Eric Raidl
Andrea Iacona
Vincenzo Crupi

An Axiomatic System for Concessive Conditionals

Abstract. According to the analysis of concessive conditionals suggested by Crupi and Iacona, a concessive conditional $p \hookrightarrow q$ is adequately formalized as a conjunction of conditionals. This paper presents a sound and complete axiomatic system for concessive conditionals so understood. The soundness and completeness proofs that will be provided rely on a method that has been employed by Raidl, Iacona, and Crupi to prove the soundness and completeness of an analogous system for evidential conditionals.

Keywords: Concessive conditional, Conditional logic, Connexive logic, Suppositional conditional, Variably strict conditional, Evidential conditional, Definable conditional, Strengthened conditional, Weak Boethius Thesis, Even if.

1. Overview

Concessive conditionals, which are typically indicated by the expression 'even if', exhibit distinctive logical features that set them apart from ordinary indicative conditionals. Imagine that the sentences (1) and (2) below are used in a situation in which Glen intends to go out for a walk and hopes for a sunny day:

(1) If the weather is good, Glen will go out

(2) Even if the weather is not good, Glen will go out

In this case, (2) differs from (1) in at least three important respects. First, from (2) one can reasonably infer that Glen will go out, because (2) seems to imply that Glen will go out no matter whether the weather is good. By contrast, (1) does not convey a similar claim, for it leaves unspecified what Glen will do in case of bad weather. Second, (2) seems to imply that 'the weather is not good' does not support 'Glen will go out'. It would be inappropriate to paraphrase (2) by saying that if the weather is bad, that is a reason for thinking that Glen will go out, or that if the weather is bad, then as a consequence Glen will go out. By contrast, such a paraphrase is

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perfectly acceptable in the case of (1). Third, (2) seems to involve some sort of asymmetry between 'the weather is good' and 'the weather is not good', in that the connection between the former and 'Glen will go out' is more natural, or less surprising, than the connection between the latter and 'Glen will go out'.

Crupi and Iacona [3] have argued that the three differences just illustrated, together with other observations about concessive conditionals that are equally plausible, pose a serious explanatory challenge, as the extant theories of concessive conditionals do not provide a satisfactory explanation of all the relevant facts. According to their analysis, which is intended to provide such explanation, a concessive conditional is a sentence of the form $(p > q) \wedge (\neg p \rhd q)$, where $>$ stands for the suppositional conditional in the Stalnaker-Lewis interpretation, and $\rhd$ stands for the evidential conditional in their own interpretation.[1] If the symbol $\hookrightarrow$ is used to characterize the concessive conditional, this is to say that $p \hookrightarrow q$ is definable as $(p > q) \wedge (\neg p \rhd q)$.[2]

In order to provide a formal treatment of concessive conditionals so understood, we will adopt a semantics that specifies the meaning of $>$ and $\rhd$ in terms of comparative measures of distance between worlds. We will assume that $p > q$ is true if and only if q is true in the closest worlds in which p is true. This is the Ramsey Test as understood in the Stalnaker-Lewis interpretation. As for $\rhd$, it will be defined in accordance with the evidential account of conditionals developed by Crupi and Iacona, that is, $p \rhd q$ is true if and only if q is true in the closest worlds in which p is true and $\neg p$ is true in the closest worlds in which $\neg q$ is true. This stronger condition, called Chrysippus Test, ensures that a suitably defined relation of incompatibility holds between p and $\neg q$.[3]

Since the two conditions that constitute the Chrysippus Test are respectively the Ramsey Test for $p > q$ and the Ramsey Test for $\neg q > \neg p$, $p \rhd q$ is definable as $(p > q) \wedge (\neg q > \neg p)$. Therefore, the analysis of $p \hookrightarrow q$ suggested by Crupi and Iacona can equally be phrased as follows: $(p > q) \wedge (\neg p > q) \wedge (\neg q > p)$. Here we will use this equivalence in order to establish a syntactic relation between $\hookrightarrow$ and $>$ that will enable us to prove our soundness and completeness results. More precisely, we will use a method that has been

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[1] Stalnaker [12], Lewis [7], Crupi and Iacona [2].

[2] Crupi and Iacona [3].

[3] Crupi and Iacona [2] spells out the notion of incompatibility and explains its relation to the intuitive understanding of support.

first developed by Raidl in a general manner and then employed by Raidl, Iacona, and Crupi for a similar system for evidential conditionals.[4]

The structure of the paper is as follows. Section 2 introduces two languages, $\mathsf{L}_>$ and $\mathsf{L}_\hookrightarrow$, which differ only in that the former includes $>$ in addition to the usual sentential connectives while the latter includes $\hookrightarrow$. Section 3 defines two functions that guarantee the intertranslatability between $\mathsf{L}_>$ and $\mathsf{L}_\hookrightarrow$. Section 4 presents the well known system VC in $\mathsf{L}_>$, and some derivable principles. Section 5 presents an axiom system in $\mathsf{L}_\hookrightarrow$ that we call CC. Section 6 draws attention to some principles that are derivable in CC. Finally, Sections 7 and 8 prove the soundness and completeness of CC by relying on the soundness and completeness of VC.

2. The Languages $\mathsf{L}_>$ and $\mathsf{L}_\hookrightarrow$

Let $\mathsf{L}_>$ be a language whose alphabet is constituted by a set of sentence letters $p, q, r, \dots$, the connectives $\neg, \supset, \wedge, \vee, >$, and the brackets $($, $)$. The formation rules of $\mathsf{L}_>$ are as follows: the sentence letters are formulas; if α is a formula, then $\neg\alpha$ is a formula; if α and β are formulas, then $(\alpha \supset \beta), (\alpha \wedge \beta), (\alpha \vee \beta), (\alpha > \beta)$ are formulas.

Let $\mathsf{L}_\hookrightarrow$ be a language whose alphabet is constituted by the same sentence letters $p, q, r, \dots$, the connectives $\neg, \supset, \wedge, \vee, \hookrightarrow$, and the brackets $($, $)$. The formation rules of $\mathsf{L}_\hookrightarrow$ are as follows: the sentence letters are formulas; if α is a formula, then $\neg\alpha$ is a formula; if α and β are formulas, then $(\alpha \supset \beta), (\alpha \wedge \beta), (\alpha \vee \beta), (\alpha \hookrightarrow \beta)$ are formulas. Basically, $\mathsf{L}_\hookrightarrow$ differs from $\mathsf{L}_>$ only in that its alphabet includes $\hookrightarrow$ instead of $>$. Their shared fragment, call it L, is a classical propositional language. We adopt the convention of not writing outer brackets in formulas. We will use the notation $\alpha \equiv \beta$ to abbreviate $(\alpha \supset \beta) \wedge (\beta \supset \alpha)$, and we will use $\top$ to refer to any classical propositional tautology, and $\bot$ to refer to any classical propositional antilogy.

The semantics of $\mathsf{L}_>$ and $\mathsf{L}_\hookrightarrow$ will be given in terms of systems of spheres, along the lines suggested by Lewis.[5]

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[4] For the method, see Raidl [9], simplified in Raidl [8]. For the logic of the evidential conditional, see Raidl, Iacona, Crupi [11] and the weaker system in Raidl [10]. In this work we follow the original formulation of the Chrysippus Test provided in Crupi and Iacona [2], also adopted in Crupi and Iacona [3]. A refined version of the test, which is largely compatible with the analysis of concessive conditionals provided in the latter work, appears in Crupi and Iacona [4].

[5] Lewis [7], pp. 14–15, 120–121.

Definition 1. Given a non-empty set W, a *system of spheres* O over W is an assignment to each $w \in W$ of a set O_w of non-empty sets of elements of W—a set of spheres around w—such that:

1. if $S \in O_w$ and $S' \in O_w$, then $S \subseteq S'$ or $S' \subseteq S$;
2. $\{w\} \in O_w$;

Clause 1 says that O_w is nested: any two spheres in O_w are such that one of them includes the other. Clause 2 implies that O_w is centered on w. Since $\{w\} \in O_w$, because we assume spheres to be non-empty, by clause 1 we have that, for every $S \in O_w$, $\{w\} \subseteq S$.

Definition 2. A *model* for $\mathsf{L}_>$ and $\mathsf{L}_\hookrightarrow$ is an ordered triple $\langle W, O, V\rangle$, where W is a non-empty set, O is a system of spheres over W, and V is a valuation function such that, for each sentence letter α and each $w \in W$, $V(\alpha, w) \in \{1, 0\}$.

$\mathsf{L}_>$ and $\mathsf{L}_\hookrightarrow$ have the same models, so they are exactly alike in this respect. The part of the semantics in which they differ is the definition of truth of a formula in a world. Let us start with $\mathsf{L}_>$. The truth of a formula α of $\mathsf{L}_>$ in a world w, which we will indicate as $w \vDash_> \alpha$, is defined as follows, where $[\alpha]_>$ is the set of worlds in which α is true according to $\vDash_>$:

Definition 3.

1. $w \vDash_> \alpha$ iff $V(\alpha, w) = 1$, for any sentence letter α;
2. $w \vDash_> \neg\alpha$ iff $w \nvDash_> \alpha$;
3. $w \vDash_> \alpha \wedge \beta$ iff $w \vDash_> \alpha$ and $\vDash_> \beta$;
4. $w \vDash_> \alpha \vee \beta$ iff $w \vDash_> \alpha$ or $w \vDash_> \beta$;
5. $w \vDash_> \alpha \supset \beta$ iff $w \nvDash_> \alpha$ or $w \vDash_> \beta$;
6. $w \vDash_> \alpha > \beta$ iff $\bigcup O_w \cap [\alpha]_> = \emptyset$ or there is $S \in O_w$ such that $\emptyset \neq S \cap [\alpha]_>$ and $S \cap [\alpha]_> \subseteq [\beta]_>$.

Note that, given clauses 2 and 6, the necessity operator $\Box$ is definable in $\mathsf{L}_>$ as follows: $\Box\,\alpha = \neg\alpha > \alpha$. To see why, assume that $\bigcup O_w \cap [\neg\alpha]_> = \emptyset$, which is the truth condition for $\Box\,\alpha$ relative to w. Then, $w \vDash_> \neg\alpha > \alpha$ by the first disjunct of clause 6. Conversely, assume that $w \vDash_> \neg\alpha > \alpha$. Then the first disjunct of clause 6 must hold, that is, $\bigcup O_w \cap [\neg\alpha]_> = \emptyset$, given that the second cannot hold by clause 2. So, from now on we will take for granted that $\Box\,\alpha$ abbreviates $\neg\alpha > \alpha$, and we will write $\Diamond$ for the dual $\neg\,\Box\,\neg$.

Now let us consider $\mathsf{L}_{\hookrightarrow}$. The truth of a formula α of $\mathsf{L}_{\hookrightarrow}$ in a world w, which we will indicate as $w \vDash_{\hookrightarrow} \alpha$, is defined as follows, where $[\alpha]_{\hookrightarrow}$ is the set of worlds in which α is true according to $\vDash_{\hookrightarrow}$:

Definition 4.

1. $w \vDash_{\hookrightarrow} \alpha$ iff $V(\alpha, w) = 1$ for any sentence letter α;
2. $w \vDash_{\hookrightarrow} \neg\alpha$ iff $w \nvDash_{\hookrightarrow} \alpha$;
3. $w \vDash_{\hookrightarrow} \alpha \wedge \beta$ iff $w \vDash_{\hookrightarrow} \alpha$ and $\vDash_{\hookrightarrow} \beta$;
4. $w \vDash_{\hookrightarrow} \alpha \vee \beta$ iff $w \vDash_{\hookrightarrow} \alpha$ or $w \vDash_{\hookrightarrow} \beta$;
5. $w \vDash_{\hookrightarrow} \alpha \supset \beta$ iff $w \nvDash_{\hookrightarrow} \alpha$ or $w \vDash_{\hookrightarrow} \beta$;
6. $w \vDash_{\hookrightarrow} \alpha \hookrightarrow \beta$ iff the following conditions are satisfied
   (a) $\bigcup O_w \cap [\alpha]_{\hookrightarrow} = \emptyset$ or there is $S \in O_w$ such that $\emptyset \neq S \cap [\alpha]_{\hookrightarrow}$ and $S \cap [\alpha]_{\hookrightarrow} \subseteq [\beta]_{\hookrightarrow}$;
   (b) $\bigcup O_w \cap [\neg\alpha]_{\hookrightarrow} = \emptyset$ or there is $S \in O_w$ such that $\emptyset \neq S \cap [\neg\alpha]_{\hookrightarrow}$ and $S \cap [\neg\alpha]_{\hookrightarrow} \subseteq [\beta]_{\hookrightarrow}$;
   (c) $\bigcup O_w \cap [\neg\beta]_{\hookrightarrow} = \emptyset$ or there is $S \in O_w$ such that $\emptyset \neq S \cap [\neg\beta]_{\hookrightarrow}$ and $S \cap [\neg\beta]_{\hookrightarrow} \subseteq [\alpha]_{\hookrightarrow}$.

Clauses 1–5 of Definition 4 are exactly like clauses 1–5 of Definition 3. This means that, as far as L is concerned, Definitions 3 and 4 yield the same results:

Fact 5. For every model, every world w, and every formula χ of L, $w \vDash_{>} \chi$ iff $w \vDash_{\hookrightarrow} \chi$.

Proof. The proof is by induction on the complexity of χ. ∎

The key difference between Definitions 3 and 4 lies in clause 6. While clause 6 of Definition 3 requires that the closest worlds in which α is true are worlds in which β is true, clause 6 of Definition 4 adds to this condition—expressed by (a)—two further conditions, (b) and (c). (b) requires that the closest worlds in which $\neg\alpha$ is true are worlds in which β is true, and (c) requires that the closest worlds in which $\neg\beta$ is true are worlds in which α is true. In other words, while clause 6 of Definition 3 expresses the Ramsey Test for $\alpha > \beta$, clause 6 of Definition 4 expresses the Ramsey test for $\alpha > \beta$ augmented by the Chrysippus Test for $\neg\alpha \rhd \beta$.

As in the case of $\mathsf{L}_{>}$, the necessity operator $\Box$ is definable as follows: $\Box\alpha = \neg\alpha \hookrightarrow \alpha$. To see why, assume that $\bigcup O_w \cap [\neg\alpha]_{\hookrightarrow} = \emptyset$. Then $w \vDash_{\hookrightarrow} \neg\alpha \hookrightarrow \alpha$ because conditions (a)–(c) of clause 6 of Definition 4 are all satisfied. The case of (a) and (c) is obvious. (b) is satisfied for the following reason:

since $\bigcup O_w \cap [\neg\alpha]_{\hookrightarrow} = \emptyset$, we have that $\bigcup O_w \subseteq [\alpha]_{\hookrightarrow}$. But since $\{w\} \in O_w$, and hence $w \in [\neg\neg\alpha]_{\hookrightarrow}$, there is a sphere S in O_w such that $\emptyset \neq S \cap [\neg\neg\alpha]_{\hookrightarrow}$ and $S \cap [\neg\neg\alpha]_{\hookrightarrow} \subseteq [\alpha]_{\hookrightarrow}$. Conversely, assume that $w \vDash_{\hookrightarrow} \neg\alpha \hookrightarrow \alpha$. Then the first disjunct of (a) must hold, that is, $\bigcup O_w \cap [\neg\alpha]_{\hookrightarrow} = \emptyset$, given that the second disjunct cannot hold by clause 2. So, from now on we will take for granted that, in $\mathsf{L}_{\hookrightarrow}$, $\Box\alpha$ abbreviates $\neg\alpha \hookrightarrow \alpha$, and we will again write $\Diamond$ for the dual $\neg\Box\neg$.

The notation that will be used for validity is the same for both languages. $\vDash_{>} \alpha$ means that α is true in every world in every model according to Definition 3. Similarly, $\vDash_{\hookrightarrow} \alpha$ means that α is true in every world in every model according to Definition 4.

3. Translation and Backtranslation

The link between $\hookrightarrow$ and $>$ can be stated in precise terms by defining a *translation function* $\circ$ that goes from $\mathsf{L}_{\hookrightarrow}$ to $\mathsf{L}_{>}$ and a *backtranslation function* $\bullet$ that goes from $\mathsf{L}_{>}$ to $\mathsf{L}_{\hookrightarrow}$.

Definition 6. Let $\circ$ be the function from $\mathsf{L}_{\hookrightarrow}$ to $\mathsf{L}_{>}$ such that:

1. $\alpha^\circ = \alpha$ if α is a sentence letter;
2. $(\neg\alpha)^\circ = \neg\alpha^\circ$;
3. $(\alpha \wedge \beta)^\circ = \alpha^\circ \wedge \beta^\circ$;
4. $(\alpha \vee \beta)^\circ = \alpha^\circ \vee \beta^\circ$;
5. $(\alpha \supset \beta)^\circ = \alpha^\circ \supset \beta^\circ$;
6. $(\alpha \hookrightarrow \beta)^\circ = (\alpha^\circ > \beta^\circ) \wedge (\neg\alpha^\circ > \beta^\circ) \wedge (\neg\beta^\circ > \alpha^\circ)$.

Clauses 1–5 entail that, whenever a formula α belongs to L, $\alpha^\circ = \alpha$. Clause 6 says that, for every formula of the form $\alpha \hookrightarrow \beta$, there is a formula of $\mathsf{L}_{>}$ that translates it, namely, $(\alpha^\circ > \beta^\circ) \wedge (\neg\alpha^\circ > \beta^\circ) \wedge (\neg\beta^\circ > \alpha^\circ)$.

The function $\circ$ is well-behaved in the following sense:

Fact 7. For every model, every world w, and every formula χ of $\mathsf{L}_{\hookrightarrow}$, $w \vDash_{\hookrightarrow} \chi$ iff $w \vDash_{>} \chi^\circ$.

Proof. The proof is by induction on the complexity of χ.
Basis. Consider the case in which χ is a sentence letter. In this case $\chi^\circ = \chi$. Then, by Fact 5, $w \vDash_{\hookrightarrow} \chi$ iff $w \vDash_{>} \chi^\circ$.
Step. Assume that the equivalence holds for any formula of complexity less than or equal to n, and that χ is a formula of complexity $n + 1$. Then five

cases are to be considered, depending on whether the main connective of χ is $\neg, \wedge, \vee, \supset$, or $\hookrightarrow$. In the first four cases, given the induction hypothesis, $w \vDash_{\hookrightarrow} \chi$ iff $w \vDash_{>} \chi^{\circ}$ because clauses 2–5 of Definition 3 are exactly like clauses 2–5 of Definition 4. So, the only case left is that in which χ has the form $\alpha \hookrightarrow \beta$. In this case, $\chi^{\circ} = (\alpha^{\circ} > \beta^{\circ}) \wedge (\neg\alpha^{\circ} > \beta^{\circ}) \wedge (\neg\beta^{\circ} > \alpha^{\circ})$. Assume that $w \vDash_{\hookrightarrow} \chi$. This means that conditions (a)–(c) of clause 6 of Definition 4 hold for α and β. By the induction hypothesis, the same conditions hold for α° and β°, with $]_{>}$ replacing $]_{\hookrightarrow}$. So, $w \vDash_{>} \alpha^{\circ} > \beta^{\circ}$ and $w \vDash_{>} \neg\alpha^{\circ} > \beta^{\circ}$ and $w \vDash_{>} \neg\beta^{\circ} > \alpha^{\circ}$, which means that $w \vDash_{>} \chi^{\circ}$. A similar reasoning in the opposite direction shows that if $w \vDash_{>} \chi^{\circ}$, then $w \vDash_{\hookrightarrow} \chi$. ■

Definition 8. Let $\bullet$ be the function from $\mathsf{L}_{>}$ to $\mathsf{L}_{\hookrightarrow}$ such that:

1. $\alpha^{\bullet} = \alpha$ if α is a sentence letter;
2. $(\neg\alpha)^{\bullet} = \neg\alpha^{\bullet}$;
3. $(\alpha \wedge \beta)^{\bullet} = \alpha^{\bullet} \wedge \beta^{\bullet}$;
4. $(\alpha \vee \beta)^{\bullet} = \alpha^{\bullet} \vee \beta^{\bullet}$;
5. $(\alpha \supset \beta)^{\bullet} = \alpha^{\bullet} \supset \beta^{\bullet}$;
6. $(\alpha > \beta)^{\bullet} = \alpha^{\bullet} \hookrightarrow (\neg\alpha^{\bullet} \vee \beta^{\bullet})$.

Clauses 1–5 entail that, whenever a formula α belongs to L, $\alpha^{\bullet} = \alpha$. Clause 6 says that, for every formula of the form $\alpha > \beta$, there is a formula that provides the translation of $\alpha > \beta$ into $\mathsf{L}_{\hookrightarrow}$, namely, $\alpha^{\bullet} \hookrightarrow (\neg\alpha^{\bullet} \vee \beta^{\bullet})$. To grasp the meaning of this clause it suffices to think that, assuming that $\alpha^{\bullet}$ and $\beta^{\bullet}$ belong to L, and thus simplify to α and β, its right-hand side can be rewritten in $\mathsf{L}_{>}$ as $(\alpha > (\neg\alpha \vee \beta)) \wedge (\neg\alpha > (\neg\alpha \vee \beta)) \wedge (\neg(\neg\alpha \vee \beta) > \alpha)$, which in turn is equivalent to $\alpha > \beta$ given Definition 3. The function $\bullet$ is well-behaved exactly in the same sense in which $\circ$ is well-behaved, although we will not prove this fact here, given that it is not necessary for our purposes.

4. The System VC

As is well known, given a language that includes $>$ in addition to the sentential connectives $\neg, \supset, \wedge, \vee$, Lewis' system VC is sound and complete with respect to his centered sphere semantics.[6] The axioms of VC are all the formulas obtained by substitution from propositional tautologies (PT) and all the formulas that instantiate the following schemas:[7]

$\alpha > \alpha$ ID

$((\alpha > \beta) \wedge (\alpha > \gamma)) \supset (\alpha > (\beta \wedge \gamma))$ AND

$((\alpha > \gamma) \wedge (\beta > \gamma)) \supset ((\alpha \vee \beta) > \gamma)$ OR

$((\alpha > \gamma) \wedge (\alpha > \beta)) \supset ((\alpha \wedge \beta) > \gamma)$ CM

$((\alpha > \gamma) \wedge \neg(\alpha > \neg\beta)) \supset ((\alpha \wedge \beta) > \gamma)$ RM

$(\alpha > \beta) \supset (\alpha \supset \beta)$ MI

$(\alpha \wedge \beta) \supset (\alpha > \beta)$ CS

The principles expressed by these schemas are respectively Identity, Conjunction of Consequents, Disjunction of Antecedents, Cautious Monotonicity, Rational Monotonicity, Material Implication, and Conjunctive Sufficiency.

The rules of inference of VC are MP — the standard Modus Ponens for $\supset$ — plus Left Logical Equivalence and Right Weakening:

$$\frac{\vdash_{\mathsf{VC}} \alpha \equiv \beta}{\vdash_{\mathsf{VC}} (\alpha > \gamma) \supset (\beta > \gamma)} \qquad \text{LLE}$$

$$\frac{\vdash_{\mathsf{VC}} \alpha \supset \beta}{\vdash_{\mathsf{VC}} (\gamma > \alpha) \supset (\gamma > \beta)} \qquad \text{RW}$$

Given that VC is sound and complete with respect to Lewis' centered sphere semantics, and that $\hookrightarrow$ and $>$ are related in the way explained, it is provable that there is a system for $\hookrightarrow$ which is sound and complete with respect to a similar semantics. Of course, this system will not have exactly the same properties as VC, given that it rests on a different account of conditionals. But the principles it displays will be reducible to principles that hold in VC, modulo the relation between $\hookrightarrow$ and $>$.

In the rest of this section we will briefly recall some useful facts about VC. First of all, the following principles are derivable in VC.

Fact 9. If $\vdash_{\mathsf{VC}} \alpha \supset \beta$ then $\vdash_{\mathsf{VC}} \alpha > \beta$ SC

Proof. We have $\alpha > \alpha$ by ID. Since $\vdash_{\mathsf{VC}} \alpha \supset \beta$, we can conclude $\alpha > \beta$ by RW. ■

[6]Compare Lewis [6].

[7]The form of the axioms and rules comes from the conditional logic of Chellas [1], most of the labelling from the non-monotonic reasoning tradition, see Kraus, Lehmann, and Magidor [5].

Fact 10. If $\vdash_{\mathsf{VC}} \alpha \equiv \beta$ then $\vdash_{\mathsf{VC}} (\gamma > \alpha) \supset (\gamma > \beta)$ RLE

Proof. Directly from RW. ∎

Fact 11. $\vdash_{\mathsf{VC}} \alpha > \top$ CN

Proof. We have $\alpha > \alpha$ by ID. Since $\vdash_{\mathsf{VC}} \alpha \supset \top$ by PT, we obtain $\alpha > \top$ by RW. ∎

Fact 12. $\vdash_{\mathsf{VC}} \Box \top$ N

Proof. By CN we have $\bot > \top$. That is $\neg\top > \top$. Hence $\Box \top$. ∎

Fact 13. $\vdash_{\mathsf{VC}} \alpha \supset (\top > \alpha)$ TI

Proof. Directly from CS. ∎

The principles expressed by the first two facts are Supraclassicality and Right Logical Equivalence. CN says that conditionals with tautological consequents always hold. N says that tautologies are necessary. TI says that truth implies the inner modality $\boxdot$, where $\boxdot\alpha = (\top > \alpha)$. Note in fact that in VC, the inner modality collapses with truth, due to MP and CS. The outer modality $\Box$, instead, is definable in the way explained in section 2, that is, $\Box\,\alpha = (\neg\alpha > \alpha)$.

The next principle, Necessary Consequent, is phrased in terms of the latter definition, as it says that conditionals with necessary consequents always hold.

Fact 14. $\vdash_{\mathsf{VC}} \Box\,\beta \supset (\alpha > \beta)$ NC

Proof. Assume $\Box\,\beta$, that is, $\neg\beta > \beta$. Since $\neg\beta > \neg\beta$ by ID, hence $\neg\beta > (\beta \wedge \neg\beta)$ by AND, we get that $\neg\beta > \bot$. Therefore, $\neg\beta > \alpha$ by RW. Together with $\neg\beta > \beta$, this yields $(\alpha \wedge \neg\beta) > \beta$ by CM. But we also have $(\alpha \wedge \beta) > \beta$ by SC (Fact 9). Combining this with $(\alpha \wedge \neg\beta) > \beta$ yields $\alpha > \beta$ by OR and LLE. ∎

Let us close with the following principle, which may be called Modularity:

Fact 15. $\vdash_{\mathsf{VC}} ((\alpha \vee \beta) > \neg\beta) \supset (((\alpha \vee \gamma) > \neg\gamma) \vee ((\gamma \vee \beta) > \neg\beta))$ Mod

Proof. Assume $(\alpha \vee \beta) > \neg\beta$. Note that $(\gamma \wedge \neg\beta) > \neg\beta$ by SC (Fact 9). Thus $(\alpha \vee \beta \vee (\gamma \wedge \neg\beta)) > \neg\beta$ by OR. Since the antecedent is equivalent to $\alpha \vee \beta \vee \gamma$, we obtain $(\alpha \vee \beta \vee \gamma) > \neg\beta$ by LLE. Moreover, either $(\alpha \vee \beta \vee \gamma) > \neg\gamma$ or $\neg((\alpha \vee \beta \vee \gamma) > \neg\gamma)$. Consider the first case. By ID we have $(\alpha \vee \beta \vee \gamma) > (\alpha \vee \beta \vee \gamma)$. Together with the previously established $(\alpha \vee \beta \vee \gamma) > \neg\beta$, we obtain $(\alpha \vee \beta \vee \gamma) > ((\alpha \vee \beta \vee \gamma) \wedge \neg\beta)$ by AND. Hence $(\alpha \vee \beta \vee \gamma) > (\alpha \vee \gamma)$

by RW. From this and the first case assumption, we obtain $(\alpha \vee \gamma) > \neg\gamma$ by CM. Consider the second case. From $\neg((\alpha \vee \beta \vee \gamma) > \neg\gamma)$ we obtain $\neg((\alpha \vee \beta \vee \gamma) > (\neg\gamma \wedge \neg\beta))$ by contraposing RW, and hence $\neg((\alpha \vee \beta \vee \gamma) > \neg(\gamma \vee \beta))$, contraposing RW again. Together with the previously established $(\alpha \vee \beta \vee \gamma) > \neg\beta$, we obtain $(\gamma \vee \beta) > \neg\beta$ by RM and LLE. ■

To understand the name of this principle it suffices to think that, according to Definition 3, $(\alpha \vee \beta) > \neg\beta$ is true when α precedes β in the sense that the closest worlds in which α is true come before the closest worlds in which β is true (unless α is impossible, in which case $(\alpha \vee \beta) > \neg\beta$ says that β is impossible as well). So the principle says that if α precedes β, then either α precedes γ or γ precedes β. This is precisely the order property called Modularity.

5. The System CC

Now we will present an axiom system in $\mathsf{L}_{\hookrightarrow}$ called CC, which stands for 'concessive conditional'. The axioms of CC are all the formulas obtained by substitution from propositional tautologies (PT) and all the formulas that instantiate the following schemas:

$\Box\top$	N
$((\alpha \hookrightarrow \beta) \wedge (\alpha \hookrightarrow \gamma)) \supset (\alpha \hookrightarrow (\beta \wedge \gamma))$	AND
$(\alpha \hookrightarrow (\neg\alpha \vee \beta)) \supset (\alpha \hookrightarrow (\neg\alpha \vee \beta \vee \gamma))$	WDW
$\alpha \supset (\top \hookrightarrow \alpha)$	TI
$\Box\beta \supset (\alpha \hookrightarrow \beta)$	NC
$((\alpha \vee \beta) \hookrightarrow \neg\beta) \supset (((\alpha \vee \gamma) \hookrightarrow \neg\gamma) \vee ((\gamma \vee \beta) \hookrightarrow \neg\beta))$	Mod
$(\alpha \hookrightarrow \beta) \supset \beta$	CT
$((\alpha \hookrightarrow (\neg\alpha \vee \beta)) \wedge (\neg\alpha \hookrightarrow (\alpha \vee \beta)) \wedge (\neg\beta \hookrightarrow (\beta \vee \alpha))) \equiv (\alpha \hookrightarrow \beta)$	CSS

AND is exactly as in VC, and N, WDW, TI, NC, and Mod are derivable in VC, as it turns out from Facts 12–15. Some of these axioms can be seen as weak replacements of principles from VC. For example, WDW is a weak replacement of RW. WDW is derivable from RW, but the reverse is not true.[8] Similarly TI is a weak replacement of CS, in that it is derivable from CS, but for the converse we need CM. This is why CS is not derivable in CC. Other principles are obtained by backtranslating principles from VC. Mod is a

[8] WDW stands for Weak Disjunctive Weakening, where Disjunctive Weakening (DW) refers to the principle $(\alpha > \beta) \supset (\alpha > (\beta \vee \gamma))$. RW is equivalent to joining DW and RLE—see Observation 1 in Raidl [10].

backtranslation of Mod.[9] The axioms presented so far express properties that $\hookrightarrow$ shares with $>$. The remaining two axioms CT and CSS, instead, express principles that do not hold in VC.[10] CT expresses the intuition illustrated in Section 1 that a concessive conditional implies its consequent. This axiom replaces MI, but it is stronger than MI: if β holds, then $\alpha \supset \beta$ holds, but not the other way round. This, among other things, makes MI derivable in CC, while CT is not derivable in VC. CSS has no obvious intuitive meaning. It is required for technical purposes and encodes the semantic definition of the concessive conditional. Indeed, if we rewrite CSS by replacing first α and β by $\alpha^\bullet$ and $\beta^\bullet$ and then applying the backtranslation to the left-hand side, we get the definition of $\hookrightarrow$ in terms of $>$.[11]

The rules of inference of CC are MP and the following:

$$\frac{\vdash_{\mathsf{CC}} \alpha \equiv \beta}{\vdash_{\mathsf{CC}} (\alpha \hookrightarrow \gamma) \supset (\beta \hookrightarrow \gamma)} \qquad \text{LLE}$$

$$\frac{\vdash_{\mathsf{CC}} \alpha \equiv \beta}{\vdash_{\mathsf{CC}} (\gamma \hookrightarrow \alpha) \supset (\gamma \hookrightarrow \beta)} \qquad \text{RLE}$$

LLE is as in VC. RLE is weaker than RW, in that it requires that α and β are provably equivalent.

6. Derivable Principles

This section draws attention to five principles that are derivable in CC, and that will prove useful in the following sections. The first principle (CN) also holds in VC (Fact 11). The second, the third, and the last of these principles are respectively backtranslations of CM, RM, and OR (which we denote by CM*, RM*, OR*). To these five principles, we add three connexive principles—Restricted Aristotle's Thesis (RAT), weak Boethius Thesis (wBT), and Restricted Aristotle's Second Thesis (RAT2)—which we also show to be derivable. At the end of the section, we provide a Table 1 with a list of further (in)valid principles for the concessive $\hookrightarrow$ in comparison to the suppositional conditional $>$.

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[9] WDW is a backtranslation of DW.

[10] Parts of CSS are however valid in semantics for VC. The valid part of CSS is the left to right implication, and the first reverse implication $(\alpha > \beta) \supset (\alpha > (\neg\alpha \vee \beta))$. The two other reverse implications are invalid. That is, the following principle survives in VC: $(\alpha > (\neg\alpha \vee \beta)) \equiv (\alpha > \beta)$.

[11] According to the terminology adopted in Raidl [9], CSS is the proper axiom of CC.

Fact 16. $\vdash_{\mathsf{CC}} \alpha \hookrightarrow \top$

Proof. We have $\Box\top$ by N. Hence $\alpha \hookrightarrow \top$ by NC. ■

In CC we could replace N by CN. This principle backtranslates ID. Note that Definition 4 does not validate ID, for conditions (b) and (c) of clause 6 are not satisfied whenever α is not necessary. This is quite plausible, for it would make little sense to take sentences such as 'Even if the weather is not good, the weather is not good' to be valid.

Fact 17. $\vdash_{\mathsf{CC}} ((\alpha \hookrightarrow (\neg\alpha \vee \gamma)) \wedge (\alpha \hookrightarrow (\neg\alpha \vee \beta))) \supset ((\alpha \wedge \beta) \hookrightarrow (\neg(\alpha \wedge \beta) \vee \gamma))$

Proof. Assume $\alpha \hookrightarrow (\neg\alpha \vee \gamma)$ and $\alpha \hookrightarrow (\neg\alpha \vee \beta)$. Either $\neg((\alpha \vee (\alpha \wedge \beta \wedge \gamma)) \hookrightarrow \neg(\alpha \wedge \beta \wedge \gamma))$ or $((\alpha \vee (\alpha \wedge \beta \wedge \gamma)) \hookrightarrow \neg(\alpha \wedge \beta \wedge \gamma))$. Consider the first case. From $\alpha \hookrightarrow (\neg\alpha \vee \gamma)$ we obtain $(\alpha \vee (\alpha \wedge \beta \wedge \neg\gamma)) \hookrightarrow (\neg\alpha \vee \gamma \vee \neg\beta)$ by LLE and WDW, which entails $(\alpha \vee (\alpha \wedge \beta \wedge \neg\gamma)) \hookrightarrow \neg(\alpha \wedge \beta \wedge \neg\gamma)$ by RLE. This together with our $\neg((\alpha \vee (\alpha \wedge \beta \wedge \gamma)) \hookrightarrow \neg(\alpha \wedge \beta \wedge \gamma))$ establishes $((\alpha \wedge \beta \wedge \gamma) \vee (\alpha \wedge \beta \wedge \neg\gamma)) \hookrightarrow \neg(\alpha \wedge \beta \wedge \neg\gamma)$ by Mod. Thus $(\alpha \wedge \beta) \hookrightarrow (\neg(\alpha \wedge \beta) \vee \gamma)$ by LLE and RLE. Consider the second case. From $(\alpha \vee (\alpha \wedge \beta \wedge \gamma)) \hookrightarrow \neg(\alpha \wedge \beta \wedge \gamma)$ we get $\alpha \hookrightarrow (\neg\alpha \vee \neg\beta \vee \neg\gamma)$ by LLE and RLE. Since we assumed $\alpha \hookrightarrow (\neg\alpha \vee \beta)$, we obtain $\alpha \hookrightarrow ((\neg\alpha \vee \neg\beta \vee \neg\gamma) \wedge (\neg\alpha \vee \beta))$ by AND. Thus $\alpha \hookrightarrow (\neg\alpha \vee (\neg\gamma \wedge \beta))$ by RLE. Therefore $\alpha \hookrightarrow (\neg\alpha \vee \neg\gamma)$ by WDW and RLE. Since we assumed $\alpha \hookrightarrow (\neg\alpha \vee \gamma)$, we obtain $\alpha \hookrightarrow \neg\alpha$ by AND and RLE. Hence $\neg\neg\alpha \hookrightarrow \neg\alpha$ by LLE. That is $\Box\neg\alpha$. Thus $(\alpha \wedge \beta) \hookrightarrow \neg\alpha$ by NC. Therefore $(\alpha \wedge \beta) \hookrightarrow (\neg(\alpha \wedge \beta) \vee \neg\alpha)$ by CSS, and hence $(\alpha \wedge \beta) \hookrightarrow (\neg(\alpha \wedge \beta) \vee \neg\alpha \vee \gamma)$ by WDW. Thus $(\alpha \wedge \beta) \hookrightarrow (\neg(\alpha \wedge \beta) \vee \gamma)$ by RLE. ■

Fact 18. $\vdash_{\mathsf{CC}} ((\alpha \hookrightarrow (\neg\alpha \vee \gamma)) \wedge \neg(\alpha \hookrightarrow (\neg\alpha \vee \neg\beta))) \supset ((\alpha \wedge \beta) \hookrightarrow (\neg(\alpha \wedge \beta) \vee \gamma))$

Proof. Assume $\alpha \hookrightarrow (\neg\alpha \vee \gamma)$ and $\neg(\alpha \hookrightarrow (\neg\alpha \vee \neg\beta))$. The first assumption entails $(\alpha \vee (\alpha \wedge \beta \wedge \neg\gamma)) \hookrightarrow \neg(\alpha \wedge \beta \wedge \neg\gamma)$, as in the proof of Fact 17. From the second assumption we obtain $\neg((\alpha \vee (\alpha \wedge \beta \wedge \gamma)) \hookrightarrow \neg(\alpha \wedge \beta \wedge \gamma))$ as follows. Suppose for reductio that $(\alpha \vee (\alpha \wedge \beta \wedge \gamma)) \hookrightarrow \neg(\alpha \wedge \beta \wedge \gamma)$. Then $\alpha \hookrightarrow (\neg\alpha \vee \neg\beta \vee \neg\gamma)$ by LLE and RLE. Since we have $\alpha \hookrightarrow (\neg\alpha \vee \gamma)$, we obtain $\alpha \hookrightarrow (\neg\alpha \vee \neg\beta)$ by AND and RLE, which contradicts the assumption. Thus we may conclude that $(\alpha \wedge \beta) \hookrightarrow (\neg(\alpha \wedge \beta) \vee \gamma)$ by the same reasoning adopted in the first case of the proof of Fact 17. ■

Fact 19. $\vdash_{\mathsf{CC}} \Box\neg\beta \supset ((\alpha \vee \beta) \hookrightarrow (\neg(\alpha \vee \beta) \vee \alpha))$

Proof. Assume $\Box\neg\beta$. Then $(\alpha \vee \beta) \hookrightarrow \neg\beta$ by NC. Thus $(\alpha \vee \beta) \hookrightarrow (\neg(\alpha \vee \beta) \vee (\alpha \wedge \neg\beta))$ by RLE, hence $(\alpha \vee \beta) \hookrightarrow (\neg(\alpha \vee \beta) \vee \alpha)$ by WDW and RLE. ■

FACT 20. $\vdash_{\mathsf{CC}} \neg\Box\neg(\alpha \vee \beta) \supset (\neg((\alpha \vee \beta) \hookrightarrow \neg\alpha) \vee \neg((\alpha \vee \beta) \hookrightarrow \neg\beta))$

PROOF. This is proved by contraposition. Assume the negation of the consequent, that is, $\neg(\neg((\alpha \vee \beta) \hookrightarrow \neg\alpha) \vee \neg((\alpha \vee \beta) \hookrightarrow \neg\beta))$. Then, $(\alpha \vee \beta) \hookrightarrow \neg\alpha$ and $(\alpha \vee \beta) \hookrightarrow \neg\beta$. By AND and RLE it follows that $(\alpha \vee \beta) \hookrightarrow \neg(\alpha \vee \beta)$. Therefore $\neg\neg(\alpha \vee \beta) \hookrightarrow \neg(\alpha \vee \beta)$ by LLE. That is $\Box\neg(\alpha \vee \beta)$. Thus $\neg\neg\Box\neg(\alpha \vee \beta)$. ■

FACT 21. $\vdash_{\mathsf{CC}} ((\alpha \hookrightarrow (\neg\alpha \vee \gamma)) \wedge (\beta \hookrightarrow (\neg\beta \vee \gamma))) \supset ((\alpha \vee \beta) \hookrightarrow (\neg(\alpha \vee \beta) \vee \gamma))$

PROOF. Assume $\alpha \hookrightarrow (\neg\alpha \vee \gamma)$ and $\beta \hookrightarrow (\neg\beta \vee \gamma)$. By LLE and RLE, the first assumption entails $((\alpha \wedge \gamma) \vee (\alpha \wedge \neg\gamma)) \hookrightarrow \neg(\alpha \wedge \neg\gamma)$, and the second entails $((\beta \wedge \gamma) \vee (\beta \wedge \neg\gamma)) \hookrightarrow \neg(\beta \wedge \neg\gamma)$. Either $\Box\neg((\alpha \wedge \neg\gamma) \vee (\beta \wedge \neg\gamma))$ or $\neg\Box\neg((\alpha \wedge \neg\gamma) \vee (\beta \wedge \neg\gamma))$, so there are two cases.

Case 1. Assume $\Box\neg((\alpha \wedge \neg\gamma) \vee (\beta \wedge \neg\gamma))$. Then, $((\alpha \wedge \gamma) \vee (\beta \wedge \gamma) \vee (\alpha \wedge \neg\gamma) \vee (\beta \wedge \neg\gamma)) \hookrightarrow (\neg((\alpha \wedge \gamma) \vee (\beta \wedge \gamma) \vee (\alpha \wedge \neg\gamma) \vee (\beta \wedge \neg\gamma)) \vee (\alpha \wedge \gamma) \vee (\beta \wedge \gamma))$ by Fact 19. Using LLE and RLE, this simplifies to $(\alpha \vee \beta) \hookrightarrow (\neg(\alpha \vee \beta) \vee (\alpha \wedge \gamma) \vee (\beta \wedge \gamma))$ and thus to $(\alpha \vee \beta) \hookrightarrow (\neg(\alpha \vee \beta) \vee ((\alpha \vee \beta) \wedge \gamma))$. Therefore, $(\alpha \vee \beta) \hookrightarrow (\neg(\alpha \vee \beta) \vee \gamma)$ by WDW and RLE.

Case 2. Assume $\neg\Box\neg((\alpha \wedge \neg\gamma) \vee (\beta \wedge \neg\gamma))$. Then, $\neg(((\alpha \wedge \neg\gamma) \vee (\beta \wedge \neg\gamma)) \hookrightarrow \neg(\alpha \wedge \neg\gamma))$ or $\neg(((\alpha \wedge \neg\gamma) \vee (\beta \wedge \neg\gamma)) \hookrightarrow \neg(\beta \wedge \neg\gamma))$ by Fact 20. Without loss of generality, we may assume the first disjunct. Let $\varphi = ((\alpha \vee \beta) \wedge \gamma)$, $\psi = (\alpha \wedge \gamma)$, $\chi = (\alpha \wedge \neg\gamma)$ and $\delta = ((\alpha \vee \beta) \wedge \neg\gamma)$ (for the second disjunct it suffices to take ψ as $(\beta \wedge \gamma)$ and χ as $(\beta \wedge \neg\gamma)$). The first disjunct amounts to $\neg(\delta \hookrightarrow \neg\chi)$, and since $\delta \equiv (\delta \vee \chi)$, LLE yields $\neg((\delta \vee \chi) \hookrightarrow \neg\chi)$. Either $\neg(\varphi \hookrightarrow \neg\varphi)$ or $\varphi \hookrightarrow \neg\varphi$. Reasoning again by cases, we show that each of them establishes the desired conclusion.

Case 2.1. Assume $\neg(\varphi \hookrightarrow \neg\varphi)$. Then $\neg((\psi \vee \varphi) \hookrightarrow \neg\varphi)$ by LLE, given that $(\psi \vee \varphi) \equiv \varphi$. Moreover, we have $(\psi \vee \chi) \hookrightarrow \neg\chi$, which follows from our initial assumption $\alpha \hookrightarrow (\neg\alpha \vee \gamma)$ by LLE and RLE, given that $(\psi \vee \chi) \equiv \alpha$ and $\neg\chi \equiv (\neg\alpha \vee \gamma)$. By Mod we obtain $(\varphi \vee \chi) \hookrightarrow \neg\chi$. But since we have $\neg((\delta \vee \chi) \hookrightarrow \neg\chi)$, we obtain $(\varphi \vee \delta) \hookrightarrow \neg\delta$ again by Mod. That is $(\alpha \vee \beta) \hookrightarrow \neg\delta$ by LLE, given that $(\varphi \vee \delta) \equiv (\alpha \vee \beta)$. Thus $(\alpha \vee \beta) \hookrightarrow \neg((\alpha \vee \beta) \wedge \neg\gamma)$ because $\delta = ((\alpha \vee \beta) \wedge \neg\gamma)$. RLE then yields $(\alpha \vee \beta) \hookrightarrow (\neg(\alpha \vee \beta) \vee \gamma)$.

Case 2.2. Assume $\varphi \hookrightarrow \neg\varphi$. Then, $\neg\neg\varphi \hookrightarrow \neg\varphi$ by LLE. Thus $\Box\neg\varphi$, that is $\Box\neg((\alpha \vee \beta) \wedge \gamma)$. Thus $\alpha \hookrightarrow \neg((\alpha \vee \beta) \wedge \gamma)$ by NC. Therefore $\alpha \hookrightarrow ((\neg\alpha \wedge \neg\beta) \vee \neg\gamma)$ by RLE. Hence $\alpha \hookrightarrow (\neg\alpha \vee \neg\gamma)$ by CSS and RLE. But since we initially assumed $\alpha \hookrightarrow (\neg\alpha \vee \gamma)$, we obtain $\alpha \hookrightarrow \neg\alpha$ by AND and RLE. Hence $\neg\neg\alpha \hookrightarrow \neg\alpha$ by LLE. In similar fashion we can establish $\neg\neg\beta \hookrightarrow \neg\beta$. Thus $\neg(\neg\alpha \wedge \neg\beta) \hookrightarrow \neg\alpha$ and $\neg(\neg\alpha \wedge \neg\beta) \hookrightarrow \neg\beta$ by NC. So we get $\neg(\neg\alpha \wedge$

$\neg\beta) \hookrightarrow (\neg\alpha \wedge \neg\beta)$ by AND. Therefore $(\alpha \vee \beta) \hookrightarrow \neg(\alpha \vee \beta)$ by LLE and RLE. This entails $(\alpha \vee \beta) \hookrightarrow (\neg(\alpha \vee \beta) \vee \gamma)$ by WDW.

A restricted version of *Aristotle's Thesis* is derivable

Fact 22. $\vdash_{\mathsf{CC}} \Diamond\alpha \supset \neg(\alpha \hookrightarrow \neg\alpha)$ RAT

Proof. Suppose $\Diamond\alpha$. That is $\neg(\neg\neg\alpha \hookrightarrow \neg\alpha)$ by definition of $\Box$ and $\Diamond = \neg\Box\neg$. Hence $\neg(\alpha \hookrightarrow \neg\alpha)$ by LLE. ■

Weak Boethius Thesis is also derivable

Fact 23. $\vdash_{\mathsf{CC}} (\alpha \hookrightarrow \beta) \supset \neg(\alpha \hookrightarrow \neg\beta)$ wBT

Proof. Suppose that $\alpha \hookrightarrow \beta$ and, for reductio, that $\alpha \hookrightarrow \neg\beta$. Then $\alpha \hookrightarrow (\beta \wedge \neg\beta)$ by AND. Hence $\alpha \hookrightarrow \bot$ by RLE. Thus $\bot$ by CT and MP. Hence the reductio assumption must be false, and therefore $\neg(\alpha \hookrightarrow \neg\beta)$. ■

Finally, a restricted version of *Aristotle's Second Thesis* is derivable

Fact 24. $\vdash_{\mathsf{CC}} (\Diamond\neg\beta \wedge (\alpha \hookrightarrow \beta)) \supset \neg(\neg\alpha \hookrightarrow \beta)$ RAT2

Proof. Suppose that $\Diamond\neg\beta$, $\alpha \hookrightarrow \beta$ and, for reductio, that $\neg\alpha \hookrightarrow \beta$. Then $\neg\beta \hookrightarrow (\beta \vee \alpha)$ by CSS and similarly $\neg\beta \hookrightarrow (\beta \vee \neg\alpha)$. Therefore $\neg\beta \hookrightarrow ((\beta \vee \alpha) \wedge (\beta \vee \neg\alpha))$ by AND. Thus $\neg\beta \hookrightarrow \beta$ by RLE. But this contradicts the assumption $\Diamond\neg\beta$ (since by RAT, $\Diamond\neg\beta$ implies $\neg(\neg\beta \hookrightarrow \neg\neg\beta)$ and thus $\neg(\neg\beta \hookrightarrow \beta)$ by RLE). Hence the reductio assumption must be false, and therefore $\neg(\neg\alpha \hookrightarrow \beta)$. ■

7. Soundness of CC

Now we will prove that CC is sound by relying on the fact that VC is sound. The key result we need is the following, where χ is any formula of $\mathsf{L}_{\hookrightarrow}$:

Fact 25. If $\vdash_{\mathsf{CC}} \chi$, then $\vdash_{\mathsf{VC}} \chi^\circ$.

Proof. The proof is by induction on the length of the proof of χ in CC.
Basis. Assume that there is a proof of χ of length 1. In this case χ is an axiom. Nine cases are possible.
Case 1: For a general proof of this case, see Lemma 1 in Raidl [8].
Case 2: χ is an instance of N. In this case χ° is a conjunction $(\neg\top > \top) \wedge (\neg\neg\top > \top) \wedge (\neg\top > \neg\top)$. In VC, $\neg\top > \top$ holds in virtue of N (Fact 12), $\neg\neg\top > \top$ in virtue of ID and LLE, and $\neg\top > \neg\top$ in virtue of ID.
Case 3: χ is an instance of AND. In this case χ° is equivalent to a material conditional with antecedent $(\alpha > \beta) \wedge (\neg\alpha > \beta) \wedge (\neg\beta > \alpha) \wedge (\alpha > \gamma) \wedge (\neg\alpha >$

Table 1. Valid ($\surd$) and invalid ($\times$) principles for the concessive ($\hookrightarrow$) and the suppositional ($>$) conditional

Form for $\hookrightarrow$	Label	$\hookrightarrow$	$>$
If $\vdash \beta \supset \gamma$ then $\vdash (\alpha \hookrightarrow \beta) \supset (\alpha \hookrightarrow \gamma)$	RW	$\times$	$\underline{\surd}$
$\alpha \hookrightarrow \alpha$	ID	$\times$	$\underline{\surd}$
$((\alpha \hookrightarrow \gamma) \wedge (\beta \hookrightarrow \gamma)) \supset ((\alpha \vee \beta) \hookrightarrow \gamma)$	OR	$\times$	$\underline{\surd}$
$((\alpha \hookrightarrow \gamma) \wedge (\alpha \hookrightarrow \beta)) \supset ((\alpha \wedge \beta) \hookrightarrow \gamma)$	CM	$\times$	$\underline{\surd}$
$((\alpha \hookrightarrow \gamma) \wedge \neg(\alpha \hookrightarrow \neg\beta)) \supset ((\alpha \wedge \beta) \hookrightarrow \gamma)$	RM	$\times$	$\underline{\surd}$
$((\alpha \vee \beta) \hookrightarrow \gamma) \supset ((\alpha \hookrightarrow \gamma) \vee (\beta \hookrightarrow \gamma))$	DR	$\times$	$\surd$
$(\alpha \wedge \beta) \supset (\alpha \hookrightarrow \beta)$	CS	$\times$	$\underline{\surd}$
$(\alpha \hookrightarrow \beta) \supset (\alpha \supset \beta)$	MI	$\surd$	$\underline{\surd}$
$\alpha \hookrightarrow \top$	CN	$\surd$	$\surd$
$\Box \top$	N	$\underline{\surd}$	$\surd$

Table 1. continued

Form for $\hookrightarrow$	Label	$\hookrightarrow$	$>$
$((\alpha \hookrightarrow \beta) \wedge (\alpha \hookrightarrow \gamma)) \supset (\alpha \hookrightarrow (\beta \wedge \gamma))$	AND	$\underline{\checkmark}$	$\checkmark$
$\alpha \supset (\top \hookrightarrow \alpha)$	TI	$\underline{\checkmark}$	$\checkmark$
$\Box \beta \supset (\alpha \hookrightarrow \beta)$	NC	$\underline{\checkmark}$	$\checkmark$
$((\alpha \vee \beta) \hookrightarrow \neg\beta) \supset (((\alpha \vee \gamma) \hookrightarrow \neg\gamma) \vee ((\gamma \vee \beta) \hookrightarrow \neg\beta))$	Mod	$\underline{\checkmark}$	$\checkmark$
If $\vdash \alpha \equiv \beta$ then $\vdash (\alpha \hookrightarrow \gamma) \supset (\beta \hookrightarrow \gamma)$	LLE	$\underline{\checkmark}$	$\underline{\checkmark}$
If $\vdash \beta \equiv \gamma$ then $\vdash (\alpha \hookrightarrow \beta) \supset (\alpha \hookrightarrow \gamma)$	RLE	$\underline{\checkmark}$	$\checkmark$
$(\alpha \hookrightarrow (\neg\alpha \vee \beta)) \supset (\alpha \hookrightarrow (\neg\alpha \vee \beta \vee \gamma))$	WDW	$\underline{\checkmark}$	$\checkmark$
$((\alpha \hookrightarrow (\neg\alpha \vee \gamma)) \wedge (\beta \hookrightarrow (\neg\beta \vee \gamma))) \supset ((\alpha \vee \beta) \hookrightarrow (\neg(\alpha \vee \beta) \vee \gamma))$	OR*	$\checkmark$	$\checkmark$
$((\alpha \hookrightarrow (\neg\alpha \vee \gamma)) \wedge (\alpha \hookrightarrow (\neg\alpha \vee \beta))) \supset ((\alpha \wedge \beta) \hookrightarrow (\neg(\alpha \wedge \beta) \vee \gamma))$	CM*	$\checkmark$	$\checkmark$
$((\alpha \hookrightarrow (\neg\alpha \vee \gamma)) \wedge \neg(\alpha \hookrightarrow (\neg\alpha \vee \neg\beta))) \supset ((\alpha \wedge \beta) \hookrightarrow (\neg(\alpha \wedge \beta) \vee \gamma))$	RM*	$\checkmark$	$\checkmark$
$\Diamond \alpha \supset \neg(\alpha \hookrightarrow \neg\alpha)$	RAT	$\checkmark$	$\checkmark$
$(\Diamond \neg\beta \wedge (\alpha \hookrightarrow \beta)) \supset \neg(\neg\alpha \hookrightarrow \beta)$	RAT2	$\checkmark$	$\times$
$(\alpha \hookrightarrow \beta) \supset \neg(\alpha \hookrightarrow \neg\beta)$	wBT	$\checkmark$	$\times$
$(\alpha \hookrightarrow \beta) \supset \beta$	CT	$\underline{\checkmark}$	$\times$
$((\alpha \hookrightarrow (\neg\alpha \vee \beta)) \wedge (\neg\alpha \hookrightarrow (\alpha \vee \beta)) \wedge (\neg\beta \hookrightarrow (\beta \vee \alpha))) \equiv (\alpha \hookrightarrow \beta)$	CSS	$\underline{\checkmark}$	$\times$

Underlined valid principles ($\underline{\checkmark}$) provide a possible axiomatization. The first block is typical for $>$, but invalid for $\hookrightarrow$. The second block is typical for both $>$ and $\hookrightarrow$. The third block are weakenings of the principles RW, OR, CM, RM from the first block. The fourth block are connexive principles of which two fail for $>$. The last block contains the crucial axioms CT and CSS of $\hookrightarrow$ which are invalid for $>$

$\gamma) \wedge (\neg\gamma > \alpha)$ and consequent $(\alpha > (\beta\wedge\gamma))\wedge(\neg\alpha > (\beta\wedge\gamma))\wedge(\neg(\beta\wedge\gamma) > \alpha)$. This conditional is provable in VC because $\alpha > \beta$ and $\alpha > \gamma$ entail $\alpha > (\beta\wedge\gamma)$ by AND, $\neg\alpha > \beta$ and $\neg\alpha > \gamma$ entail $\neg\alpha > (\beta \wedge \gamma)$ by AND, and $\neg\beta > \alpha$ and $\neg\gamma > \alpha$ entail $\neg(\beta \wedge \gamma) > \alpha$ by OR and LLE.

Case 4: χ is an instance of WDW. In this case χ° is a material conditional whose antecedent is formed by three conjuncts $\alpha > (\neg\alpha\vee\beta)$, $\neg\alpha > (\neg\alpha\vee\beta)$, $\neg(\neg\alpha \vee \beta) > \alpha$, and whose consequent is formed by three conjuncts $\alpha > (\neg\alpha\vee\beta\vee\gamma)$, $\neg\alpha > (\neg\alpha\vee\beta\vee\gamma)$, $\neg(\neg\alpha\vee\beta\vee\gamma) > \alpha$. The first conjunct of the consequent follows from $\alpha > (\neg\alpha \vee \beta)$ by RW, and the other two conjuncts of the consequent hold anyway, due to Fact 9.

Case 5: χ is an instance of TI. In this case χ° is a material conditional with antecedent α and consequent $(\top > \alpha) \wedge (\bot > \alpha) \wedge (\neg\alpha > \top)$. This is provable in VC because α entails $\alpha \wedge \top$, which entails $\top > \alpha$ by CS. $\bot > \alpha$ holds by ID and RW, and $\neg\alpha > \top$ holds by CN (Fact 11).

Case 6: χ is an instance of NC. In this case χ° is a material conditional with antecedent $(\neg\beta > \beta) \wedge (\neg\neg\beta > \beta) \wedge (\neg\beta > \neg\beta)$ and consequent $(\alpha > \beta) \wedge (\neg\alpha > \beta) \wedge (\neg\beta > \alpha)$. From $\neg\beta > \beta$ we obtain $\neg\beta > \bot$ by ID and AND. Thus $\neg\beta > \alpha$ by RW, which is the third conjunct of the consequent. The first two conjuncts also follow from $\neg\beta > \beta$, since this entails $\alpha > \beta$ and $\neg\alpha > \beta$ by NC.

Case 7: χ is an instance of Mod. In this case χ° is a material conditional with antecedent $((\alpha \vee \beta) > \neg\beta) \wedge (\neg(\alpha \vee \beta) > \neg\beta) \wedge (\neg\neg\beta > (\alpha \vee \beta))$ and a consequent composed of two disjuncts: the first is $((\alpha \vee \gamma) > \neg\gamma) \wedge (\neg(\alpha \vee \gamma) > \neg\gamma) \wedge (\neg\neg\gamma > (\alpha \vee \gamma))$, the second is $((\gamma \vee \beta) > \neg\beta) \wedge (\neg(\gamma \vee \beta) > \neg\beta)\wedge(\neg\neg\beta > (\gamma\vee\beta))$. Note that, by Fact 9, we already have $\neg(\alpha\vee\gamma) > \neg\gamma$, $\neg\neg\gamma > (\alpha \vee \gamma)$, $\neg(\gamma \vee \beta) > \neg\beta$, and $\neg\neg\beta > (\gamma \vee \beta)$. Thus we only need to establish $((\alpha \vee \gamma) > \neg\gamma) \vee ((\gamma \vee \beta) > \neg\beta)$. But this follows from the first conjunct of the antecedent, $(\alpha \vee \beta) > \neg\beta$, in virtue of Mod (Fact 15).

Case 8: χ is an instance of CT. In this case χ° is a material conditional with antecedent $(\alpha > \beta) \wedge (\neg\alpha > \beta) \wedge (\neg\beta > \alpha)$ and consequent β. This is provable in VC because, given MI, $\alpha > \beta$ entails $\alpha \supset \beta$ and $\neg\alpha > \beta$ entails $\neg\alpha \supset \beta$, from which we can derive β.

Case 9: χ is an instance of CSS. Then χ° is a material biconditional whose left-hand side is formed by the conjuncts $\alpha > (\neg\alpha \vee \beta)$, $\neg\alpha > (\neg\alpha \vee \beta)$, $\neg(\neg\alpha\vee\beta) > \alpha$, $\neg\alpha > (\alpha\vee\beta)$, $\neg\neg\alpha > (\alpha\vee\beta)$, $\neg(\alpha\vee\beta) > \neg\alpha$, $\neg\beta > (\beta\vee\alpha)$, $\neg\neg\beta > (\beta \vee \alpha)$, $\neg(\beta \vee \alpha) > \neg\beta$, and whose right-hand side is formed by the conjuncts $\alpha > \beta$, $\neg\alpha > \beta$, $\neg\beta > \alpha$. From $\alpha > (\neg\alpha \vee \beta)$, $\neg\alpha > (\alpha \vee \beta)$, and

$\neg\beta > (\beta \vee \alpha)$, we obtain $\alpha > \beta$, $\neg\alpha > \beta$ and $\neg\beta > \alpha$ by ID, AND and RW. And we can reverse the reasoning, by using RW.
Step. Assume that the condition holds for every proof of length less than or equal to n, and consider a proof of χ of length $n+1$. Then four cases are possible.

Case 1: χ is an axiom. In this case we know that $\vdash_{\mathsf{VC}} \chi^\circ$, given what has been said in the basis.

Case 2: χ is obtained by means of LLE. In this case χ° is a material conditional with antecedent $(\alpha > \gamma) \wedge (\neg\alpha > \gamma) \wedge (\neg\gamma > \alpha)$ and consequent $(\beta > \gamma) \wedge (\neg\beta > \gamma) \wedge (\neg\gamma > \beta)$, and by the induction hypothesis $\vdash_{\mathsf{VC}} \alpha \equiv \beta$ and thus we also have $\vdash_{\mathsf{VC}} \neg\alpha \equiv \neg\beta$. Since VC has LLE and RW, $\alpha > \gamma$ entails $\beta > \gamma$, $\neg\alpha > \gamma$ entails $\neg\beta > \gamma$ and $\neg\gamma > \alpha$ entails $\neg\gamma > \beta$.

Case 3: χ is obtained by means of RLE. This case is analogous to case 2.

Case 4: χ is obtained by means of MP. In this case χ is preceded by two formulas $\delta \supset \chi$ and δ in the proof. By the induction hypothesis, $\delta^\circ \supset \chi^\circ$ and δ° are provable in VC, so the same goes for χ°, given that VC has MP. ∎

THEOREM 26. *If* $\vdash_{\mathsf{CC}} \chi$, *then* $\vDash_{\hookrightarrow} \chi$.

PROOF. Assume that $\vdash_{\mathsf{CC}} \chi$. Then, by Fact 25, $\vdash_{\mathsf{VC}} \chi^\circ$. Since VC is sound, it follows that $\vDash_{>} \chi^\circ$. By Fact 7, this entails that $\vDash_{\hookrightarrow} \chi$. ∎

8. Completeness of CC

In this last section we will prove that CC is complete by relying on the fact that VC is complete. First we will show that $\bullet$ *inverts* $\circ$ in CC, namely, that for any formula χ of $\mathsf{L}_{\hookrightarrow}$, the backtranslation of the translation of χ, that is, $\chi^{\circ\bullet}$, is provably equivalent to χ in CC. Then we will show that, for any formula χ of $\mathsf{L}_{>}$, if $\vdash_{\mathsf{VC}} \chi$, then $\vdash_{\mathsf{CC}} \chi^\bullet$. The combination of these two results yields the converse of Fact 25, which suffices to establish the completeness of CC.

FACT 27. $\vdash_{\mathsf{CC}} \chi^{\circ\bullet} \equiv \chi$

PROOF. The proof is by induction on the complexity of χ.
Basis. χ is a sentence letter. In this case $\chi^{\circ\bullet} = \chi$, and $\chi \equiv \chi$ is trivially provable in CC.
Step. Assume that the condition holds for every formula of complexity less than or equal to n, and that χ has complexity $n+1$. Then five cases are to

be considered, depending on whether the main connective of χ is $\neg, \wedge, \vee, \supset$, or $\hookrightarrow$. In the first four cases, given the induction hypothesis, we obtain that $\vdash_{\mathsf{CC}} \chi^{\circ\bullet} \equiv \chi$.

In the fifth case, χ is a formula $\alpha \hookrightarrow \beta$, and $\chi^{\circ\bullet}$ is $(\alpha \hookrightarrow \beta)^{\circ\bullet}$. We also assume the induction hypothesis for α and β, that is, $\alpha^{\circ\bullet}$ is provably equivalent to α and similarly for β. In this case $(\alpha \hookrightarrow \beta)^{\circ\bullet}$ is $((\alpha^\circ > \beta^\circ) \wedge (\neg\alpha^\circ > \beta^\circ) \wedge (\neg\beta^\circ > \alpha^\circ))^\bullet$, that is, $(\alpha^{\circ\bullet} \hookrightarrow (\neg\alpha^{\circ\bullet} \vee \beta^{\circ\bullet})) \wedge (\neg\alpha^{\circ\bullet} \hookrightarrow (\neg\neg\alpha^{\circ\bullet} \vee \beta^{\circ\bullet})) \wedge (\neg\beta^{\circ\bullet} \hookrightarrow (\neg\neg\beta^{\circ\bullet} \vee \alpha^{\circ\bullet}))$. By the induction hypothesis, our remarks above and given LLE and RLE, this is provably equivalent to $(\alpha \hookrightarrow (\neg\alpha \vee \beta)) \wedge (\neg\alpha \hookrightarrow (\neg\neg\alpha \vee \beta)) \wedge (\neg\beta \hookrightarrow (\neg\neg\beta \vee \alpha))$. This is further equivalent to $(\alpha \hookrightarrow (\neg\alpha \vee \beta)) \wedge (\neg\alpha \hookrightarrow (\alpha \vee \beta)) \wedge (\neg\beta \hookrightarrow (\beta \vee \alpha))$ due to RLE. But this conjunction is equivalent to $\alpha \hookrightarrow \beta$ due to CSS. ■

FACT 28. If $\vdash_{\mathsf{VC}} \chi$, then $\vdash_{\mathsf{CC}} \chi^\bullet$.

PROOF. The proof is by induction on the length of the proof of χ in VC.
Basis. Assume that there is a proof of χ of length 1. In this case χ is an axiom. Eight cases are possible.
Case 1: χ is obtained by substitution from a propositional tautology α, where $\alpha \in \mathsf{L}$. This case is analogous to case 1 in the proof of Fact 25.
Case 2: χ is an instance of ID. In this case $\chi^\bullet$ has the form $\alpha \hookrightarrow (\neg\alpha \vee \alpha)$, so it is provable in CC by CN (Fact 16) and RLE.
Case 3: χ is an instance of AND. In this case $\chi^\bullet$ is a material conditional with antecedent $(\alpha \hookrightarrow (\neg\alpha \vee \beta)) \wedge (\alpha \hookrightarrow (\neg\alpha \vee \gamma))$ and consequent $\alpha \hookrightarrow (\neg\alpha \vee (\beta \wedge \gamma))$. But the antecedent entails $\alpha \hookrightarrow ((\neg\alpha \vee \beta) \wedge (\neg\alpha \vee \gamma))$ by AND, which entails the consequent by RLE. So, $\vdash_{\mathsf{CC}} \chi^\bullet$.
Case 4: χ is an instance of OR. In this case $\chi^\bullet$ has the form $((\alpha \hookrightarrow (\neg\alpha \vee \gamma)) \wedge (\beta \hookrightarrow (\neg\beta \vee \gamma))) \supset ((\alpha \vee \beta) \hookrightarrow (\neg(\alpha \vee \beta) \vee \gamma))$, so it is provable in CC by Fact 21.
Case 5: χ is an instance of CM. In this case $\chi^\bullet$ has the form $((\alpha \hookrightarrow (\neg\alpha \vee \gamma)) \wedge (\alpha \hookrightarrow (\neg\alpha \vee \beta))) \supset ((\alpha \wedge \beta) \hookrightarrow (\neg(\alpha \wedge \beta) \vee \gamma))$, so it is provable in CC by Fact 17.
Case 6: χ is an instance of RM. In this case $\chi^\bullet$ has the form $((\alpha \hookrightarrow (\neg\alpha \vee \gamma)) \wedge \neg(\alpha \hookrightarrow (\neg\alpha \vee \neg\beta))) \supset ((\alpha \wedge \beta) \hookrightarrow (\neg(\alpha \wedge \beta) \vee \gamma))$. So it is provable in CC by Fact 18.
Case 7: χ is an instance of MI. In this case $\chi^\bullet$ has the form $(\alpha \hookrightarrow (\neg\alpha \vee \beta)) \supset (\alpha \supset \beta)$. But in CC, from $\alpha \hookrightarrow (\neg\alpha \vee \beta)$ we get $\neg\alpha \vee \beta$ by CT, hence $\alpha \supset \beta$.
Case 8: χ is an instance of CS. In this case $\chi^\bullet$ has the form $(\alpha \wedge \beta) \supset (\alpha \hookrightarrow (\neg\alpha \vee \beta))$. In CC, from α and β we can derive $\top \hookrightarrow \alpha$ and $\top \hookrightarrow \beta$ by TI, and consequently $\top \hookrightarrow (\bot \vee \alpha)$ and $\top \hookrightarrow (\bot \vee \beta)$ by RLE. By Fact 17 we get $(\top \wedge \alpha) \hookrightarrow (\neg(\top \wedge \alpha) \vee \beta)$. This yields $\alpha \hookrightarrow (\neg\alpha \vee \beta)$ by LLE and RLE.

Step. Assume that the condition holds for every proof of length less than or equal to n, and that there is a proof of χ of length $n+1$. Then four cases are possible.
Case 1: χ is an axiom. In this case we know that $\vdash_{\mathsf{CC}} \chi^{\bullet}$, given what has been said in the basis.
Case 2: χ is obtained by means of MP. This case is analogous to case 4 in the step of the proof of Fact 25.
Case 3: χ is obtained by means of LLE. In this case $\chi^{\bullet}$ is a material conditional with antecedent $\alpha \hookrightarrow (\neg\alpha \vee \gamma)$ and consequent $\beta \hookrightarrow (\neg\beta \vee \gamma)$, and by the induction hypothesis $\vdash_{\mathsf{CC}} \alpha \equiv \beta$ and thus also $\vdash_{\mathsf{CC}} \neg\alpha \equiv \neg\beta$ as well as $\vdash_{\mathsf{CC}} (\neg\alpha \vee \gamma) \equiv (\neg\beta \vee \gamma)$. But then $\alpha \hookrightarrow (\neg\alpha \vee \gamma)$ entails $\beta \hookrightarrow (\neg\alpha \vee \gamma)$ due to LLE, which further entails $\beta \hookrightarrow (\neg\beta \vee \gamma)$ due to RLE.
Case 4: χ is obtained by RW. In this case $\chi^{\bullet}$ is a material conditional with antecedent $\gamma \hookrightarrow (\neg\gamma \vee \alpha)$ and consequent $\gamma \hookrightarrow (\neg\gamma \vee \beta)$, and by the induction hypothesis $\vdash_{\mathsf{CC}} \alpha \supset \beta$. Thus we have $\vdash_{\mathsf{CC}} \beta \equiv (\alpha \vee \beta)$. By WDW $\gamma \hookrightarrow (\neg\gamma \vee \alpha)$ implies $\gamma \hookrightarrow (\neg\gamma \vee \alpha \vee \beta)$ which due to RLE and the induction hypothesis is equivalent to $\gamma \hookrightarrow (\neg\gamma \vee \beta)$. ■

Fact 29. If $\vdash_{\mathsf{VC}} \chi^{\circ}$, then $\vdash_{\mathsf{CC}} \chi$.

Proof. Assume that $\vdash_{\mathsf{VC}} \chi^{\circ}$. By Fact 28, it follows that $\vdash_{\mathsf{CC}} \chi^{\circ\bullet}$. But Fact 27 says that $\vdash_{\mathsf{CC}} \chi^{\circ\bullet} \equiv \chi$. Therefore, $\vdash_{\mathsf{CC}} \chi$. ■

Theorem 30. *If* $\vDash_{\hookrightarrow} \chi$, *then* $\vdash_{\mathsf{CC}} \chi$.

Proof. Assume that $\vDash_{\hookrightarrow} \chi$. Then, by Fact 7, $\vDash_{>} \chi^{\circ}$. Since VC is complete, this entails that $\vdash_{\mathsf{VC}} \chi^{\circ}$. By Fact 29 it follows that $\vdash_{\mathsf{CC}} \chi$. ■

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References

[1] Chellas, B. F., Basic conditional logic, *Journal of Philosophical Logic* 4:133–153, 1975.

[2] Crupi, V., and A. Iacona, The evidential conditional, *Erkenntnis* 87:2897–2921, 2022.

[3] Crupi, V., and A. Iacona, On the logical form of concessive conditionals, *Journal of Philosophical Logic* 51:633–651, 2022.

[4] Crupi, V., and A. Iacona, Conditionals: inferentialism articulated, (manuscript), 2022.

[5] Kraus, S., D. Lehmann, and M. Magidor, Nonmonotonic reasoning, preferential models and cumulative logics, *Journal of Artificial Intelligence* 44:167–207, 1990.

[6] Lewis, D., Completeness and decidability of three logics of counterfactual conditionals, *Theoria* 37:74–85, 1971.

[7] Lewis, D., *Counterfactuals*, Blackwell, 1973.

[8] Raidl, E., Strengthened conditionals, in B. Liao, and Y. Wáng, (eds.), *Context, Conflict and Reasoning*, Springer, Singapore, 2020, pp. 139–155.

[9] Raidl, E., Definable conditionals, *Topoi* 40:87–105, 2021.

[10] Raidl, E., Three conditionals: contraposition, difference-making and dependency, in M. Blicha, and I. Sedlar, (eds.), *The Logica Yearbook 2020*, College Publications, 2021, pp. 201–217.

[11] Raidl, E., A. Iacona, and V. Crupi, The logic of the evidential conditional, *The Review of Symbolic Logic* 15(3):758–770, 2022.

[12] Stalnaker, R., A theory of conditionals, in F. Jackson, (ed.), *Conditionals*, Oxford University Press, 1991, pp. 28–45.

E. Raidl
University of Tübingen
Tübingen
Germany
eric.raidl@uni-tuebingen.de

A. Iacona, V. Crupi
University of Turin
Turin
Italy