Stochastic Volatility Models and Option Prices

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Abstract. It is an observed fact in the market that the implied volatility of traded options vary from day to day. An alternative and straightforward explanation is that the instantaneous volatility of a stock is a stochastic quantity itself. The assumptions of the Black and Scholes model no longer hold. This is, therefore, one reason why Black and Scholes prices can differ from market prices of options. Having decided to make the instantaneous volatility stochastic, it is necessary to decide what sort of process it follows. The article analyzes three stochastic volatility models and considers how stochastic volatility can be incorporated into model prices of options. The investigation of stochastic volatility influence for pricing options traded in the SEB Vilnius Bank is done.

1 Introduction

The pricing of derivative instruments, such as options is a function of the movement in the price of the underlying asset over lifetime of the option. One of the main problems of financial engineering is to develop a suitable model of financial assets dynamics. The dynamics is described as a stochastic process, and pricing models describe the stochastic dynamics of asset price changes, whether this is a change in share prices, stock indices, interest rates and so on. Louis Bachelier [1] had claimed that stock prices are actually random in 1900. Comparing trajectories of random walks and stock prices, Bachielier could not find a significant difference among them. The dynamics of asset prices are reflected by uncertain movements of their values over time. Some authors [2, 3, 4, 14] state that efficient market Hypothesis (EMH) is one possible reason for the random behavior of the asset price. The EMH basically states that past history is fully reflected in present prices and markets respond immediately to new information about the asset.

The classical approach is to specify a diffusion process for the asset price, that is, a stochastic integral or stochastic differential equation where the uncertainty is driven by Wiener process. The wide spread adoption of Wiener process as a frame work for describing changes in financial asset prices is most like due to its analytic tractability.

Unfortunately, in recent years more and more attention has been given to stochastic models of financial markets which differ from traditional models. It appears that variances and covariances are not constant over time. There is now a lot of literature on time series modeling of asset prices. The reviewed literature

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[5, 6] has revealed the following empirical properties of asset dynamics: fat tails of distribution, volatility clustering, large discrete jumps, and parameter instability.

Some classical models of asset dynamics are presented in the article and stochastic volatility models of EUR/USD exchange rate based on the data of trading options in SEB Vilnius Bank are analyzed also.

2 The Stochastic Process of Stock Prices

The modeling of the asset price is concerned with the modeling of new information arrival, which affects the price. Depending on the appearance of the so called "normal" and "rare" events, there are two basic blocks in modeling the continuous time asset price. Neftci [7] states that the main difference between the "normal" and "rare" behavior concerns the size of the events and their probability to occur. Wiener process can be used if markets are dominated by "normal" events. This is a continuous time stochastic process, where extremes occur only infrequently according to the probabilities in the tails of normal distribution. The stochastic process is written in the form of the following stochastic differential equation for the asset return: $dS_t = \mu S_t dt + \sigma S_t dW_t$, where S_t – the current price of the underlying asset, μ – the constant trend, σ – the constant volatility, W_t – the standard Wiener process. Since this process has a continuous time sample path, it does not allow for discontinuity or jumps in its values when "rare" events occur. In this case , the Poisson jump process can be useful. In particular, the time series of asset price can be modeled as the sum of continuous time diffusion process and Poisson jump processes. The stochastic differential equation for S_t is:

$$dS_t = \mu S_t dt + \sigma S_t dW_t + bS_t \sum_{j=1}^{N_t} (Y_j - 1)$$

with the following addition of variables: $Y_j - 1$ - a lognormal distributed random variable representing the jump size, N_t - jumps in interval $(0, t)$ governed by a Poisson process with parameter λt, b – constant. Jump diffusion models undoubtedly capture a real phenomenon. Yet they are rarely used in practice due to difficulty in parameter estimation.

3 Stochastic Volatility Models

Most options are priced using the Black and Scholes formula, but it is well known that the assumptions upon which this formula is based are not justified empirically [12]. In particular, return distribution may be fat tailed and its volatility is certainly not constant. It is an observed fact in financial market that the implied volatilities of traded options vary from day to day. An alternative and straightforward explanation is that the instantaneous volatility of a stock is itself a stochastic quantity. Having decided to make volatility stochastic, it is necessary to decide what sort of process follows. Take a process of the form [11]:

$$dS_t = \mu(S_t, \nu_t)dt + \sigma(S_t, \nu_t)dW_{St} \tag{1}$$

$$\sigma(S_t, \nu_t) = f(\nu_t) \tag{2}$$

$$d\nu_t = \sigma(S_t, \nu_t)dt + \beta(S_t, \nu_t)dW_{\nu t} \tag{3}$$

where W_{St}, W_ν are correlated Wiener processes with correlation coefficient ρ , i.e,

$$dW_\nu = \rho dW_{St} + \sqrt{1-p^2}dZ_t \tag{4}$$

W_{St} and Z_t are uncorrelated Wiener processes [13].

Three different stochastic volatility models will be considered, such as: Hull-White, Heston, and logarithmic Ornstein-Uhlenbeck. The Hull-White model [9] is the particular case of the model described by (1) – (4) equations. Then we have that

$$dS_t = \mu S_t dt + \sigma_t S_t dW_{St}, \qquad d\nu_t = \gamma \nu_t dt + \eta \nu_t dW_{\nu t} \tag{5}$$

where $\sigma_t = \sqrt{\nu_t}$, $\gamma < 0$, W_{St}, and $W_{\nu t}$ are uncorrelated Wiener processes. For simplicity, assume that volatility can take only two values. In this case the price of Call option is equal to $C_t = E\Big[C_{BS}\Big(t, S, K, T, \sqrt{\overline{\sigma^2}}\Big)\Big|\nu_t = \nu\Big]$ where $\overline{\sigma^2} = \frac{1}{T-t}\int_t^T f\Big(\nu_x\Big)^2 dx$, $\nu_t$is the two state Markov process [13] and

$$\overline{\sigma^2} = \begin{cases} \sigma_1^2 \text{ with probability } p \\ \sigma_2^2 \text{ with probability } 1-p \end{cases}$$

Heston's option pricing model assumes that S_t and $\nu_t$satisfies the equations [10]:

$$dS_t = \mu S_t dt + \sigma_t S_t dW_{St}, \qquad d\nu_t = \kappa\Big(\theta - \nu_t\Big)dt + \eta\sqrt{\nu_t}dW_{\nu t} \tag{6}$$

where $\sigma_t = \sqrt{\nu_t}$, κ, θ, η, ρ are constants.

The Ornstein-Uhlenbeck's stochastic volatility model is

$$dS_t = \mu S_t dt + \sigma_t S_t dW_{St}, \qquad d\nu_t = \alpha\Big(\overline{\nu} - \nu_t\Big)dt + \beta dW_{\nu t} \tag{7}$$

where $\sigma_t = exp(\nu_t)$. The empirical investigation shows that $ln\sigma_t$ follows Ornstein-Uhlenbeck process with parameters $ln\overline{\sigma}$ and $\alpha > 0$. It is usual to assume that μ, $\alpha(ln\overline{\sigma} - \nu_t)$ and β are constants. Let $\rho = 0$.

4 Estimation of Parameters of Stochastic Volatility Models

Parameters of stochastic volatility models are estimated on observations of assets and options price dynamics. There are three unknown parameters in the Hull-White model: σ_1, σ_2, and p, which are estimated by the method of least squares :

$$min \sum_{i=1}^{n} \left(C_{market_i} - C_{model}\Big(\sigma_1, \sigma_2, p, K_i\Big) \right)^2$$

where n - the number of traded options per day, C_{market_i} – the market price of ith option with strike price K_i, $C_{model}\Big(\sigma_1, \sigma_2, p, K_i\Big)$ – the price of option evaluated by Hull-White model.

The parameters of Heston and Ornstein-Uhlenbeck models are estimated applying two steps procedure [12]. At first parameters μ, κ, θ, η must be valuated. Say that $\mu = 0$, then the equation (6) in discrete case has the form:

$$R_t = \sqrt{\nu_t \tau} \varepsilon_{1t}, \qquad \nu_t = \kappa\theta\tau + \Big(1 - \kappa\tau\Big)\nu_{t-\tau} + \eta\sqrt{\nu_{t-\tau}\tau}\varepsilon_{2t}$$

where R_t - the return rate of the asset, ε_{1t} and ε_{2t} - two standard normal distributed correlated values. It is constructed the auxiliary GARCH(1,1) model:

$$R_t = \sqrt{h_t}\varepsilon_t, \qquad h_t = \omega + \alpha\varepsilon_{t-1}^2 + \beta h_{t-1}$$

Where: ε_t- normally distributed random variable with mean 0 and variance h_t. In this case it is possible to estimate only three parameters $\Big(\kappa, \theta, \eta\Big)$ and coefficient of correlation is equated zero. The theoretical return rate are matching with empirical data when parameters $\Big(\kappa, \theta, \eta\Big)$ and τ are chosen.

Applying GARCH(1,1) model, the set of optimal parameters $\widehat{B} = \Big(\omega, \alpha, \beta\Big)$ for given data of the asset return rates is obtained. Thus, the set of parameters $\Theta = \Big(\kappa, \theta, \eta\Big)$ is known for each modeled time series. The next step is to compute the vector

$$m\Big(\Theta, \widehat{B}\Big)_{3*1} = \frac{1}{N}\sum_{t=1}^{N} \frac{\delta l_t \Big(R_t(\Theta) \big| R_{t-1}(\Theta), B\Big)}{\delta B}\Bigg|_{B=\widehat{B}}, \qquad l_t = -\ln h_t - \frac{R_t^2}{2h_t}$$

where $R_t(\Theta)$ are rates of modeled returns, N – the number of modeling steps. If m equals zero, then the modeled data obtained by GARCH(1,1) model will have the same parameters as observed data. The optimal set of parameters is valuated minimizing the expression $\min_{\widehat{\Theta}} m^T\Big(\Theta, \widehat{B}\Big) I^{-1} m\Big(\Theta, \widehat{B}\Big)$ with matrix of weights $I_{3*3} = \frac{1}{N}\sum_{t=1}^{N} \frac{\delta l_t\left(R_t, B\right)}{\delta B}\frac{\delta l_t\left(R_t, B\right)}{\delta B^T}\Big|_{B=\widehat{B}}$. The matrix I is obtained estimating the gradient from the observed market data.

Having the estimations $(\widehat{\kappa}, \widehat{\theta}, \widehat{\eta})$, the coefficient of correlation ρ is calculated by the method of least squares. The error between market and model prices is minimized by the procedure

$$\min_{(p)} \sum_{i=1}^{n} \Big(C_{market_i} - C_{model}\big(\rho, K_i\big) \Big)^2$$

where n is the number of daily option prices. The procedure is repeated for option prices of each day. The parameters of Logarithmic Ornstein-Uhlenbeck model are estimated in a similar way.

5 Empirical Investigation of the Models

The prices of European call options traded on exchange rate EUR/USD in SEB Vilnius Bank (Lithuania) will be investigated. The observed data are divided into several groups according to the profitability ($S/K - 1$) and life time of options. Suppose that an option is at the money (ATM) if $S = K$ or profitability belongs to the interval (–0,5%, 0,5%.). An option is in the money (ITM) if $S > K$ or profitability belongs to the interval (0.5%, 1%) and out of the money (OTM) if $S < K$ or the profitability belongs to the interval (-1%, -0,5%). An option is deep out of the money (DOTM) if profitability is less than -1%. An option is called short term if the life time of the option is equal to 1 month, intermediate - 3 months, and long term - 6 months. Daily observations of 82 days (from 2005-01-05 till 2005-04-30, totally 969 observations) were used for investigation. The values of implied volatility depend on the profitability and are calculated applying the above described models. The graphs of volatility smile are depicted in the Fig. 1.

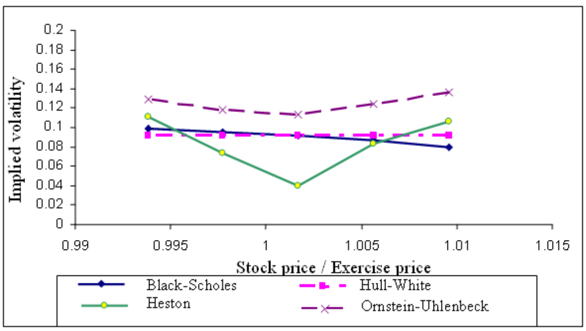

Fig. 1. Volatility smile for options of 1 month period

The implied volatility values of Hull-White model were computed by (5) equations. Thus

$$F(I) \equiv C_{BS}(I) - \widehat{p}C_{BS}(\widehat{\sigma}_1) - (1 - \widehat{p})C_{BS}(\widehat{\sigma}_2) = 0$$

where:C_{BS} – value obtained by Black-Scholes formula [8], I –implied volatility calculated by the method of Newton – Rapson.

Simulation by the method of Monte Carlo was carried out by the following algorithm:

1. Obtained paths of asset price and volatility by the Hestono and Ornstein-Uhlenbeck models.
2. Computed values of options at the end of each day: $h(S_t) = \max\{0, S_t - K\}$, $t = \overline{1, T}$.
3. The price of option is calculated by the formula: $C_{model} = e^{-r_{Base}T} E\big(h(S_T)\big)$.
4. The values of volatility are derived from the equation $C_{BS}(I) - C_{model} = 0$.

Theoretical price of an option is approximate to the market price if the obtained volatility value is close to the value of implied volatility derived from the Black-Scholes.

Models of the stochastic volatility overprice DOTM and ITM options and undervalue OTM options, except Ornstein-Uhlenbeck model which overprices all options (Table 1). This is obviously seen for short term options. The stochastic volatility models are more practical for pricing intermediate and short term options. Errors of options pricing are estimated in two ways: average relative pricing error (ARPE) and average square error (ASE).

Table 1. Comparison of implied volatilities for various models

Profit ($x = S/K - 1$)	**Model**	**Lifetime of option**			**Total**
		Short	**Interm.**	**Long**	
DOTM	Black-Scholes	0.1425	0.1209	0.1125	0.1253
($x < -0.01$)	Hull-White‘	0.1432	0.1218	0.1126	0.1259
	Heston	0.1661	0.1269	0.1039	0.1323
	Ornstein-Uhlenbeck	0.1616	0.126	0.1197	0.1358
OTM	Black-Scholes	0.1266	0.1107	0.1052	0.1141
($-0.01 < x < -0.005$)	Hull-White	0.1238	0.1106	0.1049	0.1131
	Heston	0.1194	0.0966	0.0872	0.1011
	Ornstein-Uhlenbeck	0.137	0.1219	0.1186	0.1258
ATM	Black-Scholes	0.1103	0.1012	0.0985	0.1033
($-0.005 < x < -0.005$)	Hull-White	0.1084	0.1012	0.0982	0.1026
	Heston	0.0636	0.0636	0.0661	0.063
	Ornstein-Uhlenbeck	0.1214	0.1174	0.1167	0.1185
ITM	Black-Scholes	0.0868	0.0901	0.0912	0.0894
($0.005 < x < 0.01$)	Hull-White	0.0945	0.0941	0.0934	0.094
	Heston	0.0975	0.0907	0.0603	0.0898
	Ornstein-Uhlenbeck	0.1374	0.1208	0.1175	0.1252

$$ARPE = \frac{1}{n}\sum_{i=1}^{n}\frac{\left|C_i^M - C_i\right|}{C_i}, \qquad ASE = \sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(C_i^M - C_i\right)^2}$$

where n is number of option prices, C_i and C_i^M –theoretical and market prices of options respectively. $ARPE$ and $RMSE$ are calculated with different profitability and duration of options. All the models overvalue DOTM and ITM options but the Hull-White model undervalue ITM options of all terms (Table 2). The pricing errors of ITM, DOTM, and short term options are the largest one. The Hull-White model is clearly superior comparing with the Black-Scholes and other stochastic volatility models. Relative options pricing errors of the Hull-White model are less then Black-Scholes one in 7 cases from 12. Rising duration of options their pricing errors decline. ARPE and ASE errors coincide.

Table 2. Relative errors of options pricing

Profit ($x = S/K - 1$)	**Model**	**Lifetime of option**			**Total**
		Short	**Interm.**	**Long**	
DOTM ($\boldsymbol{x < -0.01}$)	Black-Scholes	0.0381	0.0194	0.0156	0.0244
	Hull-White‘	0.0352	0.0236	0.0152	0.0246
	Heston	0.6935	0.2018	0.2064	0.3672
	Ornstein-Uhlenbeck	0.1828	0.2311	0.1682	0.194
OTM ($\boldsymbol{-0.01 < x < -0.005}$)	Black-Scholes	0.0473	0.0259	0.0175	0.0302
	Hull-White	0.0443	0.0245	0.015	0.0279
	Heston	0.3365	0.2309	0.2507	0.2726
	Ornstein-Uhlenbeck	0.1971	0.1769	0.1192	0.1644
ATM ($\boldsymbol{-0.005 < x < -0.005}$)	Black-Scholes	0.0397	0.0199	0.0171	0.0256
	Hull-White	0.038	0.0231	0.0156	0.0255
	Heston	0.3426	0.343	0.2795	0.3284
	Ornstein-Uhlenbeck	0.1884	0.1248	0.091	0.1347
ITM ($\boldsymbol{0.005 < x < 0.01}$)	Black-Scholes	0.0614	0.0348	0.0238	0.04
	Hull-White	0.0637	0.0394	0.0251	0.04
	Heston	0.3405	0.2494	0.2332	0.3077
	Ornstein-Uhlenbeck	0.1859	0.0934	0.0918	0.1236

6 Conclusions

1. Stochastic volatility models are more preferable for intermediate and long duration options.
2. In respect of profitability a stochastic volatility parameter is greater (less) then implied volatility parameter for DOTM and ITM (OTM) options.
3. All the volatility models (except Heston model) overvalue DOTM and ITM options but undervalue ATM options. The Hull – White model gives the least option pricing error and the most one gives the Heston model.
4. The Ornstein-Uhlenbeck model is suitable for pricing long term options and the Hull-White – model is relevant for various duration options.

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