/CFT correspondence in the deep infrared of the theory. It is the purpose of this paper to review this work for the proceedings of the "Corfu summer school on elementary particle Physics". Relevant literature on the Coulomb branch of $\mathcal{N} = 4$ as well as other works on current correlators within the AdS/CFT correspondence can be found in the reference list of our original paper [4] (particularly, in [5, 6]).

In order to study theories with less supersymmetry and/or broken conformal symmetry, which are closer to the theories realized in Nature, deformations of the original conjecture [1] have been studied extensively over the past years. In this paper we are concerned with the simplest modification of $\mathcal{N} = 4$ SYM by turning on vacuum expectation values (vevs) of scalar fields. The $\mathcal{N} = 4$ vector multiplet contains, beside the vector potential and four adjoint Weyl fermions, six scalars in the adjoint representation of the gauge group $U(N)$. These scalars transform in the $\mathbf{6}$ of the $SO(6)$ $\mathcal{R}$-symmetry group and maybe represented by the $N \times N$ matrices Φ^i, $i = 1, 2, \ldots, 6$. The quartic scalar potential of the theory $\sum_{i<j} \mathrm{tr}[\Phi^i, \Phi^j]^2$ has flat directions that are parametrized by six diagonal $N \times N$ matrices

$$X^i_{\mathrm{vev}} = \langle \Phi^i \rangle = \mathrm{diag}(X^i_1, X^i_2, \ldots, X^i_N) \; , \qquad \sum_{p=1}^{N} X^i_p = 0 \; . \tag{1}$$

The corresponding Coulomb branch is $(\mathbb{R}^6)^N / S_N$ and on points away from the origin the conformal symmetry is broken. The action of $\mathcal{N} = 4$ SYM contains the term $\sum_i \mathrm{tr}(D_\mu \Phi^i)^2$ which couples the gauge fields to the scalars. When the latter acquire vevs the gauge bosons become massive. It is convenient to choose the standard real basis for the $SU(N)$ generators $J_{pq} = e_{pq} - 1/N \delta_{pq} I_{N\times N}$, where the matrix elements of the matrices e_{pq} are: $(e_{pq})_{rs} = \delta_{pr}\delta_{qs}$. Hence, according to (1), we give vevs to the scalars represented by the six-dimensional vector $\vec{\Phi}$, as $\vec{\Phi} = h_p \vec{X}_p$, where $h_p = J_{pp}$ are the generators of the Cartan subalgebra of $SU(N)$. The masses of the gauge fields arise from the term $\sum_i \mathrm{tr}[\Phi^i, A_\mu]^2$. After some computation we find the (mass)^2 matrix with elements

[7, 4]

$$(M^2)_{pq} = |\vec{X}_p - \vec{X}_q|^2, \qquad p,q = 1,2,\dots,N \ , \tag{2}$$

up to a numerical factor of order 1. Hence, the masses have the geometrical interpretation as the distances between the various vev positions distributed in the scalar space $\mathbb{R}^6$. Equivalently, they are given by the masses of the strings stretched between the D3-branes located at these points. It is clear, that some of these masses may be degenerate, depending on the specific distribution of vevs.

Let us illustrate some of the features with the toy example of a discrete distribution of vevs in an polygon with N vertices located on a circle of radius r_0 in the 1-2 plane [8]

$$\vec{X}_p = (r_0 \cos\phi_p, r_0 \sin\phi_p, 0,0,0,0) \ , \quad \phi_p = 2\pi p/N \ , \quad p = 1,2,\dots,N \ . \tag{3}$$

In this case we find from (2), that [7]

$$M_n = 2r_0 \sin(\pi n/N) \ , \qquad n = 1,2,\dots,N \ , \tag{4}$$

which is an exact result for any N. The degeneracy for the zero mode is $d_N = N-1$ and for the rest $d_n = 2(N-n)$. It is easily seen that $\sum_{n=1}^N d_n = N^2 - 1$. Hence, for large N there are W-bosons with masses of order r_0 and W-bosons with light masses of order r_0/N. For more general distributions the same result holds with r_0 being replaced by the average value for the distribution of vevs.

2 Correlators from gauge theory

Since the scalars carry non-trivial $\mathcal{R}$-charge, the $\mathcal{R}$-symmetry is in general broken on the Coulomb branch. This is the well known phenomenon of spontaneous breaking of a global symmetry in field theory, and, therefore, we expect massless poles in the $\mathcal{R}$-symmetry current correlators for every broken symmetry generator, which correspond to the massless Goldstone bosons.[1] In order to investigate this issue we start with the case of unbroken $\mathcal{R}$-symmetry where the vev's corresponding to the six scalars of the theory are turned off. The $\mathcal{R}$-symmetry currents J^a_μ are bilinear in the scalar fields X^i, $i = 1,2,\dots,6$ and transform in the adjoint of $SO(6)$

$$J^{kl}_\mu = \frac{1}{g^2_{\rm YM}} T^{kl}_{ij} \mathrm{Tr}(X^i \partial_\mu X^j) \ + \ \text{fermions} \ , \tag{5}$$

where $T^{kl}_{ij} = \delta_{ki}\delta_{lj} - \delta_{kj}\delta_{ki}$ are the components of the 6×6 matrices T^{kl} of $SO(6)$. The scalars X^i, being free fields, have the following two-point function (in our conventions the field theory action has an overall factor of $1/g^2_{YM}$)

$$\langle X^i_{pq}(x) X^j_{rs}(0)\rangle = g^2_{\rm YM}\delta^{ij}(\delta_{qr}\delta_{ps} - \frac{1}{N}\delta_{pq}\delta_{rs})\frac{1}{r^2} \ , \qquad p,q,r,s = 1,2,\dots,N \ . \tag{6}$$

[1] See also the pedagogical lectures on spontaneous symmetry breaking [9] during this school.

After performing the Wick contractions we find the two-point function of the currents

$$\langle J_\mu^{ij}(x) J_\nu^{kl}(0)\rangle \sim N^2(\delta_{ik}\delta_{jl}-\delta_{il}\delta_{jk})(\Box\delta_{\mu\nu}-\partial_\mu\partial_\nu)\frac{1}{r^4} \ , \tag{7}$$

where we have kept only the leading term in the $1/N$-expansion.[2] This is indeed the correct result for the two point function which also agrees with the AdS/CFT result [5].

In the case that the symmetry is broken by turning on non-zero scalar vev's, we replace the X^i by $X^i_{\rm vev}+\delta X^i$, where $X^i_{\rm vev}$ is defined in (1) and the δX^i have the same free field two-point function as in (6). Besides the bilinear term (5) the current contains now a term linear in fluctuating fields

$$\delta J_\mu^{kl} = \frac{1}{g^2_{\rm YM}} T^{kl}_{ij}{\rm Tr}(X^i_{\rm vev}\partial_\mu \delta X^j) \ , \tag{8}$$

and the leading order correction to the conformal result (7) is

$$\langle \delta J_\mu^{ij}(x)\delta J_\nu^{kl}(0)\rangle \sim \frac{1}{g^2_{\rm YM}} H^{ij,kl}\partial_\mu\partial_\nu\frac{1}{r^2} \ , \tag{9}$$

where the group theoretical factor $H_{ij,kl}$ takes the form

$$H^{ij,kl} = \delta_{ik}A_{jl}-\delta_{jk}A_{il}-\delta_{il}A_{jk}+\delta_{jl}A_{ik} \ , \qquad A_{ij}=\sum_{p=1}^N X^i_p X^j_p \ . \tag{10}$$

It is clear that, in the ultraviolet where the vev's can be neglected, the conformal result (7) dominates, whereas in the infrared the dominant term is (9). The symmetric tensor A_{ij} is given in terms of the scalar vevs only and depends on their distribution. In the following we think of the vevs $X^i_{\rm vev}$ as defining N points in $\mathbb{R}^6$. In the large N limit such a discrete distribution can often be approximated by a continuous one, as long as we work with energies not too close to the vev values. Furthermore, we will consider situations where the distribution spans only a lower dimensional submanifold embedded in $\mathbb{R}^6$. From (9) we see that the tensor $H^{ij,kl}$ contains all the important information about the zero mass poles. It is antisymmetric in the indices ij and kl separately and symmetric under pairwise exchange. Note that A_{ij} is non-zero only if both indices i,j are along the vev-distribution. That implies that $H^{ij,kl}=0$ if all indices correspond to directions which are perpendicular to the distribution.

For the comparison with the dual supergravity that we will perform later, it is convenient to write down the general two point function of the currents in a particular form. In x-space it is given in terms of a function $G^{ij,kl}(x)$ as

$$\langle J_\mu^{ij}(x) J_\nu^{kl}(0)\rangle = \frac{N^2}{32\pi^4}(\Box\delta_{\mu\nu}-\partial_\mu\partial_\nu)\Box G^{ij,kl}(x) \ , \tag{11}$$

[2] For finite N, the N^2 factor in (7) is replaced by N^2-1 corresponding to the dimension of the $SU(N)$ group. We also note that the contribution of the fermions only affects the result by an overall N-independent numerical constant which is not important for our purposes.

where the projector ensures transversality of the correlator. In turn, the function $G^{ij,kl}(x)$ can be used to define a function $H^{ij,kl}(k)$ in momentum space as

$$G^{ij,kl}(x) = \frac{1}{4\pi^2} \int d^4k e^{ik\cdot x} \frac{H^{ij,kl}(k)}{k^2} = \frac{1}{r} \int_0^\infty dk H^{ij,kl}(k) J_1(kr) \ , \tag{12}$$

where the Bessel function $J_1(kr)$ is a result of the integration over angular coordinates. We cannot use $H^{ij,kl}(k)$ directly because the correlator in x-space is too singular to be Fourier transformed to momentum space. However, by using differential regularization one can make sense of such expressions by writing singular functions as derivatives of less singular ones and then defining the Fourier transform by formal partial integrations [10].

We also note that the momentum space version of (9) can be expressed in terms of a function $H^{ij,kl}(k)$ as

$$H^{ij,kl}(k) \sim -\frac{1}{g_{\rm YM}^2 N^2} \frac{H^{ij,kl}}{k^2} \ , \tag{13}$$

where $H^{ij,kl}$ on the r.h.s. is defined in (10). We emphasize that this is the interesting piece of the current correlator that potentially gives rise to massless poles, i.e. Goldstone bosons, depending on the details of $H^{ij,kl}$.

Some examples

The polygon: In this toy example we consider a discrete distribution of vevs in an N-polygon whose vertices lie on a circle of radius r_0 in the 1-2 plane (3). Computing the matrix elements A_{ij} using the definition (10) is straightforward, and we find that the only non-zero components are $A_{11} = A_{22} = Nr_0^2/2$. We note that in this case we obtain the same result even if we approximate the discrete distribution by a continuous uniform distribution of vevs on the circumference of the circle.

We now turn to the examples with vev distributions on a disc and on a three-sphere, which will be considered in section 3 (within a more general class of examples) from the supergravity side using the AdS/CFT correspondence. In these cases a direct comparison with the free field calculation can be performed and we will find precise agreement.

The three-sphere: For a uniform distribution on a three sphere of radius r_0 embedded in the 1-2-3-4 hyperplane it is obvious that $A_{ii} = Nr_0^2/4$, for $i = 1, 2, 3, 4$ and zero otherwise. These results are most easily derived in the continuous approximation of the distributions. Hence, using (10), (13) and the identities $g_{\rm YM}^2 = g_s$ and $R^4 = 4\pi g_s N$, we obtain

$$H^\lambda_{\rm sphere}(k) \sim -\frac{r_0^2}{R^4} \frac{\lambda}{k^2} \ , \tag{14}$$

where the parameter $\lambda = 0, \frac{1}{2}$ and 1 corresponds to currents in the transverse directions (unbroken $SO(2)$), broken currents in the coset and directions along the distribution (unbroken $SO(4)$), respectively.

The disc: For a uniform distribution on a disc in the 1-2 plane we have similarly that $A_{ii} = Nr_0^2/4$, for $i = 1, 2$ and zero otherwise. Using (10) and (13) we compute

$$H^{\lambda}_{\rm disc}(k) \sim \frac{r_0^2}{R^4}\frac{\lambda - 1}{k^2} \ , \tag{15}$$

where the parameter $\lambda = 0, \frac{1}{2}$ and 1 corresponds to currents along the distribution (unbroken $SO(2)$), broken currents in the coset and directions orthogonal to the distribution (unbroken $SO(4)$), respectively.

3 The supergravity side

On the dual supergravity side it is straightforward to write down the relevant supergravity background for arbitrary points on the Coulomb branch. The reason is that the vevs of the scalars are simply the positions of the D3-branes in the transverse six-dimensional space [11]. Furthermore, this is a BPS configuration and the full solution is simply a superposition of single D3-branes given by

$$ds^2 = \frac{1}{\sqrt{H}} dx_{||}^2 + \sqrt{H}\sum_{i=1}^{6} dx_i^2 \ , \qquad H = 4\pi g_s l_s^4 \sum_{p=1}^{N} \frac{1}{|\vec{x} - \vec{X}_p|^4} \ , \tag{16}$$

where $x_{||}$ denotes flat worlvolume directions of the D3-brane. While it is nice to have the most general solution, for practical purposes of calculating e.g. current correlators, it is not very useful and we will study a subspace of the Coulomb branch that (i) can be studied using gauged supergravity, (ii) corresponds to continuous distributions of D3-branes [12, 8] which means that the sum in (16) over localized D3-branes has to be replaced by an integral over a specific D3-brane density. This will allow us to make exact calculations in several cases, which can be compared with the results discussed at the end of section 2. In that respect, we mention the case of the two center solution, where N D3-branes are distributed evenly in two stacks of branes. This case has been studied in [13] and although it might look simpler than the ones we will encounter here, it is actually difficult to perform any computation exactly.

The advantage of (i) is that we do not have to study the ten-dimensional supergravity but we can restrict ourselves to a truncation of the full theory that describes only the fields relevant for our problem. In our case the relevant tool is five-dimensional $\mathcal{N} = 8$ gauged supergravity [14] of which we actually only need a further truncation which includes the metric, the $SO(6)$ gauge field and scalars in the coset $SL(6, \mathbb{R})/SO(6)$. The Lagrangian for these fields is

$$\mathcal{L} = \mathcal{L}_{\rm scalar} + \mathcal{L}_{\rm gauge} \ , \tag{17}$$

where $\mathcal{L}_{\rm scalar}$ denotes the pure gravity plus scalar sector and $\mathcal{L}_{\rm gauge}$ contains the kinetic term of the gauge fields together with their interactions with gravity and the scalars.

The explicit form of the gravity plus scalar Lagrangian is

$$\frac{1}{\sqrt{g}}\mathcal{L}_{\rm scalar} = \frac{1}{4}\mathcal{R} - \frac{1}{16}{\rm Tr}\left(\partial_{\widehat{\mu}} M M^{-1} \partial^{\widehat{\mu}} M M^{-1}\right) - P \ , \tag{18}$$

with the potential

$$P = -\frac{g^2}{32}\left[({\rm Tr}M)^2 - 2{\rm Tr}(M^2)\right] \ , \tag{19}$$

where g is a mass scale, which is related to the AdS_5 radius via $g = 2/R$ with $R = (4\pi g_s N)^{1/4} l_s$. The scalar fields sit in a symmetric traceless matrix M^{ij} where the indices are in the fundamental representation of $SO(6)$, i.e. these scalars transform in the $\mathbf{20'}$. Supersymmetric domain-wall solutions of (18) preserving 16 supercharges together with four-dimensional Lorentz invariance correspond to states on the Coulomb branch of $\mathcal{N} = 4$ SYM [15]. In this case the matrix M can be diagonalized using an $SO(6)$ gauge transformation and can be parametrized by six scalar fields

$$M = {\rm diag}(e^{2\beta_1}, \ldots, e^{2\beta_6}) \ , \tag{20}$$

obeying the constraint $\sum_{i=1}^{6}\beta_i = 0$. Alternatively one could use five independent scalar fields α_I, $I = 1, 2, \ldots 5$ which are related to the β_i's by $\beta_i = \sum_{I=1}^{5}\lambda_{iI}\alpha_I$, where λ_{iI} is a 6×5 matrix, with rows corresponding to the fundamental representation of $SL(6, \mathbb{R})$. The ansatz for the domain wall metric is

$$ds^2 = e^{2A(z)}(dz^2 + \eta_{\mu\nu}dx^\mu dx^\nu) = dr^2 + e^{2A(r)}\eta_{\mu\nu}dx^\mu dx^\nu \ , \tag{21}$$

where the relation between the coordinates z and r is such that $dr = -e^A dz$. The most general solutions is expressed in terms of an auxiliary function $F(g^2 z)$, in terms of which the conformal factor and the profiles of the scalars are [16]

$$e^{2A} = g^2(-F')^{2/3} \ , \qquad e^{2\beta_i} = \frac{f^{1/6}}{F - b_i} \ , \qquad f = \prod_{i=1}^{6}(F - b_i) \ , \qquad i = 1, 2, \ldots, 6 \ . \tag{22}$$

The constants of integration are ordered as $b_1 \geq b_2 \geq \ldots \geq b_6$ and the function F obeys the differential equation

$$(F')^4 = f \ . \tag{23}$$

Equating n of the integration constants b_i (or equivalently the associated scalar fields β_i) corresponds to preserving an $SO(n)$ subgroup of the original $SO(6)$ $\mathcal{R}$-symmetry group. In general the hypersurface $F = b_1$ corresponds to a curvature singularity which, however, has the physical interpretation as being the location of the distribution of D3-branes once we lift the solution to a Type IIB background. Furthermore, we can make contact with the disc and three-sphere distributions discussed in section 2. The disc corresponds to setting the integration constants $b_1 = b_2 = b_3 = b_4$ and $b_5 = b_6$ in which case the unbroken $\mathcal{R}$-symmetry group is $SO(2) \times SO(4)$ and the Goldstone bosons corresponding to the broken symmetries reside in the coset $SO(6)/(SO(2)\times SO(4))$.[3] For the three-sphere we have to choose $b_1 = b_2$ and $b_3 = b_4 = b_5 = b_6$ while the unbroken symmetry group and the coset are the same as for the disc. We note that the solutions (21), (22) can be lifted to type IIB solutions of the form (16) and that, this class of solutions does not include the uniform distribution of vevs on a circle.

Let us now add the gauge fields to the Lagrangian (18). The partial derivatives in (18) are replaced by gauge-covariant ones $\partial_{\hat\mu} M^{ij} \to \partial_{\hat\mu} M^{ij} + g(A^{ik}_{\hat\mu} M^{kj} + A^{jk}_{\hat\mu} M^{ik})$, and the gauge kinetic term has to be added

$$\frac{1}{\sqrt{g}}\mathcal{L}_{\rm gauge} = -\frac{1}{8}(M^{-1})^{ij}(M^{-1})^{kl} F^{ik}_{\hat\mu\hat\nu} F^{jl\hat\mu\hat\nu} \ , \tag{24}$$

where $A^{ij}_{\hat\mu}$ and $F^{ij}_{\hat\mu\hat\nu}$ are anti-symmetric in ij. Since we are interested in two-point functions we only keep terms in (17) and (24) which are quadratic in the fluctuations $\delta A^{ij}_{\hat\mu}$ of the gauge fields and the scalar fluctuations of the symmetric unimodular matrix δM^{ij}. Although we are only interested in the two point functions of the gauge fields we have to keep scalar fluctuations since the gauge fields couple to the off-diagonal scalar fluctuations δM^{ij}. However, there are no couplings of the gauge fields to the metric and the diagonal scalar fluctuations δM^{ii} at quadratic order. At this point it is crucial to distinguish between gauge fields that correspond to unbroken symmetries for which $\beta_i = \beta_j$ and gauge fields corresponding to broken symmetries for which $\beta_i \neq \beta_j$.

The case of unbroken symmetries is easier since $\delta A^{ij}_{\hat\mu}$ fluctuations do not couple to scalar fluctuations. We can choose a convenient gauge $\delta A_z = 0$ and the components along the world volume directions A_μ can be decomposed into a transverse and a longitudinal polarization $A_\mu = A^\perp_\mu + \partial_\mu \xi$. The equation of motion for the physical modes $A^\perp_\mu$, which obey $\partial^\mu A^\perp_\mu = 0$, takes the form of a field equation for a scalar field Φ:

$$\partial_z(e^B \partial_z \Phi) - k^2 e^B \Phi = 0 \ , \tag{25}$$

with the definition

$$B = A - 2(\beta_i + \beta_j) \ , \tag{26}$$

where we have omitted for notational convenience the i, j dependence of B. Note that we have also performed a Fourier transform in the x^μ-directions with $k^2 \equiv k_\mu k^\mu$.

For the broken symmetries $\beta_i \neq \beta_j$, however, there are couplings between scalars and gauge fields and things become more tricky. It is a highly non-trivial fact that we can decouple the scalars via the field redefinition [4]

$$A^{ij}_{\hat\mu} \to A^{ij}_{\hat\mu} + \frac{1}{g}\,\partial_{\hat\mu}\left(\frac{\delta M_{ij}}{e^{2\beta_i} - e^{2\beta_j}}\right) \ , \qquad \beta_i \neq \beta_j \ , \tag{27}$$

which has the form of an abelian gauge transformation. Of course, this is nothing but the Higgs mechanism. The Goldstone boson corresponding to the broken gauge symmetries

[3] See however the comments in the paragraph starting after (42).

are *eaten* by the gauge bosons which thereby obtain a mass. Since the gauge fields are massive now we cannot eliminate degrees of freedom by gauge fixing. In order to calculate the two-point function we couple the gauge field to an external current via $A^{ij}_{\widehat{\mu}} J^{\widehat{\mu}}_{ij}$ and require the current to be covariantly conserved $D_{\widehat{\mu}} J^{\widehat{\mu}}_{ij} = 0$. The A_z component decouples from the other modes which in turn is related to the longitudinal component ξ. Again, the decoupled equation for the transverse modes $A^{\perp}_{\mu}$, takes the form of a massive scalar field equation

$$\partial_z(e^B \partial_z \Phi) - \left(k^2 e^B + \frac{1}{4} g^2 (b_i - b_j)^2 e^{-B}\right) \Phi = 0 \ , \tag{28}$$

where the scalar Φ denotes any component of $A^{\perp}_{\mu}$. For $\beta_i = \beta_j$, which implies $b_i = b_j$, we recover from (28) eq. (25) that describes the cases with unbroken symmetry. Hence (28) is the general equation that can be used to calculate all current-current correlators.

Let us shortly pause here and admire the result. In the field theory we study the breaking of a global symmetry, which is signalled by the appearance of Goldstone bosons. In the supergravity dual the global $\mathcal{R}$-symmetry becomes a local gauge symmetry and we observe a different mechanism, namely the Higgs phenomenon. The would be Goldstone boson becomes an additional degree of freedom of the gauge field, i.e. the gauge field becomes massive. Hence, we see that in the AdS/CFT correspondence spontaneous symmetry breaking in the field theory corresponds to Higgsing of a local gauge symmetry in the dual supergravity/string theory. In the rest of these notes we complete the picture by showing how the massless poles in the examples of section 2 can be reproduced from a supergravity calculation of the current correlators using (28).

We will follow the standard procedure of [2, 3] to determine the current-current correlators using (28). We will work in Euclidean signature unless stated otherwise. In order to proceed we need a complete set of eigenfunctions of (28), which can be found explicitely only in a small number of examples, namely for the disc and three-sphere distributions [15, 17]. Furthermore, we have to keep the solutions that blow up at the AdS boundary since they correspond to operator insertions [2, 3]. Finally, we have to evaluate the on shell-value of the action $\frac{1}{\kappa^2}\int d^5x \mathcal{L}$ with $\frac{1}{\kappa^2} = \frac{N^2}{16\pi^2}$ for solutions Φ of (28). This yields the boundary term

$$-\lim_{\epsilon \to 0} \frac{N^2}{32\pi^2} e^B \Phi \partial_z \Phi \Big|_{z=\epsilon}^{z_{\max}} \equiv \frac{N^2}{16\pi^2} k^2 H(k) \ , \tag{29}$$

where we have to normalize $\Phi|_{z=\epsilon} = 1$ and take the limit $\epsilon \to 0$ in (29), which corresponds to the AdS boundary. Re-introducing Lorentz and group theory indices properly, we can present the current-current correlators in momentum space schematically as

$$\langle J^{ij}_{\mu}(k) J^{kl}_{\nu}(-k)\rangle = \frac{N^2}{8\pi^2}\left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) k^4 \tilde{G}^{ij,kl}(k) \ , \tag{30}$$

where a group theory factor and the momentum space version of the projector, which guarantees that the amplitude is transverse, have been included. The expression (30) is of course nothing but the momentum-space analogue of (11).

For the cases of D3-branes distributions over a disc and a three-sphere, which were discussed in section 2, (28) can be solved explicitly [4], but in general this is not possible. But since we are mainly interested in the Goldstone bosons we will take a shortcut by solving (28) for small k^2 in order to extract the massless poles. It is useful to rewrite the equation (28) in terms of the variable F

$$\frac{d}{dF}\left((F-b_i)(F-b_j)\frac{d\Phi}{dF}\right)-k^2\frac{(F-b_i)(F-b_j)}{f^{1/2}}\Phi-\frac{b_{ij}^2}{4(F-b_i)(F-b_j)}\Phi=0\ , \qquad (31)$$

where F and b_i were defined in equation (22) and $b_{ij}=b_i-b_j$. In order to extract the massless poles it suffices to concentrate on the limit $k^2\to 0$, where (31) can be solved exactly for any distribution. This will give the leading contribution to the two-point function of currents for large distances. At the AdS boundary $F\to\infty$ we impose the usual boundary condition $\Phi\to 1$ corresponding to a point-like source. Furthermore, we require Φ to be smooth at the singularity $F=b_1$ in the interior. In the following we use units where $g=2/R=1$.

Correlators and comparison with field theory

In order to make a comparison between field theoretical and supergravity results for the massless poles arising in the current correlators, we need the group theoretical factor $H^{ij,kl}$ defined in (10). It can be shown that [4]

$$A_{ij}=Nb_{1j}\ \delta_{ij}\ , \qquad (32)$$

where we defined $b_{ij}=b_i-b_j$. Consequently, our distributions have a diagonal matrix A_{ij}. Hence, the only non-zero independent components of the group theoretical factor $H^{ij,kl}$ are $H^{ij,ij}$. If all indices correspond to directions which are perpendicular to the distribution then $H^{ij,kl}=0$, whereas if all directions are along the distribution $H^{ij,ij}=N(b_{1j}+b_{1i})$. If we are in the coset one index is along the distribution (say i) and one is orthogonal to it (say j), then one of the above terms is missing and therefore $H^{ij,ij}=Nb_{1i}$. This agrees perfectly with the two special cases of the disc and sphere distribution that we considered before.

<u>Currents transverse to the distribution</u>: In this case the indices of the current i,j are such that $b_i=b_j=b_1$. Demanding regularity at the singularity $F=b_1$ and imposing the normalization condition at the boundary gives

$$\Phi=1\ . \qquad (33)$$

Therefore (29) gives

$$H(k)=0\ . \qquad (34)$$

As expected this agrees with the field theoretical results for vev distributions on a three-sphere (14) and on a disc (15).

Currents longitudinal to the distribution: In this case the indices of the current are such that $b_i, b_j \neq b_1$. Imposing regularity at the singularity at $F = b_1$ and the normalization condition at the boundary, we find

$$\Phi = \frac{1}{b_{ij}} \left(b_{1j} \left(\frac{F - b_i}{F - b_j} \right)^{1/2} - b_{1i} \left(\frac{F - b_j}{F - b_i} \right)^{1/2} \right) , \tag{35}$$

from which, using (29), we compute

$$H(k) = -\frac{b_{1i} + b_{1j}}{4k^2} . \tag{36}$$

A particularly interesting case is when $b_i = b_j \neq b_1$. Then the above expressions reduce to

$$\Phi = \frac{F - b_1}{F - b_i} \tag{37}$$

and

$$H(k) = -\frac{b_{1i}}{2k^2} . \tag{38}$$

The results for the sphere (14) and disc (15) distributions correspond precisely to that result with $b_{1i} = r_0^2/4$ (b_1 can be put to zero by a shift of the coordinate F), for $i = 1, 2, 3, 4$ and $i = 1, 2$, respectively.

Currents in the coset: In this case the indices of the current are $b_i = b_1$ and $b_j \neq b_1$. Proceeding as before we find

$$\Phi = \left(\frac{F - b_1}{F - b_j} \right)^{1/2} \tag{39}$$

and

$$H(k) = -\frac{b_{1i}}{4k^2} . \tag{40}$$

Setting $b_{1i} = r_0^2/4$, as above, one easily sees that the result (40) agrees with the field theory calculations for the disc (15) and the three-sphere (14), respectively.

Exact expressions for the sphere and disc distributions: In the case of a three-sphere distribution it is possible to compute exactly the two-point function for current correlators in the supergravity side [4], since (31) admits an appropriate solution in terms of a hypergeometric function. Here we denote for completeness the expression for

$$\begin{aligned} H^\lambda_{\text{sphere}}(k) &= \frac{r_0^2}{R^4} \frac{1-\lambda}{k^2} + \frac{1}{4} \Big(\psi\left((1+\Delta)/2\right) + \psi\left((1-\Delta)/2\right) + 2\gamma \Big) \\ &= -\frac{r_0^2}{R^4} \frac{\lambda}{k^2} + \frac{1}{2} \sum_{n=1}^{\infty} \frac{2n + \tilde{k}^2}{n(4n(n+1) + \tilde{k}^2)} , \end{aligned} \tag{41}$$

where $\tilde{k}^2 \equiv k^2 R^4 / r_0^2$ and $\psi(z)$ is the standard notation for the derivative of the logarithm of the Gamma function $\Gamma(z)$. This expression has the correct behaviour in the infrared for $k^2 \to 0$ that we have already exhibited. In addition it can be shown that it gives rise to the correct limit for the two-point function in the ultraviolet. The discrete spectrum of poles at $\tilde{k}^2 = -4n(n+1)$, $n = 1, 2, \ldots$, corresponds precisely to the discrete mass eigenvalues for the normalizable solutions of (31).[4]

In the case of distribution of vevs on a disc we also obtain [4]

$$
\begin{aligned}
H^{\lambda}_{\rm disc}(k) &= \frac{r_0^2}{R^4}\frac{\lambda-1}{k^2} + \frac{1}{2}\left(\psi\left((1+\Delta)/2\right)+\gamma\right) \\
&= \frac{r_0^2}{R^4}\frac{\lambda-1}{k^2} + \frac{1}{2}\int_0^\infty dt \frac{e^{-t}-e^{-\frac{\Delta+1}{2}t}}{1-e^{-t}} \ ,
\end{aligned}
\tag{42}
$$

where $\Delta = \sqrt{\tilde{k}^2+1}$. This expressions also has the correct behaviour in the infrared and ultraviolet limits. The branch cut for $\tilde{k}^2 = -1$ corresponds to a mass gap of the continuous spectrum for the associated normalizable solutions (in the Dirac sense) of (31).

We close this section by explaining a subtlety of these results. In the case of the three-sphere distribution we mentioned that the unbroken $\mathcal{R}$ symmetry is $H = SO(2)\times SO(4)$, however we found Goldstone bosons in the coset $SO(6)/H$, but also in the "unbroken" $SO(4)$ sector. The resolution of this discrepancy is that the $SO(4)$ symmetry is accidental and is caused by the approximation of a discrete distribution by a smooth distribution over a three-sphere. Clearly, the actual discrete distribution breaks this symmetry. Hence, the Goldstone bosons live in the larger coset $SO(6)/SO(2)$. Similar comments apply to the disk case where the role of $SO(2)$ and $SO(4)$ have to be interchanged. It is quite interesting that the gravity dual seems to know about this and reproduces the correct set of Goldstone bosons.

We have shown that the field theory and supergravity calculations can be in excellent agreement even beyond the conformal limit. The Goldstone bosons are sensitive to the infrared physics of the field theory, which on the dual supergravity side is captured by the interior of the geometry. However, for states on the Coulomb branch the supergravity solutions generically have naked singularities in the interior, hence such good agreement is better than one could expect. We have found that spontaneous breaking of global symmetries of the field theory translates on the supergravity side to the Higgs effect of local gauge symmetries. It sounds somewhat counter intuitive, but the massive bulk gauge fields corresponding to broken symmetries conspire to reproduce the correct spectrum of Goldstone bosons in the dual field theory.

[4]We note that in general the dilaton, transverse graviton and gauge field fluctuations have degenerate spectra for our models [18, 4]. This can be traced back to the fact that the corresponding fields belong to the same $\mathcal{N}=4$ supermutliplet. For an explicit demonstration of this, in some related cases, see [6].

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