Graphs with Hamiltonian Balls

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Abstract

For a vertex u of a graph G and an integer r, the ball of radius r centered at u is the subgraph $G_r(u)$ induced by the set of all vertices of G whose distance from u does not exceed r. We investigate the set $\mathcal{H}$ of connected graphs G with at least 3 vertices such that every ball of radius 1 in G has a Hamilton cycle. We prove that every graph G in $\mathcal{H}$ with n vertices has at least $2n-3$ edges, and every such graph with $2n-3$ edges is isomorphic to a triangulation of a polygon. We show that some well-known conditions for hamiltonicity of a graph G also guarantee that G has the following property: for each vertex u of G and each integer $r \geq 1$, the ball $G_r(u)$ has a Hamilton cycle.

1. Introduction

Interconnection between local and global properties of mathematical objects has always been a subject of investigations in different areas of mathematics. Usually by local properties of a mathematical object, for example a function, we mean its properties in balls with small radii. If a considered mathematical object is a graph, balls of radius r are defined only for integer r. For a vertex u of a graph G the ball of radius r centered at u is a subgraph of G induced by the set $M_r(u)$ of all vertices of G whose distance from u does not exceed r. This ball we denote by $G_r(u)$. In fact, for each vertex u of a connected graph G there is an integer $r(u)$ such that G is a ball of radius $r(u)$ centered at u. Note that our definition of the ball is different from the usual definition (see, for example, [12]) where the set $M_r(u)$ is considered as the ball of radius r centered at u.

The following problem arises naturally.

Problem 1.1. If each ball of radius 1 in a graph G enjoys a given property P, does G have the same property?

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For different properties the answers are different. For example, it is known [4] that if $G_1(u)$ is k-connected for each vertex u, then G is also k-connected. On the other hand, for each positive integer $m \geq 3$ there exists a graph G such that the chromatic number $\chi(G_1(u))$ is 2 for each vertex u but $\chi(G) = m$ (see [15,24]).

We consider Problem 1.1 in the case when P is the property of a graph to be hamiltonian. (A graph is called hamiltonian if it has a Hamilton cycle, that is, a cycle containing all the vertices of a graph). Let $\mathcal{H}$ denote the set of all connected graphs G with at least 3 vertices such that any ball of radius 1 in G is hamiltonian. The set $\mathcal{H}$ contains, in particular, locally hamiltonian graphs investigated in [7,19,22,23]: a graph G is called locally hamiltonian if, for each vertex u of G, the subgraph induced by the set of vertices adjacent to u is hamiltonian.

In Section 3 we give some general properties of graphs in $\mathcal{H}$. We show that every graph $G \in \mathcal{H}$ with n vertices has at least $2n-3$ edges and every graph $G \in \mathcal{H}$ with n vertices and $2n-3$ edges is a maximal outerplanar graph, that is, a graph isomorphic to a triangulation of a polygon.

We introduce and investigate a new property of a graph G: every ball of any radius in G is hamiltonian. Graphs with this property we call uniformly hamiltonian. Clearly, all uniformly hamiltonian graphs are in $\mathcal{H}$ but not every hamiltonian graph in $\mathcal{H}$ is uniformly hamiltonian. For example, the graph G in Fig.1 is hamiltonian and belongs to $\mathcal{H}$ but the subgraph $G_2(x_0)$ is not hamiltonian.

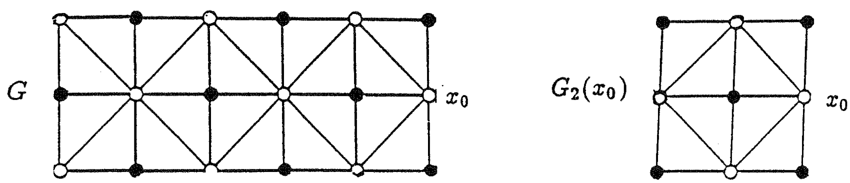

Fig.1

In Section 4 we show that some well-known conditions for hamiltonicity of a graph G guarantee also uniform hamiltonicity of G.

2. Definitions and auxiliary results

We use [3] for terminology and notation not defined here and consider simple graphs only. Let $V(G)$ and $E(G)$ denote, respectively, the vertex set and edge set of a graph G, and let $d(u, v)$ denote the distance between vertices u and v in G. For each vertex u of G and a positive integer r, we denote by $N_r(u)$ and $M_r(u)$ the sets of all $v \in V(G)$ with $d(u, v) = r$ and $d(u, v) \leq r$, respectively. The set $N_1(u)$ is called the *neighbourhood* of a vertex u. The subgraph of G induced by the set $N_1(u)$ is denoted by $< N_1(u) >$. A graph G is called *locally k-connected* if, for each vertex $u \in V(G)$, the subgraph $< N_1(u) >$ is k-connected. In other words, G is locally k-connected if, for each vertex $u \in V(G)$, the ball $G_1(u)$ is $(k+1)$-connected.

Let G be a connected graph and v be a vertex in a ball $G_r(u), r \geq 1$. We call v an *interior vertex* of $G_r(u)$ if $G_1(v)$ is a subgraph of $G_r(u)$. Clearly, every vertex in $G_{r-1}(u)$ is interior for $G_r(u)$ and if $G = G_r(u)$ then all vertices in G are interior vertices.

Let C be a cycle of a graph. We denote by $\overrightarrow{C}$ the cycle C with a given orientation, and by $\overleftarrow{C}$ the cycle C with the reverse orientation. If $u, v \in V(C)$ then $u\overrightarrow{C}v$ denote the consecutive vertices of C from u to v in the direction specified by $\overrightarrow{C}$. The same vertices, in reverse order, are given by $v\overleftarrow{C}u$. We use u^+ to denote the successor of u on $\overrightarrow{C}$ and u^- to denote its predecessor.

Analogous notation is used with respect to paths instead of cycles.

A planar graph G is called *maximal planar* if no edges can be added to G without losing planarity. A planar graph is *outerplanar* if it can be embedded in the plane so that all its vertices lie on the exterior face. A *maximal outerplanar graph*, or a mop, is an outerplanar graph such that the addition of an edge between any two non-adjacent vertices results in a non-outerplanar graph. In other words, a mop is a graph which is isomorphic to a triangulation of a polygon. Every mop has the unique Hamilton cycle which forms the exterior face. It is also known [9] that a mop with n vertices has $2n - 3$ edges and at least two vertices of degree 2. Hence every mop G with $n \geq 4$ vertices can be obtained from some mop H with $n - 1$ vertices by adding a new vertex adjacent to two consecutive vertices on the Hamilton cycle of H. (Then we say that G is obtained from H by an elementary extension). This property implies the following lemma which usually is used as a recursive characterization of a mop [14,20].

Lemma 2.1. A graph $G \neq K_3$ is a mop if and only if it can be obtained from the triangle K_3 by a sequence of elementary extensions.

3. Some properties of graphs in $\mathcal{H}$

Proposition 3.1. Every graph $G \in \mathcal{H}$ has at least $2|V(G)| - 3$ edges.

Proof. Let $C(v)$ denote a Hamilton cycle of a graph $G_1(v)$ for each $v \in V(G)$. Consider a vertex x_0 of G. If $G = G_1(x_0)$ then $|E(G)| \geq 2|N_1(x_0)| - 1 = 2|V(G)| - 3$ and the assertion of the theorem is true. Let $G_1(x_0) \neq G$ and r be an integer such that $G = G_r(x_0)$ and $G \neq G_{r-1}(x_0)$. First we will describe an algorithm which for each vertex $v \in V(G)$ constructs an edge set $E(v)$. During this algorithm a vertex is considered to be unscanned or scanned. Initially all vertices of G are unscanned. A vertex v becomes scanned if $E(v)$ is already constructed.

Step 1. Let $C(x_0) = x_0x_1...x_mx_0$. Put $E(x_0) = \emptyset, E(x_m) = \{x_mx_0\}$ and $E(x_i) = \{x_0x_i, x_ix_{i+1}\}$ for $i = 1, ..., m-1$. The vertices $x_0, x_1, ..., x_m$ are now considered to be scanned.

Step j($j \geq 2$). Assume that all vertices in $M_{j-1}(x_0)$ have already been scanned. Consider an unscanned vertex $z \in N_j(x_0)$ and a vertex $y = y(z)$ in $N_{j-1}(x_0)$ which is adjacent to z. There is a path $P = P(y, z)$ in the cycle $C(y)$ such that z lies on P, all internal vertices of P are in $N_j(x_0)$ and the origin and the terminus of P are in $N_{j-1}(x_0)$. Without loss of generality we assume that the origin of P is not y. Let $\overrightarrow{P}$ denote an oriented path obtained from P by orienting the edges in the direction from the origin to the terminal vertex. For each unscanned vertex v on $\overrightarrow{P}$ we put

$$E(v) = \{uv \in E(G)/u \in N_{j-1}(x_0)\} \cup \{vv^{-1}(\overrightarrow{P})\}$$

where $v^{-1}(\overrightarrow{P})$ is the predecessor of v on $\overrightarrow{P}$. All vertices of $\overrightarrow{P}$ are now scanned.

If there remains any unscanned vertex in $N_j(x_0)$ then we repeat Step j. Otherwise go to Step j+1 if $j < r$ and stop if $j = r$.

Clearly, $|E(v)| \geq 2$ for each $v \in V(G) \setminus \{x_0, x_m\}$ and $E(v_1) \cap E(v_2) = \emptyset$ for each pair of distinct vertices $v_1, v_2 \in V(G)$. Therefore

$$|E(G)| \geq 2|N_1(x_0)| - 1 + 2\sum_{j=2}^{r} |N_j(x_0)| = 2|V(G)| - 3.$$

The proof is complete.■

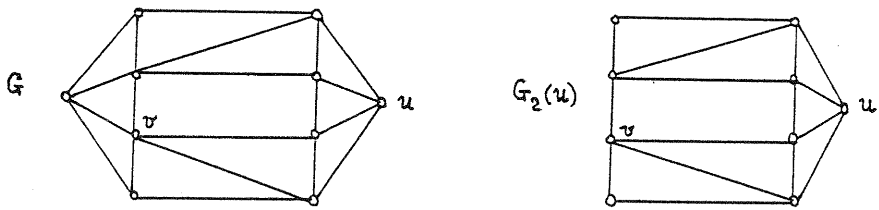

Fig. 2

It is known [4] that in a connected graph G local $(k-1)$-connectedness, $k \geq 2$, implies k-connectedness of G. Now by using the same argument, we will show that indeed in such graphs all balls are k-connected. Note that not every ball of a locally $(k-1)$–connected graph is locally $(k-1)$–connected. For example, the graph G in Fig. 2 is locally connected but in the ball $G_2(u)$ the neighbourhood of the vertex v induces a disconnected subgraph.

Proposition 3.2. If every ball of radius 1 in a connected graph G is k-connected, $k \geq 2$, then balls of any radius in G are k-connected.

Proof. It is clear that k-connectedness of $G_1(u)$ implies that $|M_r(u)| \geq k+1$ for each $u \in V(G)$ and each $r \geq 2$. Suppose that for a vertex u and an integer $r \geq 2$ the ball $G_r(u)$ is not k-connected. Then there exists a subset $S \subseteq M_r(u)$ such that $|S| \leq k-1$ and $G_r(u) - S$ is disconnected. Among all such sets S, let S_0 be one of minimum cardinality. Clearly, S_0 contains an interior vertex v of $G_r(u)$ because otherwise $G_r(u) - S_0$ is connected. The minimality of S_0 implies that there are two neighbours w_1 and w_2 of v such that w_1 and w_2 belong to different components in $G_r(u) - S$. Then the set $S_0 \cap M_1(v)$ separates w_1 and w_2 in $G_1(v)$

and $|S_0 \cap M_1(u)| \leq k-1$. This contradicts k-connectedness of $G_1(v)$. Hence all the balls in G are k-connected.∎

Corollary 3.3. Every ball in a graph $G \in \mathcal{H}$ is 2–connected.

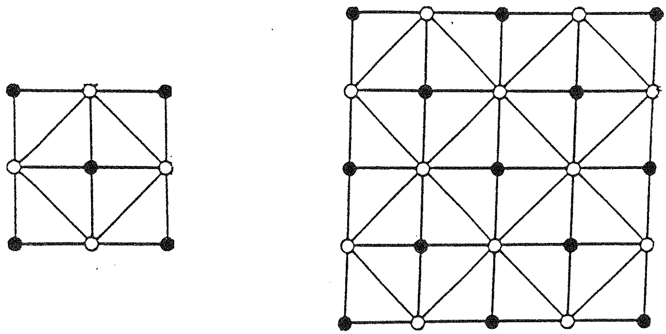

Fig. 3.

Not every graph in $\mathcal{H}$ is hamiltonian. Moreover $\mathcal{H}$ contains non-hamiltonian graphs G such that every ball of G, except G itself, is hamiltonian. Two examples of such graphs are given in Fig. 3. Non–hamiltonicity of the graph with nine vertices is evident. Non–hamiltonicity of the second graph follows from the fact that removing the twelve "light vertices" separates it into thirteen components, each containing a single "dark vertex".

However, all graphs in $\mathcal{H}$ which are minimal concerning the number of vertices are hamiltonian and even uniformly hamiltonian.

Proposition 3.4. Every graph $G \in \mathcal{H}$ with $|E(G)| = 2|V(G)| - 3$ is uniformly hamiltonian. Moreover such graphs are mops.

Proof. Let $G \in \mathcal{H}$ and $|E(G)| = 2|V(G)| - 3$. Consider a vertex x_0 of G. If $G = G_1(x_0)$ then the proposition is evident.

Assume that $G \neq G_1(x_0)$ and r be an integer such that $G_r(x_0) = G$ and $G_{r-1}(x_0) \neq G$. Using the same argument as in the proof of Theorem 3.1 we obtain that

$$|E(G)| \geq 2|N_1(x_0)| - 1 + 2\sum_{j=2}^{r} |N_j(x_0)| = 2|V(G)| - 3$$

On the other hand we have that $|E(G)| = 2|V(G)| - 3$. This implies that

(i) $|E(G)| = 2|N_1(x_0)| - 1 + 2\sum\limits_{j=2}^{r} |N_j(x_0)|$,

(ii) $|E(v)| = 2$ for each $v \in N_j(x_0)$ and $j \geq 2$,

Now by induction on j we will show that the graph $G_j(x_0)$ is a mop. For $j = 1$ this is evident. Suppose that $G_{j-1}(x_0)$ is a mop for some $j \geq 2$ and $\overrightarrow{C}$ is the

Hamilton cycle of $G_{j-1}(x_0)$ with a given orientation. Consider the set $\{\overrightarrow{P_1}, ..., \overrightarrow{P_k}\}$ of all distinct oriented paths which have been constructed in Step j of the algorithm described in the proof of Theorem 3.1. Clearly, $\overrightarrow{P_i} = b_i\overrightarrow{Q_i}a_i$ where $a_i, b_i \in N_{j-1}(x_0)$ and $V(\overrightarrow{Q_i}) \subseteq N_j(x_0)$. Furthermore, (ii) implies that b_i is the predecessor or the successor of a_i on $\overrightarrow{C}$ and $V(\overrightarrow{Q_i}) \subseteq N_j(x_0) \cap N_1(a_i)$

Using this observation and (i)-(ii) we deduce the following property.

Property. The terminal vertex v of $\overrightarrow{Q_i}$ is adjacent in $G_j(x_0)$ only to two vertices: a_i and the predecessor of v on $\overrightarrow{Q_i}$. Each non-terminal vertex v of $\overrightarrow{Q_i}$ is adjacent in $G_j(x_0)$ only to three vertices: a_i, the predecessor and the successor of v on $\overrightarrow{Q_i}$.

Using this property and taking into consideration that the neighbourhood of each vertex $v \in N_j(x_0)$ induces a path we obtain that $\{a_i, b_i\} \neq \{a_s, b_s\}$ for $1 \leq i < s \leq k$.

It is easy to see now that $G_j(x_0)$ can be obtained from $G_{j-1}(x_0)$ by a sequence of elementary extensions. Since $G_{j-1}(x_0)$ is a mop, we conclude, by Lemma 2.1, that $G_j(x_0)$ also is a mop.

Thus, every ball of G is a mop, that is, a hamiltonian graph. Therefore G is uniformly hamiltonian.∎

Corollary 3.5. A connected graph G with $n \geq 3$ vertices is a mop if and only if it has $2n - 3$ edges and, for each vertex u of G, the neighbourhood $N_1(u)$ induces a path.

Proof. If G is a mop then it is connected, $|V(G)| \geq 3$, $|E(G)| = 2|V(G)| - 3$ and it is not difficult to see that for each vertex $x \in V(G)$ the neighbourhood $N_1(x)$ induces a path. Conversely, suppose that G is a connected graph with $|V(G)| \geq 3$ and $|E(G)| = 2|V(G)| - 3$ such that for each vertex u of G, the neighbourhood $N_1(u)$ induces a path. Then a subgraph $G_1(u)$ is hamiltonian for each $u \in V(G)$ and $G \in \mathcal{H}$. Therefore, by Proposition 3.1, G is a mop.∎

For comparison, we note a characterisation of a maximal planar graph given by Skupien [23]: A connected graph G with $n \geq 3$ vertices is a maximal planar graph if and only if G is locally hamiltonian and has $3n - 6$ edges.

4. Some classes of uniformly hamiltonian graphs

Here we will consider some classes of hamiltonian graphs in $\mathcal{H}$. We will show that graphs in these classes are also uniformly hamiltonian.

1. *Powers of graphs*

For a connected graph G and an integer $t \geq 2$, G^t is the graph with $V(G^t) = V(G)$ where two vertices u and v are adjacent if and only if the distance between u and v in G does not exceed t. The graphs G^2 and G^3 are called, respectively, the *square* and the *cube* of G.

Proposition 4.1. For every connected graph G with at least 3 vertices the cube G^3 is a uniformly hamiltonian graph.

Proof. By the result of Karaganis [10] and Sekanina [21], the cube of every connected graph with at least 3 vertices is hamiltonian. Hence $(G_{3r}(u))^3$ is hamiltonian for each vertex u and each $r \geq 1$. Denote the graph G^3 by H. Clearly, $(G_{3r}(u))^3$ is a spanning subgraph of $H_r(u)$ because $V(H_r(u)) = V(G_{3r}(u)) = V((G_{3r})^3)$. Then hamiltonicity of $(G_{3r}(u))^3$ implies hamiltonicity of $H_r(u)$ for each $u \in V(G)$ and each $r \geq 1$. Therefore $H = G^3$ is uniformly hamiltonian.∎

It is well-known, due to Fleischner[8], that the square of every 2-connected graph is hamiltonian. But in the general case even hamiltonicity of a graph G does not guarantee uniform hamiltonicity of G^2. For example, if G is obtained from a cycle $x_1x_2...x_{2n}x_1$ by joining the vertices x_1 and x_n, $n \geq 8$ and $H = G^2$ then the subgraph $H_2(x_1)$ is not hamiltonian. We will indicate two cases when the square of a graph is uniformly hamiltonian.

Proposition 4.2. The square of every cycle is uniformly hamiltonian.

Proposition 4.3. If G is a connected, locally connected graph with $|V(G)| \geq 3$, then G^2 is uniformly hamiltonian.

Proof. Denote the graph G^2 by H. We have that $V(H_r(u)) = V(G_{2r}(u)) = V((G_{2r})^2)$ and $(G_{2r}(u))^2$ is a spanning subgraph of $H_r(u)$ for each vertex u and each $r \geq 1$. Since G is locally connected, $G_1(u)$ is 2-connected for each vertex $u \in V(G)$. Then, by Proposition 3.2, the ball $G_{2r}(u)$ is 2-connected for each $u \in V(G)$ and each integer $r \geq 1$. Therefore, by the result of Fleischner [8], $(G_{2r}(u))^2$ is hamiltonian. This implies that the ball $H_r(u)$ is also hamiltonian for each vertex $u \in V(G)$ and each $r \geq 1$. Therefore, $H = G^2$ is uniformly hamiltonian.∎

2. *Graphs with local Chvatal–Erdös condition*

Let $\alpha(G)$ and $k(G)$ denote the independence number and connectivity of a graph G, respectively. The following theorem is well-known.

Theorem A (Chvatal and Erdös [5]). A graph G with at least three vertices is hamiltonian if $\alpha(G) \leq k(G)$.

A local variation of the result of Chvatal and Erdös was obtained by Khachatrian [11]: A connected graph G is hamiltonian if $|V(G)| \geq 3$ and there is a positive integer $r \geq 1$ such that $\alpha(G_{r+1}(u)) \leq k(G_r(u))$ for each vertex $u \in V(G)$. Now we will show that in the case $r = 1$ this condition implies uniform hamiltonicity of G.

Theorem 4.3. A connected graph G with $|V(G)| \geq 3$ is uniformly hamiltonian if $\alpha(G_2(x)) \leq k(G_1(x))$ for each vertex $x \in V(G)$.

Proof. Suppose that there is a vertex $x \in V(G)$ and an integer $r \geq 1$ such that the ball $G_r(x)$ is not hamiltonian. The condition $\alpha(G_2(x)) \leq k(G_1(x))$ implies that x lies on a triangle. Among all cycles in $G_r(x)$ which contain x, let C be one of maximum length. Consider a vertex $y \in M_r(x) \setminus V(C)$ and a shortest (x, y)-path in $G_r(x)$. Clearly, there are two adjacent vertices v and u on this path such that $v \notin V(C)$, $v \in V(C)$ and u is an interior vertex of $G_r(x)$. Let $\overrightarrow{C}$ be the cycle C with a given orientation. We have that $2 \leq \alpha(G_2(u)) \leq k(G_1(u))$ since $vu^+ \notin E(G)$. Then, by Menger's theorem [13], in $G_1(u)$ there are k internally disjoint (v, u^+)-paths $Q_1, ..., Q_k$, where $k = k(G_1(u))$. Maximality of C implies that each Q_i has at least one common vertex with C. This means that there are paths $P_1, ..., P_k$ having initial vertex v that are pairwise disjoint, apart from v, and that share with C only their terminal vertices $v_1, ..., v_k$, respectively. Furthermore, maximality of C implies that $vv_i^+ \notin E(G)$ for each $i = 1, ..., k$. Then there is a pair i, j such that $1 \leq i < j \leq k$ and $v_i^+ v_j^+ \in E(G)$. (Otherwise in $G_2(v)$ there are $k+1$ mutually non-adjacent vertices $v, v_1, ..., v_k$ which contradicts the condition $\alpha(G_2(v) \leq k(G_1(v)))$. Since u is an interior vertex of $G_r(x)$, the paths $P_1, ..., P_k$ lie in $G_r(x)$.

Now by deleting the edges $v_i v_i^+$ and $v_j v_j^+$ from C and adding the edge $v_i^+ v_j^+$ together with the paths P_i and P_j, we obtain in $G_r(v)$ a cycle that is longer than C and contains x; a contradiction. Therefore, C is a Hamilton cycle of $G_r(x)$.■

3. *Claw–free graphs*

A graph G is called *claw–free* if G has no induced subgraph isomorphic to $K_{1,3}$. In terms of balls this means that for each vertex $x \in V(G)$ the ball $G_1(x)$ does not contain three mutually non-adjacent vertices. The following result is well–known.

Theorem B (Oberly and Sumner [16]). A connected, locally connected, claw-free graph G with $|V(G)| \geq 3$ is hamiltonian.

Clearly, local connectedness of a claw–free graph G is equivalent to the condition $\alpha(G_1(w)) \leq 2 \leq k(G_1(w))$ for each vertex $w \in V(G)$. Taking Theorem A into consideration we can reformulate Theorem B in the following way: In a connected, claw–free graph G hamiltonicity of balls of radius 1 implies hamiltonicity of G. The next result shows that indeed this implies hamiltonicity of all the balls of G.

Theorem 4.4. A connected, claw-free graph G with $|V(G)| \geq 3$ is locally connected if and only if it is uniformly hamiltonian.

Proof. If G is uniformly hamiltonian then, clearly, it is locally connected. Conversely, suppose that G is a connected, locally connected, claw-free graph but some ball $G_r(x)$ is not hamiltonian. Among all cycles in $G_r(x)$ which contain x, let C be one of maximum length. Clearly, there are two adjacent vertices v_1 and u such that $u \notin V(C)$, $v_1 \in V(C)$ and v_1 is an interior vertex of $G_r(x)$. Let $\overrightarrow{C}$ be the cycle C with a given orientation, and let $v_1, ..., v_n$ be the vertices of C occurring on $\overrightarrow{C}$ in the order of their indices. Since G is claw-free, $v_j^- v_j^+ \in E(G)$ for each $v_j \in V(C) \cap N_1(u)$. The subgraph $< N_1(v_1) >$ is connected because G is locally connected. Consider a shortest (u, v_1^+)-path Q in $< N_1(v_1) >$. Let $Q = u_1 u_2 ... u_t$ where $u_1 = v_1^+$ and $u_t = u$. Since Q is a shortest path, $u_2 \in V(C)$. Let $u_2 = v_{i_2}$. Since G is claw-free, $t \leq 4$. Moreover, $t = 4$ and $u_3 \in V(C)$. (If $t = 3$ then there is a cycle longer than C, which is obtained from C by deleting edges $v_{i_2}^- v_{i_2}, v_{i_2} v_{i_2}^+, v_1 v_1^+$ and adding edges $v_1 u, u v_{i_2}$ and $v_{i_2} v_1^+$; a contradiction. If $t = 4$ and $u_3 \notin V(C)$

then we also can extend C and obtain a contradiction, by taking instead of u the vertex u_3 and using the same argument).

Let $u_3 = v_{i_3}$. Consider a subgraph H induced by the set $\{u, v_1, v_1^-, v_{i_2}\}$. Then $v_1^- v_{i_2} \in E(G)$ since G is claw-free and $uv_1^-, uv_{i_2} \notin E(G)$. This implies that without loss of generality we can consider the case $1 < i_2 < i_3$ only.

Clearly, $v_1 v_{i_2}^- \notin E(G)$ because otherwise the cycle

$$v_1^- v_1^+ \overrightarrow{C} v_{i_2}^- v_1 u v_{i_3} v_{i_2} \overrightarrow{C} v_{i_3}^- v_{i_3}^+ \overrightarrow{C} v_1^-,$$

is longer than C and contains x. Furthermore, $v_1 v_{i_2}^+ \notin E(G)$ because otherwise the cycle

$$v_1^- v_1^+ \overrightarrow{C} v_{i_2} v_{i_3} u v_1 v_{i_2}^+ \overrightarrow{C} v_{i_3}^- v_{i_3}^+ \overrightarrow{C} v_1^-,$$

is longer than C and contains x. Then $v_{i_2}^- v_{i_2}^+ \in E(G)$ because otherwise the set $\{v_1, v_{i_2}^-, v_{i_2}, v_{i_2}^+\}$ induces a graph $K_{1,3}$. Now we obtain a cycle

$$v_1 u v_{i_3} v_{i_2} v_1^+ \overrightarrow{C} v_{i_2}^- v_{i_2}^+ \overrightarrow{C} v_{i_3}^- v_{i_3}^+ \overrightarrow{C} v_1,$$

which is longer than C and contains x; a contradiction. Therefore, C is a Hamilton cycle of $G_r(x)$.■

4. *Graphs with local Ore's condition*

A graph H is said to satisfy Ore's condition if $|V(H)| \geq 3$ and $d_H(u)+d_H(v) \geq |V(H)|$ for each pair of non-adjacent vertices u and v of H. It is well-known that every graph with Ore's condition is hamiltonian [17]. The following result was obtained in [2]

Theorem C (Asratian and Khachatrian [2]). Let G be a connected graph with at least three vertices where for each vertex $x \in V(G)$, the ball $G_1(x)$ satisfies Ore's condition. Then G is hamiltonian.

Graphs satisfying the condition of Theorem C are called graphs with *local Ore's condition.* Some properties of such graphs were investigated in [1]. Now we will indicate some classes of graphs with local Ore's condition which are also uniformly hamiltonian.

It is known due to Ore [18] that a graph G on $n \geq 3$ vertices is hamiltonian if it has at least $\frac{(n-1)(n-2)}{2} + 2$ edges. The next result shows that indeed such graphs are also uniformly hamiltonian.

Theorem 4.5. A graph G with $n \geq 3$ vertices is uniformly hamiltonian if $|E(G)| \geq \frac{(n-1)(n-2)}{2} + 2$.

Proof. Assume that $G \neq K_n$. First we will show that $G_1(x)$ is hamiltonian for each vertex x of G. Let E' denote the set of all pairs of non-adjacent vertices of G. Clearly, $|E'| \leq n-3$. If $G_1(x)$ is not a complete graph consider two non-adjacent vertices u and v in $G_1(x)$. Let

$$E'(x) = \{uv\} \cup \{xy/y \in V(G) \setminus M_1(x)\}.$$

Then $|E' \setminus E'(x)| \leq d(x) - 3$ since $E'(x) \subseteq E'$, $|E'(x)| = n - d(x)$ and $|E'| \leq n-3$. Hence,

$$d_{G_1(x)}(u) + d_{G_1(x)}(v) \geq 2(|M_1(x)| - 2) - (d(x) - 3) = |M_1(x)|,$$

which means that $G_1(x)$ satisfies Ore's condition for each x. The graph G is hamiltonian, by Ore's result [18] (and also by Theorem C). Furthermore, $G_2(x) = G$ for each $x \in V(G)$. (Otherwise there are two non-adjacent vertices u and v of G with distance $d(u,v) \geq 3$ and then $|E(G)| < \frac{(n-2)(n-3)}{2} + n = \frac{(n-1)(n-2)}{2} + 2$; a contradiction.) Hence, G is a hamiltonian graph where for each vertex $x \in V(G)$, the subgraph $G_1(x)$ is hamiltonian and $G_2(x) = G$. Therefore G is uniformly hamiltonian.■

Theorem 4.6. A graph G on $n \geq 4$ vertices is uniformly hamiltonian if

$$d(x) + d(y) \geq \frac{3n-3}{2}$$

for each pair of non-adjacent vertices x and y of G.

Proof. Assume that $G \neq K_n$. Clearly, the distance between any two non-adjacent vertices in G is 2 and, therefore, $G_2(x) = G$ for each $x \in V(G)$. We will show that $G_1(x)$ satisfies Ore condition for each $x \in V(G)$. Suppose that for a vertex x the subgraph $G_1(x)$ contains two non-adjacent vertices u and v such that $d_{G_1(x)}(u) + d_{G_1(x)}(v) < |M_1(x)|$. Then

$$\frac{3n-3}{2} \leq d_G(u) + d_G(v) < |M_1(x)| + 2(n - |M_1(x)|)$$

which implies that $d(x) \leq \frac{n}{2}$. Therefore there is a vertex y which is not adjacent to x and $d(y) \leq n-2$. Thus

$$d_G(x) + d_G(y) \leq (n-2) + \frac{n}{2} = \frac{3n-4}{2},$$

a contradiction. Therefore, $G_1(x)$ satisfies Ore's condition. The graph G is hamiltonian by Ore's result [17] (and also by Theorem C). Thus, G is a hamiltonian graph where for each $x \in V(G)$, the subgraph $G_1(x)$ is hamiltonian and $G_2(x) = G$. Therefore G is uniformly hamiltonian.■

Now we will show that the bound in Theorem 4.6 is sharp. Let $n \geq 4$ be an even integer, $U = \{u_1, u_2, ..., u_{\frac{n-2}{2}}\}$ and $V = \{v_1, v_2, ..., v_{\frac{n-2}{2}}\}$. Consider a graph G with vertex set $U \cup V \cup \{u, v\}$, such that $U \cup \{u\}$ induces a complete subgraph, $V \cup \{v\}$ induces a complete subgraph, u is adjacent to v, and each vertex of U is adjacent to each vertex of V. Clearly, $d_G(u) = d_G(v) = \frac{n-2}{2} + 1 = \frac{n}{2}$ and $d_G(w) = n-2$ for each vertex $w \in U \cup V$. Therefore the degree sum of any two non-adjacent vertices in G is $(n-2)+\frac{n}{2}=\frac{3n-4}{2}$. However G is not uniformly hamiltonian since the subgraphs $G_1(u)$ and $G_1(v)$ are not hamiltonian.

A graph G is said to satisfy Dirac's condition if $|V(G)|\geq 3$ and $d(x)\geq\frac{|V(G)|}{2}$ for each vertex $x\in V(G)$. Graphs with this condition are hamiltonian [6].

Proposition 4.7. For every $n\geq 1$ there exists a graph G with the condition $d(x)\geq\frac{1}{2}|V(G)|+n$ for each $x\in V(G)$, which is not uniformly hamiltonian.

Proof. Let $F_1,...,F_{n+2},H_1,...,H_{n+2}$ be disjoint complete graphs each on $n+1$ vertices. Construct a graph G by joining each vertex of F_i with each vertex of H_j for $i,j=1,...,n+2$. Clearly, G is a k-regular graph with $|V(G)|=2(n+1)(n+2)$ and $k=(n+1)(n+2)+n=\frac{1}{2}|V(G)|+n$. Consider a vertex $x\in V(H_1)$. Then $G_1(x)=H_1\cup F_1\cup...\cup F_{n+2}$. If we delete $n+1$ vertices of H_1 from $G_1(x)$ we obtain $n+2$ components. Hence $G_1(x)$ is not hamiltonian. Therefore, G is not uniformly hamiltonian. ∎

Proposition 4.7 shows that Dirac's condition is weak for uniform hamiltonicity. The next result gives a Dirac-type condition which guarantees uniform hamiltonicity of a graph.

Theorem 4.8. A graph G with $n\geq 3$ vertices is uniformly hamiltonian if $d(x)\geq\frac{2n-1}{3}$ for each vertex x of G.

Proof. Assume that $G\neq K_n$. Clearly, $G_2(x)=G$ for each $x\in V(G)$ and G is hamiltonian [6]. Hence it is sufficient to show that $G_1(x)$ satisfies Ore condition for each $x\in V(G)$. Suppose that for a vertex x the ball $G_1(x)$ contains two non-adjacent vertices u and v such that $d_{G_1(x)}(u)+d_{G_1(x)}(v)<|M_1(x)|$. Then

$$\frac{4n-2}{3}\leq d_G(u)+d_G(v)<|M_1(x)|+2(n-|M_1(x)|)$$

which implies that $d(x)<\frac{2n-1}{3}$; a contradiction. Therefore, $G_1(x)$ satisfies local Ore's condition and G is uniformly hamiltonian.∎

Now we will consider uniform hamiltonicity of complete m-partite graphs with $m\geq 3$.

The complete m-partite graph $K_{n_1,...,n_m}$ where $m\geq 3$ is that graph whose vertex set is partitioned into sets $V_1,...,V_m$ so that $|V_i|=n_i$ for each $i=1,...,m$ and so that uv is an edge of the graph if and only if u and v belong to distinct partite sets V_i and V_j.

Theorem 4.9. Let $G=K_{n_1,...,n_m}$ be an m-partite complete graph where $m\geq 3$ and $n_1\leq n_2\leq...\leq n_m$. Then G is uniformly hamiltonian if and only if $|V(G)|\geq 2n_m+n_{m-1}-1$, and this condition is equivalent to local Ore's condition for G.

Proof. Let $V_1,V_2,...,V_m$ denote partite sets of G and $|V_i|=n_i$ for $i=1,...,m$. Suppose that $|V(G)|\geq 2n_m+n_{m-1}-1$. We will show that for each $u\in V(G)$ the ball $G_1(u)$ satisfies Ore's condition. Consider two non-adjacent vertices x and y in

$G_1(u)$. Then $x, y \in V_i$ and $u \in V_j$ for some $i \neq j$, $|V(G_1(u))| = |V(G)| - |V_j| + 1$ and $|V(G)| - 2|V_i| - |V_j| + 1 \geq |V(G)| - 2n_m - n_{m-1} + 1 \geq 0$. Therefore

$$d_{G_1(u)}(x) + d_{G_1(u)}(y) = 2(|V(G)| - |V_i| - |V_j| + 1) \geq |V(G_1(u))|.$$

Thus, $G_1(u)$ satisfies Ore's condition and, therefore, is hamiltonian. Then, by Theorem C, G also is hamiltonian. This implies that G is uniformly hamiltonian since $G_2(u) = G$ for each $u \in V(G)$.

Conversely, suppose G is uniformly hamiltonian. Consider a vertex $v \in V_{m-1}$ and a Hamilton cycle C of $G_1(v)$. Then $1 + n_1 + ... + n_{m-2} \geq n_m$ since no two vertices of V_m appear consecutively on C. Therefore, $|V(G)| \geq 2n_m + n_{m-1} - 1$. As we have shown above, this implies that $G_1(u)$ satisfies Ore's condition for each $u \in V(G)$.■

Taking Theorems 4.5, 4.6, 4.8 and 4.9 into consideration we formulate the following conjecture.

Conjecture. Every graph with local Ore's condition is uniformly hamiltonian.

Finally we show that graphs with local Ore's condition have a ball property which is close to uniform hamiltonicity.

Theorem 4.10. Let G be a graph with $|V(G)| \geq 3$ which satisfies local Ore's condition. Then for each vertex $x \in V(G)$ and each integer $r \geq 1$ the ball $G_r(x)$ has the following property: every longest cycle in $G_r(x)$ contains all interior vertices of $G_r(x)$.

Proof. Since G satisfies local Ore's condition, $d_{G_1(w)}(u) + d_{G_1(w)}(v) \geq |M_1(w)|$ for each $w \in V(G)$ and each pair of non-adjacent vertices $u, v \in N_1(w)$. Clearly,

$$d_{G_1(w)}(u) + d_{G_1(w)}(v) = |M_1(w) \cap N_1(u) \cap N_1(v)| + |M_1(w) \cap (N_1(u) \cup N_1(v))|.$$

Then Ore's condition for the ball $G_1(w)$ is equivalent to the condition

(1) $|M_1(w) \cap N_1(u) \cap N_1(v)| \geq |M_1(w) \setminus (N_1(u) \cup N_1(v))|$

for each pair of non-adjacent vertices $u, v \in M_1(w)$. Now consider a longest cycle C in a ball $G_r(x)$. Suppose that C does not contain all interior vertices of $G_r(x)$. Then there is an interior vertex v of $G_r(x)$ outside C with $N_1(v) \cap V(C) \neq \emptyset$. Let $w_1, ..., w_k$ be the vertices of $W = N_1(v) \cap V(C)$ occurring on $\overrightarrow{C}$ in the order of their indices. Then the set $W^+ = \{w_1^+, ..., w_k^+\}$ is independent, since any two vertices in W^+ are non-adjacent. (Otherwise, W^+ contains two adjacent vertices w_i^+ and w_j^+ and then $G_r(x)$ has a cycle $w_i v w_j \overleftarrow{C} w_i^+ w_j^+ \overrightarrow{C} w_i$ which is longer than C; a contradiction).

Since $d(v, w_i^+) = 2$ for each $i = 1, ..., k$, we obtain from (1) that

(2) $\sum_{i=1}^{k} |M_1(w_i) \cap N_1(w_i^+) \cap N_1(v)| \geq \sum_{i=1}^{k} |M_1(w_i) \setminus (N_1(w_i^+) \cup N_1(v))|$.

Let $e(W, W^+)$ denote the number of edges in G with one end in W and the other in W^+. Clearly, $M_1(w_i) \cap N_1(w_i^+) \cap N_1(v) \subseteq V(C)$ for each $i = 1, ..., k$ because C is a longest cycle of $G_r(x)$ and $M_1(v) \subseteq M_r(x)$. Then

(3) $\sum_{i=1}^{k} |M_1(w_i) \cap N_1(w_i^+) \cap N_1(v)| = e(W, W^+)$

and

(4) $\sum_{i=1}^{k} |M_1(w_i) \setminus \big(N_1(w_i^+) \cup N_1(v)\big)| \geq e(W, W^+) + k$

because $v \notin W^+$ and $v \in M_1(w_i) \setminus \big(N_1(w_i^+) \cup N_1(v)\big)$ for each $i = 1, ..., k$. But (3) and (4) contradict (2). Therefore, C contains all interior vertices of the ball $G_r(x)$.■

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