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Helly-Type Theorems on the Homology of the Space of Transversals

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Abstract. In this paper we "measure" the size of the set of n-transversals of a family F of convex sets in R^{n+k} according to its homological complexity inside the corresponding Grassmannian manifold. Our main result states that the "measure" μ of the set of n-transversals of F is greater than or equal to k if and only if every $k+1$ members of F have a common point and also if and only if for some integer m, $1 \leq m \leq n$, and every subfamily F' of F with $k+2$ members, the "measure" μ of the set of m-transversals of F' is greater than or equal to k.

1. Introduction

For a family $F = \{A^1, \dots, A^d\}$ of d convex sets in R^{n+k}, let $T_n(F)$ be the set of n-transversals to F, that is, the set of all n-planes in R^{n+k} which intersect every member of F.

If X is a set of n-planes in R^{n+k}, we say that $\mu(X) \geq r$ if X has "homologically" as many n-planes as the set of n-planes through the origin in R^{n+r}. Thus, μ "measures" the homological complexity of X inside the corresponding Grassmannian manifold. We use this "measure" to prove that if subfamilies of F with few members have enough transversals of small dimension, then the whole family F has many transversals of a fixed dimension. That is, after a formal definition of μ, in Section 2, we prove in Section 3 the equivalence of the following three properties:

- Every $k+1$ members of F have a point in common.
- $\mu(T_n(F)) \geq k$.
- For some integer m where $1 \leq m \leq n$ and every subfamily F' of F with $k+2$ members, $\mu(T_m(F')) \geq k$.

The first equivalence can be thought of as a homological version of Horn and Klee's classical results [5], [6]. See also [4]. They proved that the following assertions are equivalent:

(a) Every $k+1$ members of F have a point in common.
(b) Every linear n-subspace of R^{n+k} admits a translate which is a member of $T_n(F)$.
(c) Every $(n-1)$-plane Λ lies in a member of $T_n(F)$.

First note that (b) is just assertion (c), when Λ lies at infinity. In fact, the set of all n-planes that contain Λ is a manifold embedded in the corresponding Grassmannian manifold, which represents an element of its cohomology. So, by using the product structure of the cohomology we shall prove that

$$\mu(T_n(F)) \geq k \quad \Rightarrow \quad (b) \text{ and } (c).$$

If X is a set of n-planes in R^{n+k} and for every linear n-subspace of R^{n+k} we can choose a translate which is a member of X, then $\mu(T_n(F))$ is not necessarily greater than or equal to k, unless, of course, according to our definition of μ, the choice can be done continuously. If $X = T_n(F)$, the existence of a member of $T_n(F)$ parallel to every linear n-subspace of R^{n+k} implies that we can choose this member continuously and hence that

$$\mu(T_n(F)) \geq k \quad \Leftrightarrow \quad (b) \text{ and } (c).$$

The spirit of the complete equivalences follows the topological study of the space of transversals initiated in [1] and [2].

We consider Euclidean n-space R^n and complete it to the n-projective space P^n by adding the hyperplane at infinity. Let $G(n+k, n)$ be the Grassmannian nk-manifold of all n-planes through the origin in Euclidean space R^{n+k}. Although we summarize what we need in Section 2, good references for the homology and cohomology of Grassmannian manifolds are [7], [9] and [3]; see also [8]. In this paper we use reduced Cech-homology and cohomology with Z_2-coefficients.

2. The Topology of Grassmannian Manifolds

Let $\lambda_1, \ldots, \lambda_n$ be a sequence of integers such that $0 \leq \lambda_1 \leq \cdots \leq \lambda_n \leq k$. We denote by:

(2.1) $\{\lambda_1, \ldots, \lambda_n\} = \{H \in G(n+k, n) \mid \dim\,(H \cap R^{\lambda_j + j}) \geq j,\ j = 1, \ldots, n\}$. For example, $\{0, \lambda, \ldots, \lambda\} = \{H \in G(n+k, n) \mid R^1 \subset H \subset R^{n+\lambda}\}$ and $\{k-\lambda, \ldots, k-\lambda, k\} = \{H \in G(n+k, n) \mid \dim(H \cap R^{n-1+k-\lambda}) \geq n-1\}$.

(2.2) It is known that $\{\lambda_1, \ldots, \lambda_n\} \subset G(n+k, n)$ is a closed connected λ-manifold, where $\lambda = \sum_1^n \lambda_i$, except possibly for a closed connected subset of codimension three. Thus, $H^\lambda(\{\lambda_1, \ldots, \lambda_n\}; Z_2) = Z_2 = H_\lambda(\{\lambda_1, \ldots, \lambda_n\}; Z_2)$. Let $(\lambda_1, \ldots, \lambda_n) \in H_\lambda(G(n+k, n); Z_2)$ be the λ-cycle which is induced by the inclusion $\{\lambda_1, \ldots, \lambda_n\} \subset G(n+k, n)$. These cycles are called *Schubert-cycles*. A canonical basis for $H_\lambda(G(n+$

$k, n); Z_2)$ consists of all Schubert-cycles $(\xi_1, \ldots, \xi_n)$ such that $0 \leq \xi_1 \leq \cdots \leq \xi_n \leq k$ and $\sum_1^n \xi_i = \lambda$

(2.3) We denote by $[\lambda_1, \ldots, \lambda_n] \in H^\lambda(G(n+k, n); Z_2)$ the λ-cocycle whose value is one for $(\lambda_1, \ldots, \lambda_n)$ and zero for any other Schubert-cycle of dimension λ. Thus a canonical basis for $H^\lambda(G(n+k, n); Z_2)$ consists of all Schubert-cocycles $[\xi_1, \ldots, \xi_n]$ such that $0 \leq \xi_1 \leq \cdots \leq \xi_n \leq k$ and $\sum_1^n \xi_i = \lambda$.

The isomorphism D: $H_\lambda(G(n+k, n); Z_2) \to H^{nk-\lambda}(G(n+k, n); Z_2)$ given by $D((\lambda_1, \ldots, \lambda_n)) = [k - \lambda_n, \ldots, k - \lambda_1]$ is the classical *Poincaré Duality Isomorphism*.

(2.4) By the above, if $X \subset G(n+k, n)$ is such that $X \cap \{\lambda_1, \ldots, \lambda_n\} = \emptyset$ and i_X: $X \to G(n+k, n)$ is the inclusion, then

$$i_X^*(D((\lambda_1, \ldots, \lambda_n))) = i_X^*([k - \lambda_n, \ldots, k - \lambda_1]) = 0.$$

(2.5) Let $M(n+k, n)$ be the set of all n-planes in R^{n+k}. Thus, $G(n+k, n) \subset M(n+k, n)$. We regard $M(n+k, n)$ as an open subset of $G(n+k+1, n+1)$, making the following identifications:

Let $z_0 \in R^{n+k+1} - R^{n+k}$ be a fixed point and, without loss of generality, let $G(n+k+1, n+1)$ be the space of all $(n+1)$-planes in R^{n+k+1} through z_0. We identify $H \in M(n+k, n)$ with the unique $(n+1)$-plane $H' \in G(n+k+1, n+1)$ which contains H and passes through z_0. Thus

$$G(n+k, n) \subset M(n+k, n) \subset G(n+k+1, n+1),$$

where $M(n+k, n)$ is an open subset of $G(n+k+1, n+1)$ and $G(n+k, n) \subset G(n+k+1, n+1)$ may be regarded as $\{0, k, \ldots, k\}$, the set of all $(n+1)$-planes in R^{n+k+1} which contains R^1. In other words, if j: $G(n+k, n) \to G(n+k+1, n+1)$ is the inclusion, then $j(\{\lambda_1, \ldots, \lambda_n\}) = \{0, \lambda_1, \ldots, \lambda_n\}$. So, if $0 \leq \lambda \leq k$, then $\{0, \lambda, \ldots, \lambda\}$ as a subset of $M(n+k, n)$ is the set of all n-planes H through the origin in R^{n+k} with the property that $H \subset R^{n+\lambda}$.

If $X \subset M(n+k, n)$, then i_X: $X \to G(n+k+1, n+1)$ denotes the inclusion.

(2.6) Let A be a subset of X, let i: $A \to X$ be the inclusion and let $\gamma \in H^*(X; Z_2)$. We say that γ *is zero or not zero* in A, provided $i^*(\gamma)$ is zero or not zero respectively, in $H^*(A; Z_2)$.

Now we are ready to state our main definition which captures the basic idea of having as many n-planes as the set of all n-planes through the origin in R^{n+r}.

Definition. Let $X \subset M(n+k, n) \subset G(n+k+1, n+1)$. For $0 \leq r \leq k$, we say that the "measure" of X is at least r,

$$\mu(X) \geq r,$$

if $[0, r, \ldots, r]$ is not zero in X.

It is easy to verify that if $\mu(X) \geq r$, then, for any integer $0 \leq r_0 \leq r$, $\mu(X) \geq r_0$. Furthermore, observe that if $m > 0$, then X is also naturally contained in $M(n+m+k, n)$ and the definition of the "measure" μ is independent of m.

Example 2.1. Let $F = \{A^0, \dots, A^d\}$ be a family of convex sets. We say that F has a *cycle of transversal lines* if there is a transversal line that moves continuously until it comes back to itself with the opposite orientation. Observe that F has a cycle of transversal lines if and only if $\mu(T_1(F)) \geq 1$.

The following lemma will be very useful for our purposes.

Lemma 2.1. *Let $X \subset M(n+k, n)$ be a collection of n-planes and let H be an r-plane of R^{n+k}, $1 \leq r \leq k$. If $\mu(X) \geq r$, then there is $\Gamma \in X$ such that $\pi_H(\Gamma)$ is a single point, where $\pi_H \colon R^{n+k} \to H$ is the orthogonal projection.*

Proof. Let $Y \subset M(n+k, n)$ be the set of all n-planes Γ in R^{n+k} such that $\pi_H(\Gamma)$ is a single point. As in (2.5), we regard $Y \subset M(n+k, n)$ as a subset of $G(n+k+1, n+1)$. Let Δ be the $(n+k-r)$-plane in R^{n+k+1} through z_0 orthogonal to the $(r+1)$-plane that contains H and passes through z_0. Note that $\Gamma \in Y$ if and only if the $(n+1)$-plane Γ' that contains Γ and passes through z_0 is such that $\dim(\Gamma' \cap \Delta) \geq n$. Consequently, if we regard Y as a subset of $G(n+k+1, n+1)$, by (2.1) and (2.5), $Y = \{k-r, \dots, k-r, k\}$.

We regard X as a subset of $G(n+k+1, n+1)$ and suppose that $X \cap Y = \emptyset$. Then, by (2.4), $i_X^*([0, r, \dots, r]) = 0$, which means that $[0, r, \dots, r]$ is zero in X, but this is a contradiction because $\mu(X) \geq r$. Then $X \cap Y \neq \emptyset$. This completes the proof of Lemma 2.1. □

Remark 2.1. If, in the above proof, $k = r$ and $Y \subset M(n+k, n)$ is the set of all n-planes Γ in R^{n+k} such that $\Gamma \subset \Lambda$, where Λ is an $(n-1)$-plane in P^{n+k}, then we obtain the following result. Let $X \subset M(n+k, n)$ be a collection of n-planes with the property that $\mu(X) \geq k$, then every linear n-subspace of R^{n+k} admits a translate which is a member of X; and every $(n-1)$-plane Λ lies in a member of X.

3. The Space of Transversals

Let $F = \{A^0, \dots, A^d\}$ be a family of convex sets in R^{n+k} and let $T_n(F)$, *the space of n-transversals of F*, be the subset of the Grassmannian manifold $M(n+k, n)$ of n-planes that intersect all members of F.

Before stating our first result we need the following technical lemma.

Lemma 3.1. *Let $A^0, A^1, \dots, A^k$ be $k+1$ convex sets in R^{n+k}, $n \geq 0$, such that $\bigcap_0^k A^i = \emptyset$. Then there is a k-dimensional linear subspace H of R^{n+k} with the property that $\bigcap_0^k \pi_H(A^i) = \emptyset$, where $\pi_H \colon R^{n+k} \to H$ is the orthogonal projection.*

Proof. The proof is by induction on k. If $k = 1$, the proof follows by the separation theorem for disjoint convex sets. Suppose the theorem is true for k, we will prove it for $k+1$.

Let $A^0, A^1, \dots, A^{k+1}$ be $k+2$ convex sets in R^{n+k}, such that $\bigcap_0^{k+1} A^i = \emptyset$. Since $(\bigcap_0^k A^i) \cap A^{k+1} = \emptyset$, then there is a hyperplane Λ that separates $\bigcap_0^k A^i$ from A^{k+1}.

Suppose $\bigcap_0^k A^i \subset \Lambda^-$ and $A^{k+1} \subset \Lambda^+$, where Λ^+ and Λ^- are the closed half-spaces determined by Λ. Note that $\bigcap_0^k (A^i \cap \Lambda^+) = \emptyset$.

By the induction hypothesis, there is a k-dimensional linear subspace H_0 such that $\bigcap_0^k \pi_{H_0}(A^i \cap \Lambda^+) = \emptyset$. Let H be a $(k+1)$-dimensional linear subspace containing H_0 and the one-dimensional linear subspace orthogonal to Λ. We shall prove that

$$\bigcap_0^{k+1} \pi_H(A^i) = \emptyset.$$

Assume the opposite and take $x \in \bigcap_0^{k+1} \pi_H(A^i)$. Since $x \in \pi_H(A^{k+1}) \subset \pi_H(\Lambda^+)$, then $x \in \pi_H(A^i \cap \Lambda^+)$, for $i = 0, \dots, k$, which is a contradiction because $\bigcap_0^k \pi_H(A^i \cap \Lambda^+) \neq \emptyset$ implies $\bigcap_0^k \pi_{H_0}(\pi_H(A^i \cap \Lambda^+)) = \bigcap_0^k \pi_{H_0}(A^i \cap \Lambda^+) \neq \emptyset$. □

Our first result characterizes families of convex sets with the $(k+1)$-intersection property.

Theorem 3.2. *Let $F = \{A^1, \dots, A^d\}$ be a family of d convex sets in R^{n+k}, $d \geq k+1$. Every subfamily of F with $k+1$ members has a common point if and only if*

$$\mu(T_n(F)) \geq k.$$

Proof. Suppose every subfamily of F with $k+1$ members has a common point. We start by constructing a continuous map $\psi: G(n+k, n) \to T_n(F)$ as follows: for every n-plane H through the origin, let $\pi_H: R^{n+k} \to H^\perp$ be the orthogonal projection, where $H^\perp$ is the k-plane through the origin orthogonal to H. We consider the family $\pi_H(F) = \{\pi_H(A^1), \dots, \pi_H(A^d)\}$ of d convex sets in $H^\perp$. Note that every subfamily of $\pi_H(F)$ with $k+1$ members has a common point. Therefore, by Helly's theorem, the convex set $F(H) = \bigcap_1^d \pi_H(A^i)$ is not empty. Note also that $F(H) \subset H^\perp$ depends continuously on $H \in G(n+k, n)$. Let $\psi(H)$ be the n-plane through the center of mass of $F(H)$ and orthogonal to $H^\perp$. By construction, $\psi(H) \in T_n(F)$.

Let $i: T_n(F) \to G(n+k+1, n+1)$ and note that $i\psi: G(n+k, n) \to G(n+k+1, n+1)$ is homotopic to the inclusion. Therefore, by (2.1) and (2.3), $[0, k, \dots, k]$ is not zero in $T_n(F)$ and hence $\mu(T_n(F)) \geq k$.

Suppose now $\mu(T_n(F)) \geq k$ and suppose that $\bigcap_1^{k+1} A^i = \emptyset$. By Lemma 3.1, there is a k-dimensional linear subspace H of R^{n+k} with the property that $\bigcap_1^{k+1} \pi_H(A^i) = \emptyset$, where $\pi_H: R^{n+k} \to H$ is the orthogonal projection. This is a contradiction because, by Lemma 2.1, there is $\Gamma \in T_n(F)$ such that $\pi_H(\Gamma)$ is a single point which lies in $\bigcap_1^d \pi_H(A^i)$. This completes the proof of Theorem 3.2. □

Example 3.1. For $k = 1$ and $n = 2$, Theorem 3.2 states that every two members of F have a common point if and only if for every direction there is a transversal plane to F orthogonal to it.

Our next result characterizes families of $k+2$ convex sets with the $(k+1)$-intersection property. Note that this time our transversals need not be of dimension k.

Theorem 3.3. *Let $F = \{A^1, \ldots, A^{k+2}\}$ be a family of $k+2$ convex sets in R^{n+k} and consider an integer $1 \leq m \leq n$. Every subfamily of F with $k+1$ members has a common point if and only if*

$$\mu(T_m(F)) \geq k.$$

Proof. Suppose every subfamily of F with $k+1$ members has a common point. For $i = 1, \ldots, k+2$, let $a_i \in \bigcap_{j \neq i}\{A^j \in F\} \neq \emptyset$ and let Γ be an $(m+k)$-plane containing $\Theta = \{a_1, \ldots, a_{k+2}\}$. Furthermore, for $i = 1, \ldots, k+2$, let $B^i \subset \Gamma$ be the convex hull of the set $\{a_j \in \Theta \mid i \neq j\}$. Therefore, $F' = \{B^1, \ldots, B^{k+2}\}$ is a family of convex sets in the $(m+k)$-plane Γ with the property that $T_m(F') \subset T_m(F)$ because, for $i = 1, \ldots, k+2$, $B^i \subset A^i$. By Theorem 3.2, for $n = m$, $\mu(T_m(F')) \geq k$, which immediately implies that $\mu(T_m(F)) \geq k$.

Suppose now $\mu(T_m(F)) \geq k$ and suppose $\bigcap_1^{k+1} A^i = \emptyset$. By Lemma 3.1, there is a k-dimensional linear subspace H of R^{n+k} with the property that $\bigcap_1^{k+1} \pi_H(A^i) = \emptyset$, where $\pi_H\colon R^{n+k} \to H$ is the orthogonal projection. Note now that $T_m(F) \subset M(m+(n-m+k), m)$ is a collection of m-planes in $R^{m+(n-m+k)}$ with the property that $\mu(T_m(F)) \geq k$, and H is a k-plane, $1 \leq k \leq n-m+k$. By Lemma 2.1, there is $\Gamma \in T_m(F)$ such that $\pi_H(\Gamma)$ is a single point which lies in $\bigcap_1^{k+1} \pi_H(A^i)$. This is a contradiction. □

Example 3.2. For $k = 1$ and $m = 1$, Theorem 3.3 states that three convex sets have the property that every two of them have a common point if and only if there is a cycle of transversal lines to them.

We conclude with our main result, whose proof follows immediately from Theorems 3.2 and 3.3.

Theorem 3.4. *Let $F = \{A^1, \ldots, A^d\}$ be a family of d convex sets in R^{n+k}, $d \geq k+2$, and consider an integer $1 \leq m \leq n$. Every subfamily F' of F with $k+2$ members has the property that $\mu(T_m(F')) \geq k$ if and only if $\mu(T_n(F)) \geq k$.*

Example 3.3. Following Horn and Klee's spirit, for $k = 1$, $n = 2$ and $m = 1$, Theorem 3.4 states that every three convex sets of F have a cycle of transversal lines if and only if F has transversal planes orthogonal to every direction.

Example 3.4. For $m = n$, Theorem 3.4 states that if for every subfamily F' of F with $k+2$ members and for every linear n-subspace of R^{n+k} there is a translate which is an n-transversal to F', then every linear n-subspace of R^{n+k} admits a translate which is an n-transversal to F.

Example 3.5. Let $F = \{A^1, \ldots, A^d\}$ be a family of convex sets in R^{n+k}. According to [1], F has a *virtual n-point* if there are (homologically) as many n-transversals to F as if F had a common point, that is, as many n-transversals as there are n-planes through the origin in R^{n+k}. More precisely, F has a *virtual n-point* if and only if $\mu(T_n(F)) \geq k$. For $m = n$, Theorem 3.4 states that every subfamily F' of F with $k+2$ members has a virtual n-point if and only if F has a virtual n-point.

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