MULTIPOLE EXPANSION FOR A SINGLE HELICAL CURRENT CONDUCTOR

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Abstract

Various helical coils such as helical dipole magnets, multifilamentary superconductors and superconducting strands have something in common with the helical structure. In this paper, we discuss the analytical expression for the magnetic field of a single helical current conductor. In addition, the comparison between the analytical and numerical calculations is made.

1 INTRODUCTION

The magnetic field of helical coils has been examined by several authors. [1,2,3,4,5,6] In this paper, the multipole expansion for a single helical current conductor is derived as the extension of the case for a single straight current conductor. [7] Then, the comparison between the analytical and numerical calculations is made for a single helical current conductor. [6] The Cesàro's method of summation is applied for this multipole expansion.

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3-dimensional (3D) Laplace's equation in circular cylindrical coordinates is as follows,

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} + \frac{\partial^2\psi}{\partial z^2} = 0 \qquad (1)$$

Since the winding is periodic in z with a pitch length L, the general solution is, [8]

$$\psi_h(r,\theta,z) = \sum_{n=1}^{\infty}\left(c_n' I_n(nkr) + d_n' K_n(nkr)\right)\times \left\{a_n'\cos\left(n(\theta - kz)\right) + b_n'\sin\left(n(\theta - kz)\right)\right\} + \left(e'\ln r + f'\right)\times\left(g''\theta + h''kz + i'\right) \qquad (2)$$

where $k = 2\pi/L$, and $I_n(nkr)$ and $K_n(nkr)$ are the modified Bessel functions of the first and second kind of order n, respectively. For the interior scalar potential of helical coil, we can define the following form for r < a,

$$\psi_{h,in}(r,\theta,z) = \frac{\mu_0 I}{2\pi}\sum_{n=1}^{\infty}(n-1)!\left[\frac{2}{n k a}\right]^n I_n(n k r)\times \left\{-a_n(k)\cos\left(n(\theta - k z)\right) + b_n(k)\sin\left(n(\theta - k z)\right)\right\} - \frac{\mu_0 I}{2\pi}k z \qquad (3)$$

Then, the asymptotic form for this scalar potential as $k \to 0$ ($L \to \infty$) is,

$$\lim_{k\to 0}\left[\psi_{h,in}(r,\theta,z)\right] = \psi_{2d,in}(r,\theta) \qquad (4)$$

$$\psi_{2d,in}(r,\theta) = \frac{\mu_0 I}{2\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{r}{a}\right)^n \sin\left(n(\theta - \varphi)\right) = \frac{\mu_0 I}{2\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{r}{a}\right)^n\left(-a_n\cos(n\theta) + b_n\sin(n\theta)\right) \qquad (5)$$

where $\psi_{2d,in}(r,\theta)$ is the interior scalar potential of 2D non-spiral coil. From this scalar potential, the interior (r < a) magnetic field of helical coil is,

$$\begin{cases} B_r(r,\theta,z) = -\dfrac{\partial\psi_h}{\partial r} = -\dfrac{\mu_0 I}{2\pi}\displaystyle\sum_{n=1}^{\infty} n!\left[\frac{2}{n k a}\right]^n k\, I_n'(n k r)\times \left\{-a_n(k)\cos\left(n(\theta - k z)\right) + b_n(k)\sin\left(n(\theta - k z)\right)\right\} \\ B_\theta(r,\theta,z) = -\dfrac{1}{r}\dfrac{\partial\psi_h}{\partial\theta} = -\dfrac{\mu_0 I}{2\pi}\displaystyle\sum_{n=1}^{\infty} n!\left[\frac{2}{n k a}\right]^n \frac{I_n(n k r)}{r}\times \left\{a_n(k)\sin\left(n(\theta - k z)\right) + b_n(k)\cos\left(n(\theta - k z)\right)\right\} \\ B_z(r,\theta,z) = -\dfrac{\partial\psi_h}{\partial z} = -\dfrac{\mu_0 I}{2\pi}\displaystyle\sum_{n=1}^{\infty}(-k)\, n!\left[\frac{2}{n k a}\right]^n I_n(n k r)\times \left\{a_n(k)\sin\left(n(\theta - k z)\right) + b_n(k)\cos\left(n(\theta - k z)\right)\right\} + \dfrac{\mu_0 I}{2\pi}k \end{cases} \qquad (6)$$

On the other hand, for the exterior scalar potential of helical coil, we can define the following form for r > a,

$$\psi_{h,ex}(r,\theta,z) = \frac{\mu_0 I}{2\pi}\sum_{n=1}^{\infty}(n-1)!\left[\frac{2}{n k a}\right]^n \frac{I_n'(n k a)}{K_n'(n k a)} K_n(n k r)\times \left\{-a_n(k)\cos\left(n(\theta - k z)\right) + b_n(k)\sin\left(n(\theta - k z)\right)\right\} - \frac{\mu_0 I}{2\pi}\theta \qquad (7)$$

Then, the exterior (r > a) magnetic field of helical coil is,

$$\begin{cases} B_r(r,\theta,z) = -\dfrac{\mu_0 I}{2\pi}\displaystyle\sum_{n=1}^{\infty} n!\left[\frac{2}{n k a}\right]^n k\,\frac{I_n'(n k a)}{K_n'(n k a)} K_n'(n k r)\times \left\{-a_n(k)\cos\left(n(\theta - k z)\right) + b_n(k)\sin\left(n(\theta - k z)\right)\right\} \\ B_\theta(r,\theta,z) = -\dfrac{\mu_0 I}{2\pi}\displaystyle\sum_{n=1}^{\infty} n!\left[\frac{2}{n k a}\right]^n \frac{I_n'(n k a)}{K_n'(n k a)}\frac{K_n(n k r)}{r}\times \left\{a_n(k)\sin\left(n(\theta - k z)\right) + b_n(k)\cos\left(n(\theta - k z)\right)\right\} + \dfrac{\mu_0 I}{2\pi}\dfrac{1}{r} \\ B_z(r,\theta,z) = -\dfrac{\mu_0 I}{2\pi}\displaystyle\sum_{n=1}^{\infty}(-k)\, n!\left[\frac{2}{n k a}\right]^n \frac{I_n'(n k a)}{K_n'(n k a)} K_n(n k r)\times \left\{a_n(k)\sin\left(n(\theta - k z)\right) + b_n(k)\cos\left(n(\theta - k z)\right)\right\} \end{cases} \qquad (8)$$

On the situation that the currents are confined to lie on the surface of a circular cylinder of radius a, the surface currents will give rise to a discontinuity of the components B_z, B_θ, at the interface of radius a, but the radial component B_r will pass continuously through this interface. The values of $a_n(k)$, $b_n(k)$ can be determined for the current element. Applying Ampere's law for a closed path on z=constant plane enclosing the current element at radius a, we can obtain the following equation,

$$\left(B_{\theta,out} - B_{\theta,in}\right)\big|_{r=a} = \mu_0\, j_z\, \Delta a \tag{9}$$

Then, the coefficients an(k) and bn(k) are obtained with the Wronskian relation, [9] as follows,

$$\begin{cases} a_n(k) = -\dfrac{2}{I}\dfrac{1}{(n-1)!}\left(\dfrac{n\,k\,a}{2}\right)^n k\,a^2\,K_n'(n\,k\,a)\,\Delta a \displaystyle\int j_z \sin n\theta\, d\theta \\ b_n(k) = -\dfrac{2}{I}\dfrac{1}{(n-1)!}\left(\dfrac{n\,k\,a}{2}\right)^n k\,a^2\,K_n'(n\,k\,a)\,\Delta a \displaystyle\int j_z \cos n\theta\, d\theta \end{cases} \quad \text{- - - (10)}$$

For a helical line currents : current +I , radius a, angle φ, the coefficients $a_n(k)$, $b_n(k)$ can be calculated. The following relation between the current and the current density is used with the real cross section S of the conductor and the cross section S_z (=S/sinα) of the conductor on the z=constant plane. α is the pitch of the winding so that the relationships between the above-mentioned k and α are k = 1/(a tan α). Then, we can obtain the following expression,

$$j_z = j \sin\alpha = \frac{I}{S}\sin\alpha = \frac{I}{S_z} \tag{11}$$

$$\begin{cases} a_n(k) = \dfrac{2}{(n-1)!}\left(\dfrac{n\,k\,a}{2}\right)^n \left(k\,a\,K_{n-1}(n\,k\,a) + K_n(n\,k\,a)\right)\sin n\varphi \\ b_n(k) = \dfrac{2}{(n-1)!}\left(\dfrac{n\,k\,a}{2}\right)^n \left(k\,a\,K_{n-1}(n\,k\,a) + K_n(n\,k\,a)\right)\cos n\varphi \end{cases} \quad \text{- - - (12)}$$

When n is fixed and k → 0, the limiting forms for small arguments of the modified Bessel function of the second kind of order n, $K_n(nkr)$ are as follows, [9]

$$\begin{cases} K_0(k\,a) \approx -\ln(k\,a) \\ K_n(n\,k\,a) \approx -\dfrac{1}{2}\Gamma(n)\left(\dfrac{2}{n\,k\,a}\right)^n, \qquad n \geq 1 \end{cases} \tag{13}$$

Then,

$$\begin{cases} \lim_{k\to 0}\left[a_n(k)\right] = a_n = \sin n\varphi \\ \lim_{k\to 0}\left[b_n(k)\right] = b_n = \cos n\varphi \end{cases} \tag{14}$$

As a result, the interior magnetic field of a single helical conductor with the current +I, located at radius a and angle φ is for r < a,

$$\begin{cases} B_r(r,\theta,z) = \dfrac{\mu_0 I}{\pi} k^2 a \displaystyle\sum_{n=1}^{\infty} n\,K_n'(n\,k\,a)\,I_n'(n\,k\,r) \times \\ \qquad \sin\left(n(\theta - \varphi - k\,z)\right) \\ B_\theta(r,\theta,z) = \dfrac{\mu_0 I}{\pi} k\,a \displaystyle\sum_{n=1}^{\infty} n\,K_n'(n\,k\,a)\,\dfrac{I_n(n\,k\,r)}{r} \times \\ \qquad \cos\left(n(\theta - \varphi - k\,z)\right) \\ B_z(r,\theta,z) = -\dfrac{\mu_0 I}{\pi} k^2 a \displaystyle\sum_{n=1}^{\infty} n\,K_n'(n\,k\,a)\,I_n(n\,k\,r) \times \\ \qquad \cos\left(n(\theta - \varphi - k\,z)\right) + \dfrac{\mu_0 I}{2\pi} k \end{cases} \tag{15}$$

Then, the transverse field magnitude at axis of the helical winding B(r=0,θ,z=0) is obtained as follows,

$$\begin{aligned} & B_y(r=0,\theta,z=0) \\ & = \frac{\mu_0 I}{\pi}\left(k\,a\,K_0(k\,a) + K_1(k\,a)\right) \times \\ & \left\{\frac{k}{2}\left\{\cos(2\theta - \varphi) - \cos\varphi\right\} + \frac{k}{2}\cos(2\theta - \varphi)\right\} \\ & = -\frac{\mu_0 I}{2\pi} k\left(k\,a\,K_0(k\,a) + K_1(k\,a)\right)\cos\varphi \end{aligned} \tag{16}$$

This result coincides with that obtained with the Biot and Savart's Law. [10] Similarly, the expression for the exterior (r > a) magnetic field of a single helical conductor is obtained. The above expression for the magnetic field of helical coil is the function of r and θ - kz, and is helically symmetric. However, the helical symmetry of the coil structure does not demand that the scalar potential is invariant to the transformation θ - kz = constant.

3 COMPARISON BETWEEN THE ANALYTICAL AND NUMERICAL CALCULATIONS FOR A SINGLE HELICAL CURRENT CONDUCTOR

The sum Σ to a finite order of the above expressions of the interior and exterior magnetic fields do not approach the same value at r=a, as shown in Fig.1. This discontinuity of the magnetic field at r=a can be overcome with application of the Cesàro's method of summation, as shown in Fig.2. [11] As a result, the interior (r < a) magnetic field of a single helical conductor with the current +I, located at radius a and angle φ is expressed as follows,

$$\begin{cases} B_r(r,\theta,z) = \dfrac{\mu_0 I}{\pi} k^2 a \dfrac{1}{N} \displaystyle\sum_{m=1}^{m=N}\left\{\sum_{n=1}^{n=m} n\,K_n'(n\,k\,a)\,I_n'(n\,k\,r) \times \right. \\ \qquad \left. \sin\left(n(\theta - \varphi - k\,z)\right)\right\} \\ B_\theta(r,\theta,z) = \dfrac{\mu_0 I}{\pi} k\,a \dfrac{1}{N} \displaystyle\sum_{m=1}^{m=N}\left\{\sum_{n=1}^{n=m} n\,K_n'(n\,k\,a)\,\dfrac{I_n(n\,k\,r)}{r} \times \right. \\ \qquad \left. \cos\left(n(\theta - \varphi - k\,z)\right)\right\} \\ B_z(r,\theta,z) = -\dfrac{\mu_0 I}{\pi} k^2 a \dfrac{1}{N} \displaystyle\sum_{m=1}^{m=N}\left\{\sum_{n=1}^{n=m} n\,K_n'(n\,k\,a)\,I_n(n\,k\,r) \times \right. \\ \qquad \left. \cos\left(n(\theta - \varphi - k\,z)\right)\right\} + \dfrac{\mu_0 I}{2\pi} k \end{cases} \quad \text{- - - (17)}$$

For a single helical conductor with current I = 100 A, radius a = 0.33 mm, angle φ = 30º and pitch length L = 9.51 mm, the comparison between the analytical and numerical calculations was made. Since an agreement was confirmed between the analytical and numerical calculations, the field was calculated for the whole space except for the singular point occupied by a single helical conductor, as shown in Fig.3.

4 CONCLUSION

An analytical expression for the magnetic field of a single helical coil is obtained. This expression will be useful to estimate the various electromagnetic characteristics of helical coils.

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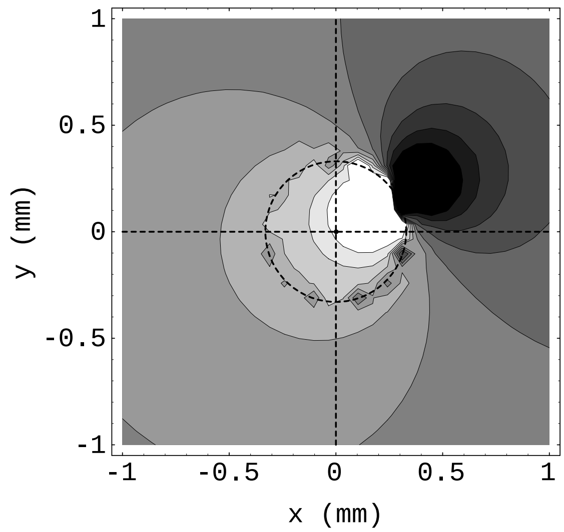

Fig. 1. Contour plot of B_Z at z=0 analytically calculated with Eq.(15) to n=20.

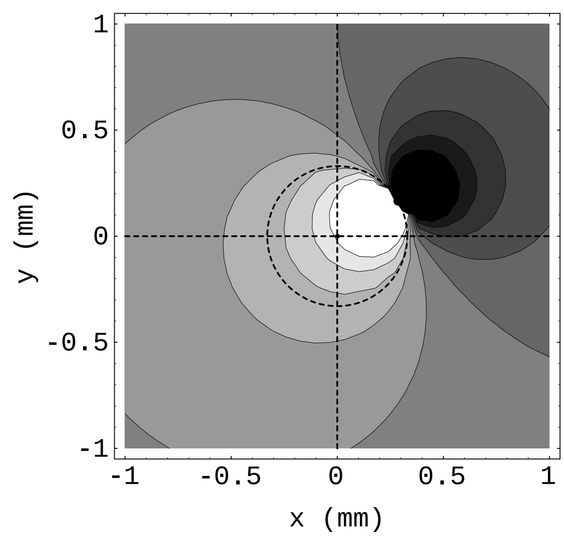

Fig. 2. Contour plot of B_Z at z=0 analytically calculated with Eq.(17) to N=20.

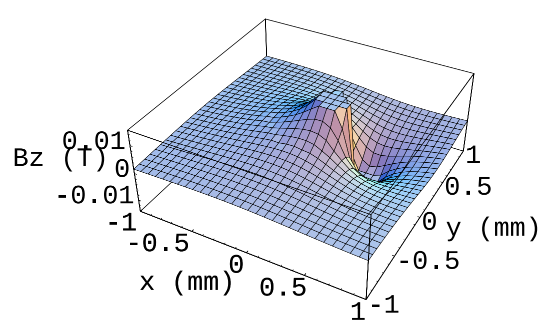

Fig. 3. 3D plot of the analytically calculated B_Z at z=0, which is consistent with the numerical calculation with Biot-Savart Law.