M5-branes, 4d gauge theory and 2d CFT

Yuji Tachikawa

based on works in collaboration with

L. F. Alday, **D. Gaiotto**,

and many others

February 2010

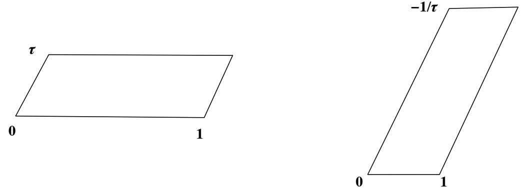
Contents

$\mathcal{N} = 4$ and M5-branes

- N M5-branes on S^1 → N D4-branes: $\mathbf{SU}(N)$ SYM in 5d
- N M5-branes on T^2 → $\mathbf{SU}(N)$ SYM in 4d
- 4d gauge coupling

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}$$

is the shape of the torus



→ S-duality!

$\mathcal{N} = 2$ and M5-branes

- Wrap N M5-branes on a more **general Riemann surface** possibly with **punctures**, to get $\mathcal{N} = 2$ superconformal field theories

$\mathcal{N} = 2$ and M5-branes

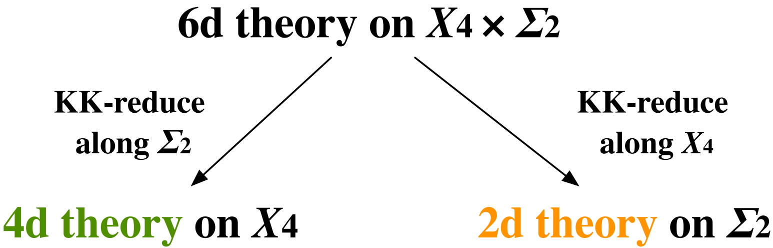
- Wrap N M5-branes on a more **general Riemann surface** possibly with **punctures**, to get $\mathcal{N} = 2$ superconformal field theories

- Anticipated in '96–'98 by [Lerche,Warner], [Klemm,Mayr,Vafa], [Witten], [Marshakov,Martellini,Morozov], [Ito,Yang], [Kapustin], ...

  but not thoroughly explored until [Gaiotto,0904.2715]

6d, 4d and 2d

- N M5-branes wrapped on $X_4 \times \Sigma_2$
- Consider the partition function Z of the 6d theory,
- furthermore suppose Z depends **only on the complex structure**.



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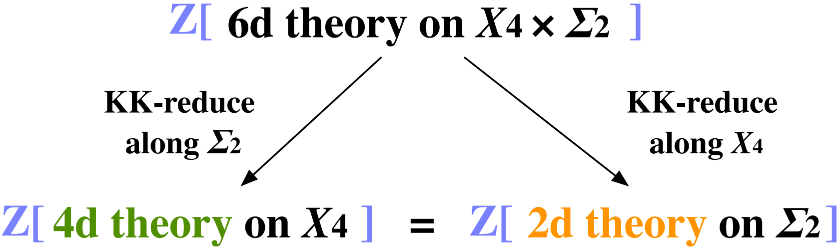

6d, 4d and 2d

- N M5-branes wrapped on $X_4 \times \Sigma_2$
- Consider the partition function Z of the 6d theory,
- furthermore suppose Z depends **only on the complex structure**.

Z[4d theory on X_4] = Z[2d theory on Σ_2]

Contents

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$\mathcal{N} = 2$ gauge theory

- Vector multiplets:

$$\phi^{i\bar{j}} \qquad \psi_{\alpha}^{i\bar{j}} \qquad F_{\mu\nu}^{i\bar{j}}$$

- $\phi = \mathbf{diag}(a_1, a_2, \ldots, a_n)$ is a SUSY configuration
- breaks $\mathbf{SU}(N)$ to $\mathbf{U}(1)^{N-1}$:

$$S = \tau_{\text{UV}}\, \mathbf{tr}\, F_{\mu\nu}F_{\mu\nu} + c.c. \longrightarrow S = \tau_{ij}(a)F_{\mu\nu}^{i}F_{\mu\nu}^{j} + c.c.$$

- In $\mathcal{N} = 2$ superspace,

$$S = \int d^4\theta \tau_{\text{UV}}\, \mathbf{tr}\, \Phi^2 + c.c. \longrightarrow S = \int d^4\theta \mathcal{F}(a) + c.c.$$

$\mathcal{F}$ is called the **prepotential**.

$$a_i, \qquad d\,F^i = 0,$$

$$a_i^D = \frac{\partial \mathcal{F}}{\partial a_i}, \qquad d\,\tau_{ij}(a) \star F^j = 0$$

Prepotential

- Perturbatively,

$$\mathcal{F}(a) = \tau_{UV} \sum a_i^2 + \sum_{i>j} (a_i - a_j)^2 \log \frac{a_i - a_j}{\Lambda_{UV}}$$

- Instanton corrections

$$\mathcal{F}(a) = \sum_{i>j} (a_i - a_j)^2 \log \frac{a_i - a_j}{\Lambda} + \sum_k \Lambda^{2Nk} f_k(a)$$

where $f_k(a)$: k-instanton correction.

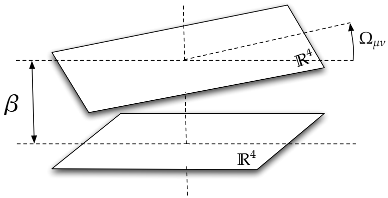
$$f_1(a) = \sum_i \prod_{j \neq i} (a_i - a_j)^{-2}.$$

- Huge literature devoted to calculate f_k.

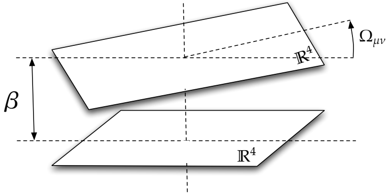
Nekrasov's partition function

$$Z(a;\epsilon_1,\epsilon_2) = \exp(\frac{\mathcal{F}(a)}{\epsilon_1\epsilon_2} + \cdots)$$

1. Take 5d version of the theory
2. Put it on a circle of length β.
3. Glue the two ends by
   - by 4d rotation $\Omega = e^{\beta\epsilon_1 L_{12} + \beta\epsilon_2 L_{34}}$ and
   - by gauge rotation $g = \mathbf{diag}(e^{\beta a_1}, \ldots, e^{\beta a_N})$
4. Take $\beta \to 0$.



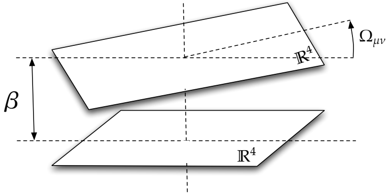
Localization



- Only half-BPS configurations contribute.
- Consider the circle as "time."
- At each slice, we have an instanton.
- As time changes, the parameters of the instanton changes.
- It's a supersymmetric QM on the instanton parameter space.

$$Z = \sum_k \Lambda^{2Nk} \operatorname*{tr}_{\text{QM Hilbert space}} g\Omega(-1)^F e^{-\beta H}$$

Localization



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$$Z = \sum_k \Lambda^{2Nk} \operatorname*{tr}_{\text{QM vacuum}} g\Omega(-1)^F$$

Localization: example

$$Z = \operatorname*{tr}_{\text{QM vacuum}} g(-1)^F$$

- Consider a SUSY particle moving on a sphere, with monopole flux n
- $\mathbf{SU}(2) = \mathbf{SO}(3)$ acts on it. Take $g = \mathbf{diag}(e^{i\theta/2}, e^{-i\theta/2})$.
- Vacua form spin n rep. →

$$Z = e^{in\theta} + e^{i(n-1)\theta} + \cdots + e^{-in\theta}$$

- Can also be calculated from the sum of two fixed point contributions

$$Z = \frac{e^{i(n+1/2)\theta}}{e^{i\theta/2} - e^{-i\theta/2}} + \frac{e^{-i(n+1/2)\theta}}{e^{-i\theta/2} - e^{i\theta/2}}$$

Nekrasov's partition function: explicit form

$$Z = \sum_k \Lambda^{2Nk} \operatorname*{\mathbf{tr}}_{\text{QM vacuum}} g\Omega(-1)^F$$

- $\mathbf{SO}(4)$ and $\mathbf{SU}(N)$ act on instanton moduli space.
- Need to understand fixed points and their neighborhood.

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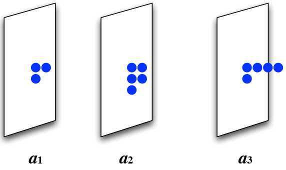

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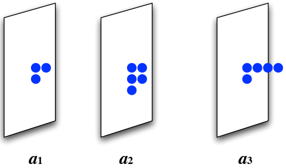


$$Z = \sum_{(Y_1, Y_2, \ldots, Y_N)} \Lambda^{2N(|Y_1|+\cdots+|Y_N|)} Z_{\{Y_i\}}(a_i; \epsilon_1, \epsilon_2)$$

Nekrasov's partition function: explicit form

$$Z(a;\epsilon_1,\epsilon_2) = \sum_{(Y_1,Y_2,\dots,Y_N)} \Lambda^{2N(|Y_1|+\cdots+|Y_N|)} Z_{\{Y_i\}}(a_i;\epsilon_1,\epsilon_2)$$

- One-instanton contribution is ($a_{ij} = a_i - a_j$, $\epsilon = \epsilon_1 + \epsilon_2$)

$$Z_1 = \sum_i \frac{1}{\epsilon_1\epsilon_2 \prod_{j\neq i} a_{ji}(\epsilon - a_{ji})}$$

- Two-instanton contribution is

$$Z_2 = \sum_{i<j} \frac{1}{(\epsilon_1\epsilon_2)^2(a_{ij}^2-\epsilon_1^2)(a_{ij}^2-\epsilon_2^2)\prod_{k\neq i,j} a_{ki}(\epsilon-a_{ki})a_{kj}(\epsilon-a_{kj})}$$
$$+\sum_i \frac{1}{2\epsilon_1\epsilon_2^2(\epsilon_2-\epsilon_1)\prod_{j\neq i} a_{ji}(\epsilon-a_{ji})(\epsilon_2-a_{ji})(\epsilon+\epsilon_2-a_{ji})}$$
$$+\sum_i \frac{1}{2\epsilon_2\epsilon_1^2(\epsilon_1-\epsilon_2)\prod_{j\neq i} a_{ji}(\epsilon-a_{ji})(\epsilon_1-a_{ji})(\epsilon+\epsilon_1-a_{ji})}$$

Recap

- Prepotential

$$\mathcal{F}(a) = \sum_{i<j} (a_i - a_j)^2 \log \frac{a_i - a_j}{\Lambda} + \sum \Lambda^{2Nk} f_k(a)$$

- Nekrasov's partition function

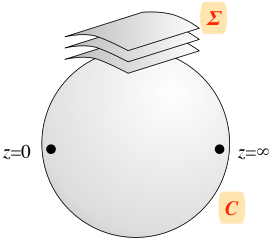
$$Z(a; \epsilon_1, \epsilon_2) = \exp\left(\frac{\mathcal{F}(a_1, \ldots, a_N)}{\epsilon_1 \epsilon_2} + \cdots\right)$$

has an explicit expression in terms of Young diagrams.

- There's a completely different way to encode $\mathcal{F}(a)$, without any expansion.

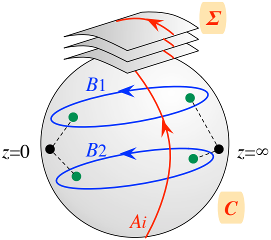
Seiberg-Witten curve

$$\Sigma: \quad \Lambda^N\left(z+\frac{1}{z}\right) = y^N + u_2 y^{N-2} + \cdots + u_N$$



- The sphere parameterized by z is **the base C**, the **Gaiotto curve**.

$$\Sigma: \quad \Lambda^N(z+\frac{1}{z}) = y^N + u_2 y^{N-2} + \cdots + u_N$$



- The sphere parameterized by z is **the base** C, the **Gaiotto curve**.
- Its N-sheeted cover is the **Seiberg-Witten curve** Σ

$$a_i = \int_{A_i} \frac{ydz}{z}, \qquad a_i^D = \int_{B_i} \frac{ydz}{z}, \quad \text{and} \quad a_i^D = \frac{\partial \mathcal{F}(a)}{\partial a_i}.$$

Seiberg-Witten curve

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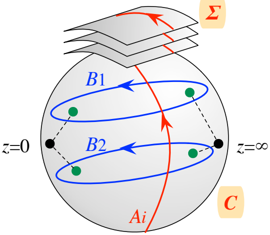


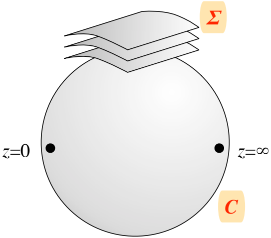
- Let $y^N + y^{N-2}u_2 + \cdots + u_N = \prod(y - y_i)$.
- Suppose $y_i \gg \Lambda$ → $y \sim y_i$ on A_i →

$$a_i = \int_{A_i} \frac{y dz}{z} = y_i + O(\Lambda).$$

M5-branes

Let $\lambda = y dz/z$. Then

$$\Lambda^N(z + \frac{1}{z}) = y^N + u_2 y^{N-2} + \cdots + u_N$$



becomes

$$\lambda^N + \phi_2(z)\lambda^{N-2} + \cdots + \phi_N(z) = 0$$

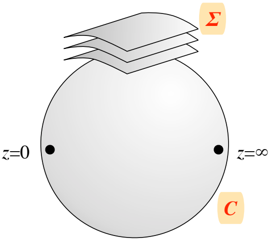
where

$$\phi_k(z) = u_k \left(\frac{dz}{z}\right)^k, \qquad \phi_N(z) = (\Lambda^N z + u_N + \frac{\Lambda^N}{z}) \left(\frac{dz}{z}\right)^k.$$

M5-branes

$$\lambda^N + \phi_2(z)\lambda^{N-2} + \cdots + \phi_N(z) = 0$$

- λ: one-form. ϕ_k: degree-k form.
- Determine $\Sigma \subset T^*C$
- Wrapping N M5-branes on Σ inside the hyperkahler T^*C.
- 11d spacetime is $\mathbb{R}^{3,1} \times T^*C \times \mathbb{R}^3$.

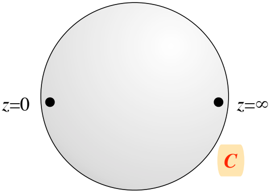


6d $\mathcal{N} = (2,0)$ theory

- Compactification of 6d $\mathcal{N} = (2,0)$ theory on C with **punctures**.
- $\phi_k(z)$ are the worldvolume fields with poles at punctures.
- $\Phi(z)$ be the hypothetical adjoint-valued field whose invariant polynomials are $\phi_k(z)$

$$0 = \lambda^N + \phi_2(z)\lambda^{N-2} + \cdots + \phi_N(z) = \det(\lambda - \Phi(z))$$

- The Seiberg-Witten differential λ is the 'eigenvalue' of $\Phi(z)$.



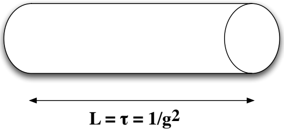
Comments

All these have been, in some sense, known from around '97.
Gaiotto's insight was

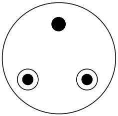
- What's important is the base C with $\phi_k(z)$.
- Punctures control the divergence of $\phi_k(z)$.
- You can put punctures at will.
- Lagrangian $\Leftrightarrow$ configuration of the punctures
- VEV $\Leftrightarrow$ $\phi_k(z)$

Rules

- a tube gives $\mathbf{SU}(N)$ gauge group



- a three-punctured sphere gives the matter field e.g.

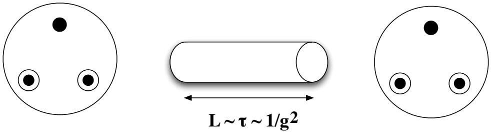


gives $N \times N$ hypermultiplets.

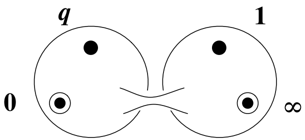
- Punctures carry flavor symmetries. $\odot$: $\mathbf{SU}(N)$ and $\bullet$: $\mathbf{U}(1)$

Practice!

- Want to make a $\mathbf{SU}(N)$ with $2N$ fundamentals?
- Take N fundamental hypers, gauge fields, another N fundamental hypers



- Connect!



- $q \sim \exp(i\tau)$.

Contents

4d CFT vs 2d CFT

- We now have a **map**

$$G_N : \quad \text{Riemann surface with punctures} \longrightarrow \quad \text{4d field theory}$$

- G_N **behaves nicely under degenerations** of the Riemann surface Σ
i.e. any thin, long tube gives a weakly coupled $\mathbf{SU}(N)$ gauge group

4d CFT vs 2d CFT

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- G_N **behaves nicely under degenerations** of the Riemann surface $\mathbf{\Sigma}$ i.e. any thin, long tube gives a weakly coupled $\mathbf{SU}(N)$ gauge group
- Take whatever physical quantity Z calculable in 4d:

$$Z : \text{ 4d field theory} \longrightarrow \text{number}$$

- Then, $Z(G_N(\mathbf{\Sigma}))$ **factorizes under degenerations** of $\mathbf{\Sigma}$,

4d CFT vs 2d CFT

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$$Z : \text{ 4d field theory} \longrightarrow \text{ number}$$

- Then, $Z(G_N(\mathbf{\Sigma}))$ **factorizes under degenerations** of $\mathbf{\Sigma}$,

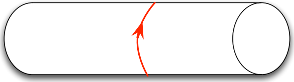
- This morally means that $Z \circ G_N$ gives a **2d CFT**.

4d CFT vs 2d CFT

- **Nekrasov's instanton partition function**
  = the **Virasoro/W_N conformal block**.
- **Full partition function**
  = the **Liouville/Toda correlation function**.
- **Superconformal Index** = a **2d TQFT**
  [Gadde-Pomoni-Rastelli-Razamat]

SU(2) vs. Liouville

- What do we get from $\mathbf{2}$ M5-branes?



- Each channel is labeled by one variable $\boldsymbol{a}$ with the identification $\boldsymbol{a} \sim -\boldsymbol{a}$.
- Three-point Interaction is non-zero for generic $\boldsymbol{a_1}$, $\boldsymbol{a_2}$ and $\boldsymbol{a_3}$

SU(2) vs. Liouville

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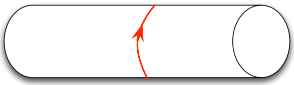
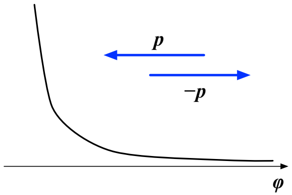


- Each channel is labeled by one variable a with the identification $a \sim -a$.
- Three-point Interaction is non-zero for generic a_1, a_2 and a_3
- Such 2d CFT is bound to be Liouville [Teschner,…]

(N.B. I learned this argument from Ari Pakman.)

Liouville

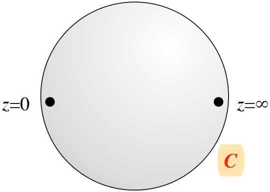


- Reflection off an exponential wall.
- The action is

$$S = \frac{1}{\pi} \int d^2x \sqrt{g} \left(|\partial_\mu \varphi|^2 + \mu e^{2b\varphi} + QR\varphi \right)$$

- $Q = b + 1/b$ and $c = 1 + 6Q^2$.

Proposal: SU(2)



- SW curve was $\lambda^2 = \phi_2(z)$.
- Liouville theory has $T(z)$.
- Both have spin-2.

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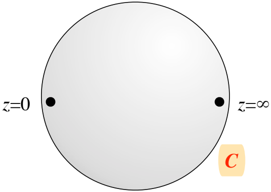

- SW curve was $\lambda^2 = \phi_2(z)$.
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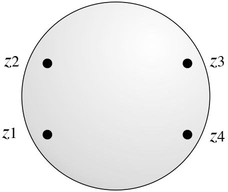
$$\langle T(z) \rangle dz^2 \to \phi_2(z) \quad \text{when} \quad \epsilon_{1,2} \to 0$$

under

$$\text{vev } a \leftrightarrow \text{momentum } p, \qquad \frac{(\epsilon_1 + \epsilon_2)^2}{\epsilon_1 \epsilon_2} \leftrightarrow Q^2$$

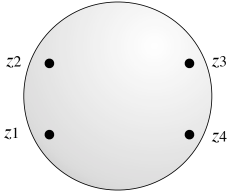
SU(2) with four flavors



$$\Longleftrightarrow \quad \langle V_{\alpha_1}(z_1) V_{\alpha_2}(z_2) V_{\alpha_3}(z_3) V_{\alpha_4}(z_4) \rangle$$

- What are the operators $V_\alpha(z)$?
- $\phi_2(z) \sim \frac{m_i^2 dz^2}{(z - z_i)^2} \quad \Longleftrightarrow \quad T(z) V_\alpha(z_i) \sim \frac{m_i^2}{(z - z_i)^2} V_\alpha(z_i)$

SU(2) with four flavors



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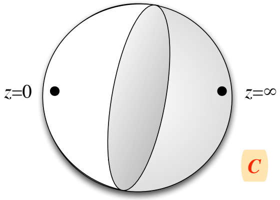
- $V_{\alpha_i}(z)$ is a **primary state** with dimension m_i^2.

Pure SU(2)



$\Longleftrightarrow \quad \langle w|w\rangle$

- What's the state $|w\rangle$?
- $\lambda^2 = \phi_2(z)$ where $\phi_2(z) \sim (\frac{u}{z^2} + \frac{\Lambda^2}{z^3})dz^2$ around $z = 0$.
- Recall

$$T(z)dz^2 \sim (\cdots + \frac{L_0}{z^2} + \frac{L_1}{z^3} + \cdots)dz^2.$$

this suggests

$$L_0|w\rangle = (Q^2 - a^2)|w\rangle, \quad L_1|w\rangle = \Lambda^2|w\rangle, \quad L_n|w\rangle = 0 \quad (n \geq 2)$$

- $|w\rangle$ is the **coherent state** of the Virasoro algebra !

Comparison to Nekrasov

- What you do: assume

$$|w\rangle = |a\rangle + cL_{-1}|a\rangle + (c'L_{-2} + c''L_{-1}^2)|a\rangle + \cdots$$

and impose

$$L_1|w\rangle = \Lambda^2|w\rangle, \qquad L_2|w\rangle = 0, \ \ldots$$

Recursively determine c, c', c'', ...

- Form $\langle w|w\rangle$. (n.b.: you need to take BPZ conjugate.)

Comparison to Nekrasov

- What you do: assume

$$|w\rangle = |a\rangle + cL_{-1}|a\rangle + (c'L_{-2} + c''L_{-1}^{2})|a\rangle + \cdots$$

and impose

$$L_1|w\rangle = \Lambda^2|w\rangle, \qquad L_2|w\rangle = 0, \ \ldots$$

Recursively determine c, c', c'', $\ldots$.
- Form $\langle w|w\rangle$. (n.b.: you need to take BPZ conjugate.)
- Magically agrees with Nekrasov's $Z(a, \epsilon_1, \epsilon_2)$.

Status: SU(2)

- Many checks.
- [Fateev-Litvinov 0912.0504] **proved** the equality

$$Z \text{ of } \mathrm{SU}(2) \text{ with massive adjoint}$$
$$= \text{torus one-point conformal block}$$

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  by showing boths sides satisfy the same non-linear relation.

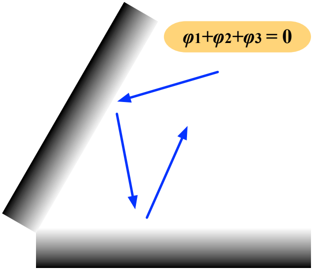
- Mathematically, the 2d/4d relation means

$$\bigoplus_k H^*_{SU(2)\times U(1)^2}(\mathcal{M}_k)$$

  has the structure of **Verma module of Virasoro algebra**.

- Not proved, but should be available in a few years...

Toda



- Waves in $\varphi_1 + \varphi_2 + \cdots + \varphi_N = 0$.
- Reflection off walls at $\varphi_i = \varphi_{i+1}$.
- The action is

$$S = \frac{1}{\pi} \int d^2x \sqrt{g} \left(|\partial_\mu \vec{\varphi}|^2 + \mu \sum_i e^{2b\vec{e}_i \cdot \vec{\varphi}} + Q\vec{\rho} \cdot \vec{\varphi} R \right)$$

- $Q = b + 1/b$ and $c = (N-1) + Q^2 N(N^2 - 1)$.

W_N symmetry

- Toda theory is not just a CFT with $T(z)$.
- Has W_N symmetry, with generators

$$W_2(z) = T(z), \quad W_3(z), \quad \ldots, \quad W_N(z)$$

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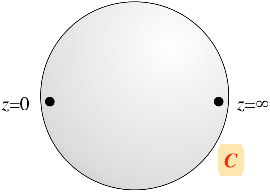
- OPE of W_3 algebra is

$$W_3(z)W_3(0) \sim \frac{c}{3z^6} + \frac{2T(0)}{z^4} + \frac{\partial T(0)}{z^3} + \frac{1}{z^2}\left[2\beta\Lambda(0) + \frac{3}{10}\partial^2 T(0)\right] + \frac{1}{z}\left[\beta\partial\Lambda(0) + \frac{1}{15}\partial^3 T(0)\right]$$

where

$$\Lambda(z) = :T(z)T(z): - \frac{3}{10}\partial^2 T(z), \qquad \beta = \frac{16}{22+5c}$$

Proposal: SU(N)



- SW curve was $\lambda^N + \phi_2(z)\lambda^{N-2} + \cdots + \phi_N(z) = 0.$
- Toda theory has $T(z), W_3(z), \cdots, W_N(z).$

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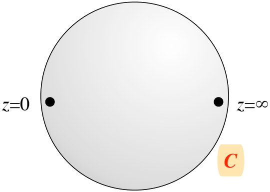

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- Toda theory has $T(z), W_3(z), \cdots, W_N(z).$

$$\langle W_k(z)\rangle dz^k \to \phi_k(z) \quad \text{when} \quad \epsilon_{1,2} \to 0$$

under vev $\vec{a} \leftrightarrow$ momentum $\vec{p}$, $\frac{\epsilon_1}{\epsilon_2} \leftrightarrow b^2$

- Weyl reflection = Toda reflection

Comparison for general N

- [Mironov-Morozov] studied $\mathbf{SU(3)}$ with six flavors
- [Taki] studied pure $\mathbf{SU(3)}$ upto 2-instanton level
- Both used nonlinear algebra
- [Kanno,Matsuo,Shiba,YT] used free-field representation and studied near punctures
- more to be done!

Summary

- 6d $\mathcal{N} = (0, 2)$ theory on C with punctures ⟶ 4d theory
- 2d CFT lives on C
- Nekrasov's partition function in 4d = 2d CFT quantity of C.

- Liouville/Toda theory is the quantization of the SW curve.