Construction of Multinomial Lattice Random Walks for Optimal Hedges*

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Abstract. In this paper, we provide a parameterization of multinomial lattice random walks which take cumulants into account. In the binomial and trinomial lattice cases, it reduces to standard results. Additionally, we show that higher order cumulants may be taken into account by using multinomial lattices with four or more branches. Finally, we outline two synthesis methods which take advantage of the multinomial lattice formulation. One is mean square optimal hedging in an incomplete market and the other involves pricing under "implied volatility" and "implied kurtosis".

1 Introduction

An important issue in pricing and hedging derivatives is the generality of the model for the underlying asset (see e.g., [4,9,10,13]) and its computational tractability. From this standpoint, modeling underlying asset dynamics on a multinomial lattice is useful (see e.g., [5,6,12,14,16,17] and the books of [8,11]) due to the existence of efficient methodologies for solving hedging and pricing problems. Moreover, multinomial lattice techniques allow one to price various types of derivative when no analytical formula is available. This paper seeks to provide a single parameterization for multinomial lattice random walks which can take higher order cumulants into account, instead of only the mean and variance. Before proceeding, we mention that there is an extensive body of literature on the subject of lattice techniques in derivative pricing and hedging, and we hope that readers will excuse our blatant omission of much of that work.

2 Construction of Multinomial Lattices

We will present a general description of a random walk on a multinomial lattice. Consider a stock market in the time interval $t \in [0,\ T]$, where traders are allowed to purchase and sell at discrete times $t_n = n\tau,\ n = 0,\ 1, \ldots, N$, where $\tau := T/N$. Let S_n denote the price of the stock at $t = t_n$, and suppose that u_n and d_n satisfy $u_n > d_n > 0$, then a multinomial tree with L branches at each node is given by

$$S_{n+1} = u_n^{L-l} d_n^{l-1} S_n,\ l = 1, \ldots L \tag{1}$$

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where $p_l,\ l = 1, \ldots L$ are the corresponding probabilities which satisfy $p_1 + \cdots + p_L = 1$. To make the multinomial tree recombine, we further assume that $u_n/d_n = c$ for all $n = 0, \ldots, N-1$ for some constant $c\ (> 1)$. One can verify that the process in (1) consists of a lattice (or a recombining multinomial tree), where the stock may achieve $n(L-1)+1$ possible prices at time $t = t_n,\ n = 0, \ldots, N$. For example, in the case of $u_n = u$ and $d_n = d$ for all $n = 1, \ldots, N-1$, the price of the stock at the k-th node from the top of the lattice is given by

$$S_n^{(k)} = u^{n(L-1)+1-k} d^{k-1} S_0,\ k = 1,\ 2, \ldots, n(L-1)+1. \tag{2}$$

2.1 Parameterization for Multinomial Lattices with Cumulants

Let X_n be the log stock return between t_n and t_{n+1} defined as

$$X_n := \ln S_{n+1} - \ln S_n,$$

and assume that each X_n is independent. Notice that

$$\ln S_N = \ln S_0 + \sum_{n=0}^{N-1} X_n.$$

We will construct multinomial lattice random walks to model stock price dynamics in terms of (local) cumulants of X_n through suitable choices of the parameters, L, N, u, d and $p_1, \ \ldots, p_L$. The m-th cumulant of X_n will be denoted by

$$\mathbb{C}\left(X_n^m\right).$$

Note that the cumulant $\mathbb{C}\left(X_n^m\right)$ is a polynomial in the moments $\mathbb{E}\left(X_n^v\right)$ with $v \leq m$, where the first and second cumulants are the mean and variance of X_n, respectively. The third and fourth cumulants are related to skewness and (Fisher) kurtosis, and are given by

$$\mathbb{C}\left(X_n^3\right) = M_n^{(3)},\ \mathbb{C}\left(X_n^4\right) = M_n^{(4)} - 3M_n^{(2)}, \tag{3}$$

where $M_n^{(m)}$ is the m-th central moment given as

$$M_n^{(m)} = \mathbb{E}\left[\left(X_n - \mathbb{E}\left(X_n\right)\right)^m\right].$$

The cumulants have an additive property when independent random variables are summed. For example, the m-th cumulant of $\sum_{n=0}^{N-1} X_n$ is just the sum of the m-th cumulants of X_n for $n = 0, \ldots, N-1$.

We will provide a parameterization of multinomial lattice random walks which take cumulants into account. Let

$$u_n := \exp\left(\frac{\nu_n}{L-1} \cdot \tau\right) \exp\left(\sqrt{\frac{\tau}{\alpha}}\right),\ d_n := \exp\left(\frac{\nu_n}{L-1} \cdot \tau\right) \exp\left(-\sqrt{\frac{\tau}{\alpha}}\right) \tag{4}$$

where $\alpha > 0$ is some constant. One can readily see that u_n/d_n is constant for all $n = 0, \ldots, N-1$ if α is fixed. With these choices for u_n and d_n, X_n may be computed as

$$X_n = \ln S_{n+1} - \ln S_n = \nu_n \tau + (L - 2l + 1)\sqrt{\frac{\tau}{\alpha}}.$$

Since we have not specified any variables in (4) yet (except τ $(= T/N)$), we have $L-1$ plus 2 unknown parameters, $p_1, \ldots, p_{L-1}$ (where p_L may be calculated as $p_L = 1 - (p_1 + \cdots + p_{L-1})$, ν_n and α. We will use these parameters to take advantage of additional information (i.e., cumulants).

Suppose that $\nu_n \tau$ is the mean of X_n, i.e., the first cumulant (mean) of X_n is

$$\mathbb{C}(X_n) = \mathbb{E}\left(X_n\right) = \nu_n \tau. \tag{5}$$

In this case,

$$\sum_{l=1}^{L} p_l (L - 2l + 1) = \mathbb{E}\left(L - 2l + 1\right) = 0, \tag{6}$$

must hold, where the expectation in (6) is taken with respect to $l = 1, \ldots, L$. With this mean value, the m-th central moment $M_n^{(m)}$ is given by

$$M_n^{(m)} = \mathbb{E}\left[(X_n - \nu_n \tau)^m\right] = \left(\sqrt{\frac{\tau}{\alpha}}\right)^m \mathbb{E}\left[(L - 2l + 1)^m\right], \tag{7}$$

and the second through fourth cumulants are computed by $\mathbb{C}(X_n^2) = M_n^{(2)}$ and the formulas in (3).

Binomial Lattice Case: To illustrate the parameterization described above, we first consider the case of $L = 2$, i.e., the binomial lattice case. Since there are already two constraints for the probabilities p_1 and p_2, i.e.,

$$p_1 + p_2 = 1, \quad \sum_{l=1}^{2} p_l (L - 2l + 1) = p_1 - p_2 = 0,$$

we obtain $p_1 = p_2 = 1/2$. Suppose that the variance of X_n is given by $\sigma_n^2 \tau$. This condition restricts $\alpha = 1$ and σ_n to be constant, i.e., $\sigma_n = \sigma$ $(n = 0, \ldots, N-1)$, and we have the binomial lattice formula provided in [11] (see also the original work of [5])

$$u_n = \exp\left(\nu_n \tau + \sigma\sqrt{\tau}\right), \quad d_n = \exp\left(\nu_n \tau - \sigma\sqrt{\tau}\right), \quad p_1 = p_2 = 1/2. \tag{8}$$

Trinomial Lattice Case: In the case of a trinomial lattice, i.e., $L = 3$, we have one more parameter p_3, and this allows us to take local volatility information into account, i.e., the second cumulant. Suppose that the second cumulant (i.e., variance) of X_n is given by $\sigma_n^2\tau$. In this case, we have

$$p_1 + p_2 + p_3 = 1, \quad 2p_1 - 2p_2 = 0, \quad 4p_1 + 4p_2 = \alpha\sigma_n^2. \tag{9}$$

where the second and third equations are obtained from (6) and (7), respectively. By solving (9) with respect to $p_1,\ p_2$ and p_3, we find

$$[p_1,\ p_2,\ p_3] = \left[\frac{\alpha\sigma_n^2}{8},\ 1 - \frac{\alpha\sigma_n^2}{4},\ \frac{\alpha\sigma_n^2}{8}\right].$$

To guarantee that these probabilities are positive, α must satisfy $0 < \alpha < 4/\sigma_n^2$. If σ_n is constant, i.e., $\sigma_n = \sigma$ $(n = 0, \ldots, N-1)$, one may use $\alpha = 4/(3\sigma^2)$, which provides a trinomial lattice formula whose up, middle, and down rates and corresponding probabilities are given by

$$u_n^2 = \exp\left(\nu_n\tau + \sigma\sqrt{3\tau}\right), \quad u_n d_n = \exp\left(\nu_n\tau\right), \quad d_n^2 = \exp\left(\nu_n\tau - \sigma\sqrt{3\tau}\right),$$
$$[p_1,\ p_2,\ p_3] = [1/6,\ 2/3,\ 1/6].$$

This also corresponds to a well known finite difference scheme.

If σ_n is a function of $(S_n,\ n)$, i.e., $\sigma_n = \sigma(S_n,\ n)$, the above formula can be modified by writing σ_n in terms of a nominal value $\hat{\sigma}$ as

$$\sigma_n = (1 + \delta_n)\hat{\sigma}.$$

Let α be chosen as $\alpha = 4/(3\hat{\sigma}^2)$. Then the up, middle, and down probabilities are given as

$$[p_1,\ p_2,\ p_3] = \left[\frac{(1+\delta_n)^2}{6},\ 1 - \frac{(1+\delta_n)^2}{3},\ \frac{(1+\delta_n)^2}{6}\right].$$

Note that the probabilities are positive as long as $-\sqrt{3} - 1 < \delta_n < \sqrt{3} - 1$.

Multinomial Lattice Case: Similarly, one can pose additional conditions given by higher order cumulants by using four or more branches in a multinomial lattice. For example, if we have third cumulant information corresponding to skewness, this imposes an additional constraint,

$$\mathbb{C}\left(X_n^3\right) = \left(\sqrt{\frac{\tau}{\alpha}}\right)^3 \mathbb{E}\left[(L - 2l + 1)^m\right] = s_n\tau\left(\sigma_n\sqrt{\tau}\right)^3,$$

where $s_n\tau$ is the skewness of X_n. This condition can be taken into account if four branches are used in the multinomial lattice, i.e., $L = 4$. If we solve four linear equations for the probabilities $p_1,\ p_2,\ p_3,\ p_4$, we obtain

$$[p_1,\ p_2,\ p_3,\ p_4] = \frac{1}{16} \times \left[-1 + \alpha\sigma_n^2\left(1 + \frac{s_n\tau\sqrt{\alpha}\sigma_n}{3}\right),\ 9 - \alpha\sigma_n^2\left(1 + s_n\tau\sqrt{\alpha}\sigma_n\right),\right.$$
$$\left. 9 + \alpha\sigma_n^2\left(-1 + s_n\tau\sqrt{\alpha}\sigma_n\right),\ -1 + \alpha\sigma_n^2\left(1 - \frac{s_n\tau\sqrt{\alpha}\sigma_n}{3}\right)\right].$$

If σ_n is constant, i.e., $\sigma_n = \sigma$ $(n = 0, \dots, N-1)$, the choice $\alpha = 4/\sigma^2$ results in the following formulas:

$$u_n^3 = \exp\left(\nu_n\tau + \frac{3\sigma}{2}\sqrt{\tau}\right), \quad u_n^2 d_n = \exp\left(\nu_n\tau + \frac{\sigma}{2}\sqrt{\tau}\right),$$

$$u_n^2 d_n = \exp\left(\nu_n\tau - \frac{\sigma}{2}\sqrt{\tau}\right), \quad d_n^3 = \exp\left(\nu_n\tau - \frac{3\sigma}{2}\sqrt{\tau}\right),$$

$$[p_1,\ p_2,\ p_3,\ p_4] = \left[3 + \frac{8}{3}s_n\tau,\ 5 - 2s_n\tau,\ 5 + 2s_n\tau, 3 - \frac{8}{3}s_n\tau\right].$$

If we additionally would like to match the 4th cumulant or "kurtosis", we should introduce a multinomial lattice with five branches, i.e., $L = 5$. Let $\kappa\tau$ denote the kurtosis of X_n, then we need

$$\mathbb{C}\left(X_n^4\right) = \frac{\tau^2}{\alpha^2}\mathbb{E}\left[(6-2l)^4\right] - 3\left(\sigma_n\sqrt{\tau}\right)^4 = \kappa_n\tau\left(\sigma_n\sqrt{\tau}\right)^4,$$

as an additional constraint. In this case, the probabilities $p_1,\ p_2,\ p_3,\ p_4,\ p_5$ can be calculated through the solution of five linear equations, and are given by

$$\begin{aligned}
[p_1,\ p_2,\ p_3,\ p_4,\ p_5] = \frac{1}{96}\Bigg[&\alpha\sigma_n^2\left(-1 + s_n\tau\sqrt{\alpha}\sigma_n + \frac{\alpha\sigma_n^2}{4}(3+\kappa_n\tau)\right),\\
&\alpha\sigma_n^2\left(16 - 2s_n\tau\sqrt{\alpha}\sigma_n - \alpha\sigma_n^2(3+\kappa_n\tau)\right),\ \frac{2}{3}\left\{64 + \alpha\sigma_n^2\left(-20 + \alpha\sigma_n^2\left(3+\kappa_n\tau\right)\right)\right\},\\
&\alpha\sigma_n^2\left(16+2s_n\tau\sqrt{\alpha}\sigma_n - \alpha\sigma_n^2(3+\kappa_n\tau)\right),\ \alpha\sigma_n^2\left(-1 - s_n\tau\sqrt{\alpha}\sigma_n + \frac{\alpha\sigma_n^2}{4}(3+\kappa_n\tau)\right)\Bigg]
\end{aligned}$$

To understand the effect of kurtosis, assume that $s_n = 0$ and $\sigma_n = \sigma$ $(n = 0, \dots, N-1)$, then we obtain

$$\begin{aligned}
[p_1,\ p_2,\ p_3,\ p_4,\ p_5] = \frac{1}{96}\Bigg[&\alpha\sigma^2\left(-1 + \frac{\alpha\sigma^2}{4}(3+\kappa_n\tau)\right),\\
&\alpha\sigma^2\left(16 - \alpha\sigma^2(3+\kappa_n\tau)\right),\ \frac{2}{3}\left\{64 + \alpha\sigma^2\left(-20 + \alpha\sigma^2\left(3+\kappa_n\tau\right)\right)\right\},\\
&\alpha\sigma^2\left(16 - \alpha\sigma^2(3+\kappa_n\tau)\right),\ \alpha\sigma^2\left(-1 + \frac{\alpha\sigma^2}{4}(3+\kappa_n\tau)\right)\Bigg]
\end{aligned}$$

In this case, all the probabilities are positive if

$$\frac{4}{\sigma^2(3+\kappa_n\tau)} < \alpha < \frac{16}{\sigma^2(3+\kappa_n\tau)}.$$

Furthermore, if we choose $\alpha = 4/\sigma^2$, then the above probabilities reduce to

$$\begin{aligned}
&[p_1,\ p_2,\ p_3,\ p_4,\ p_5] =\\
&\quad\left[\frac{1}{24}(2+\kappa_n\tau),\ \frac{1}{6}(1-\kappa_n\tau),\ \frac{1}{4}(2+\kappa_n\tau),\ \frac{1}{6}(1-\kappa_n\tau),\ \frac{1}{24}(2+\kappa_n\tau)\right]. \quad (10)
\end{aligned}$$

The up-down rates corresponding to five branches can be calculated as

$$u_n^4 = \exp\left(\nu_n\tau + 2\sigma\sqrt{\tau}\right), \quad u_n^3 d_n = \exp\left(\nu_n\tau + \sigma\sqrt{\tau}\right), \quad u_n^2 d_n^2 = \exp\left(\nu_n\tau\right)$$

$$u_n d_n^3 = \exp\left(\nu_n\tau - \sigma\sqrt{\tau}\right), \quad d_n^4 = \exp\left(\nu_n\tau - 2\sigma\sqrt{\tau}\right)$$

We first notice that the probabilities are symmetric, i.e., $p_1 = p_5$ and $p_2 = p_4$. In this formulation, p_1, p_3 and p_5 increase with larger kurtosis. On the other hand, p_2 and p_4 decrease if kurtosis increases. Therefore, this confirms that the probability distribution of X_n becomes heavy tailed under positive kurtosis.

If skewness is not zero, the formulation in (10) becomes

$$[p_1,\ p_2,\ p_3,\ p_4,\ p_5] = \left[\frac{2+\kappa_n\tau+2s_n\tau}{24}, \frac{1-\kappa_n\tau-s_n\tau}{6}, \frac{2+\kappa_n\tau}{4}, \frac{1-\kappa_n\tau+s_n\tau}{6}, \frac{2+\kappa_n\tau-2s_n\tau}{24}\right]$$

with the choice of $\alpha = 4/\sigma^2$. In this case, we readily see that the probabilities are not symmetric if $s_n \neq 0$. Moreover, positive (negative) skewness causes p_1 and p_4 to increase (decrease), and the corresponding probabilities p_5 and p_2 to decrease (increase) by an equal amount.

2.2 Parameterization with Time-Dependent Distributions

In this section, we deal directly with the stock price distribution, rather than characterizing it through cumulants. We will consider the case where the stock price distribution is available at every time t_n. Under this assumption, we show that a multinomial lattice can be constructed as follows:

1. Generate a binomial lattice to match the distribution of the stock at every time step.
2. Create a multinomial lattice based on the binomial lattice.

Let $\mathbb{P}_n(S_n)$ be the probability distribution of the stock at $t = t_n$. $\mathbb{P}_n(S_n)$ may be obtained from historical data.

We begin by using a binomial lattice to describe the stock dynamics. Consider the stock prices arranged on a binomial lattice as shown in the left side of Table 1, where the price of the stock on the k-th node from the top of the lattice is denoted by $S_n^{(k)}$. Furthermore, the probability of obtaining the price S_n^k at time n is given by $P_n^{(k)} = \mathbb{P}_n(S_n^k)$ as shown in Table 1.

Let $p_n^{(k)}$ denote the probability of moving from $S_n^{(k)}$ to $S_{n+1}^{(k)}$ (this corresponds to an "up" move). The probability for a corresponding "down" move from $S_n^{(k)}$ to $S_{n+1}^{(k+1)}$ is given by $p_{n,d}^{(k)} = 1 - p_n^{(k)}$. These probabilities are computed based on the node probabilities $P_n^{(k)}$ $(k = 1,\dots,N,\ k = 1,\dots,n+1)$ as follows: Consider the node probabilities at the n-th period, $P_n^{(k)}$ $(k = 1,\dots,n+1)$, and the node probabilities at the $(n+1)$-th period, $P_{n+1}^{(k)}$ $(k = 1,\dots,n+2)$, where

Table 1. Stock price and corresponding probability

$\cdots$	$S^{(1)}_{N-2}$	$S^{(1)}_{N-1}$	$S^{(1)}_{N}$	$\cdots$	$P^{(1)}_{N-2}$	$P^{(1)}_{N-1}$	$P^{(1)}_{N}$
$\cdots$	$S^{(2)}_{N-2}$	$S^{(2)}_{N-1}$	$S^{(2)}_{N}$	$\cdots$	$P^{(2)}_{N-2}$	$P^{(2)}_{N-1}$	$P^{(2)}_{N}$
$\cdots$	$S^{(3)}_{N-2}$	$S^{(3)}_{N-1}$	$S^{(3)}_{N}$	$\cdots$	$P^{(3)}_{N-2}$	$P^{(3)}_{N-1}$	$P^{(3)}_{N}$
$\cdots$	$S^{(4)}_{N-2}$	$S^{(4)}_{N-1}$	$S^{(4)}_{N}$	$\cdots$	$P^{(4)}_{N-2}$	$P^{(4)}_{N-1}$	$P^{(4)}_{N}$
$\ddots$	$S^{(5)}_{N-2}$	$S^{(5)}_{N-1}$	$S^{(5)}_{N}$	$\ddots$	$P^{(5)}_{N-2}$	$P^{(5)}_{N-1}$	$P^{(5)}_{N}$
		$S^{(6)}_{N-1}$	$S^{(6)}_{N}$			$P^{(6)}_{N-1}$	$P^{(6)}_{N}$
			$S^{(7)}_{N}$				$P^{(7)}_{N}$

$n \in [0,\ N-1]$. Since $p_n^{(1)}$ is the probability of obtaining $S_{n+1}^{(1)}$ given $S_n^{(1)}$, it may be calculated as

$$p_n^{(1)} = P_{n+1}^{(1)} / P_n^{(1)}. \tag{11}$$

Similarly, since the probability of obtaining $S_{n+1}^{(k)}$ given $S_n^{(k)}$ satisfies

$$P_{n+1}^{(k)} = \left(1 - p_n^{(k-1)}\right) P_n^{(k-1)} + p_n^{(k)} P_n^{(k)}$$

$p_n^{(k)}$ may be calculated as

$$p_n^{(k)} = \left(P_{n+1}^{(k)} - \left(1 - p_n^{(k-1)}\right) P_n^{(k-1)}\right) / P_n^{(k)}. \tag{12}$$

Using (11) and (12), $p_n^{(k)}$ may be computed for all $n = 0, \ldots, N-1$ and $k = 1, \ldots, n+1$. This constructs a binomial lattice matching the stock price distribution.

We may now construct a multinomial lattice based on the binomial lattice as follows: Consider a two step binomial lattice, where we suppose that the up and down rates, u and d, and probabilities p, $p_1^{(1)}$, and $p_1^{(2)}$ are specified as shown in the left side of Fig. 1. The right side of Fig. 1 is a trinomial tree, where the up, middle, and down states are given by Su^2, Sud and Sd^2. If the up, middle, and down probabilities of the tree are given by

$$p_u = p \cdot p_1^{(1)},\ p_m = p \cdot (1 - p_1^{(1)}) + (1-p) \cdot p_1^{(1)},\ p_d = (1-p) \cdot (1 - p_1^{(1)})$$

then the binomial lattice and the trinomial tree will define the same random walk as far as the initial state and final distributions are concerned, i.e., both random walks have final distributions with identical statistical properties. More generally, in a similar manner one may construct a multinomial lattice with L branches at each node based on a multi-step binomial lattice.

3 Synthesis Methods

Once we have constructed a multinomial lattice, we may apply several techniques for pricing and hedging derivatives. In this section, we demonstrate some of these

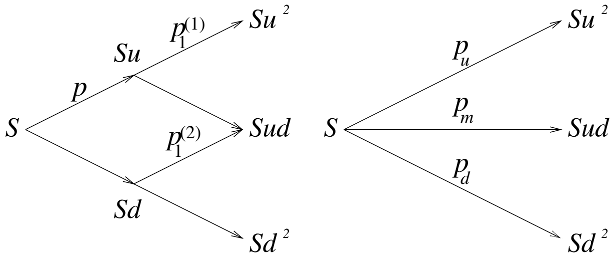

Fig. 1. Trinomial lattice construction

techniques which are used with multinomial lattices. Since most of the following ideas have been considered extensively in the literature, we merely provide a brief outline of them in this paper.

3.1 Mean Square Optimal Hedges

Mean square optimal hedging is a trading strategy which constructs a portfolio whose payoff approximates that of a derivative security as closely as possible in the mean square error sense. Although we only deal with the case of a European call option in this subsection, the same approach can be extended to other types of options, including many exotics (such as barriers, compounds, and others) and options with time optionality (such as Americans and Bermudans).

Let B_n denote the price of a (risk free) bond under the time dependent interest rate r_n where B_n satisfies

$$B_n = (1 + r_n)B_{n-1}, \quad n = 1, \dots, N. \tag{13}$$

at discrete times $t_n = n\tau,\ n = 0,\ 1, \dots, N$. Also, let C_n, $n = 0, 1, \dots, N$ denote the value of a call option with strike price K, which pays

$$C_N = (S_N - K)^+$$

at maturity $t = T$. Finally, we define a portfolio $(\delta_n, \theta_n) \in \Re^2$ indexed by time $n = 0 \dots N$, and let

$$\Omega_n := \delta_n S_n + \theta_n B_n, \quad n = 0 \dots N \tag{14}$$

be the value of the portfolio, where δ_n represents the number of shares of stock and θ_n the number of bonds held by the trader during the time interval $t \in [t_n,\ t_{n+1})$. Finally, we assume that the portfolio is self-financing;

$$\delta_{n-1} S_n + \theta_{n-1} B_n = \delta_n S_n + \theta_n B_n, \quad {}^{\forall} n = 1 \dots N. \tag{15}$$

We now introduce an optimal hedging strategy to minimize the mean square of the difference between the final payoff of the call option and the value of the portfolio (i.e. $C_N - \Omega_N$), namely mean square optimal hedging (MSOH):

$$\textbf{MSOH} \left| \begin{array}{l} \text{Minimize}: \ \mathbb{E}\left[(C_N - \Omega_N)^2 \middle| S_0, \ \Omega_0 \right] \\ \text{Subject to}: \ \delta_n \in \Re, \ n = 0, \ldots, N-1, \ \Omega_0 \in \Re \end{array} \right. \tag{16}$$

To obtain the optimal hedging strategy $\delta_k \in \Re, \ k = 0, \ldots, N-1$ and initial portfolio wealth Ω_0, dynamic programming (see e.g., [1]) may be applied once probabilities for possible outcomes for the stock have been assigned. Note that the MSOH problem can be solved very efficiently by dynamic programming if the stock process is modeled on a lattice [7].

3.2 Volatility Smile and Implied Kurtosis

We next discuss pricing models which take the "volatility smile" into account. There is a large body of literature which provides option pricing formulas for smiley options by using binomial lattices, trinomial lattices, or finite difference methods (see e.g., [6,8,14,16,17] and references therein). A common approach is to use market option data to determine a corresponding local volatility function or risk neutral probability distribution to match the volatility smile.

Another approach to modeling the local volatility function for smiley options is to take advantage of the so-called implied kurtosis [3,15]. This approach simply requires an estimate of implied kurtosis which can be extracted from the market price of options. The relation between implied kurtosis and the volatility smile is given by the following equation [3,15]:

$$\sigma(S_n, \ n) = \sigma \left[1 + \frac{\kappa \tau}{24} \left(\frac{(K - S_n)^2}{\sigma^2 S_n^2 T} - 1 \right) \right] \tag{17}$$

where σ can be thought as a "true volatility" corresponding to the variance of the stock price distribution at maturity, and κ is the (annualized) kurtosis of the stock price distribution. Therefore, this formulation provides a connection between implied volatility and a constant volatility with kurtosis.

Given the formula in (17), one can apply local volatility based pricing methods such as trinomial lattices or corresponding finite difference methods (see e.g., [8] and references therein). However it might be more suitable to use a multinomial lattice to take kurtosis (or the fourth cumulant) into account. In this case, one can directly construct a multinomial lattice with five branches as in Subsection 2.1, with constant σ and κ, instead of using a trinomial lattice with a local volatility function. A more sophisticated model can be developed by taking into consideration the time dependence of kurtosis, i.e., k_n, which provides a volatility surface curve. One can then apply standard risk neutral valuation techniques for derivatives pricing.

4 Conclusion

In this paper, we provided a parameterization of multinomial lattice random walks which take cumulants into account. In the binomial and trinomial lattice cases, this parameterization reduced to standard formulas. We showed that higher order cumulants may be taken into account by using multinomial lattices with four or more branches. Finally, we demonstrated two types of synthesis methods which take advantage of multinomial lattices: mean square optimal hedging in incomplete markets and valuation techniques which use implied volatility or kurtosis.

References

1. Bellman, R.: *Dynamic Programming*. Princeton University Press, Princeton, NJ (1957)
2. Black, F., Scholes, M.; The Pricing of Options and Corporate Liabilities. Journal of Political Economy **81** (1973) 637–654
3. Bouchaud, J-P. and M. Potters.: *Theory of Financial Risks*. Cambridge University Press (2000)
4. Cox, J.C., Ross, S.A.: The valuation of options for alternative stochastic processes. Journal of Financial Economics **3** (1976) 145–166
5. Cox, J.C., Ross, S.A., Rubinstein, M.: Option pricing: A simplified approach. Journal of Financial Economics **7** (1979) 229–263
6. Derman, E., Kani, I.: Riding on a Smile. Risk **7** (1994) 18–20
7. Fedotov, S., Mikhailov, S.: Option Pricing for Incomplete Markets via Stochastic Optimization: Transaction Costs, Adaptive Control, and Forecast. Int. J. of Theoretical and Applied Finance **4** No. 1 (2001) 179–195
8. J. Hull, *Options, Futures, and Other Derivative Securities*, 4th edition. Englewood Cliffs: Prentice-Hall, 1999.
9. Hull, J., White, A.: The Pricing of Options on Assets with Stochastic Volatilities. Journal of Finance **42** (1987) 281–300
10. Jarrow, R., Rudd, A.: Approximate option valuation for arbitrary stochastic processes. Journal of Financial Economics **10** (1982) 347–369
11. Jarrow, R., Rudd, A.: *Option Pricing*. McGraw-Hill Professional Book Group (1983)
12. Karandikar, R.L., Rachev, S.T.: A generalized binomial model and option pricing formulae for subordinated stock-price processes. Probability and Mathematical Statics **15** (1995) 427–447
13. Merton, R.C., Option pricing when underlying stock returns are discontinuous. Journal of Financial Economics **3** (1976) 125–144
14. Pirkner, C.D., Weigend, A.S., Zimmermann, H.: Extracting Risk-Neutral Densities from Option Prices Using Mixture Binomial Trees. Proc. of the IEEE/IAFE/INFORMS Conf. on Computational Intelligence for Financial Engineering (1999) 135–158
15. Potters, M., R. Cont and J-P. Bouchaud.: Financial markets as adaptive systems. Europhys. Lett. **41** No. 3 (1998) 239–244
16. Rubinstein, M.: Implied Binomial Trees. Journal of Finance **3** (1994) 771–818
17. Rubinstein, M.: On the Relation Between Binomial and Trinomial Option Pricing Models. Research Program in Finance Working Papers, ♯RPF-292, University of California at Berkeley (2000)